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On the compound Poisson risk model with dependence and a threshold dividend strategy

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ABSTRACT

In this paper, we consider the compound Poisson risk model with a threshold dividend strategy and a dependence structure modeled by a Farlie–Gumbel–Morgenstern copula. The integro-differential equations satisfied by the Gerber–Shiu functions and the expected discounted dividend payments paid until ruin respectively are derived. Further, by deriving and solving the renewal equations satisfied by the Gerber–Shiu functions and the expected discounted dividend payments, we give the explicit formulas for them.

1. Introduction

Dividend strategy for insurance risk model was initially introduced by De Finetti (1957) for a binomial model. From then on, more general barrier strategies have been studied in several papers. See e.g., Lin et al. (2003), Gerber and Shiu (2006), Lin and Pavlova (2006) and Yang and Zhang (2008) and references therein. In the ruin theory, the researchers are mainly focused on the ruin and related quantities, such as the Gerber–Shiu function, and the expected discounted dividend payments paid until ruin. Lin et al. (2003) investigate the Gerber–Shiu function for the classical risk model with a constant dividend barrier. Later on, Lin and Pavlova (2006) study the Gerber–Shiu functions and related problems for the classical compound Poisson risk model with a threshold dividend strategy. Recently, Yang and Zhang (2008) consider the Gerber–Shiu function in a Sparre Andersen model with multi-layer dividend strategy. In Li et al. (2009), the authors give a closed form expression of the expected discounted dividend function for a jump–diffusion risk process by studying a constructed fluid flow process.

In recent years, the risk model with dependence structure between inter-arrival times and claim sizes has got more and more attention since the independence of them is not well applicable from the practical point of view. Albrecher and Boxma (2004) propose an extension of the compound Poisson risk model where the distribution of a claim interval is controlled by the previous claim size. Boudreault et al. (2006) consider a risk model with a reverse dependence structure where the distribution of the next claim size depends on the last inter-arrival time. Later on, Landriault (2008) studies the risk model with interclaim-dependent claim sizes and a constant dividend barrier. Cossette et al. (2010) propose a dependence structure between the claim amounts and the inter-arrival times which is introduced through a Farlie–Gumbel–Morgenstern (FGM) copula, where the defective renewal equation for the Gerber–Shiu function is obtained and solved. Zhang and Yang (2011) consider the Gerber–Shiu function in a perturbed risk model with the similar dependence structure as Cossette et al. (2010).

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In this paper, we consider the classical risk model with a threshold strategy and dependence between claim amounts and inter-arrival times modeled by a FGM copula. Integro-differential equations for the Gerber–Shiu functions and expected discounted dividend payments paid until ruin are first derived and then solved.

The rest of the paper is organized as follows. In Section 2, we give the description of the model. Integro-differential equations satisfied by the Gerber–Shiu functions are derived and solved in Section 3. Finally, in Section 4, we discuss the expected discounted dividend payments paid until ruin.

2. Model description

Consider the compound Poisson surplus process $\{U(t), t \geq 0\}$ given by

$$U(t) = u + ct - S(t), \quad t \geq 0, \tag{1}$$

where $u(\geq 0)$ is the insurer's initial surplus, $c(>0)$ is the rate of premium income, and $S(t) = \sum_{i=1}^{N(t)} X_i$ denotes the total amount of claims. $\{X_i\}_{i=1}^{\infty}$ are independent and identically distributed (i.i.d) random variables (r.v.'s) with common distribution function (df) denoted by F_X and probability density function (pdf) denoted by f_X (with Laplace transform denoted by f_X^*), representing the successive claim amounts, and $\{N(t), t \geq 0\}$ is a Poisson process with positive parameter λ, which is assumed to be independent of $\{X_i\}_{i=1}^{\infty}$, representing the number of claims until time t. Define $\{W_j, j \geq 1\}$ to be a sequence of inter-arrival times of the Poisson process. It is known that $W_j, j \geq 1$, are i.i.d with common pdf $p(t) = \lambda e^{-\lambda t}$. Obviously, $(X_j, W_j), j \geq 1$, are i.i.d random vectors. Further, the net profit condition is given by $c > \lambda \mathbb{E}(X_1)$.

In this paper, we consider a modified model of (1), i.e., we assume that the claim amounts $\{X_i\}_{i=1}^{\infty}$ and the inter-arrival times $\{W_j, j \geq 1\}$ are not independent but with a dependence structure modeled by a FGM copula. To be specific, recalling that (e.g., Denuit et al. (2005)) the FGM copula is defined, for $\theta \in [-1, 1]$, by

$$C_\theta^{\mathrm{FGM}}(u_1, u_2) = u_1 u_2 + \theta u_1 u_2 (1 - u_1)(1 - u_2), \quad u_1, u_2 \in [0, 1],$$

we assume that, for fixed $j \geq 1$, the joint pdf of (X_j, W_j) is defined by

$$f_{X,W}(x, t) = f_X(x)\lambda e^{-\lambda t} + \theta h_X(x)(2\lambda e^{-2\lambda t} - \lambda e^{-\lambda t}), \quad x, t \geq 0,$$

where $h_X(x) = (1 - 2F_X(x))f_X(x), x \geq 0$ (with its Laplace transform denoted by h_X^*). Furthermore, we suppose that the company pays dividends to its shareholders in the following way. When the surplus is below a threshold, say $b(\geq u)$, no dividend is paid. However, when it exceeds b, dividends are paid continuously at a rate $c - \alpha$ ($\alpha \in [0, c]$ and $\alpha > \lambda \mathbb{E}(X_1)$, providing a positive safety loading factor), thus the net premium rate after dividend payments becomes α. Let $\{U_b(t), t \geq 0\}$ denote the modified surplus process under this threshold dividend strategy described above, then it satisfies the following stochastic differential equation:

$$dU_b(t) = \begin{cases} cdt - dS(t), & U_b(t) < b, \\ \alpha dt - dS(t), & U_b(t) \geq b, \end{cases} \quad t \geq 0, \tag{2}$$

with $U_b(0) = u$.

Let $T_b = \inf\{t \geq 0, U_b(t) < 0\}(\inf\{\emptyset\} = \infty)$ be the time of ruin for the risk process (2). Define, for $\delta \geq 0$,

$$m(u; \alpha; b) = \mathbb{E}\left(e^{-\delta T_b} w(U_b(T_b-), |U_b(T_b)|) I(T_b < \infty) | U_b(0) = u\right), \quad u \geq 0,$$

to be the Gerber–Shiu function, where $w(\cdot, \cdot)$ is a non-negative function, $U_b(T_b-)$ and $|U_b(T_b)|$ are two important non-negative r.v.'s in connection with the time of ruin T_b, representing the surplus immediately before ruin and the deficit at ruin, respectively, and $I(\cdot)$ denotes the indicator function.

It is worth noting that $m(u; 0; b)$ denotes the Gerber–Shiu function of the risk model (2) with the constant barrier strategy (i.e., when the surplus exceeds b, all the premiums are paid as dividends), and $m(u; \alpha; \infty)$ denotes the Gerber–Shiu function of the risk model (2) with the dependence structure and without paying any dividend.

From the threshold dividend strategy described above, we see that the dividend distributing process, denoted by $\{D(t), t \geq 0\}$, satisfies the following stochastic differential equation:

$$dD(t) = \begin{cases} (c - \alpha)dt, & U_b(t) \geq b, \\ 0, & U_b(t) < b, \end{cases} \quad t \geq 0.$$

Next, define, for $\delta \geq 0$,

$$v(u; b) = \mathbb{E}\left(\int_0^{T_b} e^{-\delta t} dD(t) | U_b(0) = u\right), \quad u \geq 0,$$

to be the expected discounted dividend payments until ruin.

For notational simplicity, in the sequel, we will use the following notation:

$$m(u; \alpha; b) = \begin{cases} m_1(u), & 0 \leq u < b, \\ m_2(u), & u \geq b, \end{cases} \qquad \begin{cases} m(u) = m(u; 0; b), \\ m(u; \infty) = m(u; \alpha; \infty), \end{cases} \qquad v(u; b) = \begin{cases} v_1(u), & 0 \leq u < b, \\ v_2(u), & u \geq b. \end{cases} \tag{3}$$

We note that $m(u; \alpha; \infty)$ has nothing to do with α.

3. Analysis of the Gerber–Shiu functions

In this section, we will investigate the Gerber–Shiu function of the risk process (2). We first give some preliminary results for the Gerber–Shiu function $m(u)$, $u \in [0, b]$, under a constant barrier strategy.

Let $\mathcal{I}$ and $\mathcal{D}$ denote, respectively, the identity and the differential operators. Using a similar approach as in Landriault (2008), we derive that $m(u)$, $u \in [0, b]$, satisfies the integro-differential equation:

$$\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)m(u)=\frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\sigma_1(u)+\frac{\lambda\theta}{c}\left(\frac{\delta}{c}\mathcal{I}-\mathcal{D}\right)\sigma_2(u),\quad 0\le u\le b, \tag{4}$$

with boundary conditions

$$m'(b)=0,\qquad m''(b)=-\frac{\lambda}{c}\sigma_1'(b)-\frac{\lambda\theta}{c}\sigma_2'(b),$$

where

$$\sigma_1(u)=\int_0^u m(u-x)f_X(x)dx+\alpha_1(u),\qquad \sigma_2(u)=\int_0^u m(u-x)h_X(x)dx+\alpha_2(u),$$

$$\alpha_1(u)=\int_u^\infty w(u,x-u)f_X(x)dx,\qquad \alpha_2(u)=\int_u^\infty w(u,x-u)h_X(x)dx.$$

From the theory of integro-differential equation, we conclude that the general solution of Eq. (4) can be given as

$$m(u)=m(u;\infty)+\gamma_1 y_1(u)+\gamma_2 y_2(u),\quad 0\le u\le b, \tag{5}$$

where γ_1 and γ_2 are two constants, and $y_1(\cdot)$ and $y_2(\cdot)$ are two functions satisfying the integro-differential equation:

$$\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)y_i(u)=\frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\int_0^u y_i(u-x)f_X(x)dx$$
$$+\frac{\lambda\theta}{c}\left(\frac{\delta}{c}\mathcal{I}-\mathcal{D}\right)\int_0^u y_i(u-x)h_X(x)dx,\quad i=1,2. \tag{6}$$

Note that $m(u;\infty)$ has been discussed extensively in the literature, see e.g., Cossette et al. (2010).

Recall the Dickson–Hipp operator T_r, $r \in \mathbb{C}$, defined by

$$T_rf(x)=\int_x^\infty e^{-r(u-x)}f(u)du,\quad x\ge 0,$$

for an integrable real-valued function f. It is shown that

$$T_sf'(b)=sT_sf(b)-f(b)$$

where $f'(x)$ denotes the derivative of $f(x)$ (see Li and Garrido (2004) for more on the properties of the Dickson–Hipp operator).

By taking the Dickson–Hipp operator $T_s(*)(0)$ (multiply with e^{-ru} and then take integral from 0 to ∞) on both sides of (6), and then taking inverse of it, we see that $y_i(\cdot)$, $i = 1, 2$, satisfy the following defective renewal functions:

$$y_i(u)=\int_0^u y_i(u-x)h(x)dx+r_i(u),\quad i=1,2, \tag{7}$$

where

$$h(u)=\frac{\lambda}{c}T_{\rho_2}f_X(u)+\frac{\lambda\theta}{c}T_{\rho_2}h_X(u)+\frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}-\rho_1\right)T_{\rho_2}T_{\rho_1}f_X(u)+\frac{\lambda\theta}{c}\left(\frac{\delta}{c}-\rho_1\right)T_{\rho_2}T_{\rho_1}h_X(u),$$

$$r_1(u)=\frac{\rho_1-\frac{3\lambda+2\delta}{c}}{\rho_1-\rho_2}e^{\rho_1 u}+\frac{\rho_2-\frac{3\lambda+2\delta}{c}}{\rho_2-\rho_1}e^{\rho_2 u},\qquad r_2(u)=\frac{1}{\rho_1-\rho_2}e^{\rho_1 u}+\frac{1}{\rho_2-\rho_1}e^{\rho_2 u},$$

with ρ_1, ρ_2 being the two different positive real roots of the Lundberg's generalized equation:

$$g(s)=\left(\frac{2\lambda+\delta}{c}-s\right)\left(\frac{\lambda+\delta}{c}-s\right)-\frac{\lambda}{c}f_X^*(s)\left(\frac{2\lambda+\delta}{c}-s\right)-\frac{\lambda\theta}{c}h_X^*(s)\left(\frac{\delta}{c}-s\right)=0. \tag{8}$$

See Cossette et al. (2010) for the analysis of the Lundberg's generalized equation.

From the renewal equation theory, we can conclude that

$$y_i(u)=\sum_{n=0}^\infty\int_0^u r_i(u-x)dH^{n*}(x),\quad i=1,2, \tag{9}$$

where $H(x)=\int_0^x h(u)du$ and $H^{n*}(x)$ denotes the n-fold convolution of $H(x)$ with itself.

Next we give the integro-differential equations satisfied by the Gerber–Shiu function $m(u; \alpha; b)$ (recall (3)).

Theorem 3.1. *The Gerber–Shiu function $m(u;\alpha;b)$ satisfies the following integro-differential equations:*

$$\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)m_1(u)=\frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)\pi_1(u)+\frac{\lambda\theta}{c}\left(\frac{\delta}{c}\mathcal{I}-\mathcal{D}\right)\pi_2(u),\quad 0\le u<b,\quad (10)$$

and

$$\begin{aligned}\left(\frac{\lambda+\delta}{\alpha}\mathcal{I}-\mathcal{D}\right)\left(\frac{2\lambda+\delta}{\alpha}\mathcal{I}-\mathcal{D}\right)m_2(u)&=\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}\mathcal{I}-\mathcal{D}\right)\\&\times\left(\int_0^{u-b}m_2(u-x)f_X(x)dx+\int_{u-b}^{u}m_1(u-x)f_X(x)dx+\int_u^\infty\omega(u,x-u)f_X(x)dx\right)+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}\mathcal{I}-\mathcal{D}\right)\\&\times\left(\int_0^{u-b}m_2(u-x)h_X(x)dx+\int_{u-b}^{u}m_1(u-x)h_X(x)dx+\int_u^\infty\omega(u,x-u)h_X(x)dx\right),\quad u\ge b,\end{aligned}\quad (11)$$

with boundary condition

$$m_1(b)=m_2(b),\quad (12)$$

where

$$\pi_1(u)=\int_0^u m_1(u-x)f_X(x)dx+\beta_1(u),\quad \beta_1(u)=\int_u^\infty\omega(u,x-u)f_X(x)dx,\quad (13)$$

$$\pi_2(u)=\int_0^u m_1(u-x)h_X(x)dx+\beta_2(u),\quad \beta_2(u)=\int_u^\infty\omega(u,x-u)h_X(x)dx.\quad (14)$$

Proof. By conditioning on the time and the amount of the first claim, we have

$$\begin{aligned}m_1(u)=&\int_0^{\frac{b-u}{c}}\int_0^{u+ct}e^{-\delta t}m(u+ct-x;\alpha;b)\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)\left(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t}\right)\right]dxdt\\&+\int_0^{\frac{b-u}{c}}\int_{u+ct}^{\infty}e^{-\delta t}\omega(u+ct,x-u-ct)\times\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)\left(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t}\right)\right]dxdt\\&+\int_{\frac{b-u}{c}}^{\infty}\int_0^{b+\alpha\left(t-\frac{b-u}{c}\right)}e^{-\delta t}m\left(b+\alpha\left(t-\frac{b-u}{c}\right)-x;\alpha;b\right)\\&\times\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t})\right]dxdt\\&+\int_{\frac{b-u}{c}}^{\infty}\int_{b+\alpha\left(t-\frac{b-u}{c}\right)}^{\infty}e^{-\delta t}\omega\left(b+\alpha\left(t-\frac{b-u}{c}\right);x-b-\alpha\left(t-\frac{b-u}{c}\right)\right)\\&\times\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t})\right]dxdt.\end{aligned}\quad (15)$$

Eq. (15) can be rewritten as

$$\begin{aligned}m_1(u)=&\lambda\int_0^{\frac{b-u}{c}}e^{-(\lambda+\delta)t}(\pi_1(u+ct)-\theta\pi_2(u+ct))dt+2\lambda\theta\int_0^{\frac{b-u}{c}}e^{-2(\lambda+\delta)t}\pi_2(u+ct)dt\\&+\lambda\int_{\frac{b-u}{c}}^{\infty}e^{-(\lambda+\delta)t}\left(\pi_1\left(b+\alpha\left(t-\frac{b-u}{c}\right)\right)-\theta\pi_2\left(b+\alpha\left(t-\frac{b-u}{c}\right)\right)\right)dt\\&+2\lambda\theta\int_{\frac{b-u}{c}}^{\infty}e^{-2(\lambda+\delta)t}\pi_2\left(b+\alpha\left(t-\frac{b-u}{c}\right)\right)dt,\end{aligned}\quad (16)$$

where $\pi_1(u)$ and $\pi_2(u)$ are given in Eqs. (13) and (14), respectively.
Letting $u+ct=v$ in Eq. (16), we further have

$$\begin{aligned}m_1(u)=&\frac{\lambda}{c}\int_u^b e^{-\frac{(\lambda+\delta)(v-u)}{c}}(\pi_1(v)-\theta\pi_2(v))dv+\frac{2\lambda\theta}{c}\int_u^b e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2(v)dv\\&+\frac{\lambda}{c}\int_b^\infty e^{-\frac{(\lambda+\delta)(v-u)}{c}}\left(\pi_1\left(b+\alpha\left(\frac{v-b}{c}\right)\right)-\theta\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)\right)dv\\&+\frac{2\lambda\theta}{c}\int_b^\infty e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)dv.\end{aligned}\quad (17)$$

Differentiating Eq. (17) with respect to (w.r.t.) u leads to

$$m_1'(u) = \frac{\lambda}{c}\frac{\lambda+\delta}{c}\int_u^b e^{-\frac{(\lambda+\delta)(v-u)}{c}}(\pi_1(v)-\theta\pi_2(v))dv + \frac{2\lambda\theta}{c}\frac{2\lambda+\delta}{c}\int_u^b e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2(v)dv + \frac{\lambda}{c}\frac{\lambda+\delta}{c}\int_b^\infty e^{-\frac{(\lambda+\delta)(v-u)}{c}}\left(\pi_1\left(b+\alpha\left(\frac{v-b}{c}\right)\right)-\theta\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)\right)dv + \frac{2\lambda\theta}{c}\frac{2\lambda+\delta}{c}\int_b^\infty e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)dv - \frac{\lambda}{c}\pi_1(u) - \frac{\lambda\theta}{c}\pi_2(u). \tag{18}$$

It follows easily from the last two equations that

$$\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)m_1(u) = -\frac{\lambda}{c}\frac{2\lambda\theta}{c}\int_u^b e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2(v)dv - \frac{\lambda}{c}\frac{2\lambda\theta}{c} \times \int_b^\infty e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)dv + \frac{\lambda}{c}\pi_1(u) + \frac{\lambda\theta}{c}\pi_2(u). \tag{19}$$

Differentiating Eq. (19) w.r.t. u yields

$$\mathcal{D}\left(\frac{\lambda+\delta}{c}\mathcal{I}-\mathcal{D}\right)m_1(u) = -\frac{2\lambda+\delta}{c}\frac{\lambda}{c}\frac{2\lambda\theta}{c}\int_u^b e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2(v)dv + \frac{\lambda}{c}\pi_1'(u) + \frac{\lambda\theta}{c}\pi_2'(u) + \frac{\lambda}{c}\frac{2\lambda\theta}{c}\pi_2(u) - \frac{2\lambda+\delta}{c}\frac{\lambda}{c}\frac{2\lambda\theta}{c}\int_b^\infty e^{-\frac{(2\lambda+\delta)(v-u)}{c}}\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)dv. \tag{20}$$

Consequently, we have from Eqs. (19) and (20) that Eq. (10) is established.

Similarly, we obtain that

$$m_2(u) = \int_0^\infty\int_0^{u+\alpha t} e^{-\delta t}m(u+\alpha t-x;b)\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)\left(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t}\right)\right]dxdt + \int_0^\infty\int_{u+\alpha t}^\infty \omega(u+\alpha t, x-u-\alpha t)\left[f_X(x)\lambda e^{-\lambda t}+\theta h_X(x)\left(2\lambda e^{-2\lambda t}-\lambda e^{-\lambda t}\right)\right]dxdt.$$

Thus, Eq. (11) can be proved by using the same arguments as Eq. (10). The proof is completed. □

Next let us recall the divided differences of a function h with respect to distinct numbers $s, r_1, r_2, \ldots$, defined recursively as follows:

$$h[r_1,s] = \frac{h(s)-h(r_1)}{s-r_1}, \quad h[r_1,r_2,s] = \frac{h[r_1,s]-h[r_1,r_2]}{s-r_2}$$
$$h[r_1,r_2,r_3,s] = \frac{h[r_1,r_2,s]h[r_1,r_2,r_3]}{s-r_3},$$

and so on. The following theorem is our main result of this section.

Theorem 3.2. *The Gerber–Shiu function $m(u;\alpha;b)$ is given by*

$$m(u;\alpha;b) = \begin{cases} m(u;\infty)+\xi_1 y_1(u)+\xi_2 y_2(u), & 0\le u<b,\\ k_1(u)+\xi_1 k_2(u)+\xi_2 k_3(u), & u\ge b,\end{cases} \tag{21}$$

where ξ_1, ξ_2 are two constants determined in Lemma 3.1 given below, $y_i(\cdot)$, $i=1,2$, are the same as in (9),

$$k_1(u) = \sum_{n=0}^\infty\left(\int_0^{u-b}A(u-x)dH_\alpha^{n*}(x)+\int_0^{u-b}dH_\alpha^{n*}(x)\int_0^b m(y;\infty)R(u-x-y)dy\right),$$
$$k_2(u) = \sum_{n=0}^\infty\int_0^{u-b}dH_\alpha^{n*}(x)\int_0^b y_1(y)R(u-x-y)dy,$$
$$k_3(u) = \sum_{n=0}^\infty\int_0^{u-b}dH_\alpha^{n*}(x)\int_0^b y_2(y)R(u-x-y)dy,$$
$$h_\alpha(u) = \frac{\lambda}{\alpha}T_{\rho_4}f_X(u)+\frac{\lambda\theta}{\alpha}T_{\rho_4}h_X(u)+\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}-\rho_3\right)T_{\rho_4}T_{\rho_3}f_X(u)+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}-\rho_3\right)T_{\rho_4}T_{\rho_3}h_X(u),$$

$$R(u)=\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}-\rho_3\right)T_{\rho_3}T_{\rho_4}f_X(u)+\frac{\lambda}{\alpha}T_{\rho_4}f_X(u)+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}-\rho_3\right)T_{\rho_3}T_{\rho_4}h_X(u)+\frac{\lambda\theta}{\alpha}T_{\rho_4}h_X(u),$$

$$A(u)=T_{\rho_3}T_{\rho_4}\left\{\frac{\lambda}{\alpha}\frac{2\lambda+\delta}{\alpha}\beta_1(u)-\frac{\lambda}{\alpha}\beta_1'(u)+\frac{\lambda\theta}{\alpha}\frac{\delta}{\alpha}\beta_2(u)-\frac{\lambda\theta}{\alpha}\beta_2'(u)\right\}(u),\qquad H_\alpha(u)=\int_0^u h_\alpha(x)dx,$$

and H_α^{n*} denotes n-fold convolution of $H_\alpha(u)$ with itself.

Proof. We note that Eq. (10) coincides with Eq. (4). Consequently, the expression of $m_1(u)$, $0\le u<b$, follows easily. Next we focus on $m_2(u)$, $u\ge b$.

Taking the Dickson–Hipp operator $T_s(*)(b)$ on both sides of Eq. (11), we obtain

$$\begin{aligned}
&-m_2'(b)+s(sT_sm_2(b)-m_2(b))-\frac{3\lambda+2\delta}{\alpha}(sT_sm_2(b)-m_2(b))+\frac{\lambda+\delta}{\alpha}\frac{2\lambda+\delta}{\alpha}T_sm_2(b)\\
&=\frac{\lambda}{\alpha}\frac{2\lambda+\delta}{\alpha}\left(f_X^*(s)T_sm_2(b)+\int_0^b m_1(y)T_sf_X(b-y)dy+T_s\beta_1(b)\right)\\
&\quad-\frac{\lambda}{\alpha}\left(sf_X^*(s)T_sm_2(b)+s\int_0^b m_1(y)T_sf_X(b-y)dy\right)-\frac{\lambda}{\alpha}\left(-\int_0^b m_1(b-y)f_X(x)dy+T_s\beta_1'(b)\right)\\
&\quad+\frac{\lambda\theta}{\alpha}\frac{\delta}{\alpha}\left(h_X^*(s)T_sm_2(b)+\int_0^b m_1(y)T_sh_X(b-y)dy+T_s\beta_2(b)\right)\\
&\quad-\frac{\lambda\theta}{\alpha}\left(sh_X^*(s)T_sm_2(b)+s\int_0^b m_1(y)T_sh_X(b-y)dy\right)-\frac{\lambda\theta}{\alpha}\left(-\int_0^b m_1(y)h_X(b-y)dy+T_s\beta_2'(b)\right),
\end{aligned}$$

which can be rewritten as

$$\begin{aligned}
&\left\{\left(\frac{2\lambda+\delta}{\alpha}-s\right)\left(\frac{\lambda+\delta}{\alpha}-s\right)-\frac{\lambda}{\alpha}f_X^*(s)\left(\frac{2\lambda+\delta}{\alpha}-s\right)-\frac{\lambda\theta}{\alpha}h_X^*(s)\left(\frac{\delta}{\alpha}-s\right)\right\}T_sm_2(b)\\
&=\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}-s\right)\int_0^b m_1(y)T_sf_X(b-y)dy+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}-s\right)\int_0^b m_1(y)T_sh_X(b-y)dy\\
&\quad+\frac{\lambda}{\alpha}\int_0^b m_1(y)f_X(b-y)dy+\frac{\lambda\theta}{\alpha}\int_0^b m_1(y)h_X(b-y)dy+m_2'(b)\\
&\quad+\left(s-\frac{3\lambda+2\delta}{\alpha}\right)m_2(b)+\frac{\lambda}{\alpha}\frac{2\lambda+\delta}{\alpha}T_s\beta_1(b)-\frac{\lambda}{\alpha}T_s\beta_1'(b)+\frac{\lambda\theta}{\alpha}\frac{\delta}{\alpha}T_s\beta_2(b)-\frac{\lambda\theta}{\alpha}T_s\beta_2'(b).
\end{aligned}\tag{22}$$

It is observed that $g_0(s):=(\frac{2\lambda+\delta}{\alpha}-s)(\frac{\lambda+\delta}{\alpha}-s)-\frac{\lambda}{\alpha}f_X^*(s)(\frac{2\lambda+\delta}{\alpha}-s)-\frac{\lambda\theta}{\alpha}h_X^*(s)(\frac{\delta}{\alpha}-s)$ has the same form as $g(s)$ in (8), and thus it has also two distinct positive roots, denoted by ρ_3 and ρ_4, respectively.

Considering the property of Dickson–Hipp operator and dividend difference operator, we derive that

$$\begin{aligned}
&\left\{\left(\frac{2\lambda+\delta}{\alpha}-s\right)\left(\frac{\lambda+\delta}{\alpha}-s\right)-\frac{\lambda}{\alpha}f_X^*(s)\left(\frac{2\lambda+\delta}{\alpha}-s\right)-\frac{\lambda\theta}{\alpha}h_X^*(s)\left(\frac{\delta}{\alpha}-s\right)\right\}T_sm_2(b)[s,\rho_3,\rho_4]\\
&\quad=(1-T_sh_\alpha(0))T_sm_2(b)\\
&\left\{\frac{\lambda}{\alpha}\int_0^b m_1(y)f_X(b-y)dy+\frac{\lambda\theta}{\alpha}\int_0^b m_1(y)h_X(b-y)dy+m_2'(b)+\left(s-\frac{3\lambda+2\delta}{\alpha}\right)m_2(b)\right\}[s,\rho_1,\rho_2]=0\\
&\left\{\frac{\lambda}{\alpha}\frac{2\lambda+\delta}{\alpha}T_s\beta_1(b)-\frac{\lambda}{\alpha}T_s\beta_1'(b)+\frac{\lambda\theta}{\alpha}\frac{\delta}{\alpha}T_s\beta_2(b)-\frac{\lambda\theta}{\alpha}T_s\beta_2'(b)\right\}[s,\rho_1,\rho_2]=T_sA(b)\\
&\left\{\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}-s\right)\int_0^b m_1(y)T_sf_X(b-y)dy+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}-s\right)\int_0^b m_1(y)T_sh_X(b-y)dy\right\}[s,\rho_3,\rho_4]\\
&\quad=\frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}-\rho_3\right)\int_0^b m_1(y)T_sT_{\rho_3}T_{\rho_4}f_X(b-y)dy+\frac{\lambda}{\alpha}\int_0^b m_1(y)T_sT_{\rho_4}f_X(b-y)dy\\
&\quad\quad+\frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}-\rho_3\right)\int_0^b m_1(y)T_sT_{\rho_3}T_{\rho_4}h_X(b-y)dy+\frac{\lambda\theta}{\alpha}\int_0^b m_1(y)T_sT_{\rho_4}h_X(b-y)dy\\
&\quad=\int_0^b m_1(y)T_sR(b-y)dy.
\end{aligned}$$

Consequently, taking divided difference operator with respect to ρ_3, ρ_4 on both sides of Eq. (22) leads to

$$(1-T_s h_\alpha(0))T_s m_2(b)=\int_0^b m_1(y)T_s R(b-y)dy+T_s A(b).$$

Taking the inverse of the Dickson–Hipp operator gives

$$m_2(u)=\int_0^{u-b} m_2(u-y)h_\alpha(y)dy+\int_{u-b}^{u} m_1(u-y)R(y)dy+A(u),\quad u\geq b. \tag{23}$$

From renewal equation theory, we conclude that

$$m_2(u)=\sum_{n=0}^{\infty}\int_0^{u-b}\left(\int_0^b m_1(y)R(u-x-y)dy+A(u-x)\right)dH_\alpha^{n*}(x),\quad u>b.$$

Inserting the expression of $m_1(u)$ given in Eq. (21) into equation above, we have that

$$m_2(u)=k_1(u)+\xi_1 k_2(u)+\xi_2 k_3(u),\quad u\geq b,$$

and thus the proof is complete. □

In the next lemma, we give system of linear equations satisfied by the constants ξ_i, $i=1,2$, appearing in Eq. (21).

Lemma 3.1. *The constants ξ_1 and ξ_2 satisfy the following system of linear equations:*

$$\begin{aligned}
&\xi_1\left(\frac{\lambda}{\alpha}c_2\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{\lambda\theta}{\alpha}c_5\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{2\lambda\theta}{\alpha}c_5\left(\frac{2\lambda+\delta}{\alpha}\right)-y_1(b)\right)\\
&\quad+\xi_2\left(\frac{\lambda}{\alpha}c_3\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{\lambda\theta}{\alpha}c_6\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{2\lambda\theta}{\alpha}c_6\left(\frac{2\lambda+\delta}{\alpha}\right)-y_2(b)\right)\\
&\ =m(b;\infty)-\frac{\lambda}{\alpha}c_1\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{\lambda\theta}{\alpha}c_4\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{2\lambda\theta}{\alpha}c_4\left(\frac{2\lambda+\delta}{\alpha}\right),
\end{aligned} \tag{24}$$

$$\xi_1\left(y_1(b)-(y_1*R)(b)\right)+\xi_2\left(y_2(b)-(y_2*R)(b)\right)=-m(b;\infty)+(m(*;\infty)*R)(b)+A(b) \tag{25}$$

where the functions $R(\cdot)$, $A(\cdot)$, $k_i(\cdot)$, $i=1,2,3$, are given as in Theorem 3.2,

$$\begin{aligned}
c_1(a)&=T_a\beta_1(b)+\int_0^b m(x;\infty)T_a f_X(b-x)dx,+\int_b^\infty e^{-a(u-b)}du\int_b^u k_1(x)f_X(u-x)dx,\\
c_2(a)&=\int_0^b y_1(x)T_a f_X(b-x)dx+\int_b^\infty e^{-a(u-b)}du\int_b^u k_2(x)f_X(u-x)dx,\\
c_3(a)&=\int_0^b y_2(x)T_a f_X(b-x)dx+\int_b^\infty e^{-a(u-b)}du\int_b^u k_3(x)f_X(u-x)dx,\\
c_4(a)&=T_a\beta_2(b)+\int_0^b m(x;\infty)T_a h_X(b-x)dx,+\int_b^\infty e^{-a(u-b)}du\int_b^u k_1(x)h_X(u-x)dx,\\
c_5(a)&=\int_0^b y_1(x)T_a h_X(b-x)dx+\int_b^\infty e^{-a(u-b)}du\int_b^u k_2(x)h_X(u-x)dx,\\
c_6(a)&=\int_0^b y_2(x)T_a h_X(b-x)dx+\int_b^\infty e^{-a(u-b)}du\int_b^u k_3(x)h_X(u-x)dx,
\end{aligned}$$

and $(y_i*R)(b)=\int_0^b y_i(b-x)R(x)dx$, $i=1,2$.

Proof. Taking Dickson–Hipp operator $T_a(*)(b)$ for $\pi_1(u)$ given in (13), we have that

$$\begin{aligned}
\int_b^\infty e^{-a(u-b)}\pi_1(u)du&=\int_b^\infty e^{-a(u-b)}\left(\int_0^u m(x;b)f_X(u-x)dx+\beta_1(u)\right)du\\
&=\int_b^\infty e^{-a(u-b)}\int_0^u m(x;b)f_X(u-x)dxdu+T_a\beta_1(b)\\
&=\int_b^\infty e^{-a(u-b)}\left(\int_0^b m_1(x)f_X(u-x)dx+\int_b^u m_2(x)f_X(u-x)dx\right)du+T_a\beta_1(b).
\end{aligned}$$

Inserting the expressions of $m_1(u)$ and $m_2(u)$ given in Theorem 3.2 into the last equation, we obtain

$$\int_b^\infty e^{-a(u-b)}\pi_1(u)du = c_1(a) + c_2(a)\xi_1 + c_3(a)\xi_2. \tag{26}$$

Similarly, we derive that

$$\int_b^\infty e^{-a(u-b)}\pi_2(u)du = c_4(a) + c_5(a)\xi_1 + c_6(a)\xi_2. \tag{27}$$

Considering Eq. (17) for $u = b$, we have

$$\begin{aligned} m_1(b) = {} & \frac{\lambda}{c}\int_b^\infty e^{-\frac{(\lambda+\delta)(v-b)}{c}}\left(\pi_1\left(b+\alpha\left(\frac{v-b}{c}\right)\right) - \theta\pi_2\left(b+\alpha\left(\frac{v-b}{c}\right)\right)\right)dv \\ & + \frac{2\lambda\theta}{c}\int_b^\infty e^{-\frac{(2\lambda+\delta)(v-b)}{c}}\pi_2\left(b+\alpha\left(b+\alpha\left(\frac{v-b}{c}\right)\right)\right)dv. \end{aligned} \tag{28}$$

Let $b + \alpha(\frac{v-b}{c}) = u$, then Eq. (28) can be rewritten as

$$m_1(b) = \frac{\lambda}{\alpha}\int_b^\infty e^{-\frac{(\lambda+\delta)(u-b)}{\alpha}}(\pi_1(u) - \theta\pi_2(u))du + \frac{2\lambda\theta}{\alpha}\int_b^\infty e^{-\frac{(2\lambda+\delta)(u-b)}{\alpha}}\pi_2(u)du. \tag{29}$$

Inserting Eqs. (26)–(27) and the expressions of $m_1(u)$ given in Theorem 3.2 into Eq. (29), we conclude that Eq. (24) is valid. Further, Eq. (25) follows from Eq. (12), Eq. (23) and the expressions of $m_1(u)$. Therefore the proof is complete. □

4. Expected discounted dividend payments

In this section, by the similar approach, we obtain an analytical expression for expected discounted dividend payments with a threshold dividend strategy and we only give the results here.

Theorem 4.1. *The expected discounted dividend payments $v(u; b)$ satisfy the following equations:*

$$\left(\frac{\lambda+\delta}{c}\mathcal{I} - \mathcal{D}\right)\left(\frac{2\lambda+\delta}{c}\mathcal{I} - \mathcal{D}\right)v_1(u) = \frac{\lambda}{c}\left(\frac{2\lambda+\delta}{c}\mathcal{I} - \mathcal{D}\right)\pi_3(u) + \frac{\lambda\theta}{c}\left(\frac{\delta}{c}\mathcal{I} - \mathcal{D}\right)\pi_4(u), \quad 0 \le u \le b,$$

and

$$\begin{aligned} & \left(\frac{\lambda+\delta}{\alpha}\mathcal{I} - \mathcal{D}\right)\left(\frac{2\lambda+\delta}{\alpha}\mathcal{I} - \mathcal{D}\right)v_2(u) \\ & = \frac{\lambda}{\alpha}\left(\frac{2\lambda+\delta}{\alpha}\mathcal{I} - \mathcal{D}\right)\left(\int_0^{u-b} v_2(u-x)f_X(x)dx + \int_{u-b}^u v_1(u-x)f_X(x)dx\right) \\ & \quad + \frac{\lambda\theta}{\alpha}\left(\frac{\delta}{\alpha}\mathcal{I} - \mathcal{D}\right)\left(\int_0^{u-b} v_2(u-x)h_X(x)dx + \int_{u-b}^u v_1(u-x)h_X(x)dx\right) + \frac{c-\alpha}{\alpha}\frac{2\lambda+\delta}{\alpha}, \quad u > b, \end{aligned}$$

with boundary condition

$$v_1(b) = v_2(b),$$

where

$$\pi_3(u) = \int_0^u v(u-x; b)f_X(x)dx, \qquad \pi_4(u) = \int_0^u v(u-x; b)h_X(x)dx.$$

Theorem 4.2. *The expected discounted dividend payments $v(u; b)$ are given by*

$$v(u; b) = \begin{cases} \eta_1 y_1(u) + \eta_2 y_2(u), & 0 \le u \le b, \\ k_0(u) + \eta_1 k_2(u) + \eta_2 k_3(u), & u \ge b, \end{cases}$$

where η_1, η_2 are two constants determined in Lemma 4.1 given below, $y_i(\cdot)$, $i = 1, 2$, are given in Eq. (9), $k_i(\cdot)$, $i = 2, 3$, can be found in Theorem 3.2 and

$$k_0(u) = \sum_{n=0}^{\infty} \frac{c-\alpha}{\alpha}\frac{2\lambda+\delta}{\alpha}\frac{1}{\rho_3\rho_4}H_\alpha^{n*}(u-b).$$

Lemma 4.1. *The constants η_1 and η_2 appearing in Theorem 4.2 satisfy the following system of linear equations:*

$$\eta_1\left(\frac{\lambda}{\alpha}c_2\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{\lambda\theta}{\alpha}c_5\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{2\lambda\theta}{\alpha}c_5\left(\frac{2\lambda+\delta}{\alpha}\right)-y_1(b)\right)$$
$$+\eta_2\left(\frac{\lambda}{\alpha}c_3\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{\lambda\theta}{\alpha}c_6\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{2\lambda\theta}{\alpha}c_6\left(\frac{2\lambda+\delta}{\alpha}\right)-y_2(b)\right)$$
$$=-\frac{\lambda}{\alpha}c_0\left(\frac{\lambda+\delta}{\alpha}\right)+\frac{\lambda\theta}{\alpha}c_7\left(\frac{\lambda+\delta}{\alpha}\right)-\frac{2\lambda\theta}{\alpha}c_7\left(\frac{2\lambda+\delta}{\alpha}\right),$$
$$\eta_1\left(y_1(b)-(y_1*R)(b)\right)+\eta_2\left(y_2(b)-(y_2*R)(b)\right)=\frac{c-\alpha}{\alpha}\frac{2\lambda+\delta}{\alpha}\frac{1}{\rho_3\rho_4},$$

where the functions $R(\cdot)$, $k_i(\cdot)$, $i=2,3$ are given as in Theorem 3.2, $k_0(\cdot)$ is given in Theorem 4.2, $c_i(\cdot)$, $i=2,3,5,6$, can be found in Lemma 3.1 and

$$c_0(a)=\int_b^\infty e^{-a(u-b)}du\int_b^u k_0(x)f_X(u-x)dx,\qquad c_7(a)=\int_b^\infty e^{-a(u-b)}du\int_b^u k_0(x)h_X(u-x)dx.$$

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