Finite K-theory spaces

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Abstract: This paper presents some new results on algebraic K-theory with finite coefficients. The argument is based on a topological construction of a space $F_mK(R)$, for any ring R and any integer $m \geq 2$, having the property that the ordinary homotopy theory of $F_mK(R)$ is isomorphic to the algebraic K-theory of R with coefficients in $\mathbb{Z}/m$: $\pi_n(F_mK(R)) \cong K_n(R;\mathbb{Z}/m)$ for $n \geq 1$. This space $F_mK(R)$ is called the mod m K-theory space of R. The paper is devoted to the investigation of several properties of the groups $K_n(R;\mathbb{Z}/m)$, for $n \in \mathbb{Z}$, and to some calculations of the integral homology of finite K-theory spaces.

1. Introduction

Algebraic K-theory of rings with finite coefficients was introduced by Browder in [6] and has been very useful. Browder used this technique in order to show that the group $K_3(\mathbb{Z}) \cong \mathbb{Z}/48$ occurs as a direct summand in $K_{8k+3}(\mathbb{Z})$, for all positive integers k (see [6, Theorem 4.8]). More recently, the complete calculation of the 2-torsion of $K_*(\mathbb{Z})$ by Rognes and Weibel (see [26] and [22]) was possible because of the application of Voevodsky's proof of the Milnor conjecture to the investigation of the Bloch-Lichtenbaum spectral sequence for computing $K_*(\mathbb{Q};\mathbb{Z}/2)$ and $K_*(\mathbb{Z};\mathbb{Z}/2)$. Several consequences of this important result are presented in [3, Section 9], in particular the complete description of the homotopy type of the K-theory space $BGL(\mathbb{Z})^+$ after completion at 2, the calculation of the 2-adic product structure of $K_*(\mathbb{Z})$ and of the module structure over the Steenrod algebra of the mod 2 cohomology of the infinite general linear group $GL(\mathbb{Z})$. Many other results in algebraic K-theory are based on arguments using K-theory with finite coefficients (see for instance Suslin's work on the K-theory of fields [23] and [24]). In [15], Karoubi and Lambre introduced the Hochschild homology with finite coefficients, constructed the Dennis trace map from Hochschild homology with finite coefficients and found an unexpected relationship with number theory. The mod m Tate-Farell-Vogel cohomology of groups has been introduced in [11] having applications to mod m algebraic K-theory.

The purpose of the present paper is to establish a couple of new important properties of the groups $K_n(R;\mathbb{Z}/m)$ for any ring R and any integer $m \geq 2$.

Our investigation of the algebraic K-theory of rings with finite coefficients is based on a topological construction of a space $F_m(X)$ associated with any loop space X and any positive integer m with the property that

$$\pi_n(F_m(X)) \cong \pi_{n+1}(X;\mathbb{Z}/m)$$

for all $n \geq 0$. This construction is presented in Section 2 and applied in Section 3 to the special case of the classifying space $BQP(R)$ of Quillen's Q-construction on the category $P(R)$ of finitely generated projective left R-modules, for any ring R. This provides the **mod m K-theory space** of the ring R, which will be denoted by $F_mK(R)$ and which satisfies

$$\pi_n(F_mK(R)) \cong \pi_{n+1}(BQP(R);\mathbb{Z}/m) \cong K_n(R;\mathbb{Z}/m)$$

for all $n \geq 1$.

The classical definition of the algebraic K-groups with coefficients in $\mathbb{Z}/m$ is a topological notion since $K_n(R;\mathbb{Z}/m)$ is defined as the n-th homotopy group of the loop space $\Omega BQP(R)$ with coefficients in $\mathbb{Z}/m$ (see [6] and [19] for the definition of homotopy groups with finite coefficients). On the other hand, Karoubi and Lambre have produced a purely algebraic definition of the group $K_1(R;\mathbb{Z}/m)$ (see [15]). We provide an algebraic definition of the group $K_2(R;\mathbb{Z}/m)$ and our argument enables us to show in Sections 4 and 5 that this algebraic definition coincides, for $n = 1$ and $n = 2$, with the usual topological definition (see Theorems 4.2 and 5.3). Using similar techniques, we introduce in Section 5 the notion of the Steinberg group $St(R;\mathbb{Z}/m)$ of a ring R with coefficients in $\mathbb{Z}/m$ which fits into the central extension

$$0 \longrightarrow K_2(R;\mathbb{Z}/m) \longrightarrow St(R;\mathbb{Z}/m) \longrightarrow E(R) \longrightarrow 1\,.$$

This Steinberg group $St(R;\mathbb{Z}/m)$ turns out to be quasi-perfect.

Section 6 extends the definition of $K_n(R;\mathbb{Z}/m)$ to the case of non positive integers n. Section 7 provides a proof of the following result (see Theorem 7.1).

The mod m Fundamental Theorem. *Let R be any unital ring and m any integer ≥ 2. For $n \geq 1$ there is a functorial isomorphism*

$$K_n(R[t,t^{-1}];\mathbb{Z}/m) \cong K_n(R;\mathbb{Z}/m) \oplus K_{n-1}(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m)\,,$$

where $NK_n(R;\mathbb{Z}/m)$ denotes the cokernel of the homomorphism $K_n(R;\mathbb{Z}/m) \to K_n(R[t];\mathbb{Z}/m)$ induced by the inclusion $R \hookrightarrow R[t]$.

In [25], Weibel proved that excision holds and Mayer-Vietoris sequences exist for mod m algebraic K-theory and $\mathbb{Z}[1/m]$-algebras. We will provide another form of the Mayer-Vietoris sequences for mod m algebraic K-theory in Section 8 (see Theorem 8.1 and Corollary 8.2).

Finally, we investigate in Section 9 homotopical properties of the connected mod m K-theory space $\overline{F_pK}(R)$ which is defined by $\overline{F_pK}(R) = F_p(BE(R)^+)$ in the special case where p is a prime number. Its homotopy groups are $\pi_n(\overline{F_mK(R)}) \cong K_{n+1}(R;\mathbb{Z}/m)$ for $n \geq 1$ and we approximate the order of its Postnikov k-invariants and prove that the $(2p-3)$-rd Postnikov section of $\overline{F_pK}(R)$ is a generalized Eilenberg-Maclane space. Consequently, we can calculate the integral homology of $\overline{F_pK}(R)$in dimensions $\leq 2p-3$ as follows (see Corollary 9.4).

Theorem. *For any unital ring R and any prime number p, one has*

$$H_i(\overline{F_pK}(R);\mathbb{Z}) \cong H_i\Big(\prod_{n=2}^{i+1} K(K_n(R;\mathbb{Z}/p), n-1);\mathbb{Z}\Big)$$

for $i \leq 2p-3$.

Throughout the paper, for an abelian group A and the multiplication by $m : A \to A$, we shall use the following notations: $A_{(m)} = ker\, m$ and $A/m = coker\, m$.

2. The homotopy fiber of the m-th power map

Let X be a loop space and m any integer ≥ 2. The goal of this first section is to investigate the main homotopical properties of the homotopy fiber $F_m(X)$ of the m-th power map $\chi^m : X \to X$. Our first result provides a strong relationship between the homotopy groups of $F_m(X)$ and the homotopy groups of X with coefficients in $\mathbb{Z}/m$. Recall that the homotopy groups of a space Y with coefficients in $\mathbb{Z}/m$ are defined by $\pi_n(Y;\mathbb{Z}/m) = [P^n(m), Y]$ for $n \geq 2$, where $P^n(m)$ denotes the Moore space $S^{n-1} \cup_m e^n$ which is the cofiber of the degree m map $S^{n-1} \to S^{n-1}$ (cf. [19]); recall that $\pi_n(Y;\mathbb{Z}/m)$ has a group structure whenever $n \geq 3$. If $X \simeq \Omega Y$ is a loop space, $\pi_2(X;\mathbb{Z}/m)$ is a group and if Y itself is a loop space, one can extend the definition to $n = 1$ by $\pi_1(X;\mathbb{Z}/m) = \pi_2(Y;\mathbb{Z}/m)$ which has a group structure.

Theorem 2.1. *Let $X \simeq \Omega Y$ be any connected loop space, m any integer ≥ 2 and $F_m(X)$ the homotopy fiber of the m-th power map $\chi^m : X \longrightarrow X$. Then there is an isomorphism*

$$\theta : \pi_n(F_m(X)) \xrightarrow{\cong} \pi_{n+1}(X;\mathbb{Z}/m)\,,$$

for all $n \geq 0$.

Proof. (We would like to thank Fred Cohen and Jérôme Scherer for discussions on that argument.) For any space Y, it is known that the pointed mapping space functor $map_*(-, Y)$ sends a pushout into a pullback and, since $map_*(*, Y) \simeq *$, sends a cofibration into a fibration (see for instance [5, p. 334, Proposition 4.1]). Therefore, if we apply this functor to the cofibration

$$S^1 \xrightarrow{\deg m} S^1 \xrightarrow{\psi} P^2(m)\,,$$

we obtain the fibration

$$map_*(P^2(m), Y) \longrightarrow map_*(S^1, Y) \longrightarrow map_*(S^1, Y)\,.$$

It is clear that $map_*(S^1, Y) \simeq \Omega Y$ and that the second map is the m-th power map $\chi^m : \Omega Y \to \Omega Y$ because it is induced by the degree m map $S^1 \to S^1$. Consequently, we have a homotopy equivalence

$$F_m(\Omega Y) \simeq map_*(P^2(m), Y)$$

and the assertion follows for any integer $n \geq 0$ from the isomorphism

$$\begin{aligned}\theta : \pi_n(F_m(\Omega Y)) &\cong \pi_n(map_*(P^2(m), Y)) \cong [S^n, map_*(P^2(m), Y)] \cong [\Sigma^n P^2(m), Y] \\ &\cong [P^{n+2}(m), Y] \cong \pi_{n+2}(Y;\mathbb{Z}/m) \cong \pi_{n+1}(\Omega Y;\mathbb{Z}/m) \cong \pi_{n+1}(X;\mathbb{Z}/m)\,.\end{aligned}$$

Remark 2.2. The corresponding statement is wrong for the *homology* of loop spaces; more precisely, the groups $H_n(F_m(X);\mathbb{Z})$ and $H_{n+1}(X;\mathbb{Z}/m)$ do not coincide in general. Consider for instance the special case where $m = p$ is a prime number and X is the Eilenberg-MacLane space $K(\mathbb{Z}, k)$ for some positive integer k: one has $F_p(K(\mathbb{Z}, k)) = K(\mathbb{Z}/p, k-1)$, but it turns out that $H_n(K(\mathbb{Z}/p, k-1);\mathbb{Z}) \ncong H_{n+1}(K(\mathbb{Z}, k);\mathbb{Z}/p)$, since it is well known that $H_n(K(\mathbb{Z}/p, k-1);\mathbb{Z})$ may contain elements of order p^r with r arbitrarily large (see [8] or [9] and [10]).

Of course, the homotopy exact sequence of the fibration $F_m(X) \longrightarrow X \xrightarrow{\chi^m} X$ and the definition of the homotopy groups with coefficients in $\mathbb{Z}/m$ (cf. [19, p. 3]) provide the exact sequences

$$\cdots \longrightarrow \pi_{n+1}(X) \xrightarrow{\cdot m} \pi_{n+1}(X) \longrightarrow \pi_n(F_m(X)) \longrightarrow \pi_n(X) \xrightarrow{\cdot m} \pi_n(X) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \pi_{n+1}(X) \xrightarrow{\cdot m} \pi_{n+1}(X) \longrightarrow \pi_{n+1}(X;\mathbb{Z}/m) \longrightarrow \pi_n(X) \xrightarrow{\cdot m} \pi_n(X) \longrightarrow \cdots\,.$$

The next proposition shows that the isomorphism θ provided by the previous theorem is actually compatible with these two exact sequences.

Proposition 2.3. *For any loop space X, any integer $m \geq 2$ and any positive integer n, the diagram*

$$\begin{array}{ccccccccc}
\pi_{n+1}(X) & \xrightarrow{\cdot m} & \pi_{n+1}(X) & \xrightarrow{\varphi_1} & \pi_n(F_m(X)) & \xrightarrow{\psi_1} & \pi_n(X) & \xrightarrow{\cdot m} & \pi_n(X) \\
\downarrow = & & \downarrow = & & \theta \downarrow \cong & & \downarrow = & & \downarrow = \\
\pi_{n+1}(X) & \xrightarrow{\cdot m} & \pi_{n+1}(X) & \xrightarrow{\varphi_2} & \pi_{n+1}(X;\mathbb{Z}/m) & \xrightarrow{\psi_2} & \pi_n(X) & \xrightarrow{\cdot m} & \pi_n(X)
\end{array}$$

is commutative.

Proof. We just have to establish the commutativity of the two middle squares in the diagram. The cofibration $S^1 \xrightarrow{\deg m} S^1 \xrightarrow{\psi} P^2(m)$ provides a cofibration

$$S^1 \xrightarrow{\psi} P^2(m) \xrightarrow{\varphi} S^2 .$$

By definition of the homotopy with coefficients, the homomrphisms φ_2 and ψ_2 are induced by the maps φ and ψ respectively. Thus, the commutativity follows from the fact that the homomorphisms φ_1 and ψ_1 are also induced by φ and ψ respectively. Indeed, φ produces a map

$$\varphi^\sharp : map_*(S^2, Y) \to map_*(P^2(m), Y) \simeq F_m(X)$$

which clearly induces the homomorphism $\varphi_1 : \pi_n(map_*(S^2, Y)) \cong \pi_{n+2}(Y) \cong \pi_{n+1}(X) \to \pi_n(F_m(X))$. Similarly, ψ provides a map

$$\psi^\sharp : F_m(X) \simeq map_*(P^2(m), Y) \to map_*(S^1, Y)$$

which induces the homomorphism ψ_1.

Remark 2.4. Let m be an integer ≥ 2, X a loop space and D a discrete space. It is then obvious that $F_m(X \times D) \simeq F_m(X)$.

Remark 2.5. Let m be an integer ≥ 2, X a loop space and $Z = \Omega X$ its loop space. It is easy to check that $F_m(Z) \simeq \Omega F_m(X)$.

3. Finite K-theory spaces

For any unital ring R, apply Quillen's Q-construction on the category $P(R)$ of finitely generated projective left R-modules and consider its classifying space $BQP(R)$ whose loop space satisfies:

$$\Omega BQP(R) \simeq BGL(R)^+ \times K_0(R) ,$$

where $BGL(R)^+$ is the space obtained by performing Quillen's plus construction on the classifying space of the infinite general linear group $GL(R)$. The higher algebraic K-groups of the ring R have been defined by Quillen in [21] as follows:

$$K_n(R) = \pi_{n+1}(BQP(R)) \cong \pi_n(\Omega BQP(R)) \quad \text{for} \quad n \geq 0 .$$

It turns out that $K_n(R) \cong \pi_n(BGL(R)^+)$ for $n \geq 1$. It is well known that the spaces $BQP(R)$ and $BGL(R)^+$ are infinite loop spaces. Browder investigated the algebraic K-groups with coefficients in $\mathbb{Z}/m$ for $n \geq 1$ (see [6]):

$$K_n(R;\mathbb{Z}/m) = \pi_{n+1}(BQP(R);\mathbb{Z}/m) \cong \pi_n(BGL(R)^+;\mathbb{Z}/m)\,.$$

Remark 3.1. The m-th power map $\chi^m : BQP(R) \to BQP(R)$ is actually induced by the map $P(R) \to P(R)$ which sends a projective module P to the direct sum of m copies of P, because the homotopy associative and commutative H-space structure of $BQP(R)$ is actually given by the map $BQP(R) \times BQP(R) \to BQP(R)$ induced by the map $P(R) \times P(R) \to P(R)$ which sends a pair of projective modules (P, P') to the direct sum $P \oplus P'$.

Definition 3.2. For any unital ring R and any integer $m \geq 2$, let us define the mod m K-theory space of R by

$$F_mK(R) = F_m(BQP(R))\,,$$

i.e., the homotopy fiber of the m-th power map $\chi^m : BQP(R) \longrightarrow BQP(R)$.

Theorem 2.1 immediately implies the following consequence.

Theorem 3.3. *For any unital ring R, any integer $m \geq 2$ and any integer $n \geq 1$, one has an isomorphism*

$$\pi_n(F_mK(R)) \cong K_n(R;\mathbb{Z}/m)\,.$$

Remark 3.4. For any unital ring R and any integer $m \geq 2$, the homotopy exact sequence of the fibration

$$F_mK(R) \longrightarrow BQP(R) \xrightarrow{\chi^m} BQP(R)$$

provides the long exact sequence of algebraic K-functors (see also Proposition 2.3)

$$\cdots \longrightarrow K_n(R) \xrightarrow{\cdot m} K_n(R) \longrightarrow K_n(R;\mathbb{Z}/m) \longrightarrow K_{n-1}(R) \xrightarrow{\cdot m} K_{n-1}(R) \longrightarrow K_{n-1}(R;\mathbb{Z}/m) \to \cdots$$
$$\longrightarrow K_2(R) \xrightarrow{\cdot m} K_2(R) \longrightarrow K_2(R;\mathbb{Z}/m) \longrightarrow K_1(R) \xrightarrow{\cdot m} K_1(R) \longrightarrow K_1(R;\mathbb{Z}/m) \longrightarrow K_0(R) \xrightarrow{\cdot m} K_0(R)\,.$$

Remark 3.5. One can also consider the space $F_m(BGL(R)^+)$ and observe that

$$F_m(BGL(R)^+) \simeq \Omega F_mK(R)$$

because Remarks 2.4 and 2.5 show that

$$F_m(BGL(R)^+) \simeq F_m(BGL(R)^+ \times K_0(R)) \simeq F_m(\Omega BQP(R)) \simeq \Omega F_m(BQP(R))\,.$$

Again, one can apply Theorem 2.1 and deduce that

$$\pi_n(F_m(BGL(R)^+)) \cong \pi_{n+1}(BGL(R)^+;\mathbb{Z}/m) \cong K_{n+1}(R;\mathbb{Z}/m)$$

for $n \geq 0$.

Notice that the spaces $F_mK(R)$ and $F_m(BGL(R)^+)$ are not necessarily connected. If one prefers to work with a connected space, one can look at the universal cover $BE(R)^+$ of $BGL(R)^+$, where $E(R)$ is the subgroup of $GL(R)$ generated by elementary matrices. The space $BE(R)^+$ is simply connected and its homotopy groups are $\pi_n(BE(R)^+) \cong K_n(R)$ for $n \geq 2$.

Definition 3.6. For any unital ring R and any integer $m \geq 2$, let us define the connected mod m K-theory space of R by

$$\overline{F_mK}(R) = F_m(BE(R)^+)\,,$$

i.e., the homotopy fiber of the m-th power map $\chi^m : BE(R)^+ \to BE(R)^+$. This space is a connected space whose homotopy groups are

$$\pi_n(\overline{F_m K(R)}) \cong K_{n+1}(R; \mathbb{Z}/m)$$

for $n \geq 1$ because of Theorem 2.1.

Proposition 3.7. *If $f : R \to S$ is a ring homomorphism, then f induces continuous maps $f_* : F_m K(R) \to F_m K(S)$ and $\overline{f}_* : \overline{F_m K}(R) \to \overline{F_m K}(S)$ for any integer $m \geq 2$.*

Proof. It is well known that the homomorphism $f : R \to S$ induces an infinite loop map $f_\sharp : BQP(R) \to BQP(S)$. Thus, we get the commutative diagram

$$\begin{array}{ccc} BQP(R) & \xrightarrow{\chi^m} & BQPR) \\ \downarrow f\sharp & & \downarrow f\sharp \\ BQP(S) & \xrightarrow{\chi^m} & BQP(S) \end{array}$$

which produces a map $f_* : F_m K(R) \to F_m K(S)$ on the homotopy fibers of the m-th power maps χ^m. The same argument provides the map $\overline{f}_*$.

The first example to consider is the case of the finite field $\mathbb{F}_q$ for a prime q.

Proposition 3.8. *For any prime number q, the* mod q *K-theory space $\overline{F_q K}(\mathbb{F}_q)$ is contractible.*

Proof. Since the K-groups of the finite field $\mathbb{F}_q$ do not contain any q-torsion (cf. [20]), one can then deduce from Definition 3.6 that all homotopy groups of $\overline{F_q K}(\mathbb{F}_q)$ vanish and that there is a homotopy equivalence $\overline{F_q K}(\mathbb{F}_q) \simeq *$.

4. Algebraic interpretation of $K_1(R; \mathbb{Z}/m)$

In [15], Karoubi and Lambre introduced algebraically a first mod m algebraic K-functor for any unital ring R and any integer $m \geq 2$, which we will denote by $K_1^{'}(R; \mathbb{Z}/m)$. Let us briefly recall the construction of this abelian group.

Let $m : P(R) \to P(R)$ be the m-th power functor which sends a projective module to a direct sum of m copies of it (cf. Remark 3.1). Then define the category $C(m)$ whose objects are triples (P, α, Q), where P and Q are objects of $P(R)$ and $\alpha : m(P) \cong m(Q)$ is an isomorphism of R-modules. A morphism $(P, \alpha, Q) \to (P', \alpha', Q')$ is a pair (f, g) of R-homomorphisms $f : P \to P'$ and $g : Q \to Q'$, such that $m(g)\alpha = \alpha' m(f)$. The sum in the category $C(m)$ is defined in a natural way and we get the abelian monoid of isomorphism classes of objects of $C(m)$. Let $K(C(m))$ be the Grothendieck group of $C(m)$ and N its subgroup generated by elements of the form

$$(P, \alpha, Q) + (Q, \beta, S) - (P, \beta\alpha, S) \,.$$

Definition 4.1. $K_1'(R; \mathbb{Z}/m) = K(C(m))/N$ for any unital ring R and any integer $m \geq 2$.

The goal of this section is to show that this algebraic definition coincides with the topological definition of the K_1-group with coefficients in $\mathbb{Z}/m$.

Theorem 4.2. *For any unital ring R and any integer $m \geq 2$, there is an isomorphism*

$$K_1'(R;\mathbb{Z}/m) \cong \pi_1(F_m K(R)) \cong K_1(R;\mathbb{Z}/m)\,.$$

Proof. Denote by $Q(m) : QP(R) \to QP(R)$ the functor induced by $m : P(R) \to P(R)$. Take the pullback $co(Q(m))$ of the following diagram of categories (look at [4, Chapter 7, Definition 3.1], for the definition of the pullback of categories and for the notation we use):

$$\begin{array}{ccc} & & QP(R) \\ & & \downarrow{\scriptstyle Q(m)} \\ QP(R) & \xrightarrow{Q(m)} & QP(R) \end{array} \tag{1}$$

We also need the pullback Y of the following diagram of spaces:

$$\begin{array}{ccc} & & BQP(R) \\ & & \downarrow{\scriptstyle BQ(m)} \\ BQP(R) & \xrightarrow{BQ(m)} & BQP(R) \end{array} \tag{2}$$

which is the geometric realization of Diagram (1). It is easily checked that the functor $Q(m)$ is an additive cofinal functor, where for unital rings A and B a functor $\varphi : QP(A) \to QP(B)$ is called cofinal if any object S of $QP(B)$ is a direct summand of $\varphi(P)$ for some object P of $QP(A)$. Since the isomorphisms in $QP(R)$ are the same as in $P(R)$, it follows from [4, Chapter 7, Theorem 5.3], that there is an exact sequence

$$K_1(R) \xrightarrow{\cdot m} K_1(R) \longrightarrow K_0'(Q(m)) \longrightarrow K_0(R) \xrightarrow{\cdot m} K_0(R)\,,$$

in which the group $K_0'(\varphi)$ defined by Bass in [4] for any additive cofinal functor φ between additive categories is in fact the group $K_1'(R;\mathbb{Z}/m)$ of Karoubi and Lambre, in the case where $\varphi = Q(m)$.

Diagrams (1) and (2) induce the commutative diagrams

$$\begin{array}{ccc} K(co(Q(m))) & \longrightarrow & K(QP(R)) \\ \downarrow & & \downarrow{\scriptstyle K(Q(m))} \\ K(QP(R)) & \xrightarrow{K(Q(m))} & K(QP(R)) \end{array} \tag{3}$$

and

$$
\begin{array}{ccc}
 & & \pi_1(F_m(BQP(R))) \\
 & & \downarrow \\
\pi_1(Y) & \longrightarrow & \pi_1(BQP(R)) \\
\downarrow & & \downarrow {\scriptstyle \pi_1(BQ(m))} \\
\pi_1(BQP(R)) & \xrightarrow{\pi_1(BQ(m))} & \pi_1(BQP(R))\,,
\end{array}
\tag{4}
$$

in which the right column is exact. Denote by

$$\Delta : K(QP(R)) \to K(co(Q(m)))$$

and

$$\Delta : \pi_1(BQP(R)) \to \pi_1(Y)$$

respectively the diagonal maps which are the split homomorphisms induced by the pullback diagrams (1) and (2). Diagrams (3) and (4) and the isomorphism $K(QP(R)) \cong \pi_1(BQP(R))$ provide a homomorphism $K(co(Q(m))) \to \pi_1(Y)$ and finally a homomorphism

$$\eta : K(co(Q(m)))/Im\Delta \longrightarrow \pi_1(Y)/Im\Delta\,.$$

Let M be the subgroup of $K(co(Q(m)))$ generated by the elements of the form

$$[(P, \alpha\alpha_1\alpha_2, Q)] + [(P, \alpha, Q)] - [(P, \alpha\alpha_1, Q)] - [(P, \alpha\alpha_2, Q)]\,,$$

where $(P, \alpha, Q) \in co(Q(m))$, i.e., $m(P) \cong m(Q)$, and $\alpha_1 \in Aut_R(m(P))$, $\alpha_2 \in Aut_R(m(Q))$. Then we have the equalities $K_0'(Q(m)) = K(co(Q(m)))/(M + Im\Delta)$ and $\eta(M) = 0$ according to [4, Chapter 7, Section 5]. Thus, we have constructed a homomorphism

$$\psi : K_1'(R; \mathbb{Z}/m) \cong K_0'(Q(m)) \longrightarrow \pi_1(F_m(BQP(R)))$$

induced by η by taking into account the obvious homomorphism $\pi_1(Y)/Im\Delta \to \pi_1(F_m(BQP(R)))$. This homomorphism ψ fits in the middle of the following commutative diagram with exact rows:

$$
\begin{array}{ccccccccc}
K_1(R) & \longrightarrow & K_1(R) & \longrightarrow & K_1'(R;\mathbb{Z}/m) & \longrightarrow & K_0(R) & \longrightarrow & K_0(R) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \psi & & \downarrow \cong & & \downarrow \cong \\
\pi_2(BQP(R)) & \longrightarrow & \pi_2(BQP(R)) & \longrightarrow & \pi_1(F_m(BQP(R))) & \longrightarrow & \pi_1(BQP(R)) & \longrightarrow & \pi_1(BQP(R))
\end{array}
$$

and the five lemma implies the assertion

$$K_1'(R; \mathbb{Z}/m) \cong \pi_1(F_m(BQP(R))) = \pi_1(F_m K(R)) \cong K_1(R; \mathbb{Z}/m)\,.$$

5. Algebraic interpretation of $K_2(R; \mathbb{Z}/m)$

Let us first define mod m algebraic K-functors for rings which are not necessarily unital (i.e., with identity).

Remark 5.1. Let R be an arbitrary ring. Consider the ring with identity $R^+ = R + \mathbb{Z}$, where the sum is as usual and the product is given by $(r,z)\cdot(r',z') = (rr' + rz' + zr', zz')$. One gets a short split exact sequence of rings

$$0 \longrightarrow R \xrightarrow{i} R^+ \xrightarrow{p} \mathbb{Z} \longrightarrow 0\,,$$

where i denotes the inclusion and p the projection. Then define the space $BQP(R)$ to be the homotopy fibre of the map $BQP(p) : BQP(R^+) \to BQP(\mathbb{Z})$. The continuous map $BQ(m) : BQP(R^+) \to BQP(R^+)$, given by the m-th power functor m, induces a continuous map $BQ(m) : BQP(R) \to BQP(R)$ and we denote again by $F_m(BQP(R))$ the homotopy fibre of this map $BQP(m)$. Now, one can define

$$K_n(R;\mathbb{Z}/m) = \pi_n(F_m(BQP(R)))$$

for any $n \geq 1$, any $m \geq 2$ and any ring R (not necessarily with identity). Moreover, the long exact sequence of algebraic K-functors given by Remark 3.4 holds for any arbitrary ring. If R is unital (with identity e), one has an isomorphism $\varphi : R^+ \xrightarrow{\cong} R \times \mathbb{Z}$ given by $\varphi(r,z) = (r + ze, z)$, and we recover Definition 3.2.

Now, let R be a unital ring and $\tau : F \longrightarrow R$ a free presentation of R. This means that F is a free ring and τ a surjective homomorphism of rings. Let us consider the short exact sequence of rings

$$0 \longrightarrow I \xrightarrow{\sigma} F \xrightarrow{\tau} R \longrightarrow 0\,,$$

where $I = ker\,\tau$ and where σ is the natural inclusion. According to Remark 3.4, we have a long exact sequence

$$\cdots \longrightarrow K_1(I) \xrightarrow{\cdot m} K_1(I) \longrightarrow K_1(I;\mathbb{Z}/m) \longrightarrow K_0(I) \xrightarrow{\cdot m} K_0(I)\,,$$

where $K_1(I;\mathbb{Z}/m)$ is defined by Remark 5.1.

Let us introduce the following algebraic definition of the K_2-functor with coefficients in $\mathbb{Z}/m$.

Definition 5.2. For any unital ring R and any integer $m \geq 2$, let $K_2'(R;\mathbb{Z}/m)$ be the pushout of the following diagram:

$$\begin{array}{ccc} K_1(I)/m & \longrightarrow & K_1'(I;\mathbb{Z}/m)\,, \\ \downarrow & & \\ K_2(R)/m & & \end{array}$$

where $K_1'(I;\mathbb{Z}/m) = ker\,(p_* : K_1'(R^+;\mathbb{Z}/m) \to K_1'(\mathbb{Z};\mathbb{Z}/m))$ and in which the homomorphism $K_1(I)/m \to K_2(R)/m$ is induced by the composition of the surjection $K_1(I) \to K_1(F,I)$ with the isomorphism $K_1(F,I) \cong K_2(R)$. By Theorem 4.2 it is clear that $K_1'(I;\mathbb{Z}/m)$ is naturally isomorphic to $K_1(I;\mathbb{Z}/m)$ and the construction of $K_2'(R;\mathbb{Z}/m)$ is purely algebraic.

Theorem 5.3. *For any unital ring R and any integer $m \geq 2$, there is an isomorphism*

$$K_2'(R;\mathbb{Z}/m) \cong \pi_2(F_m K(R)) \cong K_2(R;\mathbb{Z}/m)\,.$$

Proof. For any surjective homomorphism of rings $f : A \to B$, let us denote by $hBQP(f)$ the homotopy fibre of the map $BQP(f) : BQP(A) \to BQP(B)$ induced by f. It is known that the homotopy group $\pi_n(hBQP(f))$ is isomorphic to the relative algebraic K-functor $K_n(A,J)$ for $n \geq 1$, where $J = ker\,f$. The commutative diagram

$$\begin{array}{ccc} BQP(A) & \xrightarrow{BQP(f)} & BQP(B) \\ \downarrow{\scriptstyle BQ(m)} & & \downarrow{\scriptstyle BQ(m)} \\ BQP(A) & \xrightarrow{BQP(f)} & BQP(B) \end{array}$$

yields a continuous map $hBQP(f,m) : hBQP(f) \to hBQP(f)$ and let $F_m(hBQP(f))$ be the homotopy fibre of $hBQP(f,m)$. Define $K_n(A,J;\mathbb{Z}/m) = \pi_n(F_m(hBQP(f)))$ for $n \geq 1$. We obtain a long exact sequence of relative algebraic K-functors

$$\cdots \to K_2(A,J) \xrightarrow{\cdot m} K_2(A,J) \to K_2(A,J;\mathbb{Z}/m) \to K_1(A,J) \xrightarrow{\cdot m} K_1(A,J) \to K_1(A,J;\mathbb{Z}/m) \to K_0(J) \xrightarrow{\cdot m} K_0(J).$$

Let us return to the particular case of the free presentation $0 \longrightarrow I \xrightarrow{\sigma} F \xrightarrow{\tau} R \longrightarrow 0$ of the unital ring R and consider the commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & I^+ & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle =} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & F^+ & \longrightarrow & R^+ & \longrightarrow & 0\,, \end{array}$$

where the homomorphism $I^+ \to F^+$ is induced by the inclusion $\sigma : I \to F$, $F^+ \to R^+$ is induced by τ, and $\mathbb{Z} \to R^+$ is given by $z \mapsto (0,z)$. This diagram produces homomorphisms $\pi_n(BQP(I)) \to \pi_n(hBQP(\tau))$ and $\pi_n(F_m(BQP(I))) \to \pi_n(F_m(hBQP(\tau)))$ for $n \geq 1$, such that we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(I) & \xrightarrow{\cdot m} & K_1(I) & \longrightarrow & K_1'(I;\mathbb{Z}/m) & \longrightarrow & K_0(I) & \xrightarrow{\cdot m} & K_0(I) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle =} & & \downarrow{\scriptstyle =} \\ K_1(F,I) & \xrightarrow{\cdot m} & K_1(F,I) & \longrightarrow & K_1(F,I;\mathbb{Z}/m) & \longrightarrow & K_0(I) & \xrightarrow{\cdot m} & K_0(I)\,. \end{array}$$

Since $K_1(F,I) \cong K_2(R)$, the above diagram and Definition 5.2 provide a homomorphism $\eta : K_2'(R;\mathbb{Z}/m) \to K_1(F,I;\mathbb{Z}/m)$ and we obtain from the isomorphism $K_0(I) \cong K_1(R)$ the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_2(R) & \xrightarrow{\cdot m} & K_2(R) & \longrightarrow & K_2'(R;\mathbb{Z}/m) & \longrightarrow & K_1(R) & \xrightarrow{\cdot m} & K_1(R) \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ K_1(F,I) & \xrightarrow{\cdot m} & K_1(F,I) & \longrightarrow & K_1(F,I;\mathbb{Z}/m) & \longrightarrow & K_0(I) & \xrightarrow{\cdot m} & K_0(I)\,, \end{array}$$

which shows that η is an isomorphism. It remains to show that $K_1(F,I;\mathbb{Z}/m) \cong K_2(R;\mathbb{Z}/m)$.

It is easily checked that the homotopy fibre $hF_m(BQP(\tau))$ of the map $F_m(BQP(\tau)) : F_m(BQP(F)) \to F_m(BQP(R))$ is homeomorphic to the space $F_m(hBQP(\tau))$. Therefore, the long exact homotopy sequence of the homotopy fibration

$$hF_mBQP(\tau) \longrightarrow F_m(BQP(F)) \longrightarrow F_m(BQP(R))$$

yields the isomorphisms $K_{n+1}(R;\mathbb{Z}/m) \cong \pi_n(hF_mBQP(\tau)) \cong K_n(F,I;\mathbb{Z}/m)$ for $n \geq 1$ by taking into account that $\pi_n(F_m(BQP(F))) = 0$ for $n \geq 1$, since F is a free ring. This completes the proof.

Corollary 5.4. *The group $K_2'(R;\mathbb{Z}/m)$ does not depend on the free presentation of the ring R.*

The end of this section will be devoted to the definition of the Steinberg group modulo m. Let us start with the following topological definition.

Definition 5.5. For any unital ring R and any integer $m \geq 2$, let $St(R;\mathbb{Z}/m)$ be the pushout of the diagram

$$\begin{array}{ccc} K_2(R) & \longrightarrow & St(R) \\ \downarrow & & \\ \pi_2(F_mK(R)) \cong K_2(R;\mathbb{Z}/m) & & \end{array}$$

We get a short exact sequence

$$0 \longrightarrow K_2(R;\mathbb{Z}/m) \longrightarrow St(R;\mathbb{Z}/m) \longrightarrow E(R) \longrightarrow 1$$

which is a central extension of $E(R)$.

It is also possible to provide a purely algebraic definition of the Steinberg group $St(R;\mathbb{Z}/m)$ without the use of the group $K_2(R;\mathbb{Z}/m) \cong \pi_2(F_mK(R))$. For this, take the pushout $St_m(R)$ of the diagram

$$\begin{array}{ccc} K_2(R) & \longrightarrow & St(R) \\ \downarrow & & \\ K_2(R)/m & & \end{array}$$

One gets a short exact sequence

$$0 \longrightarrow K_2(R)/m \longrightarrow St_m(R) \longrightarrow E(R) \longrightarrow 1$$

which is a universal m-central extension of $E(R)$; that means that this extension is universal between all central extensions $\alpha : X \to E(R)$ of $E(R)$ such that $(ker\, \alpha)_{(m)} = 0$. Therefore $H_2(E(R);\mathbb{Z}/m) \cong K_2(R)/m$ (see [7]).

Definition 5.6. For any unital ring R and any integer $m \geq 2$, let us denote by $St'(R;\mathbb{Z}/m)$ the pushout of the diagram

$$\begin{array}{ccc} K_1(I)/m & \longrightarrow & K_1'(I;\mathbb{Z}/m) \\ \downarrow & & \\ St_m(R) & & \end{array}$$

where $0 \to I \xrightarrow{\sigma} F \xrightarrow{\tau} R \to 0$ is a free presentation of R and where the homomorphism $K_1(I)/m \to St_m(R)$ is the composition of the epimorphism $K_1(I)/m \to K_2(R)/m$ with the injection $K_2(R)/m \to St_m(R)$. Theorem 5.3 enables us to check easily the following assertion.

Theorem 5.7. *For any unital ring R and any integer $m \geq 2$, there is an isomorphism*

$$St'(R;\mathbb{Z}/m) \cong St(R;\mathbb{Z}/m)\,.$$

It follows that the group $St'(R;\mathbb{Z}/m)$ does not depend on the free presentation of the ring R.

The above given pushouts induce an injection $\alpha : St_m(R) \to St(R;\mathbb{Z}/m)$ such that there is an exact sequence

$$0 \longrightarrow St_m(R) \xrightarrow{\alpha} St(R;\mathbb{Z}/m) \longrightarrow (K_1(R))_{(m)} \longrightarrow 0\,.$$

Remark 5.8. The Steinberg group $St(R;\mathbb{Z}/m)$ is a quasi-perfect group, which means that its commutator subgroup is perfect. In order to prove this assertion, consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_2(R) & \longrightarrow & St(R) & \longrightarrow & E(R) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & K_2(R)/m & \longrightarrow & St_m(R) & \longrightarrow & E(R) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle =} & & \\
0 & \longrightarrow & K_2(R;\mathbb{Z}/m) & \longrightarrow & St(R;\mathbb{Z}/m) & \longrightarrow & E(R) & \longrightarrow & 0\,,
\end{array}$$

which induces the exact sequence

$$0 \longrightarrow K_2(R)/m \longrightarrow K_2(R;\mathbb{Z}/m) \times St_m(R) \xrightarrow{\beta} St(R;\mathbb{Z}/m) \longrightarrow 0\,.$$

Because α is injective and $St_m(R)$ is a perfect group, the restriction of α to the commutator subgroups

$$\alpha_\sharp : St_m(R) \longrightarrow [St(R;\mathbb{Z}/m), St(R;\mathbb{Z}/m)]$$

is injective. In order to show that $\alpha_\sharp$ is also surjective, take an element

$$xyx^{-1}y^{-1} \in [St(R;\mathbb{Z}/m), St(R;\mathbb{Z}/m)]\,,$$

choose (a,u) and (b,v) in $K_2(R;\mathbb{Z}/m) \times St_m(R)$ such that $\beta(a,u) = x$ and $\beta(b,v) = y$, and observe that

$$\beta((a,u)(b,v)(a,u)^{-1}(b,v)^{-1}) = xyx^{-1}y^{-1}$$

and $\alpha_\sharp(uvu^{-1}v^{-1}) = xyx^{-1}y^{-1}$. Therefore, $[St(R;\mathbb{Z}/m), St(R;\mathbb{Z}/m)] \cong St_m(R)$ is a perfect group.

Moreover, it follows that $(K_1(R))_{(m)}$ is the abelianization of $St(R;\mathbb{Z}/m)$.

Remark 5.9. It would be interesting to give a presentation of $St(R;\mathbb{Z}/m)$ by generators and relations extending the Steinberg relations to the $\bmod m$ case.

Remark 5.10. For any ring R, we can define the groups $K'_1(R;\mathbb{Z}/m)$ and $K'_2(R;\mathbb{Z}/m)$ as the kernels of the canonical homomorphisms $K'_1(R^+;\mathbb{Z}/m) \xrightarrow{p_*} K'_1(\mathbb{Z};\mathbb{Z}/m)$ and $K'_2(R^+;\mathbb{Z}/m) \xrightarrow{p_*} K'_2(\mathbb{Z};\mathbb{Z}/m)$ respectively, and it is then clear that Theorems 4.2 and 5.3 hold for any ring R and any integer $m \geq 2$.

6. Non positive algebraic K-functors with finite coefficients

Recall the definition of the cone $C(R)$ and of the suspension $S(R)$ of a ring R with identity (see [17, Section 1.4 and 2.3]). The ring $C(R)$ is the ring of all infinite matrices with entries in R such that each row and each column has at most finitely many non-zero entries. Denote by $J(R)$ the ideal of $C(R)$ consisting of finite matrices. We have a canonical inclusion $R \hookrightarrow J(R)$. The ring $S(R) = C(R)/J(R)$ is called the suspension of the ring R. These definitions are extended in a natural way to any ring R, not necessarily with identity, using the short exact sequence

$$0 \longrightarrow R \longrightarrow R^+ \xrightarrow{p} \mathbb{Z} \longrightarrow 0$$

and the definitions $J(R) = ker\,(J(p) : J(R^+) \to J(\mathbb{Z}))$, $C(R) = ker\,(C(p) : C(R^+) \to C(\mathbb{Z}))$ and $S(R) = ker\,(S(p) : S(R^+) \to S(\mathbb{Z}))$.

Definition 6.1. For any ring R and any integer $m \geq 2$, define

$$K_{-n}(R; \mathbb{Z}/m) = \pi_1(F_m K(S^{n+1}(R))) \cong K_1(S^{n+1}(R); \mathbb{Z}/m)$$

for $n \geq 0$, where $S^{n+1}(R) = S(S^n(R))$.

Our definition of $\mod m$ non positive algebraic K-functors differs from Weibel's definition for which $K_0(R; \mathbb{Z}/m) = K_0(R) \otimes \mathbb{Z}/m$ and $K_n(R; \mathbb{Z}/m) = 0$ for $n < 0$.

An equivalent algebraic definition of $\mod m$ non positive algebraic K-functors could be given by using Karoubi-Lambre's definition of the first $\mod m$ algebraic K-functor.

Definition 6.2. For any unital ring R, any integer $m \geq 2$ and any integer $n \geq 0$, $K'_{-n}(R; \mathbb{Z}/m) = K'_1(S^{n+1}(R); \mathbb{Z}/m)$.

Theorem 6.3. *For any unital ring R and any integer $m \geq 2$, there are a long exact sequence*

$$\cdots \longrightarrow K_2(R; \mathbb{Z}/m) \longrightarrow K_1(R) \xrightarrow{\cdot m} K_1(R) \longrightarrow K_1(R; \mathbb{Z}/m) \longrightarrow K_0(R) \xrightarrow{\cdot m} K_0(R) \longrightarrow K_0(R; \mathbb{Z}/m)$$
$$\longrightarrow K_{-1}(R) \xrightarrow{\cdot m} K_{-1}(R) \longrightarrow K_{-1}(R; \mathbb{Z}/m) \longrightarrow K_{-2}(R) \xrightarrow{\cdot m} K_{-2}(R) \longrightarrow \cdots$$

and isomorphisms

$$K_n(S(R); \mathbb{Z}/m) \cong K_{n-1}(R; \mathbb{Z}/m)$$

for $n \geq 1$.

Proof. For any unital ring R one has $K_{-n}(R) = K_0(S^n(R))$ for $n \geq 1$ by definition (see [14]). Since $K_0(C(R)) = K_1(C(R)) = 0$ (see [12] and [14]), one has $K_1(S^n(R)) \cong K_0(J(S^{n-1}(R))) \cong K_0(S^{n-1}(R)) = K_{-n+1}(R)$ for $n \geq 1$. Therefore, for any $n \geq 1$, the fibration

$$F_m(BQP(S^n(R))) \longrightarrow BQP(S^n(R)) \xrightarrow{\chi^m} BQP(S^n(R))$$

yields the following exact sequence

$$K_{-n+1}(R) \xrightarrow{\cdot m} K_{-n+1}(R) \longrightarrow K_{-n+1}(R; \mathbb{Z}/m) \longrightarrow K_{-n}(R) \xrightarrow{\cdot m} K_{-n}(R)\,.$$

By splicing these exact sequences for all $n \geq 1$ we obtain the required long exact sequence of algebraic K-functors.

The map $BGL(R)^+ \to BGL(J(R))^+$ induced by the inclusion $R \hookrightarrow J(R)$ is a homotopy equivalence (see [12, Chapter 2]) implying the isomorphism $K_n(R; \mathbb{Z}/m) \to K_n(J(R); \mathbb{Z}/m)$ for all $n \in \mathbb{Z}$, since $K_{-n}(R) \to K_{-n}(J(R))$ is an isomorphism for all $n \geq 0$ (see [13]). Therefore the short exact sequence of rings

$$0 \longrightarrow J(R) \longrightarrow C(R) \xrightarrow{\gamma} S(R) \longrightarrow 0$$

induces a homotopy fibration (see [12, Chapter 3, Section 3])

$$BQP(R) \longrightarrow BQP(C(R)) \longrightarrow BQP(S(R))\,,$$

where the left map is induced by the natural inclusion $R \hookrightarrow C(R)$, and which provides a long exact sequence of algebraic K-functors

$$\cdots \longrightarrow K_3(S(R)) \longrightarrow K_2(R) \longrightarrow K_2(C(R)) \longrightarrow K_2(S(R)) \longrightarrow$$
$$K_1(R) \longrightarrow K_1(C(R)) \longrightarrow K_1(S(R)) \longrightarrow K_0(R) \longrightarrow K_0(C(R)) \longrightarrow K_0(S(R))\,.$$

It follows that the well-defined homomorphism $K_n(R) \to \pi_{n+1}(hBQP(\gamma))$ is an isomorphism for $n \geq 1$, where $hBQP(\gamma)$ is the homotopy fibre of $BQP(\gamma)$. Therefore, the fibration

$$hF_m(BQP(\gamma)) \longrightarrow F_m(BQP(C(R))) \longrightarrow F_m(BQP(S(R)))$$

yields a long exact sequence

$$\cdots \to K_3(S(R);\mathbb{Z}/m) \to K_2(R;\mathbb{Z}/m) \to K_2(C(R);\mathbb{Z}/m) \to K_2(S(R);\mathbb{Z}/m) \to K_1(R;\mathbb{Z}/m)$$
$$\to K_1(C(R);\mathbb{Z}/m) \to K_1(S(R);\mathbb{Z}/m) \to K_0(R;\mathbb{Z}/m) \to K_0(C(R);\mathbb{Z}/m) \to K_0(S(R);\mathbb{Z}/m).$$

It is well known (see [12, Chapter 2, Section 2, D]) that the space $BGL(C(R))^+$ is contractible and it is shown in [14] that $K_n(C(R)) = 0$ for $n \leq 0$. It follows that $K_n(C(R);\mathbb{Z}/m) = 0$ for all $n \in \mathbb{Z}$. We finally deduce that the above given exact sequence implies an isomorphism $K_n(S(R);\mathbb{Z}/m) \xrightarrow{\cong} K_{n-1}(R;\mathbb{Z}/m)$ for $n \geq 1$.

Remark 6.4. Theorem 6.3 holds for any ring R, not necessarily with identity.

Remark 3.4 and Theorem 6.3 imply that $m^2 K_n(R;\mathbb{Z}/m) = 0$ for any ring R and any integer $n \in \mathbb{Z}$. The following result is actually known.

Theorem 6.5. *For any unital ring R and any integer $m \geq 2$ one has $m\,K_n(R;\mathbb{Z}/m) = 0$ if $m \not\equiv 2 \bmod 4$, and $2m\,K_n(R;\mathbb{Z}/m) = 0$ if $m \equiv 2 \bmod 4$ for all $n \geq 1$.*

This theorem was proved topologically by Browder for $n \geq 2$ (see [6, Proposition 1.5]) and later algebraically by Karoubi and Lambre for $n = 1$ (see [15]).

It is obvious that Theorem 6.5. holds for any ring R, since $K_n(R;\mathbb{Z}/m)$ is a subgroup of $K_n(R^+;\mathbb{Z}/m)$. By Definition 6.1, it is also clear that the same holds for non positive algebraic K-functors $K_n(R;\mathbb{Z}/m)$, $n \leq 0$. Moreover, observe that for an exact category $\mathcal{A}$ and for the H-space $BQ(\mathcal{A})$ as in [21], we may define $K_n(\mathcal{A};\mathbb{Z}/m) = \pi_n(F_m(BQ(\mathcal{A})))$ for $n \geq 1$ and $m \geq 2$. Then $K_n(\mathcal{A};\mathbb{Z}/m)$ is isomorphic to the mod m K-group of $\mathcal{A}$ defined in [25] for $n \geq 1$. Theorem 6.5 holds then also for $K_n(\mathcal{A};\mathbb{Z}/m), n \geq 1$ (see Theorem 2.1 of [25]). The proof of Theorem 6.5 for $K_1(\mathcal{A};\mathbb{Z}/m)$ can be found in [15].

Another immediate consequence of Theorem 6.3 is the following assertion.

Corollary 6.6. *For any ring R, any integer $m \geq 2$ and any integer $n \geq 1$, there is an isomorphism*

$$K_{-n}(R;\mathbb{Z}/m) \cong K_0(S^n(R);\mathbb{Z}/m)\,.$$

Theorem 6.7. *For any short exact sequence of rings $0 \longrightarrow R' \xrightarrow{\alpha} R \xrightarrow{\beta} R'' \longrightarrow 0$ and any integer $m \geq 2$, there is a long exact sequence*

$$K_1(R';\mathbb{Z}/m) \to K_1(R;\mathbb{Z}/m) \to K_1(R'';\mathbb{Z}/m) \xrightarrow{\delta^1}$$
$$K_0(R';\mathbb{Z}/m) \to K_0(R;\mathbb{Z}/m) \to K_0(R'';\mathbb{Z}/m) \xrightarrow{\delta^0} K_{-1}(R';\mathbb{Z}/m) \to K_{-1}(R;\mathbb{Z}/m) \to \cdots$$
$$\to K_{n+1}(R'';\mathbb{Z}/m) \xrightarrow{\delta^{n+1}} K_n(R';\mathbb{Z}/m) \to K_n(R;\mathbb{Z}/m) \to K_n(R'';\mathbb{Z}/m) \xrightarrow{\delta^n} K_{n-1}(R';\mathbb{Z}/m) \to \cdots\,,$$

for $n \leq 0$.

Proof. Since the sequence $0 \to S(R') \to S(R) \to S(R'') \to 0$ is exact, it suffices to show the exactness of

$$K_1(R';\mathbb{Z}/m) \to K_1(R;\mathbb{Z}/m) \to K_1(R'';\mathbb{Z}/m) \to K_0(R';\mathbb{Z}/m) \to K_0(R;\mathbb{Z}/m) \to K_0(R'';\mathbb{Z}/m)\,.$$

There is a continuous map $BQP(R') \to hBQP(\beta)$ such that the composite of $BQP(R') \to hBQP(\beta) \to BQP(R)$ is equal to $BQP(\alpha)$ and which induces a homomorphism $K_n(R';\mathbb{Z}/m) \to K_n(R,I;\mathbb{Z}/m)$ for all $n \geq 1$, where $I = ker\,\beta$ (see the proof of Theorem 5.3). First we will show that the sequence

$$K_1(R';\mathbb{Z}/m) \longrightarrow K_1(R;\mathbb{Z}/m) \longrightarrow K_1(R'';\mathbb{Z}/m)$$

is exact. The commutative diagram with exact rows

$$\begin{array}{ccccccccc}
K_1(R') & \longrightarrow & K_1(R') & \longrightarrow & K_1(R';\mathbb{Z}/m) & \longrightarrow & K_0(R') & \longrightarrow & K_0(R') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow = & & \downarrow = \\
K_1(R,I) & \longrightarrow & K_1(R,I) & \longrightarrow & K_1(R,I;\mathbb{Z}/m) & \longrightarrow & K_0(R') & \longrightarrow & K_0(R')\,,
\end{array}$$

in which the two left vertical arrows are surjective, implies that the map $K_1(R';\mathbb{Z}/m) \to K_1(R,I;\mathbb{Z}/m)$ is an epimorphism. The homotopy fibration

$$hF_m(BQP(\beta)) \longrightarrow F_m(BQP(R)) \longrightarrow F_m(BQP(R''))$$

yields the exact sequence

$$\cdots \longrightarrow \pi_2(F_m(BQP(R''))) \longrightarrow \pi_1(hF_m(BQP(\beta))) \longrightarrow \pi_1(F_m(BQP(R))) \longrightarrow \pi_1(F_m(BQP(R'')))\,,$$

where $\pi_1(hF_m(BQP(\beta))) = K_1(R,I;\mathbb{Z}/m)$. It follows that the sequence

$$K_1(R';\mathbb{Z}/m) \longrightarrow K_1(R;\mathbb{Z}/m) \longrightarrow K_1(R'';\mathbb{Z}/m)$$

is exact.

Then, let us consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
K_0(R') & \longrightarrow & K_0(R') & \longrightarrow & K_0(R';\mathbb{Z}/m) & \longrightarrow & K_{-1}(R') & \longrightarrow & K_{-1}(R') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_0(R,I) & \longrightarrow & K_0(R,I) & \longrightarrow & K_0(R,I;\mathbb{Z}/m) & \longrightarrow & K_{-1}(R,I) & \longrightarrow & K_{-1}(R,I)\,.
\end{array}$$

Since there are isomorphisms $K_0(R') \cong K_0(R,I)$ and $K_{-1}(R') \cong K_{-1}(R,I)$, the homomorphism $K_0(R';\mathbb{Z}/m) \to K_0(R,I;\mathbb{Z}/m) = \pi_1(F_m(hBQP(S(\beta))))$ is an isomorphism.

The homotopy fibration

$$hF_m(BQP(S(\beta))) \longrightarrow F_m(BQP(S(R))) \longrightarrow F_m(BQP(S(R'')))$$

provides an exact sequence

$$\pi_2(F_m(BQP(S(R)))) \longrightarrow \pi_2(F_m(BQP(S(R'')))) \longrightarrow$$
$$\pi_1(hF_m(BQP(S(\beta)))) \longrightarrow \pi_1(F_m(BQP(S(R)))) \longrightarrow \pi_1(F_m(BQP(S(R'))))\,.$$

Since $hF_m(BQP(S(\beta))) \simeq F_m(hBQP(S(\beta)))$, it follows that

$$\pi_1(hF_m(BQP(S(\beta)))) \cong \pi_1(F_m(hBQP(S(\beta)))) \cong K_0(R';\mathbb{Z}/m)\,.$$

Therefore the sequence

$$K_1(R;\mathbb{Z}/m) \longrightarrow K_1(R'';\mathbb{Z}/m) \longrightarrow K_0(R';\mathbb{Z}/m) \longrightarrow K_0(R;\mathbb{Z}/m) \longrightarrow K_0(R'';\mathbb{Z}/m)$$

is exact. This completes the proof.

Theorem 6.8. *There is only one (up to equivalence) sequence $(T_n, \partial_n, n \le 0)$ of functors T_n and connecting homomorphisms ∂_n from the category of rings to the category of abelian groups satisfying the following conditions:*

a) $(T_n, \partial_n, n \le 0)$ is a connected sequence of functors.
b) The functor T_0 is equivalent to the functor $K_0(-;\mathbb{Z}/m)$.
c) For any short exact sequence of rings $0 \to R' \to R \to R'' \to 0$, the sequence

$$T_0(R') \longrightarrow T_0(R) \longrightarrow T_0(R'') \xrightarrow{\partial_0} T_{-1}(R') \longrightarrow T_{-1}(R) \longrightarrow T_{-1}(R'') \xrightarrow{\partial_{-1}}$$
$$\cdots \longrightarrow T_n(R'') \xrightarrow{\partial_n} T_{n-1}(R') \longrightarrow T_{n-1}(R) \longrightarrow T_{n-1}(R'') \xrightarrow{\partial_{n-1}} T_{n-2}(R') \longrightarrow \cdots$$

is exact ($n \le 0$).
d) $T_n(R) = 0$ for $n \le 0$ and any ring R of the form $C(A)$.
e) The inclusion $R \hookrightarrow J(R)$ induces an isomorphism $T_n(R) \cong T_n(J(R))$ for all $n \le 0$ and any ring R.

Proof. The existence is obvious since Theorem 6.7 provides the sequence of functors $(K_n(-;\mathbb{Z}/m), \delta_n, n \le 0)$ which satisfies conditions (a) – (e) of the theorem. The proof of the uniqueness is standard, similar to the case of classical negative algebraic K-functors (see [16]) and left to the reader.

Beside the definitions of $K_{-n}(R;\mathbb{Z}/m)$ and $K'_{-n}(R;\mathbb{Z}/m)$ (see Definitions 6.1 and 6.2), there is another way to introduce negative algebraic K-functors with coefficients. Remember that the negative algebraic K-functors were first introduced by Bass in [4], by induction as follows:

$$K_{-n}(R) = \mathit{coker}\,(K_{1-n}(R[t]) \oplus K_{1-n}(R[t^{-1}]) \to K_{1-n}(R[t,t^{-1}]))$$

for $n \ge 1$. It was shown in [13] that both definitions are equivalent. This equivalence was realized using the homomorphism $\rho : R[t,t^{-1}] \to S(R)$ defined by

$$\rho : \sum_{i\in\mathbb{Z}} r_i t^i \longmapsto \begin{pmatrix} r_0 & r_1 & r_2 & \cdots & \cdots \\ r_{-1} & r_0 & r_1 & r_2 & \cdots \\ r_{-2} & r_{-1} & r_0 & r_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

Following Bass [4], we may also introduce an algebraic inductive definition of the negative mod m K-functors as follows.

Definition 6.9. For any ring R, any integer $m \ge 2$ and any integer $n \ge 1$, let us define

$$K^B_{-n}(R;\mathbb{Z}/m) = \mathit{coker}\,(K^B_{1-n}(R[t];\mathbb{Z}/m) \oplus K^B_{1-n}(R[t^{-1}];\mathbb{Z}/m) \to K^B_{1-n}(R[t,t^{-1}];\mathbb{Z}/m))$$

for $n \ge 1$, where $K^B_0(R;\mathbb{Z}/m) = K_0(R;\mathbb{Z}/m)$ and $R[t,t^{-1}]$ is the ring of Laurent polynomials over R.

Proposition 6.10. *For any ring R and any integer $m \ge 2$, there is an isomorphism $K^B_{-n}(R;\mathbb{Z}/m) \cong K_{-n}(R;\mathbb{Z}/m)$ for $n \ge 1$.*

Proof. It is known (see [4]) that there is an exact sequence

$$0 \longrightarrow K_0(R) \to K_0(R[t]) \oplus K_0(R[t]) \longrightarrow K_0(R[t,t^{-1}]) \xrightarrow{p_0} K_{-1}(R) \longrightarrow 0\,,$$

where p_0 splits. Since $S^n(R[t]) \cong S^n(R)[t]$, $S^n(R[t,t^{-1}]) \cong S^n(R)[t,t^{-1}]$ and $K_n(R) \cong K_0(S^n(R))$ for $n \geq 1$, this splitting holds for any negative algebraic K-functor K_n, $n \leq -1$. Denote by $S_0(R)$ the long exact sequence

$$K_0(R) \xrightarrow{m} K_0(R) \longrightarrow K_0(R;\mathbb{Z}/m) \longrightarrow K_{-1}(R) \xrightarrow{m} K_{-1}(R) \longrightarrow \cdots,$$

by $S_0^2(R[t])$ the long exact sequence $S_0(R[t]) \oplus S_0(R[t])$ and by $S_{-1}^B(R)$ the long sequence

$$K_{-1}(R) \xrightarrow{m} K_{-1}(R) \xrightarrow{\varphi_{-1}} K_{-1}^B(R;\mathbb{Z}/m) \xrightarrow{\psi_{-1}} K_{-2}(R) \xrightarrow{m} K_{-2}(R) \longrightarrow$$
$$\cdots \longrightarrow K_{-n}(R) \xrightarrow{\varphi_{-n}} K_{-n}^B(R;\mathbb{Z}/m) \xrightarrow{\psi_{-n}} K_{-n}(R) \longrightarrow \cdots,$$

where φ_{-n} and ψ_{-n} are induced by $\alpha_{-n+1}(t,t^{-1}) : K_{-n+1}(R[t,t^{-1}]) \to K_{-n+1}(R[t,t^{-1}];\mathbb{Z}/m)$ and $\beta_{-n+1}(t,t^{-1}) : K_{-n+1}(R[t,t^{-1}];\mathbb{Z}/m) \to K_{-n}(R[t,t^{-1}])$ respectively. Under these notations one has the following exact sequence of sequences

$$0 \longrightarrow S_0(R) \longrightarrow S_0^2(R[t]) \longrightarrow S_0(R[t,t^{-1}]) \longrightarrow S_{-1}^B(R) \longrightarrow 0.$$

It is easily checked that this exact sequence implies the exactness of the sequence $S_{-1}^B(R)$. On the other hand, the composite of the homomorphisms $K_0(R[t];\mathbb{Z}/m) \oplus K_0(R[t];\mathbb{Z}/m) \to K_0(R[t,t^{-1}];\mathbb{Z}/m)$ and $K_0(R[t,t^{-1}];\mathbb{Z}/m) \to K_0(S(R);\mathbb{Z}/m)$, is trivial and therefore there is a natural homomorphism

$$\omega_{-1} : K_{-1}^B(R;\mathbb{Z}/m) \longrightarrow K_0(S(R);\mathbb{Z}/m) \cong K_{-1}(R;\mathbb{Z}/m)$$

induced by the homomorphism ρ which provides the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
K_{-1}(R) & \longrightarrow & K_{-1}(R) & \longrightarrow & K_{-1}^B(R;\mathbb{Z}/m) & \longrightarrow & K_{-2}(R) & \longrightarrow & K_2(R) \\
\big\downarrow = & & \big\downarrow = & & \big\downarrow \omega_{-1} & & \big\downarrow = & & \big\downarrow = \\
K_{-1}(R) & \longrightarrow & K_{-1}(R) & \longrightarrow & K_{-1}(R;\mathbb{Z}/m) & \longrightarrow & K_{-2}(R) & \longrightarrow & K_2(R)
\end{array}$$

and shows that the groups $K_{-1}^B(R;\mathbb{Z}/m)$ and $K_{-1}(R;\mathbb{Z}/m)$ are isomorphic. Moreover for $n > 1$ we obtain isomorphisms

$$K_{-n}^B(R;\mathbb{Z}/m) \cong K_{-1}^B(S^{n-1}(R);\mathbb{Z}/m) \cong K_{-1}(S^{n-1}(R);\mathbb{Z}/m) \cong K_{-n}(R;\mathbb{Z}/m).$$

This completes the proof.

7. The mod m fundamental theorem

Confirming Weibel's Remark [25] suggesting that all the results of Quillen higher algebraic K-theory [21] hold for the mod m K-theory, this section is only devoted to the proof of the following important result.

Theorem 7.1. *Let R be any unital ring. For $n \geq 1$ there is a functorial isomorphism*

$$K_n(R[t,t^{-1}];\mathbb{Z}/m) \cong K_n(R;\mathbb{Z}/m) \oplus K_{n-1}(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m),$$

where $NK_n(R;\mathbb{Z}/m)$ denotes the cokernel of the homomorphism $K_n(R;\mathbb{Z}/m) \to K_n(R[t];\mathbb{Z}/m)$ induced by the inclusion $R \hookrightarrow R[t]$.

The property expressed by the mod m Fundamental Theorem will be called algebraic periodicity in mod m algebraic K-theory.

Proof. The homomorphism $\rho : R[t,t^{-1}] \to S(R)$, given at the end of Section 6, induces a homomorphism $K_n(R[t,t^{-1}];\mathbb{Z}/m) \to K_n(S(R);\mathbb{Z}/m)$ for $n \geq 1$. Since we have an isomorphism $K_n(S(R);\mathbb{Z}/m) \cong K_{n-1}(R;\mathbb{Z}/m)$ by Theorem 6.3, one gets natural homomorphisms

$$\rho_{n,m} : K_n(R[t,t^{-1}];\mathbb{Z}/m) \to K_{n-1}(R;\mathbb{Z}/m)$$

for $n \geq 1$. On the other hand, we have also natural homomorphisms $K_n(R;\mathbb{Z}/m) \to K_n(R[t,t^{-1}];\mathbb{Z}/m)$, $NK_n(R;\mathbb{Z}/m) \to K_n(R[t,t^{-1}];\mathbb{Z}/m)$ and $NK_n(R;\mathbb{Z}/m) \to K_n(R[t,t^{-1}];\mathbb{Z}/m)$ for $n \geq 1$, induced by the inclusions $R \hookrightarrow R[t,t^{-1}]$, $R[t] \hookrightarrow R[t,t^{-1}]$ and $R[t^{-1}] \hookrightarrow R[t,t^{-1}]$ respectively. Thus we obtain the following sequence

$$0 \longrightarrow K_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m) \longrightarrow K_n(R[t,t^{-1}];\mathbb{Z}/m) \xrightarrow{\rho_{n,m}} K_{n-1}(R;\mathbb{Z}/m) \longrightarrow 0$$

for $n \geq 1$. Our aim is to show the split exactness of this sequence.

Let us write $NK_n(R)$ for the cokernel of $K_n(R) \to K_n(R[t])$ and consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_n(R) & \xrightarrow{\cdot m} & K_n(R) & \xrightarrow{\beta_n} & K_n(R;\mathbb{Z}/m) & \xrightarrow{\alpha_n} & K_{n-1}(R) & \xrightarrow{\cdot m} & K_{n-1}(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_n(R[t]) & \xrightarrow{\cdot m} & K_n(R[t]) & \xrightarrow{\beta_n(t)} & K_n(R[t];\mathbb{Z}/m) & \xrightarrow{\alpha_n(t)} & K_{n-1}(R[t]) & \xrightarrow{\cdot m} & K_{n-1}(R[t]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
NK_n(R) & \xrightarrow{\cdot m} & NK_n(R) & \xrightarrow{\beta'_n} & NK_n(R;\mathbb{Z}/m) & \xrightarrow{\alpha'_n} & NK_{n-1}(R) & \xrightarrow{\cdot m} & NK_{n-1}(R)\,, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where the columns and the two top rows are exact. This implies clearly the exactness of the bottom row, where α'_n and β'_n are induced by $\alpha_n(t)$ and $\beta_n(t)$ respectively. Let us use the following notation for $n \geq 1$: $W_n(R) = K_n(R) \oplus NK_n(R) \oplus NK_n(R)$ and $W_n(R;\mathbb{Z}/m) = K_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m) \oplus NK_n(R;\mathbb{Z}/m)$. Then we have a long exact sequence

$$\cdots \longrightarrow W_n(R) \xrightarrow{\cdot m} W_n(R) \xrightarrow{\bar{\beta}_n} W_n(R;\mathbb{Z}/m) \xrightarrow{\bar{\alpha}_n} W_{n-1}(R) \xrightarrow{\cdot m} W_{n-1}(R) \longrightarrow \cdots\,,$$

where the homomorphisms $\bar{\alpha}_n$ and $\bar{\beta}_n$ are defined in a natural way. Let us consider the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & & 0 & & 0 & & 0 & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
W_n(R) & \xrightarrow{\cdot m} & W_n(R) & \longrightarrow & W_n(R;\mathbb{Z}/m) & \longrightarrow & W_{n-1}(R) & \xrightarrow{\cdot m} & W_{n-1}(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
K_n(R[t,t^{-1}]) & \xrightarrow{\cdot m} & K_n(R[t,t^{-1}]) & \xrightarrow{\tilde{\beta}_n} & K_n(R[t,t^{-1}];\mathbb{Z}/m) & \longrightarrow & K_{n-1}(R[t,t^{-1}]) & \xrightarrow{\cdot m} & K_{n-1}(R[t,t^{-1}]) \\
\downarrow \rho_n & & \downarrow \rho_n & & \downarrow \rho_{n,m} & & \downarrow \rho_{n-1} & & \downarrow \rho_{n-1} \\
K_{n-1}(R) & \xrightarrow{\cdot m} & K_{n-1}(R) & \xrightarrow{\beta_{n-1}} & K_{n-1}(R;\mathbb{Z}/m) & \xrightarrow{\alpha_{n-1}} & K_{n-2}(R) & \xrightarrow{\cdot m} & K_{n-2}(R)\,, \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

where the right and left two columns are functorially split. It follows that the middle column is exact. It remains to show that the homomorphism $\rho_{n,m}$ splits for $n \geq 1$. To this end, we define a homomorphism

$$\delta_{n-1,m} : K_{n-1}(R;\mathbb{Z}/m) \longrightarrow K_n(R[t,t^{-1}];\mathbb{Z}/m)$$

as follows. Because of the corresponding theorem with integral coefficients, there is a functorial homomorphism $\delta_{n-1} : K_{n-1}(R) \to K_n(R[t,t^{-1}])$, for $n \geq 1$, such that the composition $\rho_n\delta_{n-1}$ is the identity. Then, let $x \in K_{n-1}(R;\mathbb{Z}/m)$. Since $m\delta_{n-2}\alpha_{n-1}(x) = 0$, there is an element $y \in K_n(R[t,t^{-1}];\mathbb{Z}/m)$ such that $\alpha_{n-1}\rho_{n,m}(y) = \alpha_{n-1}(x)$. Consequently, there is a $z \in K_{n-1}(R)$ such that $\beta_{n-1}(z) = x - \rho_{n,m}(y)$. Finally, define

$$\delta_{n-1,m}(x) = \tilde{\beta}_n\delta_{n-1}(z) + y\,.$$

It is easily to check that we obtain a correctly defined homomorphism which satisfies

$$\rho_{n,m}\delta_{n-1,m}(x) = \rho_{n,m}\tilde{\beta}_n\delta_{n-1}(z) + \rho_{n,m}(y) = \beta_{n-1}(z) + \rho_{n,m}(y) = x\,.$$

8. The mod m Mayer-Vietoris sequence

As mentioned in the introduction, Weibel proved in [25] the existence of a Mayer-Vietoris sequence for Browder's mod m algebraic K-functors if we restrict ourselves to $\mathbb{Z}[1/m]$-algebras. We will provide another form of the Mayer-Vietoris sequence for any rings expanding the classical Mayer-Vietoris sequence of [18] to algebraic K-theory with coefficients in $\mathbb{Z}/m$ and for $\mathbb{Z}[1/m]$-algebras prolonging and completing Weibel's Mayer-Vietoris sequence.

Theorem 8.1. *Let*

$$\begin{array}{ccc}
R' & \xrightarrow{i_1} & R_1 \\
\downarrow i_2 & & \downarrow j_1 \\
R_2 & \xrightarrow{j_2} & R
\end{array}$$

be a pullback square of rings with j_2 a surjective homomorphism. Then there is a long exact sequence

$$\begin{aligned} &K_2(R';\mathbb{Z}/m) \longrightarrow K_2(R_2;\mathbb{Z}/m) \oplus K_2(R_1;\mathbb{Z}/m) \longrightarrow K_2(R;\mathbb{Z}/m) \longrightarrow \\ &\quad K_1(R';\mathbb{Z}/m) \longrightarrow K_1(R_2;\mathbb{Z}/m) \oplus K_1(R_1;\mathbb{Z}/m) \longrightarrow K_1(R;\mathbb{Z}/m) \longrightarrow \\ &\qquad K_0(R';\mathbb{Z}/m) \longrightarrow K_0(R_2;\mathbb{Z}/m) \oplus K_0(R_1;\mathbb{Z}/m) \longrightarrow K_0(R;\mathbb{R}/m) \longrightarrow K_{-1}(R';\mathbb{Z}/m) \longrightarrow \\ &\qquad\qquad K_{-1}(R_2;\mathbb{Z}/m) \oplus K_{-1}(R_1;\mathbb{Z}/m) \longrightarrow K_{-1}(R;\mathbb{Z}/m) \longrightarrow K_{-2}(R';\mathbb{Z}/m) \longrightarrow \cdots . \end{aligned}$$

Proof. Since $ker\, j_2 \cong ker\, i_2$, by splicing the two long exact sequences of Theorem 6.7 for j_2 and i_2 respectively, one obtains the following long exact sequence

$$\begin{aligned} &K_1(R';\mathbb{Z}/m) \longrightarrow K_1(R_2;\mathbb{Z}/m) \oplus K_1(R_1;\mathbb{Z}/m) \longrightarrow K_1(R;\mathbb{Z}/m) \longrightarrow \\ &\quad K_0(R';\mathbb{Z}/m) \longrightarrow K_0(R_2;\mathbb{Z}/m) \oplus K_0(R_1;\mathbb{Z}/m) \longrightarrow K_0(R;\mathbb{Z}/m) \longrightarrow K_{-1}(R';\mathbb{Z}/m) \longrightarrow \cdots . \end{aligned} \tag{5}$$

Now consider the pullback squares of topological spaces

$$\begin{array}{ccc} X & \longrightarrow & BQP(R_1) \\ \downarrow & & \downarrow \\ BQP(R_2) & \longrightarrow & BQP(R) \end{array}$$

and

$$\begin{array}{ccc} Y & \xrightarrow{q_1'} & F_m(BQP(R_1)) \\ \downarrow{\scriptstyle q_2'} & & \downarrow{\scriptstyle q_1} \\ F_m(BQP(R_2)) & \xrightarrow{q_2} & F_m(BQP(R)) \end{array} \tag{6}$$

induced by the homomorphisms j_1 and j_2. It is easily checked that $F_m(X) \simeq Y$. One has a natural continuous map $BQP(R') \to X$ which yields the commutative diagram with exact rows

$$\begin{array}{ccccccccc} K_1(R') & \longrightarrow & K_1(R') & \longrightarrow & K_1(R';\mathbb{Z}/m) & \longrightarrow & K_0(R') & \longrightarrow & K_0(R') \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \pi_2(X) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_1(F_m(X)) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X) \end{array}$$

showing that there is an isomorphism $K_1(R';\mathbb{Z}/m) \xrightarrow{\cong} \pi_1(F_m(X)) \cong \pi_1(Y)$. Diagram (16) induces the commutative diagram with exact rows

$$\begin{array}{ccccccccccc} \pi_2(hq_1') & \longrightarrow & \pi_2(Y) & \longrightarrow & K_2(R_1;\mathbb{Z}/m) & \longrightarrow & \pi_1(hq_1') & \longrightarrow & K_1(R';\mathbb{Z}/m) & \longrightarrow & K_1(R_1;\mathbb{Z}/m) \\ \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle\cong} & & \downarrow & & \downarrow \\ \pi_2(hq_2) & \longrightarrow & K_2(R_2;\mathbb{Z}/m) & \longrightarrow & K_2(R;\mathbb{Z}/m) & \longrightarrow & \pi_1(hq_2) & \longrightarrow & K_1(R_2;\mathbb{Z}/m) & \longrightarrow & K_1(R;\mathbb{Z}/m) \end{array}$$

which yields the following exact sequence

$$\pi_2(Y) \to K_2(R_2;\mathbb{Z}/m) \oplus K_2(R_1;\mathbb{Z}/m) \to K_2(R;\mathbb{Z}/m) \to K_1(R';\mathbb{Z}/m) \to K_1(R_2;\mathbb{Z}/m) \oplus K_2(R_1;\mathbb{Z}/m)\,. \quad (7)$$

On the other hand the continuous map $BQP(R') \to X$ induces the commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
K_2(R') & \longrightarrow & K_2(R') & \longrightarrow & K_2(R';\mathbb{Z}/m) & \longrightarrow & K_1(R') & \longrightarrow & K_1(R') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
\pi_2(X) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_2(F_m(X)) & \longrightarrow & \pi_1(X) & \longrightarrow & \pi_1(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}$$

which provides a surjection $K_2(R';\mathbb{Z}/m) \to \pi_2(F_m(X)) \cong \pi_2(Y)$. Therefore by gluing the exact sequences (5) and (7), we obtain the required Mayer-Vietoris sequence.

Corollary 8.2. *Let*

$$\begin{array}{ccc}
R' & \xrightarrow{\;i_1\;} & R_1 \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle j_1} \\
R_2 & \xrightarrow{\;j_2\;} & R
\end{array}$$

be a pullback square of $\mathbb{Z}[1/m]$-algebras with j_2 surjective. Then there is a long exact sequence

$$\cdots \longrightarrow K_{n+1}(R;\mathbb{Z}/m) \longrightarrow K_n(R';\mathbb{Z}/m) \longrightarrow K_n(R_2;\mathbb{Z}/m) \oplus K_n(R_1;\mathbb{Z}/m) \longrightarrow$$
$$K_n(R;\mathbb{Z}/m) \longrightarrow K_{n-1}(R';\mathbb{Z}/m) \longrightarrow \cdots$$

for all $n \in \mathbb{Z}$.

Proof. It suffices to splice the long exact sequences of Theorem 8.1 and of Corollary 1.3 of [25].

9. Postnikov invariants of finite K-theory spaces

Let us consider again the connected mod m K-theory space $\overline{F_mK}(R)$ for any unital ring R and any integer $m \geq 2$ (cf. Definition 3.6). Although the integral homology of the space $\overline{F_mK}(R)$ does not coincide with the mod m homology of the group of elementary matrices $E(R)$ (cf. Remark 2.2), the connected mod m K-theory space $\overline{F_mK}(R)$ plays an important role in the understanding of the mod m K-theory of R and it makes sense to investigate its homotopy type. In this section, we'll concentrate our attention to the special case, where $m = p$ is a prime number.

For any connected CW-complex X and any integer $n \geq 1$, let us denote by $X[n]$ the n-th Postnikov section of X (i.e., $X[n]$ is the CW-complex obtained from X by attaching cells of dimension $\geq n+2$ with the property that $\pi_i(X[n]) = 0$ for $i \geq n+1$ and $\pi_i(X[n]) \cong \pi_i(X)$ for $i \leq n$). If X is simple, the Postnikov k-invariants of X are cohomology classes $k^{n+1}(X) \in H^{n+1}(X[n-1]; \pi_n(X))$ which explain how the space X can be built up from its homotopy groups ($n \geq 2$).

We shall consider the connected mod p K-theory space $\overline{F_pK}(R)$ of any unital ring R and our purpose is to investigate the k-invariants of $\overline{F_pK}(R)$. Because of Definition 3.6, $k^{n+1}(\overline{F_pK}(R))$ is an element of the group $H^{n+1}(\overline{F_pK}(R)[n-1]; K_{n+1}(R;\mathbb{Z}/p))$.

Proposition 9.1. *For any unital ring R, any prime number p and any integer $n \geq 2$, the order of the k-invariant $k^{n+1}(\overline{F_pK}(R))$ in $H^{n+1}(\overline{F_pK}(R)[n-1]; K_{n+1}(R;\mathbb{Z}/p))$ is a power of p.*

Proof. The mod p K-theory space $\overline{F_pK}(R)$ is a connected infinite loop space, because it is the fiber of the p-th power map which is an infinite loop map. Consequently, all k-invariants of $\overline{F_pK}(R)$ are cohomology classes of finite order, according to [1] and [2]. The statement then follows from the fact that the group $K_{n+1}(R;\mathbb{Z}/p)$ is a p-torsion abelian group.

One can actually give a universal upper bound for this power of p.

Theorem 9.2. *For any unital ring R, any prime number p and any $n \geq 2$, the k-invariant $k^{n+1}(\overline{F_pK}(R))$ in $H^{n+1}(\overline{F_pK}(R)[n-1]; K_{n+1}(R;\mathbb{Z}/p))$ satisfies*
a) $k^{n+1}(\overline{F_pK}(R))$ is trivial if $2 \leq n \leq 2p-3$,
b) the order of $k^{n+1}(\overline{F_pK}(R))$ divides p^{n-2p+3} if $n \geq 2p-2$.

Proof. According to [2], the k-invariants $k^{n+1}(X)$ of a connected infinite loop space X have the following property for $n \geq 2$: the order of $k^{n+1}(X)$ in the group $H^{n+1}(X[n-1]; \pi_n(X))$ divides the integer R_n which is defined by $R_n = L_2 \cdot L_3 \cdot L_4 \cdots L_n$, where the integer L_k is the product of some primes q satisfying $2 \leq q \leq \frac{k}{2}+1$; therefore, R_n is only divisible by prime numbers $q \leq \frac{n}{2}+1$. Consequently, if $n \leq 2p-3$, p does not divide R_n and the previous theorem shows that

$$k^{n+1}(\overline{F_pK}(R)) = 0 \in H^{n+1}(\overline{F_pK}(R)[n-1]; K_{n+1}(R;\mathbb{Z}/p))\,.$$

On the other hand, if $n \geq 2p-2$, L_k can only be divisible by p if $k \geq 2p-2$ and is never divisible by p^2. One can conclude that the p-primary part of the integer R_n is at most p^{n-2p+3}. Again, Assertion (b) then follows from Proposition 9.1.

Since all k-invariants of the Postnikov section $\overline{F_pK}(R)[2p-3]$ of the connected mod p K-theory space $\overline{F_pK}(R)$ are trivial, $\overline{F_pK}(R)[2p-3]$ is a generalized Eilenberg-MacLane space (GEM) whose non-trivial homotopy groups are $\pi_n(\overline{F_pK}(R)) \cong K_{n+1}(R;\mathbb{Z}/p)$ for $1 \leq n \leq 2p-3$.

Corollary 9.3. *For any unital ring R and any prime number p,*

$$\overline{F_pK}(R)[2p-3] \simeq \prod_{n=2}^{2p-2} K(K_n(R;\mathbb{Z}/p), n-1)\,.$$

This corollary and the isomorphism $H_i(\overline{F_pK}(R);\mathbb{Z}) \cong H_i(\overline{F_pK}(R)[2p-3];\mathbb{Z})$ for $i \leq 2p-3$ enable us to deduce the following consequence on the integral homology of the connected mod p K-theory space of R:

$$H_i(\overline{F_pK}(R);\mathbb{Z}) \cong H_i\Big(\prod_{n=2}^{2p-2} K(K_n(R;\mathbb{Z}/p), n-1);\mathbb{Z}\Big)$$

for $i \leq 2p-3$. However, since $H_j(K(G,n-1);\mathbb{Z})$ vanishes if $j < n-1$, one can deduce the following assertion.

Corollary 9.4. *For any unital ring R, any prime number p and any integer $i \leq 2p-3$, one has*

$$H_i(\overline{F_pK}(R);\mathbb{Z}) \cong H_i\Big(\prod_{n=2}^{i+1} K(K_n(R;\mathbb{Z}/p), n-1);\mathbb{Z}\Big)\,.$$

This assertion is of particular interest in order to compute the integral homology of $\overline{F_pK}(R)$ in low dimensions (relatively to p) since the integral homology of Eilenberg-MacLane spaces is competely known by [8] (see also [9] and [10]).

Example 9.5. By taking $i = 2$, one can deduce that for $p \geq 3$, one has

$$\begin{aligned} H_2(\overline{F_pK}(R);\mathbb{Z}) &\cong H_2\Big(K(K_2(R;\mathbb{Z}/p),1) \times K(K_3(R;\mathbb{Z}/p),2);\mathbb{Z}\Big) \\ &\cong H_2(K(K_2(R;\mathbb{Z}/p),1);\mathbb{Z}) \oplus H_2(K(K_3(R;\mathbb{Z}/p),2);\mathbb{Z}) \end{aligned}$$

and consequently

$$H_2(\overline{F_pK}(R);\mathbb{Z}) \cong \Lambda^2(K_2(R;\mathbb{Z}/p)) \oplus K_3(R;\mathbb{Z}/p)\,,$$

where $\Lambda^2(-)$ denotes the exterior square.

The isomorphism given by Corollary 9.4 provides the following consequence.

Corollary 9.6. *For any unital ring R, any prime p and any integer $i \leq 2p-3$, the group $H_i(\overline{F_pK}(R);\mathbb{Z})$ contains $K_{i+1}(R:\mathbb{Z}/p)$ as a direct summand.*

Proof. This follows directly from Corollary 9.4 since the group $H_i(\overline{F_pK}(R);\mathbb{Z})$ contains

$$H_i(K(K_{i+1}(R;\mathbb{Z}/p),i);\mathbb{Z}) \cong K_{i+1}(R;\mathbb{Z}/p)$$

as a direct summand.

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