Integral Cohomology of 2-Local Hopf Spaces with at Most Two Non-Trivial Finite Homotopy Groups

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Abstract. In this paper we prove that a non-contractible simply-connected 2-local $H$-space $X$ with at most two non-trivial finite homotopy groups has no homology exponent, i.e. no exponent $e \geq 1$ such that $e \cdot \tilde{H}^*(X; \mathbb{Z}) = 0$.

1. Introduction

Let $X$ be a connected topological space having the homotopy type of a CW-complex. One can both consider its graded homotopy group $\pi_*(X)$ and its graded reduced integral cohomology group $\tilde{H}^*(X; \mathbb{Z})$. If an integer $h \geq 1$ such that $h \cdot \pi_*(X) = 0$ exists, then we say that $X$ has a homotopy exponent. Analogously, if an integer $e \geq 1$ such that $e \cdot \tilde{H}^*(X; \mathbb{Z}) = 0$ exists, then we say that $X$ admits a homology exponent.

A general question asked by D. Arlettaz suggests to explore the relationships between homotopy exponents and homology exponents. For instance, is it true that a space with a homotopy exponent has a homology exponent, too? In this case, how are these two exponents related? Or conversely, is it possible for a space without a homotopy exponent to admit a homology exponent?

In this paper, we focus on simply-connected 2-local $H$-spaces with one or two non-trivial finite homotopy groups. Such a space obviously admits a homotopy exponent. Our main result is the following:

**Main Theorem.** Let $X$ be a non-contractible simply-connected 2-local $H$-space with at most two non-trivial finite homotopy groups. Then $X$ has no homology exponent.

The result when $X$ is an Eilenberg-Mac Lane space is a well-known consequence of the calculations of H. Cartan [2], see Corollary 2.5. More elaborated techniques are required to prove the result when $X$ has two non-trivial homotopy groups. Section 2 establishes some preparations and investigates the situation of Eilenberg-Mac Lane spaces in detail. Some interesting examples are completely carried out in Section 3 and a proof of the main theorem is given in Section 4. We conclude the paper with some questions and comments in Section 5.
Unless otherwise specified, a space will mean a pointed, connected and simple topological space with the homotopy type of a CW-complex of finite type. We will denote by \( K(G, n) \) the Eilenberg-Mac Lane space with a single non-trivial homotopy group isomorphic to \( G \) in dimension \( n \) (\( G \) abelian if \( n \geq 2 \)).

Since we will only consider simple spaces \( X \), we will only deal with abelian fundamental groups and it will always be possible to consider the Postnikov tower built up from the Postnikov sections of \( X \) which will be denoted by \( \alpha_n : X \to X[n] \) and the \( k \)-invariants \( k^{a+1}(X) \in H^{a+1}(X[n-1]; \pi_n X) \cong [X[n-1], K(\pi_n X, n+1)] \).

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2. Transverse implications

In this introductory section we collect some well-known results that we need later in the paper, in particular results regarding the cohomology of 2-local Eilenberg-Mac Lane spaces.

A non-empty finite sequence of positive integers \( I = (a_0, \ldots, a_k) \), where \( k \) is varying, is admissible if \( a_i \geq 2a_{i+1} \) for all \( 0 \leq i \leq k - 1 \). Let \( S \) be the set of all such admissible sequences. The stable degree is a map \( \deg_{st} : S \to \mathbb{N} \) defined by \( \deg_{st}(I) = \sum_{i=0}^{k} a_i \) for all \( I = (a_0, \ldots, a_k) \in S \). The stable degree induces a grading on the set \( S \) of all admissible sequences. The excess is a map \( e : S \to \mathbb{N} \) defined by \( e(I) = 2a_0 - \deg_{st}(I) = a_0 - \sum_{i=1}^{k} a_i \) for all \( I = (a_0, \ldots, a_k) \in S \).

Let \( n \geq 1 \) and \( s \geq 1 \). Let \( \delta_i \) the connecting homomorphism in the long exact sequence of coefficients in cohomology associated to the short exact sequence

\[
\begin{array}{cccccc}
0 & \to & \mathbb{Z}/2 & \to & \mathbb{Z}/2^{n+1} & \to & \mathbb{Z}/2^s & \to & 0.
\end{array}
\]

Consider the fundamental class \( \iota_n \in H^n(K(\mathbb{Z}/2^s, n); \mathbb{Z}/2^s) \) and its mod-2 reduction \( u_n \in H^n(K(\mathbb{Z}/2^s, n); \mathbb{F}_2) \).

Convention. Let \( I = (a_0, \ldots, a_k) \) be an admissible sequence. We will write \( Sq^I u_n \) instead of \( Sq^{a_0}\cdots Sq^{a_{k-1}}\delta_{\iota_n} \) (usually denoted by \( Sq^{a_0}\cdots\delta_{\iota_n} \)) if \( a_k = 1 \) and instead of \( Sq^{a_0}\cdots Sq^{a_k} u_n \) (also denoted by \( Sq^{a_0}\cdots a_k u_n \) or \( Sq^I u_n \)) if \( a_k \neq 1 \). In particular, since \( \delta_1 = Sq^1 \) and the reduction is the identity when \( s = 1 \), we have \( Sq^I u_n = Sq^I u_n \).

J.-P. Serre [11] computed the mod-2 cohomology of Eilenberg-Mac Lane spaces and stated the following result:

**Theorem 2.1.** Let \( n \geq 1 \) and \( s \geq 1 \). The graded \( \mathbb{F}_2 \)-algebra \( H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2) \) is isomorphic to the graded polynomial \( \mathbb{F}_2 \)-algebra on generators \( Sq^I u_n \), where \( I \) covers all the admissible sequences of excess \( e(I) < n \) and \( u_n \) is the reduction of the fundamental class (see the above convention). The degree of a generator \( Sq^I u_n \) is \( \deg(Sq^I u_n) = \deg_{st}(I) + n \).

This result also reveals the \( \mathcal{A}_2 \)-algebra structure of \( H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2) \), where \( \mathcal{A}_2 \) denotes the mod-2 Steenrod algebra.
It is well known that an Eilenberg-Mac Lane space associated with an abelian group has a unique $H$-space structure up to homotopy (which can be seen as inherited from the loop space structure or from the addition law of the associated abelian group). Therefore, the differential graded $A_2$-algebra $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is also a differential graded Hopf algebra. If $H$ is a graded Hopf algebra over the field $\mathbb{F}_2$, with multiplication $\mu : H \otimes H \to H$, comultiplication $\Delta : H \to H \otimes H$, augmentation $\epsilon : H \to \mathbb{F}_2$ and unit $\eta : \mathbb{F}_2 \to H$ ($\mathbb{F}_2$ is concentrated in degree zero, see [9] for the definitions), the augmentation ideal of $H$ will be denoted by

$$\bar{H} = \ker \epsilon : H \to \mathbb{F}_2,$$

the graded module of indecomposable elements of $H$ by

$$QH = \bar{H}/\mu(\bar{H} \otimes \bar{H}) = \ker \mu : H \to \bar{H} \otimes \bar{H}$$

and the graded module of primitive elements of $H$ by

$$PH = \{x \in \bar{H} | \Delta(x) = x \otimes 1 + 1 \otimes x\} = \ker \Delta : \bar{H} \to \bar{H} \otimes \bar{H}.$$

The Milnor-Moore theorem states that there is an exact sequence of graded modules

$$0 \to P(\xi H) \to PH \to QH \to 0,$$

where $\xi H$ is the image of the Frobenius map $\xi : x \mapsto x^2$. The Hopf algebra $H$ is said to be primitively generated if $PH \to QH$ in the above exact sequence is an epimorphism.

J.-P. Serre also proved the following key result in [11]:

**Theorem 2.2.** The differential graded $A_2$-algebra $H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$ is a connected, associative, commutative and primitively generated differential graded Hopf algebra for any integer $n \geq 1$ and any integer $s \geq 1$.

It is now easy to determine the modules of primitives and indecomposables of $H^* = H^*(K(\mathbb{Z}/2^s, n); \mathbb{F}_2)$. The module of indecomposable elements is clearly given by

$$QH^* \cong \mathbb{F}_2\{Sq^I u_n | I \text{ admissible and } e(I) < n\}.$$  

Since $H^*$ is primitively generated, the Milnor-Moore theorem gives the following short exact sequence of graded $\mathbb{F}_2$-vector spaces:

$$0 \to P(\xi H^*) \to PH^* \to QH^* \to 0.$$

Therefore, every indecomposable element is primitive and every primitive element which is decomposable is a square of a primitive element. Thus we have

$$PH^* \cong \mathbb{F}_2\{(Sq^I u_n)^2 | I \text{ admissible, } e(I) < n \text{ and } i \geq 0\}.$$  

Now let us present some concepts and results on the high torsion in the integral cohomology of Eilenberg-Mac Lane spaces associated with 2-torsion groups of finite type. The material exposed here can be found with all the details in my thesis work [3]. It is mainly inspired by the work of H. Cartan in [2].

Let us start with the following key definition:
Definition 2.3. Let $X$ be a space and $\{B^r_n, d_r\}$ be its mod-2 cohomology Bockstein spectral sequence $B^r_1 \cong H^*(X; \mathbb{F}_2) \Rightarrow (H^*(X; \mathbb{Z})/\text{torsion}) \otimes \mathbb{F}_2$. Let $n$ and $r$ be two positive integers. An element $x \in B^r_n$ is said to be $\ell$-transverse if $d_r^l x^2 \neq 0 \in B^{2n}_{r+l}$ for all $0 \leq l \leq \ell$. An element $x \in B^r_n$ is said to be $\infty$-transverse, or simply transverse, if it is $\ell$-transverse for all $\ell \geq 0$. We will also speak of transverse implications of an element $x \in B^r_n$.

For instance, suppose that $x \in B^2_1$ is $\infty$-transverse and let us picture how the transverse implications of $x$ look like within the Bockstein spectral sequence:

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<table>
<thead>
<tr>
<th>$B^3_3$</th>
<th>$d_5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>\ldots \ldots \ldots</td>
<td>$x^4$</td>
</tr>
<tr>
<td>$B^2_2$</td>
<td>$d_4$</td>
</tr>
<tr>
<td>\ldots \ldots \ldots</td>
<td>$x^2$</td>
</tr>
<tr>
<td>$B^1_1$</td>
<td>$d_1$</td>
</tr>
<tr>
<td>\ldots \ldots \ldots</td>
<td>$x$</td>
</tr>
</tbody>
</table>

* 0 1 2 3 4 5 6 7 8 9 10 11 \ldots
```

Every transverse element gives rise to 2-torsion of arbitrarily high order in the integral cohomology of $X$. Actually, our strategy for disproving the existence of a homology exponent for a space will consist in exhibiting a transverse element in its mod-2 cohomology Bockstein spectral sequence.

In the special case of Eilenberg-Mac Lane spaces, we have the following result:

**Proposition 2.4.** Let $G$ be a non-trivial 2-local abelian group of finite type isomorphic to $\mathbb{Z}^\times \times \mathbb{Z}/2^{s_1} \oplus \cdots \oplus \mathbb{Z}/2^{s_l}$ and let $n \geq 2$. Consider the Eilenberg-Mac Lane space $K(G, n)$ and its mod-2 cohomology Bockstein spectral sequence $\{B^r_n, d_r\}$. Suppose that one of the following assumptions holds:

- $n$ is even and $x \in B^n_{s_j}$ is 0-transverse for any $1 \leq j \leq l$,
- $x \in P$ even $B^r_1$ is 0-transverse ($Sq^1 x \neq 0$).

Then $x$ is $\infty$-transverse.

A proof is given in [3, Theorem 1.3.2].

An algorithm explicitly computing the integral cohomology groups of such $K(G, n)$ spaces is implemented within a C++ program in [4]: the Eilenberg-Mac Lane machine. For instance, the machine produces a table for $K(\mathbb{Z}/2, 2)$ whose part in low degrees is:
The elements of order 2, 4 and 8 in degrees 3, 5 and 9 respectively are given by an \( \alpha \)-transverse element: the characteristic class \( u_2 \in H^2(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \) – which is of even degree and \( 0 \)-transverse – and its iterated squares \( u_2^n \).

Let us consider now 2-local spaces with two non-trivial finite homotopy groups.

**Corollary 2.5.** Let \( G \) be a non-trivial finite 2-torsion abelian group and let \( n \geq 2 \). The Eilenberg-Mac Lane space \( K(G, n) \) has no homology exponent.

**Proof.** According to the Künneth formula, it is sufficient to establish the result when \( G = \mathbb{Z}/2^s \) for some \( s \geq 1 \). If \( n \) is even, consider the reduction of the fundamental class \( u_n \in H^n(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \). This class survives to \( B_n^s \) and is \( 0 \)-transverse. Then \( u_n \in B_n^s \) is \( \alpha \)-transverse. If \( n \) is odd, consider the admissible sequence \((2, 1)\). Its excess is exactly 1 and therefore \( S_q^{2-1}u_n \in P_{\text{even}}H^*(K(\mathbb{Z}/2, 2); \mathbb{F}_2) \) when \( n \geq 3 \). Moreover we have \( S_q^{2-1}u_n = S_u^{3-1}u_n \) by Adem relations, which means that \( S_q^{2-1}u_n \) is \( 0 \)-transverse. Hence \( S_q^{2-1}u_n \in B_1^{n+3} \) is \( \alpha \)-transverse. \( \square \)

Let us conclude this section by proving the following crucial observation which states that the \( \alpha \)-transverse implications of an element in the cohomology of the total space of a fibration can be read in the cohomology of the fibre.

**Lemma 2.6.** Let \( j : F \to X \) be a continuous map. If \( x \in H^*(X; \mathbb{F}_2) \) is such that \( j^*(x) \neq 0 \in H^*(F; \mathbb{F}_2) \) is \( \alpha \)-transverse, then \( x \) itself is \( \alpha \)-transverse.

**Proof.** Suppose that \( x \) is not \( \alpha \)-transverse. Then there exists \( r \geq 0 \) such that \( d_r x_n^2 = 0 \). Therefore we have \( d_x x_n^2 = d_r x_n^2 j^*(x_n^2) = j^*d_r x_n^2 = 0 \), since \( d_r x_n^2 = 0 \), which contradicts \( \alpha \)-transversity of \( j^*(x) \). \( \square \)

### 3. Examples

Let us consider now 2-local spaces with two non-trivial finite homotopy groups. Recall that we want to prove that they do not have a homology exponent. This section will be devoted to two interesting examples.
Example 3.1. If such a space $X$ retracts onto an Eilenberg-MacLane space, then it is easy to deduce that $X$ has no homology exponent. This is for instance the case for the space $X$ given by the fibration

$$X \xrightarrow{i} K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 2) \xrightarrow{k} K(\mathbb{Z}/2, 4),$$

where its single non-trivial $k$-invariant

$$k \in [K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 2), K(\mathbb{Z}/2, 4)]$$

$$\cong H^4(K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 2); F_2)$$

$$\cong H^4(K(\mathbb{Z}/2, 2); F_2) \oplus H^2(\mathbb{Z}/2, 2); F_2)$$

$$\oplus F_2 \otimes H^4(K(\mathbb{Z}/2, 2); F_2)$$

$$\cong F_2 \{u_2^2 \oplus 1, u_2 \oplus v_2, 1 \oplus v_2^2\}$$

is given by $k = u_2 \oplus v_2$ where $u_2$ and $v_2$ are the fundamental classes of both copies of $K(\mathbb{Z}/2, 2)$. The space $X$ has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$ and $\pi_3(X) \cong \mathbb{Z}/2$.

**Proposition 3.2.** The space $X$ of Example 3.1 has the following properties:

1. $X$ is not a GEM (i.e. a weak product of Eilenberg-MacLane spaces),
2. $X$ is not a $H$-space,
3. $X$ retracts (weakly) onto the Eilenberg-MacLane space $K(\mathbb{Z}/2, 2)$, i.e. there exist maps $f : X \to K(\mathbb{Z}/2, 2)$ and $g : K(\mathbb{Z}/2, 2) \to X$ such that $fg \simeq \text{id}_{K(\mathbb{Z}/2, 2)}$,
4. $f^* : H^*(K(\mathbb{Z}/2, 2); F_2) \to H^*(X; F_2)$ is a monomorphism,
5. $X$ has no homology exponent.

**Proof.** The space $X$ is clearly not a GEM nor a $H$-space.

Consider the following homotopy commutative diagram based on the fibration for which $X$ is the fibre:

$$
\begin{array}{ccc}
K(\mathbb{Z}/2, 2) & \xrightarrow{i} & K(\mathbb{Z}/2, 2) \times K(\mathbb{Z}/2, 2) \xrightarrow{p_1} K(\mathbb{Z}/2, 2) \\
\uparrow{i_1} & & \downarrow{p_1} \\
K(\mathbb{Z}/2, 4) & \xrightarrow{k} & K(\mathbb{Z}/2, 4),
\end{array}
$$

where $i_1$ denotes the inclusion into the first factor, $p_1$ denotes the projection onto the first factor and $f = p_1i$. The existence of a (generally not unique) map $g$ is a consequence of the fact that $ki_1 \simeq *$. To see that $ki_1 \simeq *$, recall first that the isomorphism $[K(\mathbb{Z}/2, 2), K(\mathbb{Z}/2, 4)] \cong H^4(K(\mathbb{Z}/2, 2); F_2)$ maps $ki_1$ to $(ki_1)^*(u_4)$, where $(ki_1)^* = (i_1)^*k^* : H^4(K(\mathbb{Z}/2, 4); F_2) \to H^4(K(\mathbb{Z}/2, 2); F_2)$ and $u_4 \in H^4(K(\mathbb{Z}/2, 4); F_2)$ is the fundamental class. Now we have $(i_1)^*k^*(u_4) = (i_1)^*(u_2 \oplus v_2) = (i_1)^*(u_2 \oplus 1 \oplus v_2) = (i_1)^*(u_2 \oplus 1) \cdot (i_1)^*(1 \oplus v_2) = 0$, since $(i_1)^*(1 \oplus v_2) = 0$. Therefore $fg \simeq \text{id}$ and the result follows. $\square$
One can relate this space with other examples pointed out by F. R. Cohen and F. P. Peterson in [5]. They constructed loop maps $\Omega g : \Omega Y \to K(\mathbb{Z}/2, n)$, $n > 2$, with the property that $(\Omega g)^* : H^*(K(\mathbb{Z}/2, n); \mathbb{F}_2) \to H^*(\Omega Y; \mathbb{F}_2)$ is a monomorphism and such that $\Omega g$ does not admit a section.

Their examples are mainly provided by $\Omega \Sigma(\mathbb{R}P^\infty) \to K(\mathbb{Z}/2, n)$, the canonical multiplicative extension of Serre’s map $e : (\mathbb{R}P^\infty) \to K(\mathbb{Z}/2, n)$, and by $\Omega \Sigma BSO(3) \to K(\mathbb{Z}/2, 2)$, the canonical multiplicative extension of the second Stiefel-Whitney class in the mod-2 cohomology of $BSO(3)$ in the case $n = 2$.

The spaces $\Sigma(\mathbb{R}P^\infty)^n$ and $\Sigma BSO(3)$ have infinitely many non-trivial homotopy groups. Our space $X$ has only two. However, the loop maps $\Omega g : \Omega Y \to K(\mathbb{Z}/2, n)$ of F. R. Cohen and F. P. Peterson and our map $f : X \to K(\mathbb{Z}/2, 2)$ all induce monomorphisms.

EXAMPLE 3.3. Let us now consider a more interesting example of 2-local space with two non-trivial homotopy groups which does not admit a retract onto an Eilenberg-Mac Lane space. Therefore we will not be able to use the topological argument of the proof of Proposition 3.2 in order to prove the non-existence of a homology exponent. The main idea here is to detect $\infty$-transverse implications.

Let $X$ be the space given by the fibration

$$X \xrightarrow{k} K(\mathbb{Z}/2, 2) \xrightarrow{k} K(\mathbb{Z}/2, 4),$$

where its single non-trivial $k$-invariant

$$k \in [K(\mathbb{Z}/2, 2), K(\mathbb{Z}/2, 4)]$$

$$\cong H^4(K(\mathbb{Z}/2, 2); \mathbb{F}_2)$$

$$\cong \mathbb{F}_2\{u_2^2\}$$

is given by $k = u_2^2$ where $u_2$ is the fundamental class of $K(\mathbb{Z}/2, 2)$. The space $X$ has only two non-trivial homotopy groups $\pi_2(X) \cong \mathbb{Z}/2 \cong \pi_3(X)$.

PROPOSITION 3.4. The space $X$ of Example 3.3 has the following properties:

1. $X$ is not a GEM,
2. $X$ is an infinite loop space,
3. $X$ retracts neither onto the Eilenberg-Mac Lane space $K(\mathbb{Z}/2, 2)$, nor onto $K(\mathbb{Z}/2, 3)$,
4. However, $X$ has no homology exponent.

PROOF. The space $X$ is clearly not a GEM. It is an infinite loop space since its $k$-invariant $u_2^2$ is in the image of the $n$-fold cohomology suspension for all $n$.

In order to show that $X$ does retract neither onto $K(\mathbb{Z}/2, 2)$, nor onto $K(\mathbb{Z}/2, 3)$, let us consider the mod-2 cohomology Serre spectral sequence of the fibration.
We clearly have $Sq^1u_3$ implies that $H_4$ cohomology. Therefore we have $Sq^1g = 0$, which contradicts the fact that $k_i = *$.

Suppose now that there are maps $f : X \to K(\mathbb{Z}/2, 3)$ and $g : K(\mathbb{Z}/2, 2) \to X$ with $fg \simeq \text{id}_{K(\mathbb{Z}/2, 2)}$. The only non-trivial map $f : X \to K(\mathbb{Z}/2, 2)$ is given by the single non-trivial element $v \in H^2(X; \mathbb{F}_2)$. This forces $f \simeq i$. Therefore we have $kg \simeq k \text{id}_{K(\mathbb{Z}/2, 2)} \simeq k$ which contradicts the fact that $k_i \simeq *$.

Since $u_3$ transgresses to $u_2^3$, $Sq^1u_3$ transgresses to $Sq^1u_2^3$ which vanishes by Cartan’s formula. Therefore $Sq^1u_3 \neq 0 \in E_2^{0, 4}$ and there exists $x' \in H^4(X; \mathbb{F}_2)$ such that $x' \mapsto Sq^1u_3$ via the composition

$$j^* : H^4(X; \mathbb{F}_2) \xrightarrow{\sim} E_4^{0, 4} \oplus \cdots \oplus E_6^{0, 4} \cong H^4(K(\mathbb{Z}/2, 3); \mathbb{F}_2).$$

Set $x = Sq^2x'$. We have $j^*(x) = j^*(Sq^2x') = Sq^2j^*(x') = Sq^2u_3$ which is infinite-transverse. Thus $x$ is also infinite-transverse and $X$ cannot admit a homology exponent.

\[\square\]

4. Proof of the main result

This section is devoted to the proof of our main result. We begin to sketch a strategy allowing us to reach our aim.
Strategy. In Example 3.3, we found an element \( x \) in the mod-2 cohomology of \( X \) such that its image in the cohomology of the fibre \( j^*(x) = Sq^2 u_0 \) is non-trivial and has \( \infty \)-transverse implications; this implies for \( x \) itself to have \( \infty \)-transverse implications and prevents the existence of a homology exponent. Let us consider each of these stages in the general setting of a space \( X \) with two non-trivial homotopy groups:

(A) Every non-trivial element in \( E_2^{0,*} \) which survives to \( E_\infty^{0,*} \) gives rise to an element \( x \in H^*(X; \mathbb{F}_2) \) with a non-trivial image \( j^*(x) \) in the cohomology of the fibre. This leads us to find an element in \( E_2^{0,*} \) which transgresses to zero. The fact that transgressions in the spectral sequence commute with the Steenrod squares is very valuable for our purpose (J.-P. Serre remarked that the transgressions also commute with the connecting homomorphism \( \delta_3 \) introduced in Section 2, see [11, p. 206] and [10, p. 457]). For instance, the reduction of the characteristic class \( u_n \in E_2^{0,n} \) transgresses to a primitive element \( w \) determined by the \( k \)-invariant. Therefore, the first step will be done if we can find an admissible sequence \( I \) such that \( Sq^I u_n \neq 0 \) transgresses to \( Sq^I w = 0 \).

(B) Among all the admissible sequences \( I = (a_0, \ldots) \) of excess \( e(I) < n \) such that \( Sq^I u_n \neq 0 \) transgresses to zero, some of them are interesting because they insure on one hand that the element \( Sq^I u_n \) has a 0-transverse implication in the cohomology of the fibre and on the other hand that \( Sq^I u_n \) lies in even degree. We have seen in Proposition 2.4 that these two conditions force \( Sq^I u_n \) to have \( \infty \)-implications. It is immediate to see that such an admissible sequence “begins” with an even \( a_0 \) and has stable degree \( \deg_{st}(I) \equiv n \) (mod 2).

Let us look at the \( A_2 \)-action on the primitive elements of the cohomology of the base space. Following (A), our aim here is to find a suitable admissible sequence. Consider the following definition:

**Definition 4.1.** For all \( l \geq 1 \) define the admissible sequence

\[
\xi(l) = (2^l + 2^{l-1} - 1, \ldots, 5, 2)
\]

of stable degree \( \deg_{st}(\xi(l)) = 2^{l+1} + 2^l - 1 - 3 \) and excess \( e(\xi(l)) = l + 1 \).

**Proposition 4.2.** Let \( m \geq 2 \) and let \( G \) be a 2-torsion finite abelian group and consider the \( A_2 \)-module of primitives \( P^* H^*(K(G, m); \mathbb{F}_2) \). Let \( n \geq m + 1 \). Then we have

\[
\begin{align*}
\text{if } n \\ \text{then }
\begin{cases}
Sq^2 P^5 H^*(K(G, 2); \mathbb{F}_2) = 0. \\
Sq^3 P^6 H^*(K(G, 3); \mathbb{F}_2) = 0. \\
Sq^3 P^{n+1} H^*(K(G, m); \mathbb{F}_2) = 0 \quad \text{if } n = m + 1 \geq 4. \\
Sq^{2(n-3)} P^{n+1} H^*(K(G, m); \mathbb{F}_2) = 0 \quad \text{if } n \geq m + 2.
\end{cases}
\end{align*}
\]

**Proof.** Since \( G \) is a 2-torsion finite abelian group, we can write \( G \cong \oplus_n \mathbb{Z}/2^n \).

Suppose that \( n = m + 1 = 3 \). Let \( i_2 \in H^2(K(G, 2); G) \) be the characteristic class and consider the elements \( i_{2,a} \in H^2(K(G, 2); \mathbb{Z}/2^n) \) induced by the projections of \( G \) on each of the factors \( \mathbb{Z}/2^n \). Let \( x \in P^5 H^*(K(G, 2); \mathbb{F}_2) \). There exists an element \( y \in H^3(K(G, 2); \mathbb{F}_2) \) of the form \( y = \sum \delta_n i_{2,a} \) and such that \( x = Sq^2 y \). We have \( Sq^2 x = Sq^2 Sq^2 y = Sq^{h+1} y = \sum Sq^{s} \delta_n i_{2,a} = 0 \) since \( Sq^3 \delta_n = 0 \) for all \( s \).
Suppose that \( n = m + 1 = 4 \) and let \( x \in P^6 H^*(K(G, 3); \mathbb{F}_2) \). There exists \( y \in H^3(K(G, 3); \mathbb{F}_2) \) such that \( x = y^2 \). We then have \( Sq^3 x = Sq^3 y^2 = 0 \) by Cartan’s formula.

Suppose that \( n = m + 1 \geq 4 \) and let \( x \in P^{m+2} H^*(K(G, m); \mathbb{F}_2) \). There exists \( y \in H^m(K(G, m); \mathbb{F}_2) \) such that \( x = Sq^3 y \). We then have \( Sq^3 x = Sq^3 Sq^3 y = 0 \) by Adem relations.

Finally suppose that \( n \geq m + 2 \). For all \( m \geq 2 \) define the following subsets of the integers:

\[
M^m = \{ 2^i + 2^{i-1} | \text{ for all } i \geq m \} \quad \text{and} \quad N^m = \{ 1 + 2^{h_1} + \cdots + 2^{h_{m-1}} | h_1 \geq \ldots \geq h_{m-1} \geq 0 \}.
\]

It is a very simple arithmetic game to see that \( M^m \cap N^m = \emptyset \). Careful calculations show that for all admissible sequence \( I \), we have \( e(I) < m \) if and only if \( \text{deg}_I(I) + m \in N^m \). Thus there is no admissible sequence \( I \) of excess \( e(I) < m \) such that \( \text{deg}_I(I) + m = 2^m + 2^{m-1} \). Therefore

\[
Q^m 2^{2m+2} H^*(K(G, m); \mathbb{F}_2) = 0.
\]

Let \( x \in P^{n+1} H^*(K(G, m); \mathbb{F}_2) \). We have

\[
\text{deg}(Sq^{2n-3+2n-4} Sq^x (n-4)x) = \text{deg}(x) + \text{deg}_I(\xi(n-4)) + (2^{m-3} + 2^{m-4} - 2) = 2^{n-2} + 2^{n-3}.
\]

Since \( Q^{2n-2+2n-3} H^*(K(G, m); \mathbb{F}_2) = 0 \) when \( n \geq m + 2 \), the primitive element \( Sq^{2n-3+2n-4} Sq^x (n-4)x \) is then decomposable. Thus it is a square (maybe trivial) by Milnor-Moore. Therefore

\[
Sq^x (n-3)x = Sq^{2n-3+2n-4} Sq^x (n-4)x = \text{Sq} \text{sq} (\text{square}) = 0.
\]

We are now able to prove the main theorem.

**Theorem 4.3.** Let \( X \) be a non-contractible simply-connected 2-local \( H \)-space with at most two non-trivial finite homotopy groups. Then \( X \) has no homotopy exponent.

**Proof.** Since the case of an Eilenberg-Mac Lane space is clear, let us assume that \( X \) is a non-contractible simply-connected 2-local \( H \)-space with exactly two non-trivial homotopy groups \( \pi_m(X) \cong G \) and \( \pi_n(X) \cong H \), where \( n > m \geq 2 \) and \( G \cong \bigoplus_n \mathbb{Z}/2^n \), \( H \cong \bigoplus_h \mathbb{Z}/2^h \) are finite groups.

The space \( X \) fits into the fibrations

\[
K(H, n) \xrightarrow{j} X \xrightarrow{i} K(G, m) \xrightarrow{k} K(H, n + 1),
\]

where \( k \) is its single \( k \)-invariant. Since \( X \) is a \( H \)-space, \( k \) is a \( H \)-map.

Let \( i_m \in H^2(K(G, m); G) \) be the characteristic class and consider the elements \( t_{m, a} \in H^m(K(G, m); \mathbb{Z}/2^a) \) induced by the projections of \( G \) on each factor \( \mathbb{Z}/2^a \). Consider also all the \( u_{m, a} \in H^m(K(G, m); \mathbb{F}_2) \) given by the reduction mod 2 of the \( t_{m, a} \).
Analogously, let \( j_n \in H^n(K(H, n); H) \) be the characteristic class and consider the elements \( j_{n,b} \in H^n(K(H, n); \mathbb{Z}/2^b) \) induced by the projections of \( H \) on each factor \( \mathbb{Z}/2^b \). Consider also all the \( v_{n,b} \in H^n(K(H, n); F_2) \) given by the reduction mod 2 of the \( j_{n,b} \)'s. Moreover, pick \((t, v_n, j_n)\) among \( \{(t_b, v_{n,b}, j_{n,b}) \mid \text{for all } b\} \).

The \( E_2 \)-term of the Serre spectral sequence of the fibration

\[
K(H, n) \longrightarrow X \longrightarrow K(G, m)
\]

looks like:

\[
\begin{array}{cccccc}
\delta_{J_n,*} & 0 & \cdots & 0 & * & * \\
v_{n,*} & 0 & \cdots & 0 & * & * \\
0 & 0 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 \\
& 1 & 0 & \cdots & 0 & * & * \\
\end{array}
\]

The element \( v_n \) transgresses to \( d_{n+1}v_n \) in \( P^{n+1}H^*(K(G, m); F_2) \) which is determined by the \( k \)-invariant. Set

\[
\xi = \begin{cases} 
(2, 1) & \text{if } m = 2 \text{ and } n = 3, \\
(6, 3) & \text{if } m = 3 \text{ and } n = 4, \\
(6, 3) & \text{if } n = m + 1 \geq 5 \text{ and } n \text{ is odd}, \\
(14, 7, 3) & \text{if } n = m + 1 \geq 6 \text{ and } n \text{ is even}, \\
(2^{n-2} + 2^{n-3} - 2, \xi(n-3)) & \text{if } n \geq m + 2 \geq 4.
\end{cases}
\]

In all cases \( e(\xi) < n, \deg(Sq^\xi v_n) \) is even and \( Sq^\xi v_n \) transgresses to \( Sq^\xi d_{n+1}v_n \) which is trivial by Proposition 4.2. Let \( x \in H^*(X; F_2) \) such that \( j^*(x) = Sq^\xi v_n \).

The element \( Sq^\xi v_n \) is \( \infty \)-transverse and so is \( x \).

\[\square\]

5. Generalizations

We conclude this paper with some possible generalizations of our main result.

The first generalization in which we are interested concerns the nature of the two non-trivial homotopy groups of \( X \). We supposed them to be finite. Let us now suppose that the homotopy groups of \( X \) are of finite type. Copies of \( \mathbb{Z}(2) \), which have no torsion, may appear in this extended context.

The cohomology of Eilenberg-Mac Lane spaces associated to such groups was also computed by H. Cartan and J.-P. Serre:

\[
H^*(K(\mathbb{Z}(2), n); F_2) \cong F_2[Sq^I u_n \mid I = (a_0, ..., a_k) \text{ with } a_k \neq 1 \text{ and } e(I) < n].
\]

We say that a space \( X \) admits a torsion homology exponent if there exists an exponent for the torsion subgroup of \( H^*(X; \mathbb{Z}) \). With this definition, we have the following result on Eilenberg-Mac Lane spaces:

**Proposition 5.1.** Let \( G \) be a non-trivial \( 2 \)-torsion abelian group of finite type and \( n \geq 4 \). The Eilenberg-Mac Lane space \( K(G, n) \) has no torsion homology exponent.
Proof. By Künneth formula and Corollary 2.5, it is sufficient to suppose $G = \mathbb{Z}/(2)$. Consider the reduction of the fundamental class $u_n \in H^n(K(\mathbb{Z}/(2), n); \mathbb{F}_2)$. If $n$ is even, then $Sq^2 u_n$ is $\infty$-transverse. If $n$ is odd, then $Sq^6 u_n$ is $\infty$-transverse. □

It is not very difficult using results of H. Cartan in [2] to verify that $K(\mathbb{Z}/(2), 2)$ and $K(\mathbb{Z}/(2), 3)$ admit torsion homology exponents. So the result is the best possible in terms of connexity. The following result is an extension to spaces with at most two non-trivial homotopy groups of finite type:

**Theorem 5.2.** Let $X$ be a non-contractible 3-connected 2-local $H$-space of finite type with at most two non-trivial homotopy groups. Then $X$ has no torsion homology exponent.

The strategy and the admissible sequences $\xi$ listed in the proof of Theorem 4.3 are also suitable to prove Theorem 5.2.

W. Browder proved in [1, Theorem 6.11, p. 46] that every $H$-space of finite type which has the homotopy type of a finite CW-complex and which is simply-connected is actually 2-connected. Then, one may ask the following question.

**Question 5.3.** Let $X$ be a simply-connected 2-local $H$-space of finite type with a homology exponent. Is $X$ always 2-connected? If it is not the case for all such $H$-spaces, is it true for infinite loop spaces?

In [7] R. Levi studied the homotopy type of $p$-completed classifying spaces of the form $BG_p^\wedge$ for $G$ a finite $p$-perfect group, $p$ a prime. He constructed an algebraic analogue of Quillen’s “plus” construction for differential graded coalgebras. He then proved that the loop spaces $\Omega BG_p^\wedge$ admit integral homology exponents. More precisely, he proved that if $G$ is a finite $p$-perfect group of order $p^r \cdot m$, $m$ prime to $p$, then

$$p^r \cdot \tilde{H}_*(\Omega BG_p^\wedge; \mathbb{Z}(p)) = 0.$$  

Moreover, he proved that $BG_p^\wedge$ admits in general infinitely many non-trivial $k$-invariants, and thus in particular $\pi_* BG_p^\wedge$ is non-trivial in arbitrarily high dimensions. His method for proving this last result is based on a version of H. Miller’s theorem improved by J. Lannes and L. Schwartz [6]. This result and the results of the paper lead to the following conjecture:

**Conjecture 5.4.** Let $X$ be a connected space. If $X$ has a homology exponent, then either $X \simeq K(\pi_1(X), 1)$, or $X$ has infinitely many non-trivial $k$-invariants and, in particular, infinitely many non-trivial homotopy groups.

One can attack this conjecture by first looking at the following problem at the prime 2:

**Question 5.5.** Let $X$ be a 2-local space (of finite type) and $G$ a finite 2-torsion abelian group. If $X$ has a homology exponent, is the space map $\map_*(K(G, 2), X)$ weakly contractible?

To see that an affirmative answer to this question implies Conjecture 5.4 at the prime 2, suppose that $X$ is a 2-local space with a homology exponent and with finitely many non-trivial $k$-invariants. Then $X \simeq X[m] \times \text{GEM}$ for some integer $m$. 
Since $X$ admits a homology exponent it is a Postnikov piece. Consider then the Postnikov tower of the space $X \simeq X[m]$:  

$$
\begin{array}{c}
K(\pi_m X, m) \\
\downarrow \\
X[m-1] \\
\downarrow \\
\vdots \\
\downarrow \\
K(\pi_1 X, 1)
\end{array}
\quad
to
\begin{array}{c}
X[m] \\
\downarrow \\
X[m] \\
\downarrow \\
X[m] \\
\downarrow \\
K(\pi_{m+1} X, m+1)
\end{array}
$$

The map $j : K(\pi_m X, m) \to X[m]$ induces an isomorphism on the $m$-th homotopy groups. Therefore $\Omega^{m-2} j : K(\pi_m X, 2) \to \Omega^{m-2} X[m]$ and its adjoint map $\Omega^{m-2} K(\pi_m X, 2) \to X[m]$, which belongs to $\pi_{m-2}$ map$(K(\pi_m X, 2), X[m])$, are not nullhomotopic. This contradicts the fact that map$_* (K(\pi_m X, 2), X[m])$ is weakly contractible.

References


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