PROBABILITY
AND
MATHEMATICAL STATISTICS
Vol. 0, Fasc. 0 (0000), pp. 000–000

LIMIT PROPERTIES OF EXCEEDANCES POINT PROCESSES OF SCALED STATIONARY GAUSSIAN SEQUENCES

BY

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Abstract. We derive the limiting distributions of exceedances point processes of randomly scaled weakly dependent stationary Gaussian sequences under some mild asymptotic conditions. In the literature analogous results are available only for contracted stationary Gaussian sequences. In this paper, we include additionally the case of randomly inflated stationary Gaussian sequences with a Weibullian type random scaling. It turns out that the maxima and minima of both contracted and inflated weakly dependent stationary Gaussian sequences are asymptotically independent.

2000 AMS Mathematics Subject Classification: Primary: 60F05; Secondary: 60G15.

Key words and phrases: Stationary Gaussian sequence, exceedances point processes, maxima, minima, joint limit distribution, random contraction, random scaling, Weibullian tail behaviour.

* Research supported by the National Natural Science Foundation of China under grant 11171275 and the Natural Science Foundation Project of CQ under cstc2012jjA00029.

** Research partially supported by the Swiss National Science Foundation grant 200021-134785 and by the project RARE -318984 (a Marie Curie IRSES Fellowship within the 7th European Community Framework Programme)

1. INTRODUCTION

Let $X_n, n \geqslant 1$ be a standard stationary Gaussian sequence (ssGs) i.e., X_n's are $N(0,1)$ distributed and $\rho(n) = \mathbb{E}(X_1 X_{n+1}) = \mathbb{E}(X_j X_{n+j})$ for any $j \geqslant 1$. In the seminal contribution [3], S.M. Berman proved that the maxima $\tilde{M}_n = \max_{1 \leqslant k \leqslant n} X_k$ converges in distribution after normalization to a unit Gumbel random variable, i.e.,

$$\lim_{n\to\infty} \mathbb{P}\left(\tilde{M}_n \leqslant \tilde{a}_n x + \tilde{b}_n\right) = \exp(-\exp(-x)) =: \Lambda(x), \quad \forall x \in \mathbb{R},$$

provided that the so-called Berman condition

$$\lim_{n\to\infty} \rho(n) \ln n = 0 \tag{1.1}$$

holds, where the norming constants $\tilde{a}_n$ and $\tilde{b}_n$ are given by

$$\tilde{a}_n = \frac{1}{\sqrt{2\ln n}} \qquad \text{and} \qquad \tilde{b}_n = \sqrt{2\ln n} - \frac{\ln\ln n + \ln 4\pi}{2\sqrt{2\ln n}}.$$

Moreover, the maxima and the minima $\tilde{m}_n = \min_{1 \leqslant k \leqslant n} X_k$ are asymptotically independent, cf. [4] and [10].

In applications, commonly the observations are randomly scaled, say due to some inflation or deflation effects if financial losses are modeled, or caused by measurement errors if observations are the outcome of a certain physical experiment. Therefore, in order to model some general random scaling phenomena applicable to original data, in this paper we consider $Y = SX, Y_n = S_n X_n, n \geqslant 1$ assuming that $S, S_n, n \geqslant 1$ are independent non-negative random variables with common distribution function F being further independent of the standard Gaussian random variables $X, X_n, n \geqslant 1$.

As shown in [7] if F has a finite upper endpoint $x_F \in (0, \infty)$ and its survival function is regularly varying, then the maxima $M_n = \max_{1 \leqslant k \leqslant n} Y_k$ converge in distribution after normalization to a unit Gumbel random variable with distribution

function Λ, provided that the Berman condition holds.
If $x_F = \infty$ and $X_n, n \geqslant 1$ are iid $N(0,1)$ the convergence of maxima M_n is shown under a different normalization in [8] assuming further that F has a Weibullian tail behaviour (see below (2.1)).

The objective of the paper is twofold: first for F with a Weibullian tail behaviour, it is of interest to establish the convergence of maxima of a randomly scaled ssGs under the Berman condition; there is no result in the literature covering this case. Secondly, for both cases x_F is a positive constant, and $x_F = \infty$, we aim at establishing the same result as in [4], i.e., the asymptotic independence of maxima and minima of randomly scaled weakly dependent ssGs .
Since by using a point process approach also the joint limiting distribution of upper and lower order statistics can be easily established, we choose in this paper a point process framework considering exceedances point processes. Numerous authors dealt with the asymptotic behavior of exceedances point processes; for weakly dependent stationary sequences including Gaussian, see [10, 12, 9, 6, 1, 2, 11] and the references therein.

For $u_n(s) = a_n s + b_n, s \in \mathbb{R}$, with $a_n > 0, b_n \in \mathbb{R}$ we shall investigate the weak convergence of bivariate point processes of exceedances of levels $u_n(x)$ and $-u_n(y)$ formed by $Y_n, n \geqslant 1$. Setting $\xi_1(n) = Y_n, \xi_2(n) = -Y_n$ for $n \geqslant 1$ we define as in [14] the bivariate exceedances point processes

$$\mathbf{N}_n(\mathbf{B}, \mathbf{x}) = \sum_{d=1}^{2} \sum_{i=1}^{n} \mathrm{I}\left(\xi_d(i) > u_n(x_d), \frac{i}{n} \in B_d\right) \tag{1.2}$$

for $\mathbf{B} = \bigcup_{d=1}^{2} (B_d \times \{d\})$ with B_d the Borel set on $(0,1], d = 1, 2$, where $\mathrm{I}(\cdot)$ denotes the indicator function. The marginal point processes are defined by

$$N_{n,d}(B_d, x_d) = \sum_{i=1}^{n} \mathrm{I}\left(\xi_d(i) > u_n(x_d), \frac{i}{n} \in B_d\right), \quad d = 1, 2.$$

In order to study the weak convergence of $\mathbf{N}_n$ we need to formulate certain assumptions on the random scaling S.

Our first model concerns the case that S has a Weibullian type tail behaviour with $x_F = \infty$, whereas the second one deals with S having a regular tail behaviour at x_F. For both cases we investigate the convergence in distribution of $\mathbf{N}_n$, and further, as in [4] we prove that maxima and minima are asymptotically independent.

The rest of the paper is organized as follows. Section 2 gives the main results. Proofs and auxiliary results are displayed in Section 3.

2. MAIN RESULTS

In order to proceed with the main results we need to specify our models for the random scaling $S \geqslant 0$ with distribution function F. We consider first the case that S has a Weibullian type tail behaviour, i.e., for given positive constants L, p

$$\overline{F}(u) = \mathbb{P}\left(S > u\right) = (1 + o(1))g(u) \exp(-Lu^p), \quad u \to \infty, \tag{2.1}$$

where g is an ultimately monotone function satisfying $\lim_{t\to\infty} g(tx)/g(t) = x^{\alpha}$, $\forall x > 0$ with some $\alpha \in \mathbb{R}$. Commonly if the latter asymptotic relation holds, then g is referred to as a regularly varying function at infinity with index α. The assumption (2.1) is crucial for finding the tail asymptotics of $Y = SX$, where S and X are independent and X has an $N(0, 1)$ distribution. Indeed, in view of [1]

$$\mathbb{P}\left(Y > u\right) \sim (2 + p)^{-\frac{1}{2}} g\left(Q^{-1} u^{\frac{2}{2+p}}\right) \exp\left(-T u^{\frac{2p}{2+p}}\right), \tag{2.2}$$

as $u \to \infty$, where

$$T := 2^{-1}Q^2 + LQ^{-p}, \quad Q := (Lp)^{1/(2+p)}. \tag{2.3}$$

Hence (2.2) shows that Y has also a Weibullian type distribution. We state next our first result for this Weibullian type scaling model.

THEOREM 2.1. *Let $X_n, n \geqslant 1$ be a stationary Gaussian sequence satisfying* (1.1)*, and let $\mathbf{N}_n$ be the bivariate point process given by* (1.2) *with $S_n, n \geqslant 1$ such* *that their common distribution function F satisfies* (2.1). *If further there exist some sequences $u_n(x), n \geqslant 1, x \in \mathbb{R}$ such that for any $x \in \mathbb{R}$*

$$\lim_{n\to\infty} n\mathbb{P}\left(Y > u_n(x)\right) = \exp\left(-x\right), \tag{2.4}$$

then $\mathbf{N}_n$ converge in distribution to a Poisson process $\mathcal{N}$ on $\bigcap_{d=1}^{2}((0,1] \times \{d\})$ with intensity $\mu(\mathbf{B}) = \sum_{d=1}^{2} \exp(-x_d) m(B_d)$, where m denotes the Lebesgue measure on $(0, 1]$.

REMARK 2.1. *If* (2.1) *holds with $g(x) = Cx^{\alpha}, C > 0$, then in view of [1]*

$$\mathbb{P}\left(Y > u\right) \sim (2+p)^{-\frac{1}{2}} C Q^{-\alpha} u^{\frac{2\alpha}{2+p}} \exp\left(-Tu^{\frac{2p}{2+p}}\right), \quad u \to \infty.$$

Consequently, (2.4) *holds according to [5] p.155 with $u_n(x) = a_n x + b_n, x \in \mathbb{R}$ and Q, T as in* (2.3)*, where*

$$\begin{aligned} a_n &= \frac{2+p}{2p} T^{-\frac{2+p}{2p}} (\ln n)^{\frac{2-p}{2p}}, \\ b_n &= \left(\frac{\ln n}{T}\right)^{\frac{2+p}{2p}} + a_n \left(\frac{\alpha}{p} \ln(T^{-1} \ln n) + \ln(2+p)^{-\frac{1}{2}} C Q^{-\alpha}\right). \end{aligned}$$

Applying Theorem 2.1 we derive below the joint limiting distribution of the kth maxima and the lth minima which are stated as follows.

COROLLARY 2.1. *For positive integers k and l, let $M_n^{(k)}$ and $m_n^{(l)}$ denote the kth largest and the lth smallest of $Y_n, n \geqslant 1$, then under the conditions of Theorem 2.1, for $x, y \in \mathbb{R}$ we have*

$$\begin{aligned} & \lim_{n\to\infty} \mathbb{P}\left(M_n^{(k)} \leqslant u_n(x), m_n^{(l)} > -u_n(y)\right) \\ &= \exp\left(-\exp\left(-x\right) - \exp\left(-y\right)\right) \sum_{i=0}^{k-1} \frac{\exp\left(-ix\right)}{i!} \sum_{j=0}^{l-1} \frac{\exp\left(-jy\right)}{j!}. \end{aligned} \tag{2.5}$$

Next, we consider the case S has a finite upper endpoint, say $x_F = 1$. As in [7] we shall suppose that for any $u \in (\nu, 1)$ with some $\nu \in (0, 1)$

$$\mathbb{P}\left(S_\tau > u\right) \geqslant \mathbb{P}\left(S > u\right) \geqslant \mathbb{P}\left(S_\gamma > u\right) \tag{2.6}$$

holds with S_γ, S_τ two non-negative random variables which have a regularly varying survival function at 1 with non-negative index γ and τ, respectively. By definition $S_\alpha, \alpha \geqslant 0$ is regularly varying at 1 with index α if the distribution function of S_α has upper endpoint equal 1 and further

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(S_\alpha > 1 - x/u\right)}{\mathbb{P}\left(S_\alpha > 1 - 1/u\right)} = x^\alpha, \quad x > 0.$$

The recent contribution [7] derives the limit distribution of maxima of $Y_i, 1 \leqslant i \leqslant n$ under the following modified Berman condition

$$\lim_{n\to\infty} \rho(n)(\ln n)^{1+\Delta_\epsilon} = 0, \tag{2.7}$$

where $\Delta_\epsilon = 2(\gamma - \tau) + \epsilon$ and some $\epsilon > 0$. Our last result below extends the main finding of [7] establishing the weak convergence of the bivariate exceedances point process when S is bounded.

THEOREM 2.2. *Let $\mathbf{N}_n$ be defined as in* (1.2) *with S_n satisfying* (2.6). *If condition* (2.7) *is satisfied, then $\mathbf{N}_n$ converge in distribution as $n \to \infty$ to a Poisson process $\mathcal{N}$ on $\bigcap_{d=1}^{2}((0,1] \times \{d\})$ with intensity $\mu(\mathbf{B}) = \sum_{d=1}^{2} \exp\left(-x_d\right) m(B_d)$, where m denotes the Lebesgue measure on* $(0,1]$.

REMARK 2.2. *a) Under the assumptions of Theorem 2.2 for $x, y \in \mathbb{R}$ we have that* (2.5) *holds. Hence in particular the maxima and minima are asymptotically independent in both models for the tail behaviour of S.*
b) If S is regularly varying at 1 with some index γ, then the claim of Theorem 2.2 holds under the Berman condition, i.e., the modified Berman condition should be imposed with $\Delta_\epsilon = 0$.

3. FURTHER RESULTS AND PROOFS

LEMMA 3.1. *Let $S, Z_n, n \geqslant 1$ be independent positive random variables satisfying*

$$\exp(-\widetilde{L}_0 u^{p_1}) \leqslant \mathbb{P}\left(S > u\right) \leqslant \exp(-L_0 u^{p_1})$$

and

$$\exp(-\widetilde{L}_n u^{p_2}) \leqslant \mathbb{P}\left(Z_n > u\right) \leqslant \exp(-L_n u^{p_2})$$

for all u *large with* $p_1, p_2, \widetilde{L}_n, L_n, n \geqslant 0$ *positive constants such that* $\widetilde{L}_n, L_n \in [a,b], \forall n \geqslant 0$ *with* $a < b$ *two finite positive constants. If further* S^* *is a positive random variable independent of* $Z_n, n \geqslant 1$ *satisfying*

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(S > u\right)}{\mathbb{P}\left(S^* > u\right)} = c \in (0,\infty),$$

then we have uniformly in n *as* $u \to \infty$

$$\mathbb{P}\left(SZ_n > u\right) \sim c\mathbb{P}\left(S^* Z_n > u\right).$$

P r o o f. Let $G_n, n \geqslant 1$ be the distribution function of Z_n. By the independence of S and Z_n, for all u large

$$\begin{aligned}\overline{H}(u) &:= \mathbb{P}\left(SZ_n > u\right)\\ &\geqslant \mathbb{P}\left(S > u^{\frac{p_2}{p_1+p_2}}\right)\mathbb{P}\left(Z_n > u^{\frac{p_1}{p_1+p_2}}\right) \geqslant \exp\left(-2bu^{\frac{p_1p_2}{p_1+p_2}}\right).\end{aligned}$$

Further, for $c_1 > 0$ small enough and all u large we have

$$\begin{aligned}\int_0^{c_1 u^{\frac{p_1}{p_1+p_2}}} \mathbb{P}\left(S > \frac{u}{s}\right) dG_n(s) &\leqslant \mathbb{P}\left(S > c_1^{-1} u^{\frac{p_2}{p_1+p_2}}\right)\\ &\leqslant \exp\left(-ac_1^{-p_1} u^{\frac{p_1p_2}{p_1+p_2}}\right) = o(\overline{H}(u))\end{aligned}$$

and for some large $c_2 > 0$

$$\begin{aligned}\int_{c_2 u^{\frac{p_1}{p_1+p_2}}}^{\infty} \mathbb{P}\left(S > \frac{u}{s}\right) dG_n(s) &\leqslant \mathbb{P}\left(Z_n > c_2 u^{\frac{p_1}{p_1+p_2}}\right)\\ &\leqslant \exp\left(-ac_2^{p_2} u^{\frac{p_1p_2}{p_1+p_2}}\right) = o(\overline{H}(u)).\end{aligned}$$

Therefore, for $\delta_u = c_1 u^{p_1/(p_1+p_2)}, \lambda_u = c_2 u^{p_1/(p_1+p_2)}$ we have

$$\mathbb{P}\left(SZ_n > u\right) \sim \int_{\delta_u}^{\lambda_u} \mathbb{P}\left(S > u/s\right) dG_n(s).$$

Since further $\lim_{u\to\infty} u/\lambda_u = \infty$, for any $s \in [\delta_u, \lambda_u]$ we have $u/s \geqslant u/\lambda_u \to \infty$ as $u \to \infty$. Consequently for any $\varepsilon > 0$, $s \in [\delta_u, \lambda_u]$

$$c(1-\varepsilon) \leqslant \frac{\mathbb{P}\left(S > u/s\right)}{\mathbb{P}\left(S^* > u/s\right)} \leqslant c(1+\varepsilon)$$

holds uniformly in n for all u large implying

$$\mathbb{P}\left(SZ_n > u\right) \sim c \int\limits_{\delta_u}^{\lambda_u} \mathbb{P}\left(S^* > u/s\right) \, dG_n(s) \sim \mathbb{P}\left(S^* Z_n > u\right)$$

as $u \to \infty$ holds also uniformly in n, and thus the claim follows. ∎

LEMMA 3.2. *Let $L_n, n \geqslant 1$ be as in Lemma 3.1 and let $Z_n, n \geqslant 1$ be positive random variables such that*

$$\overline{G}_n(z) := \mathbb{P}\left(Z_n > z\right) = \exp\left(-L_n z^q\right)$$

for some $q > 0$ and all $z > 0$. If further $Z_n, n \geqslant 1$ are independent of a non-negative random variable S which satisfies (2.1)*, then we have uniformly in n*

$$\mathbb{P}\left(SZ_n > u\right) \sim \sqrt{\frac{2\pi Lp}{p+q}} A_n^{\frac{p}{2}} u^{\frac{pq}{2(p+q)}} g\left(A_n u^{\frac{q}{p+q}}\right) \exp\left(-D_n u^{\frac{pq}{p+q}}\right) \tag{3.1}$$

as $u \to \infty$, where $D_n = \left(L + Lpq^{-1}\right) A_n^p$ and $A_n = (qL_n)^{\frac{1}{p+q}} (Lp)^{-\frac{1}{p+q}}$.

P r o o f. If (2.1) holds, by Lemma 3.1, we have for all u large

$$\begin{aligned}
\mathbb{P}\left(SZ_n > u\right) &= \int\limits_0^\infty \mathbb{P}\left(Z_n > \frac{u}{s}\right) dF(s) \sim \int\limits_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \mathbb{P}\left(Z_n > \frac{u}{s}\right) dF(s) \\
&\sim \int\limits_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \exp\left(-L_n u^q s^{-q}\right) \, dF(s) \\
&\sim \int\limits_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \exp\left(-L_n u^q s^{-q}\right) \, d(g(s) \exp\left(-Ls^p\right)).
\end{aligned}$$

Using similar arguments as in the proof of Theorem 2.1 in [8] we obtain as $u \to \infty$

$$\begin{aligned}
& \mathbb{P}\left(SZ_n > u\right) \\
\sim\ & Lp \int\limits_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} s^{p-1} g(s) \exp\left(-L_n u^q s^{-q} - L s^p\right)\, ds \\
=\ & Lp A_n^p u^{\frac{qp}{p+q}} \\
& \times \int\limits_{c_1 A_n}^{c_2 A_n} z^{p-1} g\left(A_n u^{\frac{q}{p+q}} z\right) \exp\left(-A_n^p u^{\frac{pq}{p+q}} \left(Lpq^{-1} z^{-q} + L z^p\right)\right)\, dz \\
\sim\ & \sqrt{\frac{2\pi Lp}{p+q}} A_n^{\frac{p}{2}} u^{\frac{pq}{2(p+q)}} g\left(A_n u^{\frac{q}{p+q}}\right) \exp\left(-D_n u^{\frac{pq}{p+q}}\right),
\end{aligned}$$

where $D_n = \left(L + Lpq^{-1}\right) A_n^p$ and $A_n = (qL_n)^{\frac{1}{p+q}} (Lp)^{-\frac{1}{p+q}}$, and thus the proof is complete. ∎

LEMMA 3.3. *Assume that the distribution function F of random variable S satisfies* (2.1)*, and further* (2.4) *holds, then we have*

$$n \sum_{k=1}^{n-1} |\rho(k)| \int\limits_0^\infty \int\limits_0^\infty \exp\left(-\frac{(\tilde{u}_n/s)^2 + (\tilde{u}_n/t)^2}{2(1+|\rho(k)|)}\right) dF(s)\, dF(t) \to 0$$

as $n \to \infty$, where $\tilde{u}_n = u_n(x)$.

Proof. Using similar arguments as in Lemma 4.3.2 in [10], let $\iota_n = [n^\beta]$ and $\sigma = \max_{k \geqslant 1} |\rho(k)| < 1$, where β is any positive constant such that $\beta < 2(1+\sigma)^{-\frac{p}{2+p}} - 1$. According to (2.4) and (2.2) we have

$$\exp\left(-T \tilde{u}_n^{\frac{2p}{2+p}}\right) \sim \mathcal{C} g^{-1}\left(Q^{-1} \tilde{u}_n^{\frac{2}{2+p}}\right) n^{-1} \qquad \tilde{u}_n \sim \left(\frac{\ln n}{T}\right)^{\frac{2+p}{2p}},$$

where T and Q are defined in (2.3), and $\mathcal{C}$ is a positive constant which may change from line to line.

By (3.1) with $q = 2$ and $L_k = 1/2(1+|\rho(k)|)$ and split the sum into two parts, i.e.,

$$
\begin{aligned}
& n\sum_{k=1}^{n-1}|\rho(k)|\int_0^\infty\int_0^\infty \exp\left(-\frac{(\tilde{u}_n/s)^2+(\tilde{u}_n/t)^2}{2(1+|\rho(k)|)}\right)dF(s)\,dF(t)\\
\leqslant\ & \mathcal{C}n\sum_{k=1}^{n-1}|\rho(k)|\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(A_k\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2(1+|\rho(k)|)^{-\frac{p}{2+p}}T\tilde{u}_n^{\frac{2p}{2+p}}\right)\\
=\ & \mathcal{C}n\left(\sum_{k=1}^{\iota_n}+\sum_{k=\iota_n+1}^{n-1}\right)|\rho(k)|\tilde{u}_n^{\frac{2p}{2+p}}\\
& \times g^2\left(A_k\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2(1+|\rho(k)|)^{-\frac{p}{2+p}}T\tilde{u}_n^{\frac{2p}{2+p}}\right).
\end{aligned}
$$

Since $g(\cdot)$ is ultimately monotone, assume without loss of generality that it is ultimately increasing. By the assumption that $g(\cdot)$ is a regularly varying function at infinity with index α, using Potter bound see e.g., [13], [6] for arbitrary $\varepsilon>0$, $k\geqslant 1$ we have

$$
g\left(A_k\tilde{u}_n^{\frac{2}{2+p}}\right)\leqslant g\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\leqslant \mathcal{C}\tilde{u}_n^{\frac{2(\alpha+\varepsilon)}{2+p}}
$$

for all n large. Hence the first part is dominated by

$$
\begin{aligned}
& \mathcal{C}nn^{\beta}\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2(1+\sigma)^{-\frac{p}{2+p}}T\tilde{u}_n^{\frac{2p}{2+p}}\right)\\
=\ & \mathcal{C}n^{1+\beta}\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\left(\exp\left(-T\tilde{u}_n^{\frac{2p}{2+p}}\right)\right)^{2(1+\sigma)^{-\frac{p}{2+p}}}\\
\leqslant\ & \mathcal{C}n^{1+\beta}\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\left(g\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)n\right)^{-2(1+\sigma)^{-\frac{p}{2+p}}}\\
\leqslant\ & \mathcal{C}n^{1+\beta-2(1+\sigma)^{-\frac{p}{2+p}}}(\ln n)^{1+\frac{2(\alpha+\varepsilon)}{p}(1-(1+\sigma)^{-\frac{p}{2+p}})}\to 0
\end{aligned}
$$

as $n\to\infty$ since $1+\beta-2(1+\sigma)^{-\frac{p}{2+p}}<0$. Next set $\sigma(l)=\max_{k\geqslant l}|\rho(k)|<1$. We may further write

$$
\begin{aligned}
& \mathcal{C}n\sum_{k=\iota_n+1}^{n-1}|\rho(k)|\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(A_k\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2(1+|\rho(k)|)^{-\frac{p}{2+p}}T\tilde{u}_n^{\frac{2p}{2+p}}\right)\\
\leqslant\ & \mathcal{C}n^2\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2(1+\sigma(\iota_n))^{-\frac{p}{2+p}}T\tilde{u}_n^{\frac{2p}{2+p}}\right)\\
\leqslant\ & \mathcal{C}n^2\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}}g^2\left(Q^{-1}\tilde{u}_n^{\frac{2}{2+p}}\right)\exp\left(-2T\tilde{u}_n^{\frac{2p}{2+p}}\right)\exp\left(2T\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}}\right)\\
\leqslant\ & \mathcal{C}\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}}\exp\left(2T\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}}\right).
\end{aligned}
$$

Using now (1.1) as $n \to \infty$

$$\sigma(\iota_n)\tilde{u}_n^{\frac{2p}{2+p}} \sim T^{-1}\sigma(\iota_n)\ln n \leqslant T^{-1} \max_{k \geqslant \iota_n} |\rho(k)| \ln n \to 0$$

the exponential term above tends to one and the remaining product tends to zero and thus the proof is complete. ■

LEMMA 3.4. *Let $X_n, n \geqslant 1$ be a ssGs satisfying* (1.1)*, and let $S_n, n \geqslant 1$ be independent random variables satisfying* (2.1) *being further independent of X_n. Additionally, assume that the survival function of $Y_n = S_n X_n$ satisfy* (2.4)*. Further if $0 < \theta < 1$ and I_n is an interval containing $k_n \sim \theta n$ members, we have*

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \Big| \mathbb{P}\left(-u_n(y) < m(I_n) \leqslant M(I_n) \leqslant u_n(x)\right) - \exp\left(-\theta(\exp(-x) + \exp(-y))\right)\Big| = 0,$$

where $M(I_n) = \max_{i\in I_n} Y_i$ and $m(I_n) = \min_{i\in I_n} Y_i$.

P r o o f. Let $Z_n, n \geqslant 1$ be independent random variables with the same distribution as X_1 and define $\mathfrak{M}_n = \max_{1\leqslant i\leqslant n} S_i Z_i$ and $\mathfrak{m}_n = \min_{1\leqslant i\leqslant n} S_i Z_i$. For $x, y \in \mathbb{R}$, using assumption (2.4), i.e.,

$$\lim_{n\to\infty} n\mathbb{P}\left(S_1 Z_1 > u_n(x)\right) = \exp(-x), \tag{3.2}$$

$$\lim_{n\to\infty} n\mathbb{P}\left(S_1 Z_1 \leqslant -u_n(y)\right) = \exp(-y) \tag{3.3}$$

and by Theorem 1.8.2 in [10] we have

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \Big| \mathbb{P}\left(-u_n(y) < \mathfrak{m}_n \leqslant \mathfrak{M}_n \leqslant u_n(x)\right) - \Lambda(x)\Lambda(y)\Big| = 0. \tag{3.4}$$

Further if (2.1) holds, since $S_n, n \geqslant 1$ are independent with common distribution function F by a direct application of Berman inequality (see [12]) and Lemma 3.3 we obtain

$$\begin{aligned}
& \Big| \mathbb{P}\left(-u_n(y) < m_n \leqslant M_n \leqslant u_n(x)\right) - \mathbb{P}\left(-u_n(y) < \mathfrak{m}_n \leqslant \mathfrak{M}_n \leqslant u_n(x)\right)\Big| \\
\leqslant\ & \int_{[0,\infty]^n} \left| \mathbb{P}\left(\bigcap_{k=1}^{n} \left\{ -\frac{u_n(y)}{s_k} < X_k \leqslant \frac{u_n(x)}{s_k} \right\}\right) \right. \\
& \qquad \left. - \mathbb{P}\left(\bigcap_{k=1}^{n} \left\{ -\frac{u_n(y)}{s_k} < Z_k \leqslant \frac{u_n(x)}{s_k} \right\}\right) \right| dF(s_1)\cdots dF(s_n) \\
\leqslant\ & \mathcal{C} n \sum_{k=1}^{n-1} \int_0^\infty \int_0^\infty |\rho(k)| \exp\left(-\frac{(w_n/s)^2 + (w_n/t)^2}{2(1+|\rho(k)|)}\right) dF(s)\, dF(t) \\
\to\ & 0, \quad n \to \infty,
\end{aligned}$$

where $w_n = \min(|u_n(x)|, |u_n(y)|)$. Thus by (3.4) we have

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \left|\mathbb{P}\left(-u_n(y) < m_n \leqslant M_n \leqslant u_n(x)\right) - \Lambda(x)\Lambda(y)\right| = 0.$$

Now let $v_n = u_{[n/\theta]}$, using (3.2) and (3.3) we get

$$\lim_{n\to\infty} n\mathbb{P}\left(S_1 Z_1 > v_n(x)\right) = \theta \exp\left(-x\right)$$

and

$$\lim_{n\to\infty} n\mathbb{P}\left(S_1 Z_1 \leqslant -v_n(y)\right) = \theta \exp\left(-y\right),$$

hence

$$\begin{aligned}(3.5)\qquad \lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \Big|\mathbb{P}\left(-v_n(y) < m_n \leqslant M_n \leqslant v_n(x)\right) \\ - \exp\left(-\theta(\exp\left(-x\right) + \exp\left(-y\right))\right)\Big| = 0.\end{aligned}$$

Since $S_n, n \geqslant 1$ are independent and have a common distribution function F, by the stationarity of $X_n, n \geqslant 1$

$$\begin{aligned} & \mathbb{P}\left(-u_n(y) < m(I_n) \leqslant M(I_n) \leqslant u_n(x)\right) \\ = & \mathbb{P}\left(\bigcap_{i\in I_n} \{-u_n(y) < S_i X_i \leqslant u_n(x)\}\right) \\ = & \int_{(0,\infty)^{k_n}} \mathbb{P}\left(\bigcap_{i=1}^{k_n} \left\{-\frac{u_n(y)}{s_i} < X_i \leqslant \frac{u_n(x)}{s_i}\right\}\right) dF(s_1)\cdots dF(s_{k_n}) \\ = & \mathbb{P}\left(-u_n(y) < m_{k_n} \leqslant M_{k_n} \leqslant u_n(x)\right). \end{aligned}$$

Hence, replacing n by k_n in (3.5) establishes the claim . ∎

REMARK 3.1. *Under the conditions of Lemma 3.4, we have*

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\mathbb{P}\left(M(I_n) \leqslant u_n(x)\right) - \exp\left(-\theta \exp\left(-x\right)\right)\right| = 0.$$

LEMMA 3.5. *Let $I_1, I_2, \ldots, I_l$ (with l a fixed number) be disjoint subintervals of $\{1, 2, \ldots, n\}$ such that I_i has $k_{n,i} \sim \theta_i n$ elements, where θ_i are fixed positive constants with $\theta := \sum_{i=1}^{l} \theta_i \leqslant 1$. Then, under the assumptions of Lemma 3.4, we have*

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=1}^{l} \{-u_n(y) < m(I_i) \leqslant M(I_i) \leqslant u_n(x)\}\right) \\ - \prod_{i=1}^{l} \mathbb{P}\left(-u_n(y) < m(I_i) \leqslant M(I_i) \leqslant u_n(x)\right) \to 0\end{aligned}$$

as $n \to \infty$.

Proof. Since $X_n, n \geqslant 1$ is a stationary random sequence, using Berman's inequality and Lemma 3.3, we have

$$\begin{aligned}
& \left|\mathbb{P}\left(\bigcap_{i=1}^{l}\{-u_n(y) < m(I_i) \leqslant M(I_i) \leqslant u_n(x)\}\right)\right. \\
& \qquad \left. - \prod_{i=1}^{l}\mathbb{P}\left(-u_n(y) < m(I_i) \leqslant M(I_i) \leqslant u_n(x)\right)\right| \\
= & \left|\mathbb{P}\left(\bigcap_{i=1}^{l}\bigcap_{j\in I_i}\{-u_n(y) < S_jX_j \leqslant u_n(x)\}\right)\right. \\
& \qquad \left. - \prod_{i=1}^{l}\mathbb{P}\left(\bigcap_{j\in I_i}\{-u_n(y) < S_jX_j \leqslant u_n(x)\}\right)\right| \\
\leqslant & \int\limits_{(0,\infty)^{\hat{\theta}_l}} \left|\mathbb{P}\left(\bigcap_{i=1}^{l}\hat{A}_i\right) - \prod_{i=1}^{l}\mathbb{P}\left(\hat{A}_i\right)\right| dF(s_1)\cdots dF(s_{\hat{\theta}_l}) \\
\leqslant & \hat{\theta}_l \sum_{k=1}^{\hat{\theta}_l} \int\limits_0^\infty \int\limits_0^\infty |\rho(k)| \exp\left(-\frac{(w_n/s)^2 + (w_n/t)^2}{2(1+|\rho(k)|)}\right) dF(s)dF(t) \\
\to & \ 0, \quad n \to \infty,
\end{aligned}$$

where $\hat{A}_i = \bigcap_{j=\hat{\theta}_{i-1}+1}^{\hat{\theta}_i}\left\{-\frac{u_n(y)}{s_j} < X_j \leqslant \frac{u_n(x)}{s_j}\right\}$ with

$$\hat{\theta}_i = \sum_{j=1}^{i}[\theta_j n], \quad \hat{\theta}_0 = 0, \quad w_n = \min(|u_n(x)|, |u_n(y)|),$$

hence the proof is complete. ■

REMARK 3.2. *Under the conditions of Lemma 3.5, we have*

$$\lim_{n\to\infty}\left|\mathbb{P}\left(\bigcap_{i=1}^{l}\{M(I_i) \leqslant u_n(x)\}\right) - \prod_{i=1}^{l}\mathbb{P}\left(M(I_i) \leqslant u_n(x)\right)\right| = 0.$$

Proof of Theorem 2.1. In view of [14]we need first to prove that the marginal point processes of $N_{n,d}$ converge weakly to a Poisson process N_d with intensity $\exp(-x_d)$, $d = 1, 2$. By Theorem A.1 in [10] for $N_{n,1}(B_1, x_1)$, it is sufficient to show that as $n \to \infty$

$$(P_1).\quad \mathbb{E}\left(N_{n,1}((s,t],x_1)\right) \to \mathbb{E}\left(N_1((s,t],x_1)\right) = (t-s)\exp\left(-x_1\right),\ 0 < s < t \leqslant 1;$$

$$(P_2).\quad \mathbb{P}\left(\bigcap_{i=1}^{k}\{N_{n,1}((s_i,t_i],x_1)=0\}\right) \to \mathbb{P}\left(\bigcap_{i=1}^{k}\{N_1((s_i,t_i],x_1)=0\}\right) = \exp\left(-\sum_{i=1}^{k}(t_i-s_i)\exp\left(-x_1\right)\right),$$

where $0 < s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots \leqslant s_k < t_k \leqslant 1$.
We have

$$\begin{aligned}\mathbb{E}\left(N_{n,1}((s,t],x_1)\right) &= \mathbb{E}\left(\sum_{i/n\in(s,t]} \mathrm{I}(S_iX_i > u_n(x_1))\right)\\ &= \sum_{i/n\in(s,t]} \mathbb{P}\left(S_iX_i > u_n(x_1)\right)\\ &\to (t-s)\exp\left(-x_1\right) = \mathbb{E}\left(N_1((s,t],x_1)\right)\end{aligned}$$

as $n \to \infty$, where the above convergence follows from (2.4).
In order to show (P_2) note first that for $0 < s < t \leqslant 1$

$$\mathbb{P}\left(N_{n,1}((s,t],x_1)=0\right) = \mathbb{P}\left(M(I_n) \leqslant u_n(x_1)\right),$$

where $I_n = \{[sn]+1,\ldots,[tn]\}$. Further, I_n contains k_n integers with $k_n = [tn] - [sn] \sim (t-s)n$ as $n \to \infty$. Thus, in view of Remark 3.1 with $\theta = t-s < 1$ we have as $n \to \infty$

$$\mathbb{P}\left(N_{n,1}((s,t],x_1)=0\right) \to \exp\left(-(t-s)\exp\left(-x_1\right)\right). \tag{3.6}$$

Next, let E_i be the set of integers $\{[s_in]+1,\ldots,[t_in]\}$ with $0 < s_1 < t_1 \leqslant s_2 < t_2 \leqslant \cdots \leqslant s_k < t_k \leqslant 1$, then we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{i=1}^{k}\{N_{n,1}((s_i,t_i],x_1)=0\}\right) &= \mathbb{P}\left(\bigcap_{i=1}^{k}\{M(E_i) \leqslant u_n(x_1)\}\right)\\ = \prod_{i=1}^{k}\mathbb{P}\left(N_{n,1}((s_i,t_i],x_1)=0\right)&\\ &+\left(\mathbb{P}\left(\bigcap_{i=1}^{k}\{M(E_i)\leqslant u_n(x_1)\}\right) - \prod_{i=1}^{k}\mathbb{P}\left(M(E_i)\leqslant u_n(x_1)\right)\right).\end{aligned}$$

Using (3.6), the first term converges to $\exp\left(-\sum_{i=1}^{k}(t_i-s_i)\exp\left(-x_1\right)\right)$ as $n \to \infty$. By Remark 3.2 the modulus of the remaining difference of terms tends to

0. Consequently, $N_{n,1}$ converge weakly to a Poisson process N_1 with intensity $\exp(-x_1)$. Since $Y_i \stackrel{d}{=} -Y_i$, $N_{n,2}$ also converge weakly to a Poisson process N_2 with intensity $\exp(-x_2)$.

Now define the avoidance function of $\mathbf{N}_n$ as

$$F_{\mathbf{N}_n}(\mathbf{B}) = \mathbb{P}\left(N_{n,1}(B_1, x_1) = 0, N_{n,2}(B_2, x_2) = 0\right),$$

where B_1 and B_2 are defined below. To get the main result, it suffices to prove that

$$\lim_{n\to\infty} F_{\mathbf{N}_n}(\mathbf{B})$$

exists for all $\mathbf{B} = \bigcup_{d=1}^{2} \bigcup_{j=1}^{r} (B_{dj} \times \{d\})$, where r arbitrary positive integers, $B_{dj} = (s_{dj}, t_{dj}]$, $0 < s_{d1} < t_{d1} \leqslant s_{d2} < t_{d2} \leqslant \ldots \leqslant s_{dr} < t_{dr} \leqslant 1$, and $B_1 = \bigcup_{j=1}^{r} B_{1j}$, $B_2 = \bigcup_{j=1}^{r} B_{2j}$. We will show that

$$\lim_{n\to\infty} F_{\mathbf{N}_n}(\mathbf{B}) = \exp\left(-m(B_1)\exp(-x_1) - m(B_2)\exp(-x_2)\right).$$

For simplicity we only consider the case $B_1 \subset B_2$; other cases are similar. First consider the case $n(B_2 \setminus B_1) = o(n)$, i.e., $m(B_1) = m(B_2)$. Obviously,

$$\begin{aligned} 0 &\leqslant \mathbb{P}\left(-u_n(x_2) < Y_k \leqslant u_n(x_1), k/n \in B_1\right) \\ &\qquad -\mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\ &\leqslant \sum_{l:l/n \in B_2\setminus B_1} \mathbb{P}\left(-Y_l > u_n(x_2)\right) \to 0 \end{aligned}$$

as $n \to \infty$. Consequently, by Lemma 3.4 and 3.5, we have

$$\begin{aligned} &\lim_{n\to\infty} \mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\ = &\lim_{n\to\infty} \mathbb{P}\left(-u_n(x_2) < Y_k \leqslant u_n(x_1), k/n \in B_1\right) \\ = &\prod_{j=1}^{r} \exp\left(-(t_{1j} - s_{1j})\exp(-x_1)\right) \prod_{j=1}^{r} \exp\left(-(t_{1j} - s_{1j})\exp(-x_2)\right) \\ = &\exp\left(-m(B_1)\exp(-x_1) - m(B_2)\exp(-x_2)\right). \end{aligned}$$

It suffices to prove the case of $n(B_2 \setminus B_1) = O(n)$. Noting that any $z > 0$

$$\begin{aligned} &\mathbb{P}\left(-u_n(x_2) < Y_k \leqslant u_n(x_1), k/n \in B_1;\right. \\ &\qquad \left. -u_n(x_2) < Y_i \leqslant u_n(z), i/n \in B_2 \setminus B_1\right) \\ \leqslant\ &\mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\ \leqslant\ &\mathbb{P}\left(-u_n(x_2) < Y_k \leqslant u_n(x_1), k/n \in B_1;\right. \\ &\qquad \left. -u_n(x_2) < Y_i \leqslant u_n(z), i/n \in B_2 \setminus B_1\right) \\ &\qquad\quad +\mathbb{P}\left(\max(Y_i, i/n \in B_2 \setminus B_1) > u_n(z)\right) \\ =\ &\mathbb{P}\left(-u_n(x_2) < Y_k \leqslant u_n(x_1), k/n \in B_1;\right. \\ &\qquad \left. -u_n(x_2) < Y_i \leqslant u_n(z), i/n \in B_2 \setminus B_1\right) \\ &\qquad\quad + \left(1 - \mathbb{P}\left(\max(Y_i, i/n \in B_2 \setminus B_1) \leqslant u_n(z)\right)\right). \end{aligned}$$

Applying Lemma 3.4 and 3.5 once again, we obtain

$$\begin{aligned}
& \exp\left(-m(B_1)\left(\exp(-x_1)+\exp(-x_2)\right)\right) \\
& \times \exp\left(-m(B_2\setminus B_1)\left(\exp(-z)+\exp(-x_2)\right)\right) \\
\leqslant\ & \liminf_{n\to\infty} \mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\
\leqslant\ & \limsup_{n\to\infty} \mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\
\leqslant\ & \exp\left(-m(B_1)\left(\exp(-x_1)+\exp(-x_2)\right)\right) \\
& \times \exp\left(-m(B_2\setminus B_1)\left(\exp(-z)+\exp(-x_2)\right)\right) \\
& \quad + \left(1-\exp\left(-m(B_2\setminus B_1)\exp(-z)\right)\right).
\end{aligned}$$

Hence, letting $z \to \infty$ we have

$$\begin{aligned}
& \lim_{n\to\infty} \mathbb{P}\left(Y_k \leqslant u_n(x_1), k/n \in B_1; -Y_l \leqslant u_n(x_2), l/n \in B_2\right) \\
=\ & \exp\left(-m(B_1)\exp(-x_1) - m(B_2)\exp(-x_2)\right).
\end{aligned}$$

This establishes the proof. ∎

P r o o f o f C o r o l l a r y 2.1. Notice that

$$\begin{aligned}
& \mathbb{P}\left(M_n^{(k)} \leqslant u_n(x), m_n^{(l)} > -u_n(y)\right) \\
=\ & \mathbb{P}\left(N_{n,1}((0,1],x) \leqslant k-1, N_{n,2}((0,1],y) \leqslant l-1\right).
\end{aligned}$$

Hence the proof follows by Theorem 2.1. ∎

P r o o f o f T h e o r e m 2.2. By Lemma 3.3 of [7] we have if under the condition (2.7)

$$\lim_{n\to\infty} n \sum_{k=1}^{n-1} |\rho(k)| \int_0^1 \int_0^1 \exp\left(-\frac{(u_n(x)/s)^2 + (u_n(x)/t)^2}{2(1+|\rho(k)|)}\right) dF(s)\, dF(t) = 0.$$

Consequntly, Lemma 3.4 and 3.5 also hold for S_n satisfy (2.6). Hence the proof follows by utilizing similar arguments as in the proof of Theorem 2.1. ∎

Acknowledgement: We are in dept to the referees for several suggestions and corrections which improved this paper significantly.

REFERENCES

[1] M. Arendarczyk and K. Dębicki, *Asymptotics of supremum distribution of a Gaussian process over a Weibullian time,* Bernoulli, 17 (2011), pp. 194–210.

[2] M. Arendarczyk and K. Dębicki, *Exact asymptotics of supremum of a stationary Gaussian process over a random interval,* Statist. Probab. Lett. 82 (2012), pp. 645–652.

[3] S.M. Berman, *Limit theorems for the maximum term in stationary sequences,* Ann. Math. Statist. 35 (1964), pp. 502–516.

[4] R.A. Davis, *Maxima and minima of stationary sequences,* Ann. Probab. 7 (1979), pp. 453–460.

[5] P. Embrechts, C. Klüppelberg and T. Mikosch, *Modelling Extremal Events: for Insurance and Finance*, Springer Verlag, Berlin, 1997.

[6] M. Falk, J. Hüsler and R.-D. Reiss, *Laws of Small Numbers: Extremes and Rare Events, 3rd ed.,* In DMV Seminar, volume 23, Birkhäuser, Basel, 2010.

[7] E. Hashorva and Z. Weng, *Limit laws for maxima of contracted stationary Gaussian sequences,* Comm. Statist. Theory Methods, in press, (2013).

[8] E. Hashorva and Z. Weng, *Tail asymptotic of Weibull-type risks,* Statistics, (2013), http://dx.doi.org/10.1080/02331888.2013.800520.

[9] A. Hu, Z. Peng and Y. Qi, *Limit laws for maxima of contracted stationary Gaussian sequences,* Metrika, 70 (2009), pp. 279–295.

[10] M.R. Leadbetter, G. Lindgren and H. Rootzén, *Extremes and related properties of random sequences and processes,* Springer Verlag, New York, 1983.

[11] Z. Peng, J. Tong and Z. Weng, *Joint limit distributions of exceedances point processes and partial sums of Gaussian vector sequence*, Acta Math. Sin. (Engl. Ser.), 8 (2012), pp. 1647–1662.

[12] V.I. Piterbarg, *Asymptotic Methods in the Theory of Gaussian Processes and Fields,* volume 148 of Translations of Mathematical Monographs, American Mathematical Society, Providence, RI, 1996.

[13] S.I. Resnick, *Extreme Values, Regular Variation, and Point Processes*, volume 4 of Applied Probability, A Series of the Applied Probability Trust, Springer-Verlag, New York, 1987.

[14] M. Wiśniewski, *On extreme-order statistics and point processes of exceedances in multivariate stationary Gaussian sequences*, Statist. Probab. Lett. 29 (1996), pp. 55–59.

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Received on 2013;
revised version on xx.xx.xxxx