LIMIT PROPERTIES OF EXCEEDANCES POINT PROCESSES OF SCALED
STATIONARY GAUSSIAN SEQUENCES

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Abstract. We derive the limiting distributions of exceedances point
processes of randomly scaled weakly dependent stationary Gaussian se-
quencies under some mild asymptotic conditions. In the literature analogous
results are available only for contracted stationary Gaussian sequences. In
this paper, we include additionally the case of randomly inflated stationary
Gaussian sequences with a Weibullian type random scaling. It turns out that
the maxima and minima of both contracted and inflated weakly dependent
stationary Gaussian sequences are asymptotically independent.

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1. INTRODUCTION

Let $X_n, n \geq 1$ be a standard stationary Gaussian sequence (ssGs) i.e., $X_n$’s are $N(0, 1)$ distributed and $\rho(n) = \mathbb{E}(X_1 X_{n+1}) = \mathbb{E}(X_j X_{n+j})$ for any $j \geq 1$. In the seminal contribution [3], S.M. Berman proved that the maxima $\tilde{M}_n = \max_{1 \leq k \leq n} X_k$ converges in distribution after normalization to a unit Gumbel random variable, i.e.,

$$\lim_{n \to \infty} \mathbb{P}(\tilde{M}_n \leq \tilde{a}_n x + \tilde{b}_n) = \exp(-\exp(-x)) =: \Lambda(x), \quad \forall x \in \mathbb{R},$$

provided that the so-called Berman condition

$$\lim_{n \to \infty} \rho(n) \ln n = 0 \quad \text{(1.1)}$$

holds, where the norming constants $\tilde{a}_n$ and $\tilde{b}_n$ are given by

$$\tilde{a}_n = \frac{1}{\sqrt{2 \ln n}} \quad \text{and} \quad \tilde{b}_n = \sqrt{2 \ln n} - \frac{\ln \ln n + \ln 4\pi}{2\sqrt{2 \ln n}}.$$

Moreover, the maxima and the minima $\tilde{m}_n = \min_{1 \leq k \leq n} X_k$ are asymptotically independent, cf. [4] and [10].

In applications, commonly the observations are randomly scaled, say due to some inflation or deflation effects if financial losses are modeled, or caused by measurement errors if observations are the outcome of a certain physical experiment. Therefore, in order to model some general random scaling phenomena applicable to original data, in this paper we consider $Y = SX, Y_n = S_n X_n, n \geq 1$ assuming that $S, S_n, n \geq 1$ are independent non-negative random variables with common distribution function $F$ being further independent of the standard Gaussian random variables $X, X_n, n \geq 1$.

As shown in [7] if $F$ has a finite upper endpoint $x_F \in (0, \infty)$ and its survival function is regularly varying, then the maxima $M_n = \max_{1 \leq k \leq n} Y_k$ converge in distribution after normalization to a unit Gumbel random variable with distribution
function Λ, provided that the Berman condition holds. If \( x_F = \infty \) and \( X_n, n \geq 1 \) are iid \( N(0, 1) \) the convergence of maxima \( M_n \) is shown under a different normalization in [8] assuming further that \( F \) has a Weibullian tail behaviour (see below (2.1)).

The objective of the paper is twofold: first for \( F \) with a Weibullian tail behaviour, it is of interest to establish the convergence of maxima of a randomly scaled ssGs under the Berman condition; there is no result in the literature covering this case. Secondly, for both cases \( x_F \) is a positive constant, and \( x_F = \infty \), we aim at establishing the same result as in [4], i.e., the asymptotic independence of maxima and minima of randomly scaled weakly dependent ssGs.

Since by using a point process approach also the joint limiting distribution of upper and lower order statistics can be easily established, we choose in this paper a point process framework considering exceedances point processes. Numerous authors dealt with the asymptotic behavior of exceedances point processes; for weakly dependent stationary sequences including Gaussian, see [10, 12, 9, 6, 1, 2, 11] and the references therein.

For \( u_n(s) = a_n s + b_n, s \in \mathbb{R} \), with \( a_n > 0, b_n \in \mathbb{R} \) we shall investigate the weak convergence of bivariate point processes of exceedances of levels \( u_n(x) \) and \( -u_n(y) \) formed by \( Y_n, n \geq 1 \). Setting \( \xi_1(n) = Y_n, \xi_2(n) = -Y_n \) for \( n \geq 1 \) we define as in [14] the bivariate exceedances point processes

\[
(1.2) \quad N_n(B, x) = \sum_{d=1}^{2} \sum_{i=1}^{n} I(\xi_d(i) > u_n(x_d), \frac{i}{n} \in B_d)
\]

for \( B = \bigcup_{d=1}^{2} (B_d \times \{d\}) \) with \( B_d \) the Borel set on \((0, 1], d = 1, 2\), where \( I(\cdot) \) denotes the indicator function. The marginal point processes are defined by

\[
N_{n,d}(B_d; x_d) = \sum_{i=1}^{n} I(\xi_d(i) > u_n(x_d), \frac{i}{n} \in B_d), \quad d = 1, 2.
\]

In order to study the weak convergence of \( N_n \) we need to formulate certain assumptions on the random scaling \( S \).
Our first model concerns the case that \( S \) has a Weibullian type tail behaviour with \( x_F = \infty \), whereas the second one deals with \( S \) having a regular tail behaviour at \( x_F \). For both cases we investigate the convergence in distribution of \( N_n \), and further, as in [4] we prove that maxima and minima are asymptotically independent.

The rest of the paper is organized as follows. Section 2 gives the main results. Proofs and auxiliary results are displayed in Section 3.

2. MAIN RESULTS

In order to proceed with the main results we need to specify our models for the random scaling \( S \geq 0 \) with distribution function \( F \). We consider first the case that \( S \) has a Weibullian type tail behaviour, i.e., for given positive constants \( L, p \)

\[
\overline{F}(u) = P(S > u) = (1 + o(1))g(u)\exp(-Lu^p), \quad u \to \infty, 
\]

where \( g \) is an ultimately monotone function satisfying \( \lim_{t \to \infty} g(tx)/g(t) = x^{\alpha} \), \( \forall x > 0 \) with some \( \alpha \in \mathbb{R} \). Commonly if the latter asymptotic relation holds, then \( g \) is referred to as a regularly varying function at infinity with index \( \alpha \). The assumption (2.1) is crucial for finding the tail asymptotics of \( Y = SX \), where \( S \) and \( X \) are independent and \( X \) has an \( N(0, 1) \) distribution. Indeed, in view of [1]

\[
P(Y > u) \sim (2 + p)^{-\frac{1}{2}} g \left( Q^{-1}u^{\frac{2}{1+p}} \right) \exp \left( -Tu^{2/p} \right), 
\]

as \( u \to \infty \), where

\[
T := 2^{-1}Q^2 + LQ^{-p}, \quad Q := (Lp)^{1/(2+p)}.
\]

Hence (2.2) shows that \( Y \) has also a Weibullian type distribution. We state next our first result for this Weibullian type scaling model.

THEOREM 2.1. Let \( X_n, n \geq 1 \) be a stationary Gaussian sequence satisfying (1.1), and let \( N_n \) be the bivariate point process given by (1.2) with \( S_n, n \geq 1 \) such
that their common distribution function $F$ satisfies (2.1). If further there exist some sequences $u_n(x), n \geq 1, x \in \mathbb{R}$ such that for any $x \in \mathbb{R}$

\[
\lim_{n \to \infty} n \mathbb{P} (Y > u_n(x)) = \exp (-x),
\]

then $N_n$ converge in distribution to a Poisson process $N$ on $\bigcap_{d=1}^{2} ((0, 1] \times \{d\})$ with intensity $\mu (B) = \sum_{d=1}^{2} \exp (-x_d)m(B_d)$, where $m$ denotes the Lebesgue measure on $(0, 1]$.

**Remark 2.1.** If (2.1) holds with $g(x) = Cx^\alpha, C > 0$, then in view of [1]

\[\mathbb{P} (Y > u) \sim (2 + p)^{-\frac{1}{2}} CQ^{-\alpha}u^{\frac{2\alpha}{x^p}} \exp \left( -Tu^{\frac{2\alpha}{x^p}} \right), \quad u \to \infty.\]

Consequently, (2.4) holds according to [5] p.155 with $u_n(x) = a_n x + b_n, x \in \mathbb{R}$ and $Q, T$ as in (2.3), where

\[
a_n = \frac{2 + p}{2p} T^{-\frac{2+p}{2p}} (\ln n)^{\frac{2-p}{2p}},
\]

\[
b_n = \left( \frac{\ln n}{T} \right)^{\frac{2+p}{2p}} + a_n \left( \frac{\alpha}{p} \ln(T^{-1} \ln n) + \ln(2 + p)^{-\frac{1}{2}} CQ^{-\alpha} \right).
\]

Applying Theorem 2.1 we derive below the joint limiting distribution of the $k$th maxima and the $l$th minima which are stated as follows.

**Corollary 2.1.** For positive integers $k$ and $l$, let $M_n^{(k)}$ and $m_n^{(l)}$ denote the $k$th largest and the $l$th smallest of $Y_n, n \geq 1$, then under the conditions of Theorem 2.1, for $x, y \in \mathbb{R}$ we have

\[
\lim_{n \to \infty} \mathbb{P} \left( M_n^{(k)} \leq u_n(x), m_n^{(l)} > -u_n(y) \right) = \exp \left( -\exp (-x) - \exp (-y) \right) \sum_{i=0}^{k-1} \frac{\exp (-ix)}{i!} \sum_{j=0}^{l-1} \frac{\exp (-jy)}{j!}.
\]

Next, we consider the case $S$ has a finite upper endpoint, say $x_F = 1$. As in [7] we shall suppose that for any $u \in (\nu, 1)$ with some $\nu \in (0, 1)$

\[
\mathbb{P} (S_\tau > u) \geq \mathbb{P} (S > u) \geq \mathbb{P} (S_{\gamma} > u)
\]
holds with $S_\gamma, S_\tau$ two non-negative random variables which have a regularly varying survival function at 1 with non-negative index $\gamma$ and $\tau$, respectively. By definition $S_\alpha, \alpha \geq 0$ is regularly varying at 1 with index $\alpha$ if the distribution function of $S_\alpha$ has upper endpoint equal 1 and further
\[
\lim_{u \to \infty} \frac{P(S_\alpha > 1 - x/u)}{P(S_\alpha > 1 - 1/u)} = x^\alpha, \quad x > 0.
\]
The recent contribution [7] derives the limit distribution of maxima of $Y_i, 1 \leq i \leq n$ under the following modified Berman condition
\[(2.7) \quad \lim_{n \to \infty} \rho(n)(\ln n)^{1 + \Delta_\epsilon} = 0,
\]
where $\Delta_\epsilon = 2(\gamma - \tau) + \epsilon$ and some $\epsilon > 0$. Our last result below extends the main finding of [7] establishing the weak convergence of the bivariate exceedances point process when $S$ is bounded.

**Theorem 2.2.** Let $N_n$ be defined as in (1.2) with $S_n$ satisfying (2.6). If condition (2.7) is satisfied, then $N_n$ converge in distribution as $n \to \infty$ to a Poisson process $\mathcal{N}$ on $\bigcap_{d=1}^2 ((0, 1] \times \{d\})$ with intensity $\mu(B) = \sum_{d=1}^2 \exp(-x_d) m(B_d)$, where $m$ denotes the Lebesgue measure on $(0, 1]$.

**Remark 2.2.** a) Under the assumptions of Theorem 2.2 for $x, y \in \mathbb{R}$ we have that (2.5) holds. Hence in particular the maxima and minima are asymptotically independent in both models for the tail behaviour of $S$.

b) If $S$ is regularly varying at 1 with some index $\gamma$, then the claim of Theorem 2.2 holds under the Berman condition, i.e., the modified Berman condition should be imposed with $\Delta_\epsilon = 0$.

### 3. Further Results and Proofs

**Lemma 3.1.** Let $S, Z_n, n \geq 1$ be independent positive random variables satisfying
\[
\exp(-\tilde{L}_0 u^{p_1}) \leq P(S > u) \leq \exp(-L_0 u^{p_1})
\]
and
\[ \exp(-\tilde{L}_n u^{p_2}) \leq P(Z_n > u) \leq \exp(-L_n u^{p_2}) \]
for all \( u \) large with \( p_1, p_2, \tilde{L}_n, L_n, n \geq 0 \) positive constants such that \( \tilde{L}_n, L_n \in [a, b], \forall n \geq 0 \) with \( a < b \) two finite positive constants. If further \( S^* \) is a positive random variable independent of \( Z_n, n \geq 1 \) satisfying
\[
\lim_{u \to \infty} \frac{P(S > u)}{P(S^* > u)} = c \in (0, \infty),
\]
then we have uniformly in \( n \) as \( u \to \infty \)
\[ P(SZ_n > u) \sim c P(S^* Z_n > u). \]

Proof. Let \( G_n, n \geq 1 \) be the distribution function of \( Z_n \). By the independence of \( S \) and \( Z_n \), for all \( u \) large
\[
\overline{H}(u) := P(SZ_n > u) \geq P(S > u) P(Z_n > u) \geq \exp(-2bu^{p_1/(p_1+p_2)}).
\]

Further, for \( c_1 > 0 \) small enough and all \( u \) large we have
\[
\int_0^{c_1 u^{p_1/(p_1+p_2)}} P(S > u/s) dG_n(s) \leq \exp(-ac_1 u^{p_1/(p_1+p_2)}) = o(\overline{H}(u))
\]
and for some large \( c_2 > 0 \)
\[
\int_{c_2 u^{p_1/(p_1+p_2)}}^{\infty} P(S > u/s) dG_n(s) \leq \exp(-ac_2 u^{p_1/(p_1+p_2)}) = o(\overline{H}(u)).
\]

Therefore, for \( \delta_u = c_1 u^{p_1/(p_1+p_2)}, \lambda_u = c_2 u^{p_1/(p_1+p_2)} \) we have
\[
P(SZ_n > u) \sim \int_{\delta_u}^{\lambda_u} P(S > u/s) dG_n(s).
\]

Since further \( \lim_{u \to \infty} u/\lambda_u = \infty \), for any \( s \in [\delta_u, \lambda_u] \) we have \( u/s \geq u/\lambda_u \to \infty \) as \( u \to \infty \). Consequently for any \( \varepsilon > 0, s \in [\delta_u, \lambda_u] \)
\[
c(1 - \varepsilon) \leq \frac{P(S > u/s)}{P(S^* > u/s)} \leq c(1 + \varepsilon)
\]
holds uniformly in $n$ for all $u$ large implying

$$
\mathbb{P}(SZ_n > u) \sim c \int_{\delta_n}^{\lambda_n} \mathbb{P}(S^* > u/s) \, dG_n(s) \sim \mathbb{P}(S^* Z_n > u)
$$

as $u \to \infty$ holds also uniformly in $n$, and thus the claim follows. ■

**Lemma 3.2.** Let $L_{n}, n \geq 1$ be as in Lemma 3.1 and let $Z_{n}, n \geq 1$ be positive random variables such that

$$
\overline{G}_n(z) := \mathbb{P}(Z_n > z) = \exp (-L_n z^q)
$$

for some $q > 0$ and all $z > 0$. If further $Z_{n}, n \geq 1$ are independent of a non-negative random variable $S$ which satisfies (2.1), then we have uniformly in $n$

$$
(3.1) \mathbb{P}(SZ_n > u) \sim \sqrt{\frac{2\pi L_p}{p + q} A_n^{\frac{q}{p+q}} g \left( A_n u^{\frac{q}{p+q}} \right) \exp \left(-D_n u^{\frac{pq}{p+q}}\right)}
$$

as $u \to \infty$, where $D_n = \left(L + Lpq^{-1}\right) A_n^p$ and $A_n = \left(qL_n\right)^{\frac{1}{p+q}} \left(Lp\right)^{-\frac{1}{p+q}}$.

**Proof.** If (2.1) holds, by Lemma 3.1, we have for all $u$ large

$$
\mathbb{P}(SZ_n > u) = \int_{0}^{\infty} \mathbb{P}\left(\frac{Z_n}{s} > \frac{u}{s}\right) \, dF(s) \sim \int_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \mathbb{P}\left(\frac{Z_n}{s} > \frac{u}{s}\right) \, dF(s)
$$

$$
\sim \int_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \exp \left(-L_n s^{-q}\right) \, dF(s)
$$

$$
\sim \int_{c_1 u^{\frac{q}{p+q}}}^{c_2 u^{\frac{q}{p+q}}} \exp \left(-L_n u^q s^{-q}\right) \, d(g(s) \exp (-L s^p)).
$$
Using similar arguments as in the proof of Theorem 2.1 in [8] we obtain as $u \to \infty$

$$P(SZ_n > u) \sim Lp \int_{c_1u^{p/q}}^{c_2u^{p/q}} s^{p-1} g(s) \exp \left( -L_n u^q s - Ls^p \right) ds$$

$$= Lp A_n^p u^{p/q} \int_{c_1A_n}^{c_2A_n} z^{p-1} g\left( A_n u^{q/p} z \right) \exp \left( -A_n^p u^{q/p} (Lpq^{-1} z^{-q} + L z^p) \right) dz$$

$$\sim \sqrt{\frac{2\pi}{p+q}} A_n^p u^{p/q} g\left( A_n u^{q/p} \right) \exp \left( -D_n u^{p/q} \right),$$

where $D_n = \left( L + Lpq^{-1} \right) A_n^p$ and $A_n = (qL_n)^{1/p} (Lp)^{-1/p}$, and thus the proof is complete.

**Lemma 3.3.** Assume that the distribution function $F$ of random variable $S$ satisfies (2.1), and further (2.4) holds, then we have

$$n \sum_{k=1}^{n-1} |\rho(k)| \int_{0}^{\infty} \int_{0}^{\infty} \exp \left( -\frac{(\tilde{u}_n/s)^2 + (\tilde{u}_n/t)^2}{2(1 + |\rho(k)|)} \right) dF(s) dF(t) \to 0$$

as $n \to \infty$, where $\tilde{u}_n = u_n(x)$.

**Proof.** Using similar arguments as in Lemma 4.3.2 in [10], let $\epsilon_n = \lceil n^\beta \rceil$ and $\sigma = \max_{k \geq 1} |\rho(k)| < 1$, where $\beta$ is any positive constant such that $\beta < 2(1 + \sigma)^{-2/p} - 1$. According to (2.4) and (2.2) we have

$$\exp \left( -T\tilde{u}_n^{2/p} \right) \sim C g^{-1} \left( Q^{-1} \tilde{u}_n^{2/p} \right) n^{-1} \tilde{u}_n \sim \left( \frac{\ln n}{T} \right)^{2/p},$$

where $T$ and $Q$ are defined in (2.3), and $C$ is a positive constant which may change from line to line.

By (3.1) with $q = 2$ and $L_k = 1/2(1 + |\rho(k)|)$ and split the sum into two
\[ n \sum_{k=1}^{n-1} |\rho(k)| \int_0^\infty \int_0^\infty \exp \left( -\frac{(\hat{u}_n/s)^2 + (\hat{u}_n/t)^2}{2(1+|\rho(k)|)} \right) \ dF(s) \ dF(t) \]
\[ \leq C n \sum_{k=1}^{n-1} |\rho(k)| \hat{u}_n^{2p} \ g \left( A_k \hat{u}_n^{2p} \right) \exp \left( -2(1+|\rho(k)|)^{-\frac{p}{2+p}} T \hat{u}_n^{2p} \right) \]
\[ = C n \left( \sum_{k=\tau_n+1}^{n} + \sum_{k=1}^{n-1} \right) |\rho(k)| \hat{u}_n^{2p} \ g \left( A_k \hat{u}_n^{2p} \right) \exp \left( -2(1+|\rho(k)|)^{-\frac{p}{2+p}} T \hat{u}_n^{2p} \right). \]

Since \( g(\cdot) \) is ultimately monotone, assume without loss of generality that it is ultimately increasing. By the assumption that \( g(\cdot) \) is a regularly varying function at infinity with index \( \alpha \), using Potter bound see e.g., [13], [6] for arbitrary \( \varepsilon > 0 \), \( k \geq 1 \) we have
\[ g \left( A_k \hat{u}_n^{2p} \right) \leq g \left( Q^{-1} \hat{u}_n^{2p} \right) \leq C \hat{u}_n^{2(\alpha+\varepsilon)/2p} \]
for all \( n \) large. Hence the first part is dominated by
\[ C n \hat{u}_n^{2p} \ g^2 \left( Q^{-1} \hat{u}_n^{2p} \right) \exp \left( -2(1+\sigma)^{-\frac{p}{2+p}} T \hat{u}_n^{2p} \right) \]
\[ = C n^{1+\beta} \hat{u}_n^{2p} \ g^2 \left( Q^{-1} \hat{u}_n^{2p} \right) \left( \exp \left( -T \hat{u}_n^{2p} \right) \right)^{(1+\sigma)^{-\frac{p}{2+p}}} \]
\[ \leq C n^{1+\beta} \hat{u}_n^{2p} \ g^2 \left( Q^{-1} \hat{u}_n^{2p} \right) \left( g \left( Q^{-1} \hat{u}_n^{2p} \right) n \right)^{(1+\sigma)^{-\frac{p}{2+p}}} \]
\[ \leq C n^{1+\beta-2(1+\sigma)^{-\frac{p}{2+p}}} (\ln n)^{1+2(\alpha+\varepsilon)/(1+\varepsilon)(1+\sigma)^{-\frac{p}{2+p}}} \to 0 \]
as \( n \to \infty \) since \( 1 + \beta - 2(1+\sigma)^{-\frac{p}{2+p}} < 0 \). Next set \( \sigma(l) = \max_{k \geq 1} |\rho(k)| < 1 \).

We may further write
\[ C n \sum_{k=\tau_n+1}^{n-1} |\rho(k)| \hat{u}_n^{2p} \ g^2 \left( A_k \hat{u}_n^{2p} \right) \exp \left( -2(1+|\rho(k)|)^{-\frac{p}{2+p}} T \hat{u}_n^{2p} \right) \]
\[ \leq C n^{2} \sigma(\tau_n) \hat{u}_n^{2p} \ g^2 \left( Q^{-1} \hat{u}_n^{2p} \right) \exp \left( -2(1+\sigma(\tau_n))^{-\frac{p}{2+p}} T \hat{u}_n^{2p} \right) \]
\[ \leq C n^{2} \sigma(\tau_n) \hat{u}_n^{2p} \ g^2 \left( Q^{-1} \hat{u}_n^{2p} \right) \exp \left( -2T \hat{u}_n^{2p} \right) \exp \left( 2T \sigma(\tau_n) \hat{u}_n^{2p} \right) \]
\[ \leq C \sigma(\tau_n) \hat{u}_n^{2p} \exp \left( 2T \sigma(\tau_n) \hat{u}_n^{2p} \right). \]
Using now (1.1) as \( n \to \infty \)
\[
\sigma(t_n) u_n^{2p} \sim T^{-1} \sigma(t_n) \ln n \leq T^{-1} \max_{k \geq t_n} |\rho(k)| \ln n \to 0
\]
the exponential term above tends to one and the remaining product tends to zero and thus the proof is complete. ■

**Lemma 3.4.** Let \( X_n, n \geq 1 \) be a ssGs satisfying (1.1), and let \( S_n, n \geq 1 \) be independent random variables satisfying (2.1) being further independent of \( X_n \). Additionally, assume that the survival function of \( Y_n = S_n X_n \) satisfy (2.4). Further if \( 0 < \theta < 1 \) and \( I_n \) is an interval containing \( k_n = \theta n \) members, we have
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} |\mathbb{P} (-u_n(y) < m(I_n) \leq M(I_n) \leq u_n(x)) - \exp(-\theta(\exp(-x) + \exp(-y)))| = 0,
\]
where \( M(I_n) = \max_{i \in I_n} Y_i \) and \( m(I_n) = \min_{i \in I_n} Y_i \).

**Proof.** Let \( Z_n, n \geq 1 \) be independent random variables with the same distribution as \( X_1 \) and define \( \mathfrak{M}_n = \max_{1 \leq i \leq n} S_i Z_i \) and \( m_n = \min_{1 \leq i \leq n} S_i Z_i \). For \( x, y \in \mathbb{R} \), using assumption (2.4), i.e.,
\[
(3.2) \quad \lim_{n \to \infty} n \mathbb{P}(S_i Z_1 > u_n(x)) = \exp(-x),
\]
\[
(3.3) \quad \lim_{n \to \infty} n \mathbb{P}(S_i Z_1 \leq -u_n(y)) = \exp(-y)
\]
and by Theorem 1.8.2 in [10] we have
\[
(3.4) \quad \lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} |\mathbb{P} (-u_n(y) < m_n \leq \mathfrak{M}_n \leq u_n(x)) - \Lambda(x) \Lambda(y)| = 0.
\]
Further if (2.1) holds, since \( S_n, n \geq 1 \) are independent with common distribution function \( F \) by a direct application of Berman inequality (see [12]) and Lemma 3.3 we obtain
\[
\begin{align*}
&\left| \mathbb{P}(-u_n(y) < m_n \leq M_n \leq u_n(x)) - \mathbb{P}(-u_n(y) < m_n \leq \mathfrak{M}_n \leq u_n(x)) \right| \\
&\leq \int_{[0, \infty]^n} \left| \mathbb{P} \left( \bigcap_{k=1}^n \left\{ \frac{-u_n(y)}{s_k} < X_k \leq \frac{u_n(x)}{s_k} \right\} \right) - \mathbb{P} \left( \bigcap_{k=1}^n \left\{ \frac{-u_n(y)}{s_k} < Z_k \leq \frac{u_n(x)}{s_k} \right\} \right) \right| \ dF(s_1) \cdots dF(s_n) \\
&\leq \mathcal{C} n \sum_{k=1}^{n-1} \int_0^{\infty} \int_0^{\infty} |\rho(k)| \exp \left( -\frac{(w_n/s)^2 + (w_n/t)^2}{2(1 + |\rho(k)|)} \right) dF(s) dF(t) \\
&\to 0, \quad n \to \infty,
\end{align*}
\]
where $w_n = \min(|u_n(x)|, |u_n(y)|)$. Thus by (3.4) we have
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \Pr (-u_n(y) < m_n \leq M \leq u_n(x)) - \Lambda(x)\Lambda(y) \right| = 0.
\]

Now let $v_n = u_{[n/\theta]}$, using (3.2) and (3.3) we get
\[
\lim_{n \to \infty} n \Pr (S Z > v_n(x)) = \theta \exp (-x)
\]
and
\[
\lim_{n \to \infty} n \Pr (S Z < -v_n(y)) = \theta \exp (-y),
\]
hence
\[
\lim_{n \to \infty} \sup_{x, y \in \mathbb{R}} \left| \Pr (-v_n(y) \leq m_n \leq M \leq u_n(x)) - \exp (-\theta (\exp (-x) + \exp (-y))) \right| = 0.
\]

Since $S_n, n \geq 1$ are independent and have a common distribution function $F$, by the stationarity of $X_n, n \geq 1$
\[
\Pr (-u_n(y) < m(I_n) \leq M(I_n) \leq u_n(x))
= \Pr \left( \bigcap_{i \in I_n} \{-u_n(y) < S_i X_i \leq u_n(x)\} \right)
= \int_{(0, \infty)^k_n} \Pr \left( \bigcap_{i=1}^{k_n} \left\{ \frac{u_n(y)}{s_i} < X_i \leq \frac{u_n(x)}{s_i} \right\} \right) dF(s_1) \cdots dF(s_{k_n})
= \Pr (-u_n(y) < m_{k_n} \leq M_{k_n} \leq u_n(x)).
\]
Hence, replacing $n$ by $k_n$ in (3.5) establishes the claim.

\begin{remark}
Under the conditions of Lemma 3.4, we have
\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| \Pr (M(I_n) \leq u_n(x)) - \exp (-\theta \exp (-x)) \right| = 0.
\]
\end{remark}

\begin{lemma}
Let $I_1, I_2, \ldots, I_l$ (with $l$ a fixed number) be disjoint subintervals of \{1, 2, \ldots, n\} such that $I_i$ has $k_{n,i} \sim \theta_i n$ elements, where $\theta_i$ are fixed positive constants with $\theta := \sum_{i=1}^l \theta_i \leq 1$. Then, under the assumptions of Lemma 3.4, we have
\[
\Pr \left( \bigcap_{i=1}^l \{-u_n(y) < m(I_i) \leq M(I_i) \leq u_n(x)\} \right)
- \prod_{i=1}^l \Pr (-u_n(y) < m(I_i) \leq M(I_i) \leq u_n(x)) \to 0
\]
\end{lemma}
as \( n \to \infty \).

**Proof.** Since \( X_n, n \geq 1 \) is a stationary random sequence, using Berman’s inequality and Lemma 3.3, we have

\[
\left| \mathbb{P}\left( \bigcap_{i=1}^{l} \{ -u_n(y) < m(I_i) \leq u_n(x) \} \right) - \prod_{i=1}^{l} \mathbb{P}\left( -u_n(y) < m(I_i) \leq u_n(x) \right) \right|
\]

\[
= \mathbb{P}\left( \bigcap_{i=1}^{l} \bigcap_{j \in I_i} \{ -u_n(y) < S_j X_j \leq u_n(x) \} \right) - \prod_{i=1}^{l} \mathbb{P}\left( \bigcap_{j \in I_i} \{ -u_n(y) < S_j X_j \leq u_n(x) \} \right)
\]

\[
\leq \int_{0}^{\hat{\theta}_l} \left| \mathbb{P}\left( \bigcap_{i=1}^{l} \hat{A}_i \right) - \prod_{i=1}^{l} \mathbb{P}\left( \hat{A}_i \right) \right| dF(s_1) \cdots dF(s_{\hat{\theta}_l})
\]

\[
\leq \hat{\theta}_i \sum_{k=1}^{\infty} \int_{0}^{\hat{\theta}_i} \left| \rho(k) \right| \exp \left( -\frac{(w_n/s)^2 + (w_n/t)^2}{2(1 + |\rho(k)|)} \right) dF(s) dF(t)
\]

\[
\to 0, \quad n \to \infty,
\]

where \( \hat{A}_i = \bigcap_{j=\hat{\theta}_{i-1}+1}^{\hat{\theta}_i} \left\{ -\frac{u_n(y)}{s_j} < X_j \leq \frac{u_n(x)}{s_j} \right\} \) with

\[
\hat{\theta}_i = \sum_{j=1}^{i} |\theta_j|, \quad \hat{\theta}_0 = 0, \quad w_n = \min(|u_n(x)|, |u_n(y)|),
\]

hence the proof is complete. \( \blacksquare \)

**Remark 3.2.** Under the conditions of Lemma 3.5, we have

\[
\lim_{n \to \infty} \left| \mathbb{P}\left( \bigcap_{i=1}^{l} \{ M(I_i) \leq u_n(x) \} \right) - \prod_{i=1}^{l} \mathbb{P}\left( M(I_i) \leq u_n(x) \right) \right| = 0.
\]

**Proof of Theorem 2.1.** In view of [14] we need first to prove that the marginal point processes of \( N_{n,d} \) converge weakly to a Poisson process \( N_d \) with intensity \( \exp(-x_d) \), \( d = 1, 2 \). By Theorem A.1 in [10] for \( N_{n,1}(B_1, x_1) \), it is suf-
icient to show that as $n \to \infty$

(P1). $\mathbb{E}(N_{n,1}((s, t], x_1))$ 

\[ \to \mathbb{E}(N_1((s, t], x_1)) = (t - s) \exp \left( -x_1 \right), \quad 0 < s < t \leq 1; \]

(P2). $\mathbb{P} \left( \bigcap_{i=1}^{k} \{ N_{n,1}((s_i, t_i], x_1) = 0 \} \right)$ 

\[ \to \mathbb{P} \left( \bigcap_{i=1}^{k} \{ N_1((s_i, t_i], x_1) = 0 \} \right) = \exp \left( -\sum_{i=1}^{k} (t_i - s_i) \exp \left( -x_1 \right) \right), \]

where $0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_k < t_k \leq 1$.

We have

\[ \mathbb{E}(N_{n,1}((s, t], x_1)) = \mathbb{E} \left( \sum_{i/n \in (s, t]} I(S_i X_i > u_n(x_1)) \right) \]

\[ = \sum_{i/n \in (s, t]} \mathbb{P}(S_i X_i > u_n(x_1)) \]

\[ \to (t-s) \exp \left( -x_1 \right) = \mathbb{E}(N_1((s, t], x_1)) \]

as $n \to \infty$, where the above convergence follows from (2.4).

In order to show (P2) note first that for $0 < s < t \leq 1$

\[ \mathbb{P}(N_{n,1}((s, t], x_1) = 0) = \mathbb{P}(M(I_n) \leq u_n(x_1)), \]

where $I_n = \{[sn] + 1, \ldots, [tn]\}$. Further, $I_n$ contains $k_n$ integers with $k_n = [tn] - [sn] \sim (t-s)n$ as $n \to \infty$. Thus, in view of Remark 3.1 with $\theta = t-s < 1$ we have as $n \to \infty$

(3.6) $\mathbb{P}(N_{n,1}((s, t], x_1) = 0) \to \exp \left( -(t-s) \exp \left( -x_1 \right) \right)$. 

Next, let $E_i$ be the set of integers $\{[s_i n] + 1, \ldots, [t_i n]\}$ with $0 < s_1 < t_1 \leq s_2 < t_2 \leq \cdots \leq s_k < t_k \leq 1$, then we have

\[ \mathbb{P} \left( \bigcap_{i=1}^{k} \{ N_{n,1}((s_i, t_i], x_1) = 0 \} \right) = \mathbb{P} \left( \bigcap_{i=1}^{k} \{ M(E_i) \leq u_n(x_1) \} \right) \]

\[ = \prod_{i=1}^{k} \mathbb{P}(N_{n,1}((s_i, t_i], x_1) = 0) \]

\[ + \left( \mathbb{P} \left( \bigcap_{i=1}^{k} \{ M(E_i) \leq u_n(x_1) \} \right) - \prod_{i=1}^{k} \mathbb{P}(M(E_i) \leq u_n(x_1)) \right). \]

Using (3.6), the first term converges to $\exp \left( -\sum_{i=1}^{k} (t_i - s_i) \exp \left( -x_1 \right) \right)$ as $n \to \infty$. By Remark 3.2 the modulus of the remaining difference of terms tends to
0. Consequently, $N_{n,1}$ converge weakly to a Poisson process $N_1$ with intensity $\exp(-x_1)$. Since $Y_i \sim -Y_i$, $N_{n,2}$ also converge weakly to a Poisson process $N_2$ with intensity $\exp(-x_2)$.

Now define the avoidance function of $N_n$ as

$$F_{N_n}(B) = \mathbb{P}(N_{n,1}(B_1, x_1) = 0, N_{n,2}(B_2, x_2) = 0),$$

where $B_1$ and $B_2$ are defined below. To get the main result, it suffices to prove that

$$\lim_{n \to \infty} F_{N_n}(B)$$

exists for all $B = \bigcup_{d=1}^{2} \bigcup_{j=1}^{r}(B_{dj} \times \{d\})$, where $r$ arbitrary positive integers, $B_{dj} = (s_{dj}, t_{dj})$, $0 < s_{d1} < t_{d1} \leq s_{d2} < t_{d2} \leq \ldots \leq s_{dr} < t_{dr} \leq 1$, and $B_1 = \bigcup_{j=1}^{r} B_{1j}, B_2 = \bigcup_{j=1}^{r} B_{2j}$. We will show that

$$\lim_{n \to \infty} F_{N_n}(B) = \exp(-m(B_1) \exp(-x_1) - m(B_2) \exp(-x_2)).$$

For simplicity we only consider the case $B_1 \subset B_2$; other cases are similar. First consider the case $n(B_2 \setminus B_1) = o(n)$, i.e., $m(B_1) = m(B_2)$. Obviously,

$$0 \leq \mathbb{P}(-u_n(x_2) < Y_i \leq u_n(x_1), k/n \in B_1) 
-\mathbb{P}(Y_i \leq u_n(x_1), k/n \in B_1; -Y_i \leq u_n(x_2), l/n \in B_2) 
\leq \sum_{l,l/n \in B_2 \setminus B_1} \mathbb{P}(-Y_i > u_n(x_2)) \to 0$$

as $n \to \infty$. Consequently, by Lemma 3.4 and 3.5, we have

$$\lim_{n \to \infty} \mathbb{P}(Y_k \leq u_n(x_1), k/n \in B_1; -Y_i \leq u_n(x_2), l/n \in B_2) = \lim_{n \to \infty} \mathbb{P}(-u_n(x_2) < Y_k \leq u_n(x_1), k/n \in B_1) = \prod_{j=1}^{r} \exp(-(t_{1j} - s_{1j}) \exp(-x_1)) \prod_{j=1}^{r} \exp(-(t_{1j} - s_{1j}) \exp(-x_2)) = \exp(-m(B_1) \exp(-x_1) - m(B_2) \exp(-x_2)).$$

It suffices to prove the case of $n(B_2 \setminus B_1) = O(n)$. Noting that any $z > 0$

$$\mathbb{P}(-u_n(x_2) < Y_k \leq u_n(x_1), k/n \in B_1; -u_n(x_2) < Y_i \leq u_n(z), i/n \in B_2 \setminus B_1)$$

$$\leq \mathbb{P}(Y_k \leq u_n(x_1), k/n \in B_1; -Y_i \leq u_n(x_2), l/n \in B_2)$$

$$\leq \mathbb{P}(-u_n(x_2) < Y_k \leq u_n(x_1), k/n \in B_1; -u_n(x_2) < Y_i \leq u_n(z), i/n \in B_2 \setminus B_1)$$

$$+ \mathbb{P}(\max(Y_i, i/n \in B_2 \setminus B_1) > u_n(z))$$

$$= \mathbb{P}(-u_n(x_2) < Y_k \leq u_n(x_1), k/n \in B_1; -u_n(x_2) < Y_i \leq u_n(z), i/n \in B_2 \setminus B_1)$$

$$+ (1 - \mathbb{P}(\max(Y_i, i/n \in B_2 \setminus B_1) \leq u_n(z))).$$
Applying Lemma 3.4 and 3.5 once again, we obtain
\[
\exp (-m(B_1) (\exp (-x_1) + \exp (-x_2))) \\
\times \exp (-m(B_2 \setminus B_1) (\exp (-z) + \exp (-x_2))) \\
\leq \liminf_{n \to \infty} P(Y_k \leq u_n(x_1), k/n \in B_1; -Y_l \leq u_n(x_2), l/n \in B_2) \\
\leq \limsup_{n \to \infty} P(Y_k \leq u_n(x_1), k/n \in B_1; -Y_l \leq u_n(x_2), l/n \in B_2) \\
\leq \exp (-m(B_1) (\exp (-x_1) + \exp (-x_2))) \\
\times \exp (-m(B_2 \setminus B_1) (\exp (-z) + \exp (-x_2))) \\
+ (1 - \exp (-m(B_2 \setminus B_1) \exp (-z))).
\]
Hence, letting \( z \to \infty \) we have
\[
\lim_{n \to \infty} P(Y_k \leq u_n(x_1), k/n \in B_1; -Y_l \leq u_n(x_2), l/n \in B_2) \\
= \exp (-m(B_1) \exp (-x_1) - m(B_2) \exp (-x_2)).
\]
This establishes the proof. ■

**Proof of Corollary 2.1.** Notice that
\[
P \left( M^{(k)}_n \leq u_n(x), m^{(l)}_n > -u_n(y) \right) \\
= P \left( N_{n,1}((0, 1], x) \leq k - 1, N_{n,2}((0, 1], y) \leq l - 1 \right).
\]
Hence the proof follows by Theorem 2.1. ■

**Proof of Theorem 2.2.** By Lemma 3.3 of [7] we have if under the condition (2.7)
\[
\lim_{n \to \infty} n \sum_{k=1}^{n-1} \rho(k) \int_0^1 \int_0^1 \exp \left( -\frac{(u_n(x)/s)^2 + (u_n(x)/t)^2}{2(1 + |\rho(k)|)} \right) dF(s) dF(t) = 0.
\]
Consequently, Lemma 3.4 and 3.5 also hold for \( S_n \) satisfy (2.6). Hence the proof follows by utilizing similar arguments as in the proof of Theorem 2.1. ■

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