axioms

MDPI

Article

On the Use of the Generalized Littlewood Theorem Concerning Integrals of the Logarithm of Analytical Functions for the Calculation of Infinite Sums and the Analysis of Zeroes of Analytical Functions

Sergey Sekatskii [1,2]

[1] Laboratory of Biological Electron Microscopy, IPHYS, BSP 419, Ecole Polytechnique Fédérale de Lausanne, 1015 Lausanne, Switzerland; serguei.sekatski@epfl.ch; Tel.: +41-21-693-0445

[2] Department of Fundamental Biology, Faculty of Biology and Medicine, University of Lausanne, 1011 Lausanne, Switzerland

Abstract: Recently, we have established and used the generalized Littlewood theorem concerning contour integrals of the logarithm of an analytical function to obtain a few new criteria equivalent to the Riemann hypothesis. Here, the same theorem is applied to calculate certain infinite sums and study the properties of zeroes of a few analytical functions. On many occasions, this enables to facilitate the obtaining of known results thus having important methodological meaning. Additionally, some new results, to the best of our knowledge, are also obtained in this way. For example, we established new properties of the sum of inverse zeroes of a digamma function, new formulae for the sums $\sum \frac{k_i}{\rho_i^2}$ for zeroes ρ_i of incomplete gamma and Riemann zeta functions having the order k_i (These results can be straightforwardly generalized for the sums $\sum \frac{k_i}{\rho_i^n}$ with integer $n > 2$, and so on.)

Keywords: generalized Littlewood theorem; logarithm of an analytical function; zeroes and poles of analytical function; infinite sums

MSC: 30E20; 30C15; 33B20; 33B99

check for updates

Citation: Sekatskii, S. On the Use of the Generalized Littlewood Theorem Concerning Integrals of the Logarithm of Analytical Functions for the Calculation of Infinite Sums and the Analysis of Zeroes of Analytical Functions. *Axioms* **2023**, *12*, 68. https://doi.org/10.3390/axioms12010068

Academic Editors: Serkan Araci, Mehmet Acikgoz and Ugur Duran

Received: 1 December 2022
Revised: 30 December 2022
Accepted: 3 January 2023
Published: 7 January 2023

1. Introduction

The generalized Littlewood theorem concerning contour integrals of the logarithm of analytic function is the following statement [1,2]:

Theorem 1. (The Generalized Littlewood theorem). *Let C denote the rectangle bounded by the lines $x = X_1, x = X_2, y = Y_1, y = Y_2$, where $X_1 < X_2, Y_1 < Y_2$ and let f(z) be analytic and non-zero on C and meromorphic inside it, and also let g(z) be analytic on C and meromorphic inside it. Let F(z) = ln(f(z)) be the logarithm defined as follows: we start with a particular determination on $x = X_2$, and obtain the value at other points by continuous variation along y = const from $\ln(X_2 + iy)$. If, however, this path would cross a zero or pole of f(z), we take F(z) to be $F(z \pm i0)$ according to whether we approach the path from above or below. Let $\widetilde{F}(z) = \ln(f(z))$ be the logarithm defined by continuous variation along any smooth curve fully lying inside the contour, which avoids all poles and zeroes of f(z) and starts from the same particular determination $x = X_2$. Suppose that the poles and zeroes of the functions f(z) and g(z) do not coincide. Then:*

$$\int_C F(z)g(z)dz = 2\pi i\left(\sum_{\rho_g} res(g(\rho_g)\cdot\widetilde{F}(\rho_g)) - \sum_{\rho_f^0}\int_{X_1+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz + \sum_{\rho_f^{pole}}\int_{X_1+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz\right) \quad (1)$$

where the sum is over all ρ_g, which are poles of the function g(z) lying inside C, and all $\rho_f^0 = X_\rho^0 + Y_\rho^0$, which are zeroes of the function f(z) are counted, taking into account their multiplicities (that is *the corresponding term being multiplied by m for a zero of the order m), which lie inside C, and all* $\rho_f^{pole} = X_\rho^{pole} + Y_\rho^{pole}$*, which are poles of the function f(z) are counted, taking into account their multiplicities, and which lie inside C. The assumption is that all relevant integrals on the right-hand side of the equality exist.*

The proof of this theorem [2] is very close to the proof of the standard Littlewood theorem corresponding to the case $g(z) = 1$; see, e.g., Ref. [3]. In this note, we apply this theorem for certain particular cases when the contour integral $\int_C F(z)g(z)dz$ disappears (tends to zero) if the contour tends to infinity, that is, when $X_1, Y_1 \to +\infty, X_2, Y_2 \to -\infty$. This means that Equation (1) takes the form:

$$\sum_{\rho_f^0} \int_{-\infty+iY_\rho^0}^{X_\rho^0+iY_\rho^0} g(z)dz - \sum_{\rho_f^{pole}} \int_{-\infty+iY_\rho^{pole}}^{X_\rho^{pole}+iY_\rho^{pole}} g(z)dz = \sum_{\rho_g} res(g(\rho_g) \cdot F(\rho_g)) \tag{2}$$

If the integrals here can be calculated explicitly, in this way one obtains equalities involving finite or infinite sums (this last case is the most interesting one).

Certainly, different methods to calculate infinite sums, and the use of these sums to study zeroes of analytical functions are well known; see, e.g., Refs. [3–10]. However, the proposed approach in many instances materially facilitates these calculations. We also believe that the current paper is useful not only from the methodological and pedagogical point of view; it seems that in many cases, such sums are more difficult or even impossible to calculate using other methods. Let us give here a few examples (there are many more in the text). We show that if zeroes ρ_i of the digamma function with $i = 0, 1, 2, 3 \ldots$ (all of them are real and simple) are arranged in decreasing order $\rho_i > \rho_1 > \rho_2 > \ldots$, then $\lim_{N\to\infty}(\ln N + \sum_{n=0}^{N} \frac{1}{\rho_i}) = 0$. In other places of the paper, we show, e.g., that the equality $\sum \frac{k_i}{\rho_i^2} = 0$ holds for the solutions ρ_i (k_i is the order of the corresponding solution) of the equation $\exp(z) - a - z - \frac{z^2}{2} = 0$, where a is any complex number not equal to 1, and so on. New formulae for the sums $\sum \frac{k_i}{\rho_i^2}$ for zeroes ρ_i having the order k_i of incomplete gamma and Riemann zeta functions (these results can be straightforwardly generalized for the sums $\sum \frac{k_i}{\rho_i^n}$ with integer $n > 2$) are also established.

Finally, it might be also worthwhile to note that some of our research was fulfilled using the approach discussed in the paper, and concerning the analysis of the non-trivial zeroes of the Riemann zeta function, was recently included in the corresponding chapter of the Encyclopedia of Mathematics and its Applications [11].

2. Application of the Generalized Littlewood Theorem to Calculate Certain Infinite Sums

Let us start with a few examples. First, we present almost trivial ones just for illustrative purposes. In a sense, the most natural function to start with is $f(z) = \cos(\pi z)$ or similar. Evidently, for large $|z|$, $\mathrm{Im}(z) \neq 0, \ln(\cos(\pi z)) = O(z)$, and thus if we take $g(z) = 1/z^3$, the contour integral $\int_C \frac{\ln(\cos(\pi z))}{z^3} dz$ for $X_1, Y_1 \to +\infty, X_2, Y_2 \to -\infty$ vanishes (when constructing the contour, one just should avoid such values of X_1 and X_2, which give $\cos(\pm\pi X_{1,2}) = 0$). The zeroes of the function f(z) lie at $z_\rho = 1/2 \pm n$ where n is any integer, and $\int_{-\infty}^{n+1/2} \frac{1}{z^3} dz = -\frac{1}{2(n+1/2)^2}$. Further, the integrand has a single pole of the third order at z = 0, and thus the corresponding residue contribution is $\frac{1}{2}\frac{d^2 \ln(\cos(\pi z))}{dz^2}|_{z=0} = -\frac{\pi^2}{2}$. Hence, the application of Equation (2) gives $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(n+1/2)^2} = \pi^2$ which is, of course, extremely well-known especially, if recast as $\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \pi^2/8$.

A more general integral, viz. $\int\limits_C \frac{\ln(\sin(az+b))}{(z+c)^3} dz$, where a, b, and c are arbitrary complex numbers, similarly gives $\sum\limits_{n=-\infty}^{\infty} \frac{1}{(\pi n - b + ac)^2} = \frac{1}{\sin^2(b-ac)}$. This is example number 6.1.27 in [12]. Of course, here, $\pi n - b + ac$ for any n should not be equal to zero. Similarly as above, when constructing the proof (demonstrating the disappearance of the contour integral in the limit of infinitely large contours), the values of X_1 and X_2 corresponding to zeroes of $\sin(aX_i + b)$ should be avoided; we will not explicitly mention this anymore. It is instructing to compare this above proof with much more complicated and less transparent ones given on four pages, pp. 219–222, in the famous Bromwich book [4].

Another possibility is the use of the gamma function, which has simple poles at z = 0, −1, −2, ... and has no zeroes [13]. For illustration, let us consider the rather general case $\prod\limits_{n=1}^{\infty} \frac{P(n)}{Q(n)}$, that is the estimation of $\sum\limits_{n=1}^{\infty} \ln \frac{P(n)}{Q(n)}$, where $P(z) = \sum\limits_{i=0}^{p} a_i z^{-i}$ and $Q(z) = \sum\limits_{i=0}^{q} b_i z^{-i}$. We require $a_0 = b_0 = 1$ and, for convergence, $a_1 = b_1$. We factorize $P(z) = a_p(-1)^p \prod\limits_{i=1}^{p} \left(\frac{1}{r_i} - \frac{1}{z}\right)$ and $Q(z) = b_q(-1)^q \prod\limits_{i=1}^{q} \left(\frac{1}{s_i} - \frac{1}{z}\right)$, where the roots r_i, s_i are not necessarily different. Certainly, zero is not a root of our polynomials, and we require that the positive integers 1, 2, 3 ... also are not among their roots. Trivially,

$$f(z) := \ln\left(\frac{P(z)}{Q(z)}\right)' = \sum_{i=1}^{p} \frac{r_i}{z(z-r_i)} - \sum_{i=1}^{q} \frac{s_i}{z(z-s_i)} = \sum_{i=1}^{p} \left(\frac{1}{z-r_i} - \frac{1}{z}\right) - \sum_{i=1}^{q} \left(\frac{1}{z-s_i} - \frac{1}{z}\right)$$

and we consider the contour integral $\int\limits_C f(z) \ln(\Gamma(1-z)) dz$. Condition $a_1 = b_1$ ensures the necessary asymptotic $O(z^{-3})$ for large $|z|$. Pole at $z = 0$ of Equation (2) contributes nothing due to $\ln(\Gamma(1)) = 0$, while poles at r_i, s_j contribute $\ln(\Gamma(1-r_i))$ and $-\ln(\Gamma(1-s_j))$. Thus, we have:

$$\prod_{n=1}^{\infty} \frac{P(n)}{Q(n)} = \frac{\prod\limits_{i=1}^{q} \Gamma(1-s_i)}{\prod\limits_{i=1}^{p} \Gamma(1-r_i)} \tag{3}$$

Certainly, $P(z) = 1 - \frac{a^2}{z^2}$, $Q(z) = 1$, and $a \neq n$ reduces to the well-known example number 89.5.11 in [5]: $\prod\limits_{n=1}^{\infty} (1 - \frac{a^2}{n^2}) = \frac{1}{\pi a} \sin(\pi a)$, which can be easily obtained by our method also starting from the contour integral $\int\limits_C \frac{2a^2 \ln(\Gamma(1-z))}{z(z^2-a^2)}$ or $\int\limits_C \frac{2a^2 \ln(\frac{\sin(\pi z)}{z})}{z(z^2-a^2)}$ (here $\frac{2a^2}{z(z^2-a^2)} = \frac{d}{dz} \ln(1 - \frac{a^2}{z^2})$).

Remark 1. *In Weisstein, Eric W. "Infinite Product", from* MathWorld *A Wolfram Web Resource, https://mathworld.wolfram.com/InfiniteProduct.html (accessed on 30 November 2022), we find without references* $\prod\limits_{n=1}^{\infty} \frac{P(n)}{Q(n)} = \frac{b_q \prod\limits_{i=1}^{q} \Gamma(-s_i)}{a_p \prod\limits_{i=1}^{p} \Gamma(-r_i)}$. *This is the same as (3) due to the known equality* $(-r_i)\Gamma(-r_i) = \Gamma(1-r_i)$ *[13] and conditions* $a_0 = b_0 = 1$*, which require* $a_p(-1)^p \prod\limits_{i=1}^{p} \frac{1}{r_i} = 1$ *and* $b_q(-1)^q \prod\limits_{i=1}^{q} \frac{1}{s_i} = 1$.

Now, we can consider the case where $z = k = 1, 2, 3 \ldots$ is a simple root of, say, the polynomial $P(z)$. We will denote it as the first root r_1. We begin with a contour integral

$$\int\limits_C f(z)\ln((k-z)\cdot\Gamma(1-z))dz$$

so that the function under the logarithm sign is regular at $z = k$. The left-hand side of Equation (2) changes to $\sum\limits_{n=1,n\neq k}^{\infty} \ln\frac{P(n)}{Q(n)}$ while at right-hand side for each pole $z = 0$ in Equation (2), we have a contribution $-\ln k$, that is $-p\ln k$ totally; and contribution at the pole $z = r_i$ not equal to k of $f(z)$ changes to $\ln((k-r_i)\cdot\Gamma(1-r_i))$. At $z = k$, the following takes place. We know that in the vicinity of $1-k$, the residue of the gamma function is $\frac{(-1)^{k-1}}{(k-1)!}$ hence, for $z = k + \delta$ with small positive δ:

$$(k-z)\Gamma(1-z) = -\delta\Gamma(1-k-\delta) \sim \frac{(-1)^{k-1}}{(k-1)!}$$

and we have a contribution $\ln(\frac{(-1)^{k-1}}{(k-1)!})$. Thus, the total contribution of the zeroes of $P(z)$ is:

$$-p\ln k + \ln(\prod_{i=2}^{p}(k-r_i)\Gamma(1-r_i)) + \ln\frac{(-1)^{k-1}}{(k-1)!}$$

and that of zeroes of $Q(z)$ is $q\ln k - \ln(\prod\limits_{i=1}^{p}(k-s_i)\Gamma(1-s_i))$. Collecting everything together, we thus proved the following proposition.

Proposition 1. *Let* $P(z) = \sum\limits_{i=0}^{p} a_i z^{-i}$ *and* $Q(z) = \sum\limits_{i=0}^{q} b_i z^{-i}$, $a_0 = b_0 = 1$, *and* $a_1 = b_1$. *Let* r_1, $\ldots$, r_p *be roots of* $P(z)$ *and* $s_1, \ldots, s_q$ *be roots of* $Q(z)$, *not necessarily different. Let* $r_1 = k$, *where* k *is a positive integer and all other roots are not equal to* k *and any other positive integer. Then:*

$$\prod_{n=1,n\neq k}^{\infty}\frac{P(n)}{Q(n)} = (-1)^{p-q}k^{p-q}(k-1)!\frac{\prod\limits_{i=1}^{q}(k-s_i)\Gamma(1-s_i)}{\prod\limits_{i=2}^{p}(k-r_i)\Gamma(1-r_i)} \tag{4}$$

For example, for $\prod\limits_{n=1,n\neq k}^{\infty}(1-\frac{k^2}{n^2})$, we obtain:

$$\prod_{n=1,n\neq k}^{\infty}(1-\frac{k^2}{n^2}) = \frac{(-1)^{k-1}}{2} \tag{5}$$

The generalization for the case when, in addition to $r_1 = k$, some other roots are equal to integers, is straightforward.

If the series with the alternating signs are involved, the use of functions of the type $\ln(\tan)(az+b)$ or $\ln\frac{\Gamma(-z/2+1/2)}{\Gamma(-z/2+1)}$ might be helpful, as they have alternating zeroes and poles. As an example, let us consider the series $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{n^2a^2-b^2}$; number 6.1.39 in [12]. We need to use $g(z) = \frac{d}{dz}\frac{1}{a^2z^2-b^2} = -\frac{2z}{a^2(z-b/a)^2(z+b/a)^2}$ hence, we analyze the integral $\int\limits_C \ln(\tan(\pi z/2))g(z)dz$. Its disappearance for an infinitely large contour is certain and we obtain, taking into account the existence of two poles of the second order of the function g(z) at $z = \pm b/a$: $-\sum\limits_{n=-\infty}^{\infty} \frac{(-1)^n}{n^2a^2-b^2} - \frac{\pi}{ab\sin(\pi b/a)} = 0$. Thus the known result $\sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n^2a^2-b^2} = -\frac{1}{2b^2} - \frac{\pi}{2ab\sin(\pi b/a)}$.

Numerous other applications of this approach to sum infinite series can be constructed.

Another potentially useful approach is the possibility to explore the functions of more complicated arguments. For example, analyzing $\int\limits_C \frac{\ln(\Gamma(-z^2))}{(z-c)^4} dz$ we, given the simple poles at $z = \pm\sqrt{1}, \pm\sqrt{2}\ldots$ and the double pole at $z = 0$, have:

$$\sum_{n=1}^{\infty} \left(\frac{1}{(\sqrt{n}-c)^3} + \frac{1}{(\sqrt{n}+c)^3}\right) - \frac{2}{c^3} = \frac{1}{2}\frac{d^3}{dz^3}\ln(\Gamma(-z^2))|_{z=c}$$

and thus:

$$\sum_{n=1}^{\infty} \left(\frac{1}{(\sqrt{n}-c)^3} + \frac{1}{(\sqrt{n}+c)^3}\right) - \frac{2}{c^3} = 6c\Psi_1(-c^2) - 4c^3\Psi_2(-c^2) \tag{6}$$

Here, $\Psi_i(z)$ with $i = 1, 2$ are polygamma functions [13].

For example, for c = i:

$$\sum_{n=1}^{\infty} \left(\frac{1}{(\sqrt{n}-i)^3} + \frac{1}{(\sqrt{n}+i)^3}\right) = 2i + 6i\Psi_1(1) + 4i\Psi_2(1) = (2 + \pi^2 - 8\zeta(3))i \cong 2.253i \tag{7}$$

The interesting cases c = n can be analyzed by putting $c = \sqrt{n} + x$ and considering the limit $x \to 0$. Let us look at an example of $c = 1 + x$ so that $-c^2 = -1 - 2x - x^2$. First, we have, starting from the general formula:

$$\Psi(-n+x) = -\frac{1}{x} + H_n^{(1)} - \gamma + \sum_{k=1}^{\infty} (H_n^{(k+1)} + (-1)^{k+1}\varsigma(k+1))x^k,$$

where $H_n^{(k)} = \sum\limits_{l=1}^{n} \frac{1}{l^k}$ are the generalized harmonic numbers:

$$\Psi(-1+x) = -\frac{1}{x} + 1 - \gamma + (1+\zeta(2))x + (1-\zeta(3))x^2 + O(x^3),$$

thus:

$$\Psi_1(-1+x) = \frac{1}{x^2} + 1 + \zeta(2) + 2(1-\zeta(3))x + O(x^2)$$

and

$$\Psi_2(-1+x) = -\frac{2}{x^3} + 2(1-\zeta(3)) + O(x).$$

Further:

$$\Psi_1(-c^2) = \Psi(-1+(-2x-x^2)) = \frac{1}{(2x+x^2)^2} + 1 + \zeta(2) + O(x) = \frac{1}{4x^2} - \frac{1}{4x} + \frac{19}{16} + \zeta(2) + O(x)$$

and

$$c\Psi_1(-c^2) = (1+x)\Psi_1(-1+(-2x-x^2)) = (1+x)\left(\frac{1}{4x^2} - \frac{1}{4x} + \frac{19}{16} + \zeta(2) + O(x)\right) = \frac{1}{4x^2} + \frac{15}{16} + \zeta(2) + O(x).$$

Similarly:

$$\begin{aligned}\Psi_2(-c^2) &= \Psi_2(-1+(-2x-x^2)) = \tfrac{2}{(2x+x^2)^3} + 2(1-\zeta(3)) + O(x) \\ &= \tfrac{2}{8x^3(1+\frac{1}{2}x)^3} + 2(1-\zeta(3)) + O(x) = \tfrac{1}{4x^3} - \tfrac{3}{8x^2} + \tfrac{3}{8x} + \tfrac{27}{16} - 2\zeta(3) + O(x)\end{aligned}$$

and

$$c^3\Psi_2(-c^2) = (1+x)^3\Psi_2(-1+(-2x-x^2)) =$$
$$(1+3x+3x^2+x^3)(\tfrac{1}{4x^3} - \tfrac{3}{8x^2} + \tfrac{3}{8x} + \tfrac{27}{16} - 2\zeta(3) + O(x)) =$$
$$\tfrac{1}{4x^3} + \tfrac{3}{8x^2} + \tfrac{31}{16} - 2\zeta(3) + O(x).$$

The application of Equation (6) gives:

$$\sum_{n=2}^{\infty}\left(\frac{1}{(\sqrt{n}-1)^3} + \frac{1}{(-\sqrt{n}-1)^3}\right) - \frac{1}{x^3} - \frac{1}{8} - 2 =$$
$$\frac{3}{2x^2} + \frac{45}{8} + \pi^2 - \frac{1}{x^3} - \frac{3}{2x^2} - \frac{31}{4} + 8\zeta(3) + O(x)$$

Finally:

$$\sum_{n=2}^{\infty}\left(\frac{1}{(\sqrt{n}-1)^3} + \frac{1}{(\sqrt{n}+1)^3}\right) = \pi^2 + 8\zeta(3) \cong 19.486 \tag{8}$$

Equations (7) and (8) were tested numerically by calculating the sums and using $\zeta(3) = 1.202057$ [14].

The following trick also deserves to be noted. For odd functions, not only integer powers but half-integer powers also may be used as arguments because the single-valued character of the function placed under the logarithm sign can be assured by selecting the function of the type $\ln\frac{\sin(z^{n+1/2})}{z^{n+1/2}}$ or $\ln\frac{\tan(z^{n+1/2})}{z^{n+1/2}}$. (We have $\frac{\sin(z^{n+1/2})}{z^{n+1/2}} = \sum_{k=0}^{\infty}(-1)^k\frac{(z^{n+1/2})^{2k+1}}{(2k+1)!z^{n+1/2}} = \sum_{k=0}^{\infty}(-1)^k\frac{z^{2kn+k}}{(2k+1)!}$, etc.) As an example, we, starting from $\int_C \frac{1}{(z-a)^3}\ln\left(\frac{\sin(z^{3/2})}{z^{3/2}}\right)dz$, obtain

$$\sum_{n=1}^{\infty}\left(\frac{1}{((\pi n)^{2/3}-a)^2} + \frac{1}{((\pi n)^{2/3}e^{2\pi i/3}-a)^2} + \frac{1}{((\pi n)^{2/3}e^{-2\pi i/3}-a)^2}\right) = -\frac{3}{4\sqrt{a}}\cot(a^{3/2}) + \frac{9}{4}\frac{a}{\sin^2(a^{3/2})} - \frac{3}{2a^2} \tag{9}$$

(Integral $\int_C \frac{1}{(z-a)^3}\ln\left(\frac{\sin(z^{1/2})}{z^{1/2}}\right)dz$ as a starting point does not bring anything new.)

Interesting relations can be also obtained if the function $\ln(f(z)-a)$ is considered. Of course, the function f(z)-a has the same poles as f(z) but other zeroes ρ_i of the order k_i, where $f(\rho_i) = a$. The most natural first example here is $\ln(e^z - a)$, where for simplicity, we will take a real positive. The roots of $e^z - a = 0$ are $z = \ln a \pm 2\pi ni$, where n is an integer or zero, and thus we have, when analyzing $\int_C \frac{\ln(e^z-a)}{z^3}dz$: $\sum_{n=-\infty}^{\infty}\frac{1}{(\ln a+2\pi ni)^2} - \frac{a}{(1-a)^2} = 0$. This is convenient to be recast as $\sum_{n=-\infty}^{\infty}\frac{1}{(b+2\pi ni)^2} = \frac{e^b}{(1-e^b)^2}$, where b is any real not equal to 0.

This result can be, of course, obtained "in a standard fashion" starting from $\int_C \frac{\ln(\sinh(z/2))}{(z+b)^3}dz$. However, the same idea can be interestingly used to study the solutions of the equation $\Gamma(z) = a$ exploring the contour integral $\int_C \frac{\ln((\Gamma(z)-a)\cdot z)}{z^3}dz$, or solutions of $J_n(z) = a$ exploring the contour integral $\int_C \frac{\ln((J_n(z)-a)\cdot z)}{z^3}dz$, etc.

From now on, we will use the following easy Lemma, see, e.g., Ref. [7].

Lemma 1. *Let f(z) be an analytical function defined on the whole complex plane, except possibly a countable set of points. Let this function also be presented in some vicinity of the point z = 0 by the Taylor expansion $f(z) = 1 + a_1z + a_2z^2 + \ldots$ and such that the contour integral $\int_C \frac{\ln(f(z))}{z^3}dz$ tends to zero when contour C tends to infinity (see Theorem 1 for the details). Then, for the sum over zeroes $\rho_{i,0}$ having order k_i and poles $\rho_{i,pole}$ having order l_i of the function f(z), we have:*

$$\sum\left(\frac{k_i}{\rho_{i,0}^2} - \frac{l_i}{\rho_{i,pole}^2}\right) = a_1^2 - 2a_2 \tag{10}$$

Proof. We trivially have in some vicinity of the point $z = 0$, the Taylor expansion:

$$\ln(f(z)) = a_1 z + \frac{1}{2}(-a_1^2 + 2a_2)z^2 + \dots,$$

and now the direct application of the Theorem 1 to the integral $\int\limits_C \frac{\ln(f(z))}{z^3}dz$ gives the statement of the Lemma. □

From:

$$\Gamma(z) = \frac{1}{z} - \gamma + (\frac{1}{2}\gamma^2 + \frac{\pi^2}{12})z + \dots$$

we have

$$z(\Gamma(z) - a) = 1 - (\gamma + a)z + (\frac{1}{2}\gamma^2 + \frac{\pi^2}{12})z^2 + \dots,$$

so that the application of Lemma 1 immediately gives:

$$\sum \frac{k_i}{\rho_{i,0}^2} - \frac{\pi^2}{6} = (\gamma + a)^2 - \gamma^2 - \frac{\pi^2}{6}$$

and $\sum \frac{k_i}{\rho_{i,0}^2} - \frac{\pi^2}{6} = 2a\gamma + a^2$.

Remark 2. *We have amusingly* $\sum \frac{k_i}{\rho_i^2} = 0$ *for the solutions* ρ_i *of the equation* $\Gamma(z) = -2$. *(For a pair of the complex conjugate zeroes* $\rho_{1,2} = \sigma \pm it$, *one has* $\frac{1}{\rho_1^2} + \frac{1}{\rho_2^2} = \frac{2(\sigma^2 - t^2)}{(\sigma^2 + t^2)^2}$*). As follows from our analysis, this is not an isolated property but the other way around; similar properties can be established for many analytical functions. For example, equality* $\sum \frac{k_i}{\rho_i^2} = 0$ *holds for the solutions* ρ_i *of the following equations:*

$$\exp(z) - a - z - \frac{z^2}{2} = 0\ (a \neq 1),\ \sin(z) - a - z = 0\ (a \neq 0),$$

$$z^{-\alpha} J_\alpha(z) - a + \frac{z^2}{4\Gamma(2 + \alpha)} = 0\ (a \neq 1,\ J_\alpha(z)$$

is Bessel function), and so force.

3. Application of the Generalized Littlewood Theorem to Analyze Zeroes of Analytical Functions

A similar technique can be applied to calculate different sums related to zeroes of other analytical functions; that is, in a sense, not so much to study the infinite sums over the known numbers (such as integers), but to analyze the properties of zeroes themselves. Numerous examples of the kind are known, see, e.g., Ref. [7] and references therein. In particular, sums over non-trivial zeroes of the Riemann zeta function, see, e.g., Ref. [14] for the general discussion of this function, were extensively studied and some of them were attempted to be used to test the Riemann hypothesis [15–17]; see also a brief discussion of these results in the corresponding chapter of the Encyclopedia of Mathematics and its Applications [11].

By historical reasons, these sums are commonly expressed via Stieltjes constants γ_n of the Laurent expansion of the Riemann zeta function at $z = 1$: $\zeta(z) = \frac{1}{z-1} + \sum\limits_{n=0}^{\infty} \frac{(-1)^n}{n!}\gamma_n(z-1)^n$ [14], so let us follow this line. Due to the symmetry of the non-trivial Riemann zeta function zeroes ρ_i, having order k_i, with respect to the axis $\mathrm{Re} z = 1/2$ [14], we have $\sum \frac{k_i}{\rho_i^2} = \sum \frac{k_i}{(\rho-1)_i^2}$

thus, we can use the contour integral $\int_C \frac{1}{z^3}\ln(\zeta(z-1)\cdot(z-1))dz$ to calculate $\sum\frac{k_i}{\rho_i^2}$. Noting that:

$$\zeta(z-1)\cdot(z-1) = 1+\gamma(z-1)-\gamma_1(z-1)^2+O((z-1)^3),$$

we, applying Lemma 1, obtain the well-known:

$$\sum\frac{k_i}{\rho_i^2} = 1-\frac{\pi^2}{8}+\gamma^2+2\gamma_1 \tag{11}$$

which is our first illustration. Here:

$$\frac{\pi^2}{8}-1 = \sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}$$

is the sum over the trivial zeroes of the $\zeta(z-1)$ function.

Remark 3. *We can use the functional equation* $\zeta(s) = 2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s)$ *[14] to pass from the derivatives of* $\ln((s-1)\zeta(s-1))$ *to those of* $\ln(\zeta(s))$*. Clearly:*

$$\ln''\zeta(s) = \ln''(\sin(\pi s/2)\cdot\zeta(1-s))+\ln''(\Gamma(1-s)),$$

and further

$$\ln''(\Gamma(1-s))|_{s=0} = \Psi_1(1) = \frac{\pi^2}{6}$$

$$\begin{aligned}\sin(\pi s/2)\cdot\zeta(1-s) &= (\tfrac{\pi s}{2}-\tfrac{\pi^3s^3}{24})(-\tfrac{1}{s}+\gamma+\gamma_1 s)+O(s^3)\\ &= -\tfrac{\pi}{2}+\tfrac{\pi}{2}\gamma s+(\tfrac{\pi^3}{24}+\tfrac{\pi}{2}\gamma_1)+O(s^3),\end{aligned}$$

thus

$$\ln(-\frac{\pi}{2}+\frac{\pi}{2}\gamma s+(\frac{\pi^3}{24}+\frac{\pi}{2}\gamma_1)+O(s^3)) = \ln(-\frac{\pi}{2})+\ln(1-\gamma s-(\frac{\pi^2}{12}+\gamma_1)s^2)+O(s^3)).$$

We have:

$$\ln(-\frac{\pi}{2}+\frac{\pi}{2}\gamma s+(\frac{\pi^3}{24}+\frac{\pi}{2}\gamma_1)+O(s^3)) = \ln(-\frac{\pi}{2})+\ln(1-\gamma s-(\frac{\pi^2}{12}+\gamma_1)s^2)+O(s^3)).$$

Hence, from Equation (2), we see that the sum over all zeroes and poles of $\zeta(s)$ *is equal to* $\gamma^2-\frac{\pi^2}{6}+2\gamma_1$*. In this sum,*$-1+\frac{\pi^2}{24}$*is the contribution of the trivial zeroes at s =* $-2,-4,-6\ldots$ *together with the simple pole at s = 1; hence, we restore Formula (11).*

Alternatively, the whole function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$*, whose only zeroes are the non-trivial Riemann zeroes and* $\xi(s) = \xi(1-s)$ [14] *can be used for such calculations.*

3.1. Incomplete Gamma Function

As not such a trivial example, let us consider the incomplete gamma function, and then the incomplete Riemann zeta function. The asymptotic of these functions for large arguments, see, e.g., [18,19], enables us to achieve zero values of the contour integrals used when the contour tends to infinity, and we will not repeat this anymore.

The incomplete gamma function is defined for Res > 0 and real positive z as $\gamma(s,z) = \int_0^z t^{s-1}e^{-t}dt$. For fixed s in the same range Res > 0, the Taylor expansion of exponent and subsequent integration readily give the well-known formula:

$$z^{-s}\gamma(s,z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!(s+n)}z^n \tag{12}$$

If $s \neq 0, -1, -2, \ldots$, this absolutely converged series defines the whole function on the full complex z-plane, and from this expression the sums $\sum\limits_{\rho} \frac{k_i}{\rho_i^m}$, where ρ_i is a zero of the order k_i, can be easily determined for all integers $m \geq 2$ using the technique of the paper. For example, from the first terms $z^{-s}\gamma(s,z) = \frac{1}{s} - \frac{z}{s+1} + \frac{z^2}{2(s+2)} + O(z^2)$, we, applying the Lemma 1, immediately obtain:

$$\sum_{\rho_i} \frac{k_i}{\rho_i^2} = \frac{s^2}{(s+1)^2} - \frac{s}{s+2} = -\frac{s}{(s+1)^2(s+2)} \tag{13}$$

This sum is never equal to zero.

An incomplete Riemann zeta function is defined for Res > 1 and real positive z as $F(s,z) = \frac{1}{\Gamma(s)} \int\limits_0^z \frac{t^{s-1}}{\exp(t)-1} dt$. For a moment, we regard it as a function of z for some fixed s. We have the following series converging for $|t| < 2\pi$ and with the convention $B_1 = -1/2$; certainly, all other larger odd Bernoulli numbers are equal to zero [20]:

$$\frac{t}{e^t - 1} = \sum_{n=1}^{\infty} \frac{B_n}{n!} t^n \tag{14}$$

Thus, for $|z| < 2\pi$

$$z^{1-s} F(s,z)\Gamma(s) = \sum_{n=0}^{\infty} \frac{B_n z^n}{n!(s+n-1)} \tag{15}$$

here $s \neq 1, 0, -1, -2, \ldots$, (The series representation for larger values of $|z|$ is somewhat more complicated, see, e.g., Ref. [18], but this question is irrelevant to us now.)

Equation (15) enables us to find sums $\sum\limits_{\rho} \frac{k_i}{\rho_i^m}$, where, as usual, ρ_i is a zero of incomplete gamma function of the order k_i, for all integers $m \geq 2$. (Note, that already from the first terms of this development, Equation (15),$z^{1-s}F(s,z)\Gamma(s) = \frac{1}{s-1} - \frac{z}{2s} + O(z^2)$ we see that for small s, the z one obtains zero at $z \cong -2\,s$, and this root evidently dominates such sums for the case for small s and z; see below.) From

$$z^{1-s} F(s,z)\Gamma(s)(s-1) = 1 - \frac{z(s-1)}{2s} + \frac{z^2(s-1)}{12(s+1)} + O(z^3),$$

we obtain using Lemma 1:

$$\sum_{\rho_i} \frac{k_i}{\rho_i^2} = \frac{(s-1)^2}{4s^2} - \frac{s-1}{6(s+1)} = -\frac{(s^2-3)(s-1)}{12s^2(s+1)} \tag{16}$$

Incomplete Riemann zeta function at certain occasions indeed has double zeroes, see [18,19] and references therein; hence, k_i is not superfluous here and below. For small $|s|$, the sum is dominated by zero at $z \simeq -2s$. It, amusingly, is equal to zero for $s_{1,2} = \pm\sqrt{3}$.

A more interesting and important question is the analysis of the corresponding sums, when the functions at hand are understood as functions of s for some fixed z. Equation (12) immediately reveals the presence of simple poles of the incomplete gamma function at s = 0, −1, −2 ... Further, from this equation, we have for $|s| < 1$:

$$z^{-s}\gamma(s,z) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot n!} z^n \left(1 - \frac{s}{n} + \frac{s^2}{n^2} - \ldots\right),$$

and this absolutely converging series can be rearranged to give:

$$z^{-s} s\gamma(s,z) = 1 + [-z + \frac{z^2}{2 \cdot 2!} - \frac{z^3}{3 \cdot 3!} + \ldots]s + [z - \frac{z^2}{2^2 \cdot 2!} + \frac{z^3}{3^2 \cdot 3!} - \ldots]s^2 + O(s^3);$$

a series which converges for any z for $|s| < 1$ (their continuation for larger s is irrelevant for our purposes). Thus, we should define the functions $a_m(z) := (-1)^m \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^m n!} z^n$ to write $z^{-s} s\gamma(s,z) = 1 + \sum_{n=1}^{\infty} a_n(z) s^n$. (One can add the convention $a_0(z) \equiv 1$ and write $z^{-s} s\gamma(s,z) = \sum_{n=0}^{\infty} a_n(z) s^n$.)

We obtain, using the same technique as above:

$$\sum \frac{k_i}{\rho_i^2} = a_1^2(z) - 2a_2(z) + \frac{\pi^2}{6} \tag{17}$$

Here, again the term $\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the contribution of the simple poles lying at $s = -1, -2, -3 \dots$ For z tending to zero, this sum tends to be $\pi^2/6 = 1.6449 \dots$ This reflects, e.g., the known fact that for real $z < 0$, the incomplete gamma function has two real zeroes in each interval $-2n < s < 2-2n$ for $n = 1, 2, 3 \dots$, as well as known statements concerning zeroes for the case $z > 0$; see [18,19] and references therein. For z tending to zero, zeroes ρ_i tend to $-1, -2, -3 \dots$.

Remark 4. *"Almost elementary" functions can be used here. We know:*

$$Ein(z) := \int_0^z \frac{1 - e^{-t}}{t} dt = z - \frac{z^2}{2! \cdot 2} + \frac{z^3}{3! \cdot 3} - \dots,$$

thus $a_1(z) = -Ein(z)$, *and further*

$$\int_0^z \frac{Ein(t)}{t} dt = z - \frac{z^2}{2! \cdot 2^2} + \frac{z^3}{3! \cdot 3^2} - \dots = a_2(z).$$

In general, for $k = 1, 2, 3 \dots$ $a_{k+1}(z) = (-1)^k \int_0^z \frac{a_k(t)}{t} dt$. *Of course, it is also* $a_1(z) = \int_0^z \frac{a_0(t)}{t} dt$, *with* $a_0(t) = 1 - e^{-t} = t - \frac{t^2}{2!} + \frac{t^3}{3!} - \dots$.

3.2. Incomplete Riemann Zeta Function

Analogous consideration is applicable for incomplete Riemann zeta function. Let us now, for Re $s > 1$, define the function:

$$Q(s,z) := \Gamma(s) F(s,z) = \int_0^z \frac{t^{s-1}}{\exp(t) - 1} dt.$$

Certainly, with a variable change $t = zy$, if Re $s > 1$:

$$Q(s,z) = \int_0^1 (zy)^{s-1} \frac{z}{\exp(zy) - 1} dy = z^{s-1} \int_0^1 y^{s-2} \frac{zy}{\exp(zy) - 1} dy,$$

thus, substituting for

$$|zy| < 2\pi \; \frac{zy}{e^{zy} - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} (zy)^n,$$

again $B_1 = -1/2$, and integrating, we obtain $z^{1-s}Q(s,z) = \sum\limits_{n=0}^{\infty} \frac{B_n z^n}{n!(n+s-1)}$. For $|z| < 2\pi$, this is usable for all s except the simple poles at $s = 1, 0, -1, -2 \ldots$. Further, for

$$|s| < 1: \ z^{1-s}sQ(s,z) = -s - s^2 - s^3 - \ldots - \tfrac{z}{2} + \sum_{n=2}^{\infty} \tfrac{B_n z^n s}{n!(n+s-1)}$$
$$= -s - s^2 - s^3 - \ldots - \tfrac{z}{2} + \sum_{n=2}^{\infty} \tfrac{B_n z^n s}{n!(n-1)}(1 - \tfrac{s}{n-1} + \tfrac{s^2}{(n-1)^2} - \ldots),$$

so that

$$-z^{1-s}sQ(s,z) = \frac{z}{2} + [1 - \sum_{n=2}^{\infty} \frac{B_n z^n}{n!(n-1)}]s + [1 + \sum_{n=2}^{\infty} \frac{B_n z^n}{n!(n-1)^2}]s^2 + \ldots$$

This is valid for $|z| < 2\pi$, $|s| < 1$ and can be used to search for the sum of powers of inverse zeroes for such relatively small values of z. When z tends to zero, the sum is dominated by the zero at $s \simeq -z/2$.

For a larger z analytic continuation of the series, over z^n is required. This can be obtained similarly to the remark above. Let us define, for $|t| < 2\pi$ and $k = 1, 2, 3 \ldots$:

$$b_k(z) := 1 + (-1)^k \sum_{n=2}^{\infty} \frac{B_n z^n}{n!(n-1)^k}.$$

We write, for $t \neq 0$, $|t| < 2\pi$: $\frac{1}{t(e^t-1)} = \sum\limits_{n=0}^{\infty} \frac{B_n}{n!} t^{n-2}$, with $B_1 = -1/2$, and obtain:

$$\int_0^z \left(\frac{1}{t(e^t-1)} - \frac{1}{t^2} + \frac{1}{2t}\right)dt = \sum_{n=2}^{\infty} \frac{B_n}{n!(n-1)} z^{n-1}.$$

Thus:

$$b_1(z) := 1 - \sum_{n=2}^{\infty} \frac{B_n}{n!(n-1)} z^n = 1 - z\int_0^z \left(\frac{1}{t(e^t-1)} - \frac{1}{t^2} + \frac{1}{2t}\right)dt. \quad (18)$$

Further, for $k = 1, 2, 3 \ldots$:

$$(-1)^k \sum_{n=2}^{\infty} \frac{B_n}{n!(n-1)^k} z^{n-2} = \frac{b_k(z) - 1}{z^2}$$

and we have, by integration and multiplication on z:

$$(-1)^k \sum_{n=2}^{\infty} \frac{B_n}{n!(n-1)^k} z^n = z\int_0^z \frac{b_k(t) - 1}{t^2} dt$$

where

$$b_{k+1}(z) = 1 - \int_0^z \frac{b_k(t) - 1}{t^2} dt \quad (19)$$

Now, we can write:

$$-z^{1-s}sQ(s,z) = \frac{z}{2} + b_1(z)s + b_2(z)s^2 + b_3(z)s^3 + \ldots, \quad (20)$$

with the aforementioned functions $b_k(z)$ defined by (18) and (19) for the whole complex plane, except the points $z = 2\pi i n$, where n is a positive or negative integer not equal to zero. The question of the convergence of series (20) for $|z| < 2\pi$ is naturally posed. We will not study this question in detail. Fortunately, for our current purposes, it is enough that these series, for each z, converge in some vicinity at the point $s = 0$. For this, it is sufficient that for any z, some positive number $A(z)$ (possibly very large), such that asymptotically for large $|z|$ $|b_k|$ is not larger than A^k exists. Then the series at the question converge for $|s| < 1/A$. This is the case. From (18), we have for large $|z|$ the asymptotic

$$b_1(z) \sim -\frac{1}{2 \cdot 1!} z \ln z,$$

and the application of (19) gives the asymptotic:

$$b_2(z) \sim \frac{1}{2 \cdot 2!} z \ln^2 z, \; b_3(z) \sim -\frac{1}{2 \cdot 3!} z \ln^3 z,$$

etc.

Thus, the application of Lemma 1 to the function

$$-2z^{-s} sQ(s,z) = 1 + 2z^{-1} b_1(z) s + 2z^{-1} b_2(z) s^2 + \dots$$

solves the problem of finding the expression for the following sum of zeroes of the function $sQ(s,z) = s\Gamma(s)F(s,z) = s\int_0^z \frac{t^{s-1}}{\exp(t)-1} dt$:

$$\sum_\rho \frac{k_i}{\rho_i^2} = \frac{4}{z^2} b_1^2(z) - \frac{4}{z} b_2(z) + \frac{\pi^2}{6} + 1 \tag{21}$$

Here, $\frac{\pi^2}{6} + 1$ is the contribution of poles of the function $sQ(s, z)$ lying at 1, -1, -2, $-3 \dots$.

4. Zeroes of Polygamma Functions

Recently, in an interesting paper [7] Mezö and Hoffman established numerous sums, including zeroes of the digamma function $\Psi(z)$ and its generalizations. All those results can be obtained by our method, but it makes no sense to repeat. Instead, to finish the Section, we concentrate on some additional possibilities brought in by the generalized Littlewood theorem.

Just to illustrate the above point, let us consider the contour integral:

$$\int_C \frac{1}{z^3} \ln(\Psi(z) \cdot (-z)) dz.$$

We know the Laurent expansion [13,20]:

$$\Psi(x) = -\frac{1}{x} - \gamma + \sum_{n=1}^{\infty} (-1)^{n-1} \zeta(n+1) x^n \tag{22}$$

where $\ln(\Psi(z) \cdot (-z)) = \gamma z + (-\frac{\gamma^2}{2} - \zeta(2)) z^2 + O(z^3)$. A known asymptotic of the digamma function for large z, $\Psi = O(\ln z)$ [13] guarantees that the contour integral value tends to zero when $X \to \infty$. Thus, the application of the generalized Littlewood theorem gives $0 = -\frac{\gamma^2}{2} - \zeta(2) + \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{\rho_n^2} - \frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$. Here, the sum $-\frac{1}{2} \sum_{n=1}^{\infty} \frac{1}{n^2}$ is the contribution of simple poles of the function $\Psi(x) \cdot (-x)$ at the points $z = -1, -2, -3, \dots$ Remembering $\zeta(2) = \frac{\pi^2}{6}$, we thus have proven the relation $\sum_{i=0}^{\infty} \frac{1}{\rho_i^2} = \gamma^2 + \frac{\pi^2}{2}$ in [7]. Here, we partly retain the notation of [7] and denote zeroes of the digamma function; they all are real and simple, as ρ_i with $i = 1, 2, 3 \dots$ arranged in decreasing order, and $\rho_0 = 1.461632 \dots$ is the only one positive zero.

However, the aforementioned asymptotic of the digamma function shows that the value of the contour integral $\int_C \frac{1}{z^2}\ln(\Psi(z)\cdot(-z))dz$ is also equal to zero when the contour tends to infinity. With this, the application of the generalized Littlewood theorem, which, "on equal footing", treats zeroes and poles, gives:

$$\frac{1}{\rho_0}+\sum_{n=1}^{\infty}(\frac{1}{\rho_n}+\frac{1}{n})=-\gamma \tag{23}$$

Now, during the summation, we need to group zeroes of the digamma function and factors $1/n$ in pairs (of course, there are many possibilities to do so). Recalling the definition $\gamma=\lim_{N\to\infty}(\ln N-\sum_{n=1}^{N}\frac{1}{n})$ [13], we can write a relation:

$$\lim_{N\to\infty}(\ln N+\sum_{n=0}^{N}\frac{1}{\rho_i})=0 \tag{24}$$

Analyzing the contour integral $\int_C \frac{1}{(z-p)^2}\ln(\Psi(z))dz$, with p not equal to $0,-1,-2,\dots$, and not coinciding with any zero of the digamma function, we obtain:

$$\sum_{n=1}^{\infty}(\frac{1}{\rho_n-p}+\frac{1}{n+p})=-\frac{\Psi\prime(p)}{\Psi(p)} \tag{25}$$

In particular, using the known expressions for the polygamma function on integers, for positive integer k, we write:

$$\sum_{n=1}^{\infty}(\frac{1}{\rho_n-k}+\frac{1}{n+k})=\frac{\gamma-H_{k-1}}{H_{k-1}^{(2)}-\frac{\pi^2}{6}},$$

where $H_k^{(l)}=\sum_{k=1}^{k-1}\frac{1}{k^l}$ are harmonic numbers.

The case when p coincides with some pole or zero does not make a real problem. Let us consider, for example:

$$\int_C \frac{1}{(z+n)^2}\ln(-\Psi(z)\cdot(z+n))dz.$$

We know:

$$\Psi(-n+x)=-\frac{1}{x}+H_n-\gamma+\sum_{k=1}^{\infty}(H_n^{(k+1)}+(-1)^{k+1}\zeta(k+1))x^k$$

where

$$-x\Psi(-n+x)=1+(\gamma-H_n)x-\sum_{k=1}^{\infty}(H_n^{(k+1)}+(-1)^{k+1}\zeta(k+1))x^{k+1}$$

and

$$\frac{1}{\rho_k+k}+\sum_{n=1,n\neq k}^{\infty}(\frac{1}{\rho_n+k}+\frac{1}{n-k})=H_n-\gamma \tag{26}$$

In the same fashion, from:

$$\int_C \frac{1}{(z-\rho_k)^2}\ln(\Psi(z)/(z-\rho_k))dz,$$

we obtain

$$\sum_{n=0,n\neq k}^{\infty}\left\{\frac{1}{\rho_n-\rho_k}+\frac{1}{n+\rho_k}\right\}=-\frac{1}{2}\frac{\Psi\prime\prime(\rho_k)}{\Psi\prime(\rho_k)} \tag{27}$$

Similarly, for Barnes's analog of the digamma function $\Psi_G(z)$, see [7]:

$$\sum_{n=1}^{\infty}\left\{\frac{1}{\beta_n}+\frac{1}{n}\right\}=-\gamma-\frac{1}{2}\ln(2\pi)+\frac{1}{2} \tag{28}$$

Here, β_n are zeroes of the Barnes digamma function; they all are real and simple [7]. For clarity of presentation, we count them in simple decreasing order, so that $\beta_{1,2}$ are positive roots while others are negative. We used Lemma 3.1 from [7]:

$$\ln(z\Psi_G(z))=(\gamma+\frac{1}{2}\ln(2\pi)-\frac{1}{2})z+O(z^2)$$

here.

Along the same line, zeroes of polygamma functions can be analyzed. For example, the trigamma function is defined, whenever appropriate, as:

$$\Psi\prime(z)=\frac{d\Psi}{dz}=\sum_{n=1}^{\infty}\frac{1}{(n+z)^2}.$$

More generally $\Psi^{(n)}(z)=\frac{d^n\Psi}{dz^n}=\sum\limits_{k=1}^{\infty}\frac{(-1)^{k+1}k!}{(k+z)^{k+1}}$. From Equation (22), we have:

$$\Psi^{(n)}(z)=\frac{(-1)^{n+1}n!}{z^{n+1}}+\sum_{k=0}^{\infty}(-1)^{n-1+k}\frac{(n+k)!}{k!}\zeta(n+k+1)z^k \tag{29}$$

Correspondingly:

$$\frac{1}{n!}\Psi^{(n)}(z)(-1)^{n+1}z^{n+1}=1+\sum_{k=0}^{\infty}(-1)^kC_{n+k}^k\zeta(n+k+1)z^{k+n+1}$$

where $C_{n+k}^k=\frac{(n+k)!}{n!k!}$ is a binomial coefficient.

First, we consider:

$$\int_C\frac{1}{z^2}\ln(\Psi^{(n)}(z)\cdot(-1)^{n+1}z^{n+1}))dz.$$

The point $z = 0$ is regular but we have poles of the $n+1$ order at $z = -1, -2, -3\ldots$ Thus, we obtain the relation $\sum\limits_{m=1}^{\infty}(\frac{k_{n,m}}{\eta_{n,m}}+\frac{n+1}{m})=0$. (Certainly, if we suppose that all zeroes are simple, then $\sum\limits_{k=1}^{\infty}(\frac{1}{\eta_{n,k}}+\frac{n+1}{k})=0$. The simplicity of all such zeroes, however, apparently is not proven.)

Similarly, for:

$$\int_C\frac{1}{z^{j+1}}\ln(\Psi^{(n)}(z)\cdot(-1)^{n+1}z^{n+1}))dz$$

with $2\le j\le n$ we have:

$$\sum_{k=1}^{\infty}(\frac{l_{n,k}}{\eta_{n,k}^j}-\frac{n+1}{(-k)^j})=0.$$

That is:

$$\sum_{k=1}^{\infty} \frac{l_{n,k}}{\eta_{n,r}^{j}} = (-1)^{j}(n+1)\zeta(j).$$

Finally, let us consider:

$$\int_{C} \frac{1}{z^{n+2}} \ln(\Psi^{(n)}(z) \cdot (-1)^{n+1} z^{n+1}))dz.$$

We know:

$$\ln(\Psi^{(n)}(z)(-1)^{n+1}z^{n+1}) = \zeta(n+1)z^{n+1} + O(z^{n+2})$$

so that

$$\varsigma(n+1) + \frac{1}{n+1}\sum_{k=1}^{\infty}(\frac{l_{n,k}}{\eta_{n,k}^{n+1}} - \frac{n+1}{(-k)^{n+1}}) = 0.$$

For odd n, thus simply $\sum\limits_{k=1}^{\infty} \frac{l_{n,k}}{\eta_{n,k}^{n+1}} = 0$. For even: $\sum\limits_{k=1}^{\infty} \frac{l_{n,k}}{\eta_{n,k}^{n+1}} = -2(n+1)\zeta(n+1)$. Sums, including larger powers of zeroes, such as $\sum\limits_{k=1}^{\infty} \frac{l_{n,k}}{\eta_{n,k}^{n+1+l}}$ with l = 1, 2, 3 . . . , can be obtained in the same fashion.

For clarity of presentation, we collect most of these results with the following theorem.

Theorem 2. *Let $\eta_{n,k}$ be zeroes of the polygamma function $\Psi^{(n)}(z)$, taken in increasing order of their module, and $l_{n,k}$ is an order of such zero. Then:*

- *for n = 1, 2, 3 . . .* $\sum\limits_{k=1}^{\infty}(\frac{l_{n,k}}{\eta_{n,k}} + \frac{n+1}{k}) = 0$
- *for n= 2, 3,4 . . . and* $2 \le j \le n$: $\sum\limits_{k=1}^{\infty} \frac{l_{n,k}}{\eta_{n,r}^{j}} = (-1)^{j}(n+1)\zeta(j)$
- *for n = 1, 2, 3 . . .* $\sum\limits_{k=1}^{\infty} \frac{l_{2n+1,k}}{\eta_{2n+1,k}^{2n+2}} = 0$, $\sum\limits_{k=1}^{\infty} \frac{l_{2n,k}}{\eta_{2n,k}^{2n+1}} = -2(2n+1)\zeta(2n+1)$.

5. Conclusions

We showed that the generalized Littlewood theorem concerning contour integrals of the logarithm of analytical function can be successfully applied to sum certain infinite series and analyze certain properties of zeroes and poles of certain analytical functions. In the paper, we have considered a few examples, and there is no doubt that numerous other applications of this approach will be found. One of the quite prospective fields for this, in the author's opinion, is elliptical functions [13,20], where, in particular, the asymptotic for large $|z|$ apparently should enable to establish theorems concerning the sums of the type $\sum \frac{k_i}{\rho_i}$, somewhat similarly to the digamma function considered in the paper.

Funding: This research received no external funding.

Data Availability Statement: Not applicable.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Sekatskii, S.K.; Beltraminelli, S.; Merlini, D. On equalities involving integrals of the logarithm of the Riemann ζ-function and equivalent to the Riemann hypothesis. *Ukr. Math. J.* **2012**, *64*, 218–228. [CrossRef]
2. Sekatskii, S.K.; Beltraminelli, S.; Merlini, D. On Equalities Involving Integrals of the Logarithm of the Riemann ζ-function with Exponential Weight which are Equivalent to the Riemann Hypothesis. *Int. J. Analysis* **2015**, 980728.
3. Titchmarsh, E.C. *The Theory of Functions*; Oxford Univ. Press: Oxford, UK, 1939.
4. Bromwich, T.J.I. *An Introduction to the Theory of Infinite Series*; McMillan and Co.: London, UK, 1926.
5. Knopp, K. *Theory and Application of Infinite Series (Dover Books on Mathematics)*; Dover Pub.: Mineola, NY, USA, 1990.
6. Hirschman, I.I. *Infinite Series (Dover Books on Mathematics)*; Dover Pub.: Mineola, NY, USA, 2014.

7. Mezö, I.; Hoffman, M.E. Zeroes of digamma function and its Barnes G-function analogue. *Int. Transform. Spec. Funct.* **2017**, *28*, 846–858. [CrossRef]
8. Flajolet, P.; Salvy, M. Euler sums and contour integral representations. *Exp. Math.* **1998**, *7*, 15–35. [CrossRef]
9. De Doelder, P.J. On some series containing $\Psi(x) - \Psi(y)$ and $(\Psi(x) - \Psi(y))^2$ for some values of x and y. *J. Comp. Appl. Math.* **1991**, *37*, 125–141. [CrossRef]
10. Mező, I. Some infinite sums arising from the Weierstrass Product Theorem. *Appl. Math. Comp.* **2013**, *219*, 9838–9846. [CrossRef]
11. Broughan, K.A. *Equivalents of the Riemann Hypothesis*; Cambridge Univ. Press: Cambridge, UK, 2017; Volume 2: Encyclopedia of Mathematics and Its Applications; p. 165.
12. Hansen, E.R. *A Table of Series and Products*; Prentice Hall: Englewood Cliffs, NJ, USA, 1975.
13. Bateman, H. *Higher Transcendental Functions*; McGraw-Hill Book Comp.: New York, NY, USA, 1953.
14. Titchmarsh, E.C.; Heath-Brown, E.R. *The Theory of the Riemann Zeta-Function*; Clarendon Press: Oxford, UK, 1988.
15. Li, X.-E. The positivity of a sequence of numbers and the Riemann hypothesis. *J. Numb. Theory* **1997**, *65*, 325–333. [CrossRef]
16. Sekatskii, S.K. Generalized Bombieri—Lagarias' theorem and generalized Li's criterion with its arithmetic interpretation. *Ukr. Math. J.* **2014**, *66*, 371–383. [CrossRef]
17. Sekatskii, S.K. On the generalized Li's criterion equivalent to the Riemann hypothesis and its first applications. In *Schrödinger Operators, Spectral Analysis and Number Theory: In Memory of Erik Balslev*; Springer Proceedings in Mathematics & Statistics; Albeverio, S., Balslev, A., Weder, R., Eds.; Springer Nature: Cham, Switzerland, 2021; pp. 241–254. [CrossRef]
18. Kölbig, K.S. Complex zeros of an incomplete Riemann zeta function and of the incomplete gamma function. *Math. Comp.* **1970**, *24*, 679–696. [CrossRef]
19. Kölbig, K.S. On the zeros of the incomplete gamma function. *Math. Comp.* **1972**, *26*, 751–755. [CrossRef]
20. Abramowitz, M.; Stegun, I.A. *Handbook of Mathematical Functions*; Dover Publications: New York, NY, USA, 1964.