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Asymptotic Expansion of Gaussian Chaos via Probabilistic Approach

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Abstract For a centered d-dimensional Gaussian random vector $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ and a homogeneous function $h : \mathbb{R}^d \to \mathbb{R}$ we derive asymptotic expansions for the tail of the Gaussian chaos $h(\boldsymbol{\xi})$ given the function h is sufficiently smooth. Three challenging instances of the Gaussian chaos are the determinant of a Gaussian matrix, the Gaussian orthogonal ensemble and the diameter of random Gaussian clouds. Using a direct probabilistic asymptotic method, we investigate both the asymptotic behaviour of the tail distribution of $h(\boldsymbol{\xi})$ and its density at infinity and then discuss possible extensions for some general $\boldsymbol{\xi}$ with polar representation.

Keywords Wiener chaos · polynomial chaos · Gaussian chaos · multidimensional normal distribution · subexponential distribution · determinant of a random matrix · Gaussian orthogonal ensemble · diameter of random Gaussian clouds · max-domain of attraction

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1 Introduction and Main Results

Let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d)$ be a centered Gaussian random vector in $\mathbb{R}^d$, $d \geq 2$, with covariance matrix B, $B_{ij} := \mathbb{E}\xi_i\xi_j$. Let $h : \mathbb{R}^d \to \mathbb{R}$ be a homogeneous function of order $\alpha > 0$, that is, $h(x\boldsymbol{t}) = x^\alpha h(\boldsymbol{t})$ for all $x > 0$ and $\boldsymbol{t} = (t_1, \ldots, t_d) \in \mathbb{R}^d$. Simple examples for h are $h(\boldsymbol{t}) = \prod_{i=1}^d |t_i|^{\gamma_i}$ of order $\alpha = \gamma_1 + \ldots + \gamma_d$ and $h(\boldsymbol{t}) = \sum_{i=1}^d |t_i|^\alpha$ also of order α.

We say that the random variable $h(\boldsymbol{\xi})$ is a *Gaussian chaos of order* α. In the literature, the term Gaussian chaos of integer order α is traditionally reserved for the case where g is a homogeneous polynomial of degree α—this case goes back to Wiener (1938) where polynomial chaos processes were first time introduced—and by this reason it is spoken about as Wiener chaos in the Gaussian case. Here we follow the extended version of the term Gaussian chaos.

This contribution is concerned with the asymptotic behavior of the tail distribution of Gaussian chaos $h(\boldsymbol{\xi})$ and its density at infinity. We suppose that h is not negative, that is, for some $\boldsymbol{x}$, $h(\boldsymbol{x}) > 0$, otherwise our problem is trivial. The important contributions in this area are Hanson and Wright (1971) where an upper rough bound is obtained for the tail of $h(\boldsymbol{\xi})$ in the case of polynomial h of degree 2, Borell (1978, Theorem 2.2), Arcones and Giné (1993, Corollary 4.4), Janson (1997, Theorem 6.12), and Latała (1999, 2006) where some lower and upper bounds are derived in the case of polynomial h of general degree $\alpha \geq 2$ (see also Lehec (2011)). A closely related study is devoted to the derivation of lower and upper bounds for the distribution of the multiple Wiener–Itô integrals with respect to a white noise, see Major (2005, 2007). In all these papers the estimation of the distribution tail of $h(\boldsymbol{\xi})$ is based on upper bounds for the moments of $h(\boldsymbol{\xi})$; clearly this technique cannot help with exact asymptotics for the tails.

In this contribution we shall focus on the Gaussian framework, so the random vector $\boldsymbol{\xi}$ introduced above is equal in distribution to $\sqrt{B}\boldsymbol{\eta}$ where the coordinates of $\boldsymbol{\eta} = (\eta_1, \ldots, \eta_d)$ are independent with standard normal distribution. Thus for any x positive

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} = \mathbb{P}\{h(\sqrt{B}\boldsymbol{\eta}) > x\} = \mathbb{P}\{g(\boldsymbol{\eta}) > x\}, \tag{1}$$

with $g(\boldsymbol{u}) = h(\sqrt{B}\boldsymbol{u})$. The function $g : \mathbb{R}^d \to \mathbb{R}$ is homogeneous of order α as h is. In this standard way the problem for a general covariance matrix may be reduced to that with identity matrix.

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The distribution of $g(\boldsymbol{\eta})$ may contain an atom at zero point, $\mathbb{P}\{g(\boldsymbol{\eta}) = 0\} \ge 0$. Remarkably, the distribution of $g(\boldsymbol{\eta})$ restricted to $\mathbb{R} \setminus \{0\}$ always possesses a density function $p_{g(\boldsymbol{\eta})}(x)$, $x \ne 0$, and the following representation

$$p_{g(\boldsymbol{\eta})}(x) = \frac{1}{\alpha x}\big(\mathbb{E}\{\|\boldsymbol{\eta}\|^2; g(\boldsymbol{\eta}) > x\} - d \cdot \mathbb{P}\{g(\boldsymbol{\eta}) > x\}\big) \tag{2}$$

is valid for any $x > 0$, see Lemma 5 below.

Motivated by (1) and (2), in the following we shall formulate our results for the Gaussian chaos $g(\boldsymbol{\eta})$.

For the Gaussian chaos, at least two approaches are available for the asymptotic analysis of the tail distribution. The first approach is based on the asymptotic Laplace method and the second one exploits the rotation invariance of the standard normal distribution in $\mathbb{R}^d$ and may be regarded as a probabilistic approach. In the present paper we follow the probabilistic approach which is particularly convenient for study of the elliptic chaos, see Theorem 3 below. Earlier in our short note [20], we suggested to follow the asymptotic Laplace method in order to derive tail asymptotics for $g(\boldsymbol{\eta})$. Notice that the Laplace method gives less information on the most probable event where chaos large deviations occur. The advantage of the Laplace method is that it is easily applicable to so-called Weibullian chaos; the corresponding results will be presented in a forthcoming paper.

So, in this contribution our analysis is based on the rotation invariance of the standard normal distribution. That is, for a d-dimensional centered Gaussian random vector $\boldsymbol{\eta}$ with identity covariance matrix, the polar representation

$$\boldsymbol{\eta} \stackrel{d}{=} \chi\boldsymbol{\zeta} \tag{3}$$

holds in distribution where χ and $\boldsymbol{\zeta}$ are independent, $\chi^2 = \sum_{i=1}^d \eta_i^2$ has χ^2-distribution with d degrees of freedom and $\boldsymbol{\zeta}$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1} \subset \mathbb{R}^d$. Hence by the homogeneity property of h for any $x > 0$ we have

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \mathbb{P}\{\chi^\alpha g(\boldsymbol{\zeta}) > x\}. \tag{4}$$

In the sequel g is assumed to be continuous, so that $g(\boldsymbol{\zeta})$ is a non-negative bounded random variable. The random variable χ^α has the density function

$$p_{\chi^\alpha}(x) = \frac{1}{\alpha 2^{d/2-1}\Gamma(d/2)} x^{d/\alpha-1} e^{-x^{2/\alpha}/2}, \quad x > 0, \tag{5}$$

of Weibullian type with index $2/\alpha$ which is subexponential density if $\alpha > 2$, see e.g., Foss et al. [14, Sect. 4.3]. By this reason, the tail behaviour of the product $\chi^\alpha g(\boldsymbol{\zeta})$ heavily depends on the maximum of the function g on the unit sphere $\mathbb{S}_{d-1}$. Denote

$$\hat{g} := \max_{\boldsymbol{v}\in\mathbb{S}_{d-1}} g(\boldsymbol{v}) \quad \text{and} \quad \mathcal{M} := \{\boldsymbol{v} \in \mathbb{S}_{d-1} : g(\boldsymbol{v}) = \hat{g}\}.$$

We shall consider two different cases of the structure of the set $\mathcal{M}$:

(i) $\mathcal{M}$ consists of a finite number of isolated points.
(ii) $\mathcal{M}$ is a sufficiently smooth manifold of positive dimension m, $1 \le m \le d-2$, on the unit sphere.

In the second case we assume that $\mathcal{M}$ has no boundary which particularly assumes that $m \ne d-1$. This restriction comes from the observation that the existence of a boundary of the set of the points of maximum $\mathcal{M}$ strongly contradicts the condition that the function g is at least twice continuously differentiable with non-degenerate approaching of its maximum.

1.1 The case of finite $\mathcal{M}$

Here we consider a homogeneous continuous function $g : \mathbb{R}^d \to \mathbb{R}$ of order $\alpha > 0$ such that $\mathcal{M}$ consists of a finite number of points, say

$$\mathcal{M} = \{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}.$$

Let in the following $g \in C^2(\mathbb{R}^d \setminus \{\boldsymbol{0}\})$. For every point $\boldsymbol{v} \in \mathbb{S}_{d-1}$, denote by $g''_{d-1}(\boldsymbol{v})$ a Hessian matrix at point $\boldsymbol{v}$ of the function g restricted to the hyperplane tangent to the sphere $\mathbb{S}_{d-1}$ at point $\boldsymbol{v}$, that is, restricted to the hyperplane $\boldsymbol{v} + \mathcal{L}$ where $\mathcal{L} = \{\boldsymbol{u} \in \mathbb{R}^d : (\boldsymbol{u}, \boldsymbol{v}) = 0\}$. More precisely, we fix an orthogonal system of vectors in $\mathcal{L}$, say $\boldsymbol{u}_1, \dots, \boldsymbol{u}_{d-1}$ and consider the function $g_{d-1}(t_1, \dots, t_{d-1}) := g(\boldsymbol{v} + t_1\boldsymbol{u}_1 + \dots + t_{d-1}\boldsymbol{u}_{d-1})$ whose Hessian matrix is denoted by $g''_{d-1}(\boldsymbol{v})$.

Assume that for every $j = 1, \dots, k$

$$\det\Big(\frac{g''_{d-1}(\boldsymbol{v}_j)}{\alpha\hat{g}} - I_{d-1}\Big) < 0, \tag{6}$$

where I_n stands for the identity matrix of size n. As follows from Lemma 6,

$$g''_{d-1}(\boldsymbol{v}_j) - \hat{g}\alpha I_{d-1}$$

is just a Hessian matrix of the function g along the unit sphere at point $\boldsymbol{v}_j$; the latter is explained in more detail after Theorem 1. The condition (6) says that $\boldsymbol{v}_j \in \mathbb{S}_{d-1}$ is the point of non-degenerate maximum of the function g restricted to $\mathbb{S}_{d-1}$. Then the following result holds.

Theorem 1 *Let $g \in C^{2r+2}(\mathbb{R}^d \setminus \{\mathbf{0}\})$ for some $r \geq 0$. Then the following asymptotical expansion takes place, as $x \to \infty$:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = (x/\hat{g})^{-1/\alpha} e^{-(x/\hat{g})^{2/\alpha}/2}\Big(h_0 + \sum_{i=1}^{r} h_i x^{-2i/\alpha} + o(x^{-2r/\alpha})\Big), \tag{7}$$

where coefficients $h_0, \ldots, h_r \in \mathbb{R}$ only depend on α, $\hat{g}$, and derivatives of $g(\boldsymbol{\varphi})$ at points $\boldsymbol{\varphi}_j$ (the definition of $g(\boldsymbol{\varphi})$ see below after (12)); in particular,

$$h_0 := \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{k} \Big|\det\Big(\frac{g''_{d-1}(\boldsymbol{v}_j)}{\alpha\hat{g}} - I_{d-1}\Big)\Big|^{-1/2} \tag{8}$$

Moreover, the density function of $g(\boldsymbol{\eta})$ satisfies the following relation, as $x \to \infty$:

$$p_{g(\boldsymbol{\eta})}(x) = (x/\hat{g})^{1/\alpha-1} e^{-(x/\hat{g})^{2/\alpha}/2}\Big(\frac{h_0}{\alpha\hat{g}} + \sum_{i=1}^{r} \tilde{h}_i x^{-2i/\alpha} + o(x^{-2r/\alpha})\Big). \tag{9}$$

Notice that if $\alpha = 2$, then the tail distribution of $g(\boldsymbol{\eta})$ is asymptotically proportional to its density function $p_{g(\boldsymbol{\eta})}(x)$ as $x \to \infty$ with multiplier $2\hat{g}$.

It is essential assumption that the function g is at least in C^2 and that its Hessian along the unit sphere is non-degenerate on $\mathcal{M}$. If it is not so, then the tail asymptotics may be quite specific and requires additional investigation—especially in the case where $g \notin C^2$. An example of natural Gaussian chaos with degenerate Hessian along the unit sphere is discussed below, see Example 9 in Section 2.

Sometimes it is more convenient to pass to some local coordinates on the sphere. Let $V_j \subset \mathbb{R}^d$ be a neighborhood of the point $\boldsymbol{v}_j \in \mathcal{M}$ and let h_j be a twice differentiable bijection from the open cube $(0,2)^d$ to V_j such that h_j is a bijection from $\{\boldsymbol{z} \in (0,2)^d : z_d = 1\}$ to $V_j \cap \mathbb{S}_{d-1}$. Denote by

$$(g \circ h_j)''_{d-1}(\boldsymbol{z}) := \left[\frac{\partial^2 (g \circ h_j)(\boldsymbol{z})}{\partial z_i \partial z_l}\right]_{i,l=1,\ldots,d-1}, \quad \boldsymbol{z} \in (0,1)^{d-1},$$

the Hessian matrix of $g \circ h_j$ restricted to the first $d-1$ coordinates—it is a $(d-1) \times (d-1)$ matrix—and write $\boldsymbol{z}_j \in (0,2)^{d-1} \times \{1\}$ for a point satisfying $h_j(\boldsymbol{z}_j) = \boldsymbol{v}_j$. We will prove in Lemma 6 that, at every point $\boldsymbol{v}_j \in \mathcal{M}$, the following equality holds:

$$\det(g''_{d-1}(\boldsymbol{v}_j) - (\alpha\hat{g}) I_{d-1}) = \frac{\det(g \circ h_j)''_{d-1}(\boldsymbol{z}_j)}{(\det J_j(\boldsymbol{z}_j))^2}, \tag{10}$$

where $J_j(\boldsymbol{u})$ is the Jacobian matrix of h_j. Then the representation (8) for the constant h_0 can be rewritten in terms of local coordinates as follows:

$$h_0 := \frac{1}{\sqrt{2\pi}} (\alpha\hat{g})^{\frac{d-1}{2}} \sum_{j=1}^{k} \frac{|\det J_j(\boldsymbol{z}_j)|}{\sqrt{|\det(g \circ h_j)''_{d-1}(\boldsymbol{z}_j)|}}. \tag{11}$$

A particular example is given by the hyperspherical coordinates, $\boldsymbol{v} = (r, \boldsymbol{\varphi})$, with Jacobian

$$\det J(r, \boldsymbol{\varphi}) = r^{d-1} \sin^{d-2}\varphi_1 \ldots \sin\varphi_{d-2} = r^{d-1} \det J(1, \boldsymbol{\varphi}),$$

where $J(r, \boldsymbol{\varphi})$ stands for the Jacobian matrix, that is,

$$\begin{aligned}
v_1 &= r\cos\varphi_1 \\
v_2 &= r\sin\varphi_1\cos\varphi_2 \\
&\ldots \\
v_{d-1} &= r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{d-2}\cos\varphi_{d-1} \\
v_d &= r\sin\varphi_1\sin\varphi_2\ldots\sin\varphi_{d-2}\sin\varphi_{d-1},
\end{aligned} \tag{12}$$

with $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{d-1}) \in \varPi_{d-1} := [0,\pi)^{d-2} \times [0, 2\pi)$ the angular coordinates of $\boldsymbol{v}$, $r = \|\boldsymbol{v}\|$. As usual, the topology in the set $\varPi_{d-1}$ is induced by the topology on the unit sphere, in particular, all points of $\varPi_{d-1}$ are inner points. Changing in such a way variables, we have (we set $g(\boldsymbol{\varphi}) = g(\boldsymbol{v}/\|\boldsymbol{v}\|)$; the function $g(\boldsymbol{\varphi})$ is continuous too; hereinafter we denote by the same symbol g two formally different functions, on $\mathbb{R}^d$ and on $\varPi_{d-1}$, but this hopefully does not lead to any confusion)

$$\hat{g} = \max_{\boldsymbol{\varphi} \in \varPi_{d-1}} g(\boldsymbol{\varphi}) \quad \text{and} \quad \mathcal{M}_\varphi := \{\boldsymbol{\varphi} \in \varPi_{d-1} : g(\boldsymbol{\varphi}) = \hat{g}\}.$$

Denote by $g''(\boldsymbol{\varphi})$ the Hessian matrix of $g(\varphi_1, \ldots, \varphi_{d-1})$. So, in the particular case of hyperspherical coordinates, the equality (11) implies that

$$h_0 := \frac{1}{\sqrt{2\pi}} (\alpha\hat{g})^{\frac{d-1}{2}} \sum_{j=1}^{k} \frac{|\det J(1, \boldsymbol{\varphi}_j)|}{\sqrt{|\det g''(\boldsymbol{\varphi}_j)|}}. \tag{13}$$

Then the condition (6) requires that the maximum of the function $g(\boldsymbol{\varphi})$, $\boldsymbol{\varphi} \in \Pi_{d-1}$, is non-degenerate at points $\boldsymbol{\varphi}_1, \dots, \boldsymbol{\varphi}_k$.

Some results for the case of isolated points of maximum may be also found in Breitung and Richter [9]. It seems that the proofs of the asymptotic expansions of Theorems 4 and 5 in [9] cannot be considered as self-contained. For instance, it is only proven in [9, Lemma 3] that $\gamma_2 = 0$. In order to prove their Theorems 4 and 5 as they are stated, it is necessary to prove that all $\gamma_{2m=0}$. In addition, at the beginning of their proofs of Theorems 4 and 5 Breitung and Richter [9] write that all coefficients a_i for odd i are zero because of Theorem 4.5 in Fedoryuk [13, p. 82]. This argument does not help because Theorem 4.5 in Fedoryuk [13, p. 82] is not about asymptotic expansion of $F(A;(1+z)^{1/2})$ what is required by the authors, so that it is irrelevant to the issue considered.

1.2 The case of a manifold

Now consider the case where $\mathcal{M} \subset \mathbb{S}_{d-1}$ is for some $m \in \{1, \dots, d-2\}$ an m-dimensional manifold of finite volume and has no boundary.

Fix some $r \in \mathbb{Z}^+$. We assume that the manifold $\mathcal{M}$ is C^{2r+2}-smooth.

We suppose that the rank of the matrix $A_{d-1}(\boldsymbol{v}) := \frac{g''_{d-1}(\boldsymbol{v})}{\hat{g}\alpha} - I_{d-1}$ of size $d-1$ is equal to $d-1-m$ for every $\boldsymbol{v} \in \mathcal{M}$. Denote by $\det\big(\frac{g''_{d-1-m}(\boldsymbol{v})}{\hat{g}\alpha} - I_{d-1-m}\big)$ any non-zero $(d-1-m)$-minor of the matrix $A_{d-1}(\boldsymbol{v})$; notice that all $(d-1-m)$-minors are equal one to another, by using orthogonal transform and set

$$h_0 := \frac{1}{(2\pi)^{\frac{m+1}{2}}} \int_{\mathcal{M}} \left|\det\Big(\frac{g''_{d-1-m}(\boldsymbol{v})}{\hat{g}\alpha} - I_{d-1-m}\Big)\right|^{-1/2} \mathrm{d}V, \tag{14}$$

where dV is the volume element of $\mathcal{M} \subset \mathbb{S}_{d-1}$.

Theorem 2 *Assume that the above conditions on $\mathcal{M}$ are fulfilled and that $g \in C^{2r+2}(\mathbb{R}^d \setminus \{\mathbf{0}\})$. Then the following asymptotical expansion takes place, as $x \to \infty$:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = (x/\hat{g})^{\frac{m-1}{\alpha}} e^{-(x/\hat{g})^{2/\alpha}/2} \Big(h_0 + \sum_{i=1}^{r} h_i x^{-2i/\alpha} + o(x^{-2r/\alpha})\Big), \tag{15}$$

where coefficients $h_1, \dots, h_r \in \mathbb{R}$ only depend on α, $\hat{g}$, and derivatives of $g(\boldsymbol{\varphi})$ on $\mathcal{M}_\varphi$.

Moreover, the density function of $g(\boldsymbol{\eta})$ satisfies the following relation, as $x \to \infty$:

$$p_{g(\boldsymbol{\eta})}(x) = (x/\hat{g})^{\frac{m+1}{\alpha}-1} e^{-(x/\hat{g})^{2/\alpha}/2} \Big(\frac{h_0}{\alpha\hat{g}} + \sum_{i=1}^{r} \tilde{h}_i x^{-2i/\alpha} + o(x^{-2r/\alpha})\Big). \tag{16}$$

Notice that if the manifold $\mathcal{M}$ has boundary points, then asymptotic expansion becomes more complicated. In general, boundary points have no impact on the leading constant h_0. Boundary makes strong contribution on further terms. For instance, if $d = 3$ and $\mathcal{M}$ is a line-segment on the unit sphere $\mathbb{S}_2$, then the term $x^{-1/\alpha}$ appears in the parentheses of the expansions (15) and (16); the corresponding calculations in the neighborhood of the boundary may be rather specific compared to those in Lemmas 2 and 3 below. The main reason for this comes from the fact that, in most cases, the function g is not in C^2 on the boundary; this is clearly demonstrated by the function $g(x) = -x^2$ for $x \le 0$ and $g(x) = 0$ for $x \ge 0$.

By the same reasons as in the case of finite $\mathcal{M}$ we have the following representation for the constant h_0 in terms of the spherical coordinates:

$$h_0 := \frac{1}{(2\pi)^{\frac{m+1}{2}}} (\alpha\hat{g})^{\frac{d-1-m}{2}} \int_{\mathcal{M}_\varphi} \frac{|\det J(1,\boldsymbol{\varphi})|}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}} \mathrm{d}V_{\boldsymbol{\varphi}}, \tag{17}$$

where $\mathrm{d}V_{\boldsymbol{\varphi}}$ is the volume element of $\mathcal{M}_\varphi \subset \Pi_{d-1}$.

The organisation of the rest of the paper is as follows. In Section 2 we discuss our main results and provide several examples that concern different cases for the dimension of $\mathcal{M}$. Proofs of the main results are presented in Sections 3 and 4.

In [20], a preliminary version of Theorem 2 was announced. Precisely, the relation

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = (x/\hat{g})^{\frac{m-1}{\alpha}} e^{-(x/\hat{g})^{2/\alpha}/2} \big(h_0 + O(x^{-2/\alpha})\big) \quad \text{as } x \to \infty$$

was stated without proof under the assumption that the function g is three times differentiable. It was suggested to follow the asymptotic Laplace method in order to derive this relation. One of the goals of the present paper is to provide a self-contained geometric proof of asymptotic expansion with $r+1$ terms under correct smoothness conditions.

1.3 Elliptical chaos

Before presenting several examples we show how our results can be extended for elliptical chaos or more generally for the chaos of polar random vectors. Consider therefore in the following $\boldsymbol{\xi}$ such that (22) holds with $\chi > 0$ some random variable being independent of $\boldsymbol{\zeta}$. Crucial properties used in the Gaussian case are **a)** χ has distribution function in the Gumbel max-domain of attraction, and **b)** the random vector $\boldsymbol{\zeta}$ has a $d-1$ dimensional subvector which possesses a positive density function. The first property a) means that for any $t \in \mathbb{R}$

$$\mathbb{P}\{\chi > x + t/w(x)\} \sim e^{-t}\mathbb{P}\{\chi > x\} \quad \text{as } x \uparrow x_+, \tag{18}$$

with w a positive scaling function and x_+ the upper endpoint of the distribution function of χ (in the Gaussian case $w(x) = x$, and $x_+ = \infty$). We abbreviate (18) as $\chi \in GMDA(w, x_+)$. Condition (18) is satisfied by a large class of random variables, for instance if χ is such that

$$\mathbb{P}\{\chi > x\} \sim c_1 x^a e^{-c_2 x^\beta} \quad \text{as } x \to \infty$$

for some c_1, c_2, β positive and $a \in \mathbb{R}$, then (18) holds with $w(x) = \beta c_2 x^{\beta-1}$. Notice that $c\chi^\beta$ is in the Gumbel max-domain of attraction for any $c > 0$ and $\beta > 0$ if and only if χ is in the Gumbel max-domain of attraction.

In order to relax the assumption on $\boldsymbol{\zeta}$ note first that in hyperspherical coordinates the random vector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d) \in \mathbb{S}_{d-1}$ can be written as $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{d-1}) \in \Pi_{d-1}$. Since $\boldsymbol{\zeta}$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1}$ in $\mathbb{R}^d$, the density function of the random vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{d-1}) \in \Pi_{d-1}$ equals $\frac{|\det J(1,\boldsymbol{\varphi})|}{\operatorname{mes}\mathbb{S}_{d-1}}$, $\boldsymbol{\varphi} \in \Pi_{d-1}$.

If χ is some positive random variable, then $\boldsymbol{\xi}$ is an elliptically symmetric random vector. When χ^2 is chi-square distributed with d degrees of freedom we recover as special case of elliptically symmetric random vectors the Gaussian ones. In particular for the Gaussian case we have

$$\mathbb{P}\{\chi^\alpha > x\} \sim \frac{1}{2^{d/2-1}\Gamma(d/2)} x^{(d-2)/\alpha} e^{-x^{2/\alpha}/2} \quad \text{as } x \to \infty \tag{19}$$

implying that χ^α is in the Gumbel MDA with scaling function $w(x) = x^{2/\alpha-1}/\alpha$, $x > 0$. If we relax our assumption on the distribution function of χ and simply assume (19) our previous results cannot be immediately re-formulated since the Gaussianity does not hold anymore. It turns out that even the larger class of elliptically symmetric random vectors for which χ satisfies (19) is a strong (unnecessary) restriction for the derivation of the tail asymptotics of $g(\boldsymbol{\eta})$. Indeed, we shall drop in the following the explicit distributional assumption on $\boldsymbol{\zeta}$ assuming only that $\boldsymbol{\nu}$ possesses a positive bounded continuous density function, say $p_\nu(\boldsymbol{\varphi})$. Next, we present the counterpart of Theorem 2, i.e., as therein we shall impose the same conditions on $\mathcal{M}$.

Theorem 3 *Assume that $g \in C^2(\mathbb{R}^d \setminus \{\mathbf{0}\})$. If $\boldsymbol{\zeta}$ is such that $\chi^\alpha \in GMDA(w, x_+)$ and further the random vector $\boldsymbol{\nu}$ has a positive bounded continuous density function $p_\nu(\boldsymbol{\varphi})$, then*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \sim \frac{h_0}{(xw(x/\hat{g}))^{\frac{d-1-m}{2}}} \mathbb{P}\{\chi^\alpha > x/\hat{g}\} \tag{20}$$

as $x \uparrow \hat{g}x_+$, where

$$h_0 = (2\pi\hat{g}^2)^{\frac{d-1-m}{2}} \int_{\mathcal{M}_\varphi} \frac{p_\nu(\boldsymbol{\varphi})}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}} \mathrm{d}V_{\boldsymbol{\varphi}} \in (0, \infty). \tag{21}$$

In particular $g(\boldsymbol{\eta}) \in GMDA(w, \hat{g}x_+)$.

Higher order asymptotic expansions for $\mathbb{P}\{g(\boldsymbol{\eta}) > x\}$—when $g \in C^{2r+2}$ and $\mathbb{P}\{\chi^\alpha > x\}$ possesses a suitable expansion—can also be derived; the asymptotics of the density of $g(\boldsymbol{\eta})$ can be given in terms of $\mathbb{P}\{g(\boldsymbol{\eta}) > x\}$ and the scaling function w if further χ^α possesses a bounded density function such that $\mathbb{P}\{\chi^\alpha > x\} \sim p_{\chi^\alpha}(x)/w(x)$ as $x \to x_+$. We shall omit these results here. Additionally the tail asymptotics of $g(\boldsymbol{\eta})$ can be found also (using similar arguments as in the proofs of Theorem 3) if instead of χ in the Gumbel max-domain of attraction we assume that χ is in the Weibull max-domain of attraction, i.e.,

$$\mathbb{P}\{\chi > x_+ - s/x\} \sim s^\gamma \mathbb{P}\{\chi > x_+ - 1/x\} \quad \text{as } x \to \infty,$$

for any $s > 0$ with $\gamma \geq 0$ some given constant. As in the Gumbel max-domain of attraction case, χ^β is in the Weibull max-domain of attraction for some $\beta > 0$ if and only if χ is in the Weibull max-domain of attraction, see Resnick (1987) for details on max-domain of attractions.

2 Discussion and Examples

In view of Theorems 1 and 2, the Gaussian chaos is a subexponential random variable if $\alpha > 2$ (under the assumptions therein). Subexponentiality of random variables is an important concept with various applications, see e.g., Embrechts et al. [12], Foss et al. [14]. It is possible to show subexponentiality of Gaussian chaos under some weak conditions on the homogeneous function h. As follows from the polar representation (3), the Gaussian random vector

$$\boldsymbol{\xi} = \sqrt{B}\boldsymbol{\eta} \stackrel{d}{=} \chi\sqrt{B}\boldsymbol{\zeta} \tag{22}$$

has covariance matrix B. Hence by the homogeneity property of h for any $x > 0$ we have

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} = \mathbb{P}\{\chi^\alpha h(\sqrt{B}\boldsymbol{\zeta}) > x\}. \tag{23}$$

Assuming that $h(\sqrt{B}\boldsymbol{\zeta})$ is a positive bounded random variable, then in view of Cline and Samorodnitsky [10, Corollary 2.5] the random variable $h(\boldsymbol{\xi})$ is subexponential if $\alpha > 2$ because then χ^α has the density function (5) which is of Weibullian type with index $2/\alpha < 1$ and subexponential by this reason.

As follows from the representation (23), for h bounded on $\mathbb{S}_{d-1}$, that is, $\hat{h} := \max\{h(\boldsymbol{u}) : \|\boldsymbol{u}\| = 1\} < \infty$,

$$\begin{aligned}\mathbb{P}\{h(\boldsymbol{\xi}) > x\} &\le \mathbb{P}\{\chi^\alpha > x/\hat{h}\} \\ &\le \frac{1}{\alpha 2^{d/2-1}\Gamma(d/2)}\int_{x/\hat{h}}^\infty y^{d/\alpha-1}e^{-y^{2/\alpha}/2}\mathrm{d}y.\end{aligned} \tag{24}$$

This upper bound is explicit and valid for any homogeneous function h as determined above. For the special case of *decoupled polynomial chaos*, upper bounds are known that are universal in d but less explicit in x, see e.g., Latała [31, Corollary 1]. See also upper bound by Arcones and Giné [3, Theorem 4.3] where the case of general polynomial chaos is considered; the corresponding upper bound is not always better in d than (24) and it is less explicit.

If (with necessity discontinuous) a function h is unbounded on the unit sphere $\mathbb{S}_{d-1}$, then it is possible that $\mathbb{P}\{h(\sqrt{B}\boldsymbol{\zeta}) > x\} > 0$ for any x. Two cases of interest, which are also simple to deal with are $h(\sqrt{B}\boldsymbol{\zeta})$ is regularly varying with index $\gamma > 0$ and $h(\sqrt{B}\boldsymbol{\zeta})$ has a Weibullian tail.

In the first case where the tail of $h(\sqrt{B}\boldsymbol{\zeta})$ is heavier than that of χ, by Breiman's theorem (see [6])

$$\mathbb{P}\{h(\boldsymbol{\xi}) > x\} \sim \mathbb{E}\{\chi^{\gamma\alpha}\}\mathbb{P}\{h(\sqrt{B}\boldsymbol{\zeta}) > x\} \quad \text{as } x \to \infty.$$

Also, in view of Jacobsen et al. (2009) the converse of the above holds, i.e., if $h(\boldsymbol{\xi})$ is a regularly varying random variable, then $h(\sqrt{B}\boldsymbol{\zeta})$ is regularly varying too, with the same index. The second case that $h(\sqrt{B}\boldsymbol{\zeta})$ has a Weibullian tail can be handled by applying Lemma 3.2 in Arendarczyk and Dębicki (2011).

Notice that if the Gaussian vector $\boldsymbol{\xi}$ with covariance matrix B has a singular distribution, so that $\det B = 0$, then $\boldsymbol{\xi}$ is valued in the linear subspace $\mathcal{L} := \{\sqrt{B}\boldsymbol{u} : \boldsymbol{u} \in \mathbb{R}^d\}$ of lower dimension $d^* < d$. Therefore, it is necessary to proceed to a Gaussian random vector $\boldsymbol{\xi}^*$ in $\mathcal{L}$ of dimension d^* and to a new function $h^*(u) := h(u)$ defined on $\mathcal{L}$. In this way the problem is reduced to that with non-degenerate Gaussian distribution. For example, let $d = 2$ and $h(u_1, u_2) = u_2^4 - u_1^4/2$. Let further η_1 and η_2 be two independent $N(0,1)$ random variables, and set $\boldsymbol{\xi}_1 = (\eta_1, \eta_2)$ and $\boldsymbol{\xi}_2 = (\eta_2, \eta_2)$. Then the tail of $h(\boldsymbol{\xi}_1) = h(\eta_1, \eta_2) = \eta_2^4 - \eta_1^4/2$ is equivalent to that of η_2^4 which is much heavier than the tail of $h(\boldsymbol{\xi}_2) = h(\eta_2, \eta_2) = \eta_2^4/2$.

Example 1. Consider the chaos $g(\boldsymbol{\eta}) = |\eta_1|^\alpha + \ldots + |\eta_d|^\alpha$ of order $\alpha > 0$.

If $\alpha = 2$, so that we deal with χ^2-distribution with d degrees of freedom, then $\hat{g} = 1$ and $\mathcal{M}$ is the whole unit sphere $\mathbb{S}_{d-1}$ (which is a manifold of dimension $d-1$) and as known, the density function of $g(\boldsymbol{\eta})$ equals $\frac{1}{2^{d/2}\Gamma(d/2)}x^{d/2-1}e^{-x/2}$.

If $\alpha < 2$, then the function $y_1^{\alpha/2} + \ldots + y_d^{\alpha/2}$ is concave and, therefore, its maximum on the set $y_1 + \ldots + y_d = 1$, $y_i \ge 0$, is attained at the point $y_1 = \ldots = y_d = 1/d$. Hence, $\hat{g} = d^{1-\alpha/2}$ and $\mathcal{M}$ consists of 2^d points $(\pm 1/\sqrt{d}, \ldots, \pm 1/\sqrt{d})$ (which is a manifold of zero dimension). Then, by (9), the density function of $g(\boldsymbol{\eta})$ is equal to

$$c_2 x^{1/\alpha-1}e^{-d^{1-2/\alpha}x^{2/\alpha}/2}(1 + O(x^{-2/\alpha})) \quad \text{as } x \to \infty,$$

where

$$c_2 = 2^d\frac{1}{\alpha\hat{g}^{1/\alpha}}\frac{1}{\sqrt{2\pi}\sqrt{\left|\det\left(\frac{g''_{d-1}(\boldsymbol{v}_j)}{\alpha\hat{g}} - I_{d-1}\right)\right|}},$$

where $g''_{d-1}(\boldsymbol{v}_j) = \frac{\alpha(\alpha-1)}{d^{\alpha/2-1}}I_{d-1}$, $j \le 2^d$. Therefore,

$$c_2 = \frac{2^d}{\alpha d^{1/\alpha-1/2}\sqrt{2\pi}(2-\alpha)^{(d-1)/2}};$$

this is a special case of the result by Rootzén (1987, see (6.1)); see also Theorem 1.1 and Example 1.3 in Balkema et al. (1993). We see that the power of x in front of the exponent is changing together with the dimension of the manifold $\mathcal{M}$.

If $\alpha > 2$, then $\hat{g} = 1$ and $\mathcal{M}$ consists of $2d$ points $(0, \ldots, 0, \pm 1, 0, \ldots, 0)$ (which is again a manifold of zero dimension) and, by Theorem 1 with $r = 1$, the density function of $g(\boldsymbol{\eta})$ is equal to

$$c_3 x^{1/\alpha-1}e^{-x^{2/\alpha}/2}(1 + O(x^{-2/\alpha})) \quad \text{as } x \to \infty.$$

Here the matrix $g''_{d-1}(\boldsymbol{v}_j)$ is zero at every point $\boldsymbol{v}_j$ implying

$$c_3 = \frac{h_0}{\alpha} = \frac{2d}{\alpha\sqrt{2\pi}}.$$

In this case $|\eta_i|^\alpha$ has subexponential density

$$\frac{2}{\alpha\sqrt{2\pi}} x^{1/\alpha-1} e^{-x^{2/\alpha}/2}, \quad x > 0,$$

and the density function of $g(\boldsymbol{\eta})$ is asymptotically equivalent to d multiple of the density function of $|\eta_i|^\alpha$ as $x \to \infty$, see Foss et al. (2011, Chapter 4). Here the observation that $\mathcal{M}$ consists of d points $(0, \dots, 0, 1, 0, \dots, 0)$ is nothing else than the principle of a single big jump in the theory of subexponential distributions, see Foss et al. (2011, Section 3.1).

An equivalent way is to consider the L_α-norm $g(\boldsymbol{\eta}) = (|\eta_1|^\alpha + \ldots + |\eta_d|^\alpha)^{1/\alpha}$ of Gaussian vector $\boldsymbol{\eta}$ which delivers an example of Gaussian chaos of order 1. It may be naturally extended for a general Minkowski functional $h : \mathbb{R}^d \to \mathbb{R}^+$ where $h(\boldsymbol{\eta})$ is again a Gaussian chaos of order 1. Earlier tail behavior of Minkowski's type of Gaussian chaos was studied by Pap and Richter (1988).

Example 2. (Product of two Gaussian random variables ξ_1 and ξ_2) Here we consider the case $d = 2$ and assume without loss of generality that $\mathrm{Var}\xi_1 = \mathrm{Var}\xi_2 = 1$. Denote the correlation coefficient by ρ, $\rho \neq -1$. Then

$$B = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}, \quad \sqrt{B} = \frac{\sqrt{1+\rho}}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{\sqrt{1-\rho}}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

and $\boldsymbol{\xi}$ has the same distribution as $\sqrt{B}\boldsymbol{\eta}$. We consider the product

$$h(\xi_1, \xi_2) := \xi_1\xi_2 = g(\eta_1, \eta_2) = \frac{\rho}{2}(\eta_1^2 + \eta_2^2) + \eta_1\eta_2,$$

so that $\alpha = 2$ and $g(\boldsymbol{u}) = \rho(u_1^2 + u_2^2)/2 + u_1u_2$. Given $u_1^2 + u_2^2 = 1$, the maximum of u_1u_2 is attained on $\mathcal{M} = \{(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})\}$ and equals $\hat{g} = (1+\rho)/2$. At both points of the maximum we have $g''_{2-1}(\boldsymbol{v}_j) = \rho - 1$. Calculating h_0 we obtain for $\rho \neq -1$, as $x \to \infty$,

$$\mathbb{P}\{\xi_1\xi_2 > x\} = \frac{1+\rho}{\sqrt{2\pi}} x^{-1/2} e^{-x/(1+\rho)} \big(1 + O(1/x)\big),$$
$$p_{\xi_1\xi_2}(x) = \frac{1}{\sqrt{2\pi}} x^{-1/2} e^{-x/(1+\rho)} \big(1 + O(1/x)\big).$$

Example 3. (Product of independent Gaussian random variables) Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$ be a standard Gaussian standard vector, that is, its components are $N(0,1)$ independent random variables. Taking $g(\boldsymbol{u}) = u_1 \ldots u_d$ we have $\alpha = d$ and further $\hat{g} = 1/d^{d/2}$ since

$$\mathcal{M} = \{(\pm 1/\sqrt{d}, \dots, \pm 1/\sqrt{d}) \text{ with even number of negative coordinates}\},$$

which consists of 2^{d-1} points (the product $u_1 \ldots u_d$ should be positive). Further, in the spherical coordinates

$$g(\boldsymbol{\varphi}) = \sin^{d-1}\varphi_1 \ldots \sin\varphi_{d-1} \cos\varphi_1 \ldots \cos\varphi_{d-1}.$$

For instance, at the point $(1/\sqrt{d}, \dots, 1/\sqrt{d})$ we have $\cos\varphi_i = \sqrt{\frac{1}{d-i+1}}$ and $\sin\varphi_i = \sqrt{\frac{d-i}{d-i+1}}$, so that $\det J(1, \boldsymbol{\varphi}) = \sqrt{(d-1)!}/d^{(d-2)/2}$ at these points. Additional calculations show, at any point $\boldsymbol{\varphi} \in \mathcal{M}_\varphi$,

$$g''_{\varphi_i\varphi_i}(\boldsymbol{\varphi}) = -2g(\boldsymbol{\varphi})(d-i+1) = -\frac{2(d-i+1)}{d^{d/2}}, \quad g''_{\varphi_i\varphi_j}(\boldsymbol{\varphi}) = 0 \quad \text{for } i \neq j,$$

which yields $|\det g''(\boldsymbol{\varphi})| = 2^{d-1}d!/d^{d(d-1)/2}$. In this way we get the following answer

$$p_{\eta_1\ldots\eta_d}(x) = \frac{2^{(d-1)/2}}{\sqrt{2\pi d}} x^{1/d-1} e^{-dx^{2/d}/2} \big(1 + O(x^{-2/d})\big) \quad \text{as } x \to \infty.$$

The intuition behind this asymptotic behaviour is the following (see e.g., Sornette (1998)): asymptotically, the tail of the product is controlled by the realisations where all terms are of the same order; therefore $p_{\eta_1\ldots\eta_d}(x)$ is, up to the leading order, just the product of the d marginal density functions, evaluated at $x^{1/d}$.

Example 4. (Product of components of a Gaussian vector) Let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) = \sqrt{B}\boldsymbol{\eta}$ be a Gaussian vector with mean zero and with covariance matrix B and consider $h(\boldsymbol{u}) = u_1 \ldots u_d$. Further, decompose the symmetric positive-semidefinite matrix $\sqrt{B}$ as $\sqrt{B} = Q^{\mathrm{T}}DQ$ where Q is an orthogonal matrix (the rows of which are eigenvectors of $\sqrt{B}$), and D is diagonal (having the eigenvalues of $\sqrt{B}$ on the diagonal). Making use of the representation (3) we deduce

$$h(\boldsymbol{\xi}) = h(\sqrt{B}\boldsymbol{\eta}) = \chi^\alpha h(\sqrt{B}\boldsymbol{\zeta}) = \chi^\alpha h(Q^{\mathrm{T}}DQ\boldsymbol{\zeta}).$$

Since Q is orthogonal the random vector $\boldsymbol{\zeta}^* := Q\boldsymbol{\zeta}$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1}$. Therefore, $D\boldsymbol{\zeta}^*$ is distributed on the ellipsoid E with the semi-principal axes of lengths equal to the diagonal elements of D. The product of coordinates of $Q^{\mathrm{T}}\boldsymbol{v}$ on $\boldsymbol{v} \in E$ has only finite number of points of maximum; as above, denote this maximum by $\hat{g}$.

It is not clear how to identify the set $\mathcal{M}$ and the constants $\hat{g}$ and h_0 explicitly, in terms of the covariance matrix B, but we may guarantee that, by Theorem 1 with $r = 1$

$$p_{\xi_1 \dots \xi_d}(x) = \mathrm{const} \cdot x^{1/d-1} e^{-(x/\hat{g})^{2/d}/2} \big(1 + O(x^{-2/d})\big) \quad \text{as } x \to \infty.$$

Example 5. (Quadratic forms of independent $N(0,1)$ random variables) Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$ be as in the previous section and let $g(\boldsymbol{\eta}) = \sum_{i=1}^d a_i \eta_i^2$ where the constants $a_i \in \mathbb{R}$ are such that

$$a_1 \le a_2 \le \dots \le a_{d-m} < a_{d-m+1} = \dots = a_d = a, \quad a > 0.$$

If $m = d$, then $g(\boldsymbol{\eta})/a$ is a chi-square random variable. So we consider the case $m \le d-1$. Since

$$g(\boldsymbol{u}) = \sum_{i=1}^{d-m} a_i u_i^2 + a \sum_{i=d-m+1}^{d} u_i^2$$

and all $a_i < a$ for $i \le d-m$, the maximum of $g(\boldsymbol{u})$ given $\|\boldsymbol{u}\| = 1$ is attained at any point $\boldsymbol{u}$ such that $u_{d-m+1}^2 + \dots + u_d^2 = 1$ and $u_1 = \dots = u_{d-m} = 0$ implying $\hat{g} = a$. We have further

$$g(\boldsymbol{\varphi}) = a_1 \cos^2 \varphi_1 + \sum_{i=2}^{d-m} a_i \sin^2 \varphi_1 \dots \sin^2 \varphi_{i-1} \cos^2 \varphi_i + a \sin^2 \varphi_1 \dots \sin^2 \varphi_{d-m}.$$

The set $\mathcal{M}_\varphi$ of dimension $m-1$ is the sub-parallelepiped of Π_{d-1}, namely, $\mathcal{M}_\varphi = \{\boldsymbol{\varphi} \in \Pi_{d-1} : \varphi_1 = \dots = \varphi_{d-m} = \pi/2\}$ for $m \ge 2$, its inverse image $\mathcal{M}$ is a unit sphere $\mathbb{S}_{m-1}$; in the case $m = 1$ it consists of two points $(\pi/2, \dots, \pi/2, \pi/2)$ and $(\pi/2, \dots, \pi/2, 3\pi/2)$. For $\boldsymbol{\varphi} \in \mathcal{M}_\varphi$, the matrix $g''_{d-1}(\boldsymbol{\varphi})$ is diagonal with first entries $2(a_i - a)$ for $i = 1, \dots, d-m$ and zeros on the rest of diagonal, so

$$|\det g''_{d-m}(\boldsymbol{\varphi})| = 2^{d-m} \prod_{i=1}^{d-m} (a - a_i)$$

does not depend on $\boldsymbol{\varphi} \in \mathcal{M}_\varphi$. Therefore,

$$\int_{\mathcal{M}_\varphi} \frac{|\det J(1, \boldsymbol{\varphi})|}{\sqrt{|\det g''_{d-m}(\boldsymbol{\varphi})|}} \mathrm{d}V_{\boldsymbol{\varphi}} = \frac{\mathrm{mes}\, \mathcal{M}}{2^{\frac{d-m}{2}} \prod_{i=1}^{d-m} \sqrt{a - a_i}}.$$

Taking into account that $\mathrm{mes}\, \mathcal{M} = \mathrm{mes}\, \mathbb{S}_{m-1} = 2\pi^{m/2}/\Gamma(m/2)$ we finally deduce, as $x \to \infty$

$$p_{\sum_{i=1}^d a_i \eta_i^2}(x) = \frac{1}{a 2^{m/2} \Gamma(m/2)} \prod_{i=1}^{d-m} \frac{1}{\sqrt{1 - a_i/a}} (x/a)^{m/2-1} e^{-x/2a} (1 + O(1/x)), \tag{25}$$

which agrees (for the first order asymptotics) with Hoeffding (1964) (see also Zolotarev (1961), Imkeller (1994), Piterbarg (1994, 1996), Hüsler et al. (2002)).

Example 6. (Scalar product of Gaussian random vectors) Closely related to Example 5 is the scalar product of two independent Gaussian random vectors, namely we consider the Gaussian chaos $g(\boldsymbol{\eta}, \boldsymbol{\eta}^*) = \sum_{i=1}^d a_i \eta_i \eta_i^*$ with η_i, η_i^*, $i \le d$, independent $N(0,1)$ random variables. Indeed, since $\eta_i \eta_i^*$ coincides in distribution with

$$\frac{\eta_i + \eta_i^*}{\sqrt{2}} \frac{\eta_i - \eta_i^*}{\sqrt{2}} = \frac{\eta_i^2 - \eta_i^{*2}}{2}$$

we have the equality in distribution

$$g(\boldsymbol{\eta}, \boldsymbol{\eta}^*) \stackrel{d}{=} \frac{1}{2} \Big(\sum_{i=1}^d a_i \eta_i^2 - \sum_{i=1}^d a_i \eta_i^{*2} \Big).$$

Therefore, if

$$|a_1| \le |a_2| \le \dots \le |a_{d-m}| < |a_{d-m+1}| = \dots = |a_d| = a, \quad a > 0,$$

then the asymptotics of the density given by (25) is applicable, and we have as $x \to \infty$

$$p_{g(\boldsymbol{\eta}, \boldsymbol{\eta}^*)}(x) = \frac{1}{a 2^{d/2} \Gamma(m/2)} \prod_{i=1}^{d-m} \frac{1}{\sqrt{1 - a_i^2/a^2}} (x/a)^{m/2-1} e^{-x/a} (1 + O(1/x)).$$

We note that results for the scalar products of Gaussian random variables are derived in Ivanoff and Weber (1998) and Hashorva et al. (2012).

Example 7. (Determinant of a random Gaussian matrix) Let $A = [A_{ij}]_{i,j=1}^n$ be a random square matrix of order n whose entries A_{ij} are independent $N(0,1)$ random variables. Then its determinant is the following function of A_{ij}:

$$\det A = \sum_{\sigma \in S_n} \mathrm{sgn}(\sigma) \prod_{i=1}^n A_{i,\sigma_i},$$

where S_n is the set of all permutations of the set $\{1, 2, \dots, n\}$ and $\operatorname{sgn}(\sigma)$ denotes the signature of $\sigma \in S_n$. Clearly, the determinant $g(A) := \det A$ is a continuous homogeneous function of order $\alpha = n$. Here we have $d = n^2$.

The determinant of the matrix A represents the (oriented) volume of the parallelepiped generated by the vectors $\boldsymbol{A}_i := (A_{i1}, \dots, A_{in})$, $i = 1, \dots, n$. Given

$$\sum_{i=1}^{n} \|\boldsymbol{A}_i\|^2 = 1,$$

the maximal volume of this parallelepiped is attained on orthogonal vectors $\boldsymbol{A}_i$ which are of the same length, that is, on the n-dimensional cube with side of length $1/\sqrt{n}$. Therefore,

$$\hat{g} := \max_{A: \sum_{i,j=1}^{n} A_{ij}^2 = 1} \det A = n^{-n/2}.$$

The manifold consisting of points where the maximum $\hat{g}$ of $g(A)$, $A \in \mathbb{S}_{n^2-1}$, is attained, that is,

$$\mathcal{M} := \left\{A : \det A > 0, \|\boldsymbol{A}_1\| = \ldots = \|\boldsymbol{A}_n\| = 1/\sqrt{n} \text{ and } \boldsymbol{A}_1, \dots, \boldsymbol{A}_n \text{ are orthogonal}\right\}$$

has dimension $m = (n^2 - n)/2$. Therefore, by Theorem 2,

$$\mathbb{P}\{\det A > x\} = c x^{\frac{n-1}{2} - \frac{1}{n}} e^{-n x^{2/n}/2} (1 + O(x^{-2/n})) \quad \text{as } x \to \infty, \tag{26}$$

for some $c = c(n) > 0$; the computation of this constant is questionable. This answer agrees with Theorem 10.1.4(i) by Barbe (2003) in the exponential term and gives the correct power term.

Another way to show this result is to recall from Prékopa (1967, Theorem 2) that

$$\det A \stackrel{d}{=} \prod_{i=1}^{n} \chi_i^2, \tag{27}$$

where $\chi_1^2, \dots, \chi_n^2$ are independent random variables and χ_i^2 is chi-square distributed with i degrees of freedom. In the case $n = 2$ it easily follows by conditioning on η_1 and η_3:

$$\det A^2 \stackrel{d}{=} (\eta_1\eta_2 + \eta_3\eta_4)^2 \stackrel{d}{=} \eta_5^2(\eta_6^2 + \eta_7^2) =: \chi_1^2 \chi_2^2,$$

where $\eta_1, \dots, \eta_7$ are independent $N(0, 1)$ random variables. The representation (27) provides an alternative way of deducing the tail asymptotics of $\det A$, since we can readily apply Lemma 3.2 in [1].

Example 8. (Gaussian orthogonal ensemble) Now let $A = [A_{ij}]_{i,j=1}^n$ be a random square symmetric matrix of order n whose random entries A_{ij} are independent for $1 \le i \le j \le n$. Let $A_{ij} = \eta_{ij}$ for $j > i$ and $A_{ii} = \sigma\eta_{ii}$ where η_{ij} are independent standard random variables and $\sigma > 1$. In the special case $\sigma = \sqrt{2}$ the matrix A is called the *Gaussian orthogonal ensemble.*

Here the determinant $g(A) := \det A$ is again a continuous homogeneous function of order $\alpha = n$; $d = (n^2 + n)/2$. Due to the coefficients $\sigma > 1$ on the diagonal, the maximal volume of the corresponding parallelepiped is attained on the orthogonal vectors $(\pm\sigma/\sqrt{n}, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, \pm\sigma/\sqrt{n})$ with even number of minuses. Hence, $\hat{g} = (\sigma^2/n)^{n/2}$. Since $\mathcal{M}$ is finite, we apply Theorem 1 and deduce that

$$\mathbb{P}\{\det A > x\} \sim c x^{-1/n} e^{-n x^{2/n}/2\sigma^2}$$

as $x \to \infty$, for some $c = c(n) > 0$.

An alternative approach for computing asymptotics of the tail of the Gaussian orthogonal ensemble (where $\sigma = \sqrt{2}$) is to make use of the fact that the joint density function of the eigenvalues $\lambda_1(A) \le \ldots \le \lambda_n(A)$ is known and is equal to

$$c' e^{-\|y\|^2/4} \mathbb{I}\{y_1 \le \ldots \le y_n\} \prod_{i<j} (y_j - y_i),$$

with some explicitly known normalising constant $c' = c'(n) > 0$, see e.g., Theorem 2.5.2 in Anderson et al. (2010). Clearly this approach is more complicated from computational point of view because of the singularity of the product $\prod_{i<j}(y_j - y_i)$ on the diagonal $y_1 = \ldots = y_n$.

Indeed, there are Gaussian chaoses where Theorems 1 and 2 are not straightforward applicable because of degeneracy of their Hessian on the set $\mathcal{M}$ of extremal points. This is exactly the case of diameter of a random Gaussian chaos which is discussed next.

Example 9. (The diameter of a random Gaussian cloud) Let $\boldsymbol{\eta}_k = (\eta_{k1}, \dots, \eta_{km})$, $k = 1, \dots, n$, be i.i.d. random vectors in $\mathbb{R}^m$; here η_{kl}, $k = 1, \dots, n$, $l = 1, \dots, m$, are independent $N(0, 1)$ random variables. The set of random points $\{\boldsymbol{\eta}_k, k \le n\}$ may be called the *Gaussian cloud.* The problem is how to approximate the distribution of its *diameter*

$$D_n = \max_{1 \le k \le l \le n} \|\boldsymbol{\eta}_k - \boldsymbol{\eta}_l\|.$$

In [29], Matthews and Rukhin study limit behavior of D_n^2 as $n \to \infty$. Here we discuss the problem of estimation of the tail of D_n for a fixed n. First of all notice that it is equivalent to tail estimation of $g(\boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_n) := D_n^2$ which represents a smooth Gaussian chaos of order $\alpha = 2$, with $d = mn$. Since the cases $m = 1$ and $m \ge 2$ are different, consider them separately.

First consider the case of dimension 1, $m = 1$. For any $k \neq l$, introduce T_{kl} as the set of all vectors $(v_1, \ldots, v_n) \in \mathbb{R}^n$ such that $v_i = 0$ for all $i \notin \{k, l\}$, $v_k = -v_l$ and $v_k = \pm 1/\sqrt{2}$. Then, given $v_1^2 + \ldots + \eta_n^2 = 1$, the maximal value $\hat{g}$ of D_n^2 is attained on $\mathcal{M} = \cup_{k \neq l} T_{kl}$; this set consists of $n(n-1)$ points. In particular,

$$\hat{g} = \max_{\sum_{k=1}^n v_k^2 = 1} \max_{1 \leq k \leq l \leq n} (v_k - v_l)^2 = (1/\sqrt{2} - (-1/\sqrt{2}))^2 = 2.$$

Therefore, by Theorem 1,

$$\mathbb{P}\{D_n^2 > x\} = h_0 (x/2)^{-1/2} e^{-x/4} (1 + O(1/x)) \quad \text{as } x \to \infty,$$

where

$$h_0 := \frac{1}{\sqrt{2\pi}} \sum_{(v_1, \ldots, v_n) \in \cup_{k,l} T_{kl}} \Big| \det\Big(\frac{g''_{d-1}(v_1, \ldots, v_n)}{4} - I_{d-1} \Big) \Big|^{-1/2}.$$

The latter sum consists of $n(n-1)$ equal terms. Consider a typical representative, $\boldsymbol{V}_0 := (1/\sqrt{2}, -1/\sqrt{2}, , 0, \ldots, 0)$, which contains $n-2$ zeros. Consider the following orthogonal system of vectors $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_{n-1}$ in the hyperplane $\mathcal{L} := \{\boldsymbol{V} \in \mathbb{R}^n : (\boldsymbol{V}, \boldsymbol{V}_0) = 0\}$: the vector $\boldsymbol{E}_1 := (1/\sqrt{2}, 1/\sqrt{2}, 0, \ldots, 0) \in \mathbb{R}^n$ plus an orthogonal system $\boldsymbol{E}_2, \ldots, \boldsymbol{E}_{n-1}$ in $\{(0, 0, v_3, \ldots, v_n) \in \mathbb{R}^n\}$. Since the function $g(v_1, \ldots, v_n)$ is equal to $(v_1 - v_2)^2$ in some neighborhood of the point $\boldsymbol{V}_0$, the Hessian of the function g at point $\boldsymbol{V}_0$ is the following square matrix of size n

$$g'' = \begin{pmatrix} 2 & -2 & 0 & \ldots & 0 \\ -2 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

Then the Hessian of the function g restricted to the hyperplane $\boldsymbol{V}_0 + \mathcal{L}$ is zero square matrix of size $n-1$, because its entries are equal to $(g'' \boldsymbol{E}_i, \boldsymbol{E}_j)$. Hence we conclude that $h_0 = \frac{1}{\sqrt{2\pi}} n(n-1)$, so that in dimension 1

$$\mathbb{P}\{D_n > x\} = \mathbb{P}\{D_n^2 > x^2\} = \frac{n(n-1)}{\sqrt{\pi} x} e^{-x^2/4} (1 + O(1/x^2)) \quad \text{as } x \to \infty.$$

Next, we show that in dimension greater than 1 the situation is more complicated. For any $k \neq l$, introduce T_{kl} as the set of all vectors $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \in \mathbb{R}^{mn}$ such that $\boldsymbol{v}_i = \boldsymbol{0} \in \mathbb{R}^m$ for all $i \notin \{k, l\}$, $\boldsymbol{v}_k = -\boldsymbol{v}_l \in \mathbb{R}^m$ and $\boldsymbol{v}_{ki} = \pm 1/\sqrt{2m}$ for all $i \leq m$. Then, given

$$\|\boldsymbol{v}_1\|^2 + \ldots + \|\boldsymbol{v}_n\|^2 = 1,$$

the maximal value $\hat{g}$ of D_n^2 is attained on $\mathcal{M} = \cup_{k \neq l} T_{kl}$; this set consists of $2^m \frac{n(n-1)}{2}$ points. In particular,

$$\hat{g} = \max_{\sum_{k=1}^n \sum_{i=1}^m \boldsymbol{v}_{ki}^2 = 1} \max_{1 \leq k \leq l \leq n} \|\boldsymbol{v}_k - \boldsymbol{v}_l\|^2 = \sum_{i=1}^m (1/\sqrt{2m} - (-1/\sqrt{2m}))^2 = 2$$

independently of m and n. In order to apply Theorem 1, we need to compute the following constant

$$h_0 := \frac{1}{\sqrt{2\pi}} \sum_{(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \in \cup_{k,l} T_{kl}} \Big| \det\Big(\frac{g''_{d-1}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)}{4} - I_{d-1} \Big) \Big|^{-1/2}.$$

The latter sum consists of equal terms. Consider a typical representative,

$$\boldsymbol{V}_0 := (\boldsymbol{1}/\sqrt{2m}, -\boldsymbol{1}/\sqrt{2m}, \boldsymbol{0}, \ldots, \boldsymbol{0}),$$

which contains m coordinates equal to $1/\sqrt{2m}$, m coordinates equal to $-1/\sqrt{2m}$, and $m(n-2)$ zeros.

Consider the following orthogonal system of vectors $\boldsymbol{E}_1, \ldots, \boldsymbol{E}_{mn-1}$ in the hyperplane $\mathcal{L} := \{\boldsymbol{V} \in \mathbb{R}^{mn} : (\boldsymbol{V}, \boldsymbol{V}_0) = 0\}$:

$$\begin{pmatrix} \boldsymbol{E}_1 \\ \boldsymbol{E}_2 \\ \ldots \\ \boldsymbol{E}_{m-1} \\ \boldsymbol{E}_m \\ \boldsymbol{E}_{m+1} \\ \ldots \\ \boldsymbol{E}_{2m-2} \\ \boldsymbol{E}_{2m-1} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{0} & \boldsymbol{0} \ldots \boldsymbol{0} \\ \boldsymbol{e}_2 & \boldsymbol{0} & \boldsymbol{0} \ldots \boldsymbol{0} \\ \ldots & & \\ \boldsymbol{e}_{m-1} & \boldsymbol{0} & \boldsymbol{0} \ldots \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_1 & \boldsymbol{0} \ldots \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e}_2 & \boldsymbol{0} \ldots \boldsymbol{0} \\ \ldots & & \\ \boldsymbol{0} & \boldsymbol{e}_{m-1} & \boldsymbol{0} \ldots \boldsymbol{0} \\ \boldsymbol{1}/\sqrt{2m} & \boldsymbol{1}/\sqrt{2m} & \boldsymbol{0} \ldots \boldsymbol{0} \end{pmatrix}$$

where $\boldsymbol{e}_k = \frac{1}{\sqrt{k(k+1)}}(1,\dots,1,-k,0,\dots,0)$ with k units and $m-k-1$ zeros; plus an orthogonal system $\boldsymbol{E}_{2m}, \dots, \boldsymbol{E}_{mn-1}$ in $\{(\mathbf{0},\mathbf{0},v_{2m+1},\dots,v_{mn}) \in \mathbb{R}^{mn}\}$. Since $g(\boldsymbol{v}_1,\dots,\boldsymbol{v}_n) = \|\boldsymbol{v}_1-\boldsymbol{v}_2\|^2$ in some neighborhood of the point $\boldsymbol{V}_0$, the Hessian of the function g at point $\boldsymbol{V}_0$ is the following square matrix of size mn

$$g'' = 2\begin{pmatrix} I_m & -I_m & & & \\ -I_m & I_m & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix},$$

with $m(n-2)$ zero diagonal entries. Computing the entries of the Hessian of the function g restricted to the hyperplane $\boldsymbol{V}_0 + \mathcal{L}$ via $(g''\boldsymbol{E}_i, \boldsymbol{E}_j)$ we get that

$$g''_{d-1} = 2\begin{pmatrix} I_{m-1} & -I_{m-1} & & & \\ -I_{m-1} & I_{m-1} & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}, \quad \frac{g''_{d-1}}{4} - I_{d-1} = \frac{1}{2}\begin{pmatrix} -I_{m-1} & -I_{m-1} & & & \\ -I_{m-1} & -I_{m-1} & & & \\ & & -2 & & \\ & & & \ddots & \\ & & & & -2 \end{pmatrix}.$$

Hence we conclude that $\det(g''_{d-1}/4 - I_{d-1}) = 0$ in the case $m \ge 2$, so that $\mathbb{P}\{D_n > x\}xe^{x^2/4} \to \infty$ as $x \to \infty$. This example calls for study of degenerated Hessians but we do not concern this question in the current paper.

Similar examples can be given for spherical chaos by applying our Theorem 3. The calculation of h_0 therein is readily obtained using the results of previous examples. Note that the determination of $\hat{g}$ and the parameter m is the same as for the Gaussian chaos. In order to avoid repetition we present only the case of Examples 1 and 7; the remaining cases can be easily extended by studying the next two examples.

Example 10. Let $\boldsymbol{\eta}$ be d-dimensional random vector which is spherically distributed, such that (1) holds with χ a positive random radius. Consider $g(\boldsymbol{\eta}) = |\eta_1|^\alpha + \ldots + |\eta_d|^\alpha$ with $\alpha > 0$ given and assume that χ^α has distribution function in the Gumbel max-domain of attraction with some scaling function w.

If $\alpha \in (0,2)$, then by Example 1 we have $m = 0$ and $\hat{g} = d^{1-\alpha/2}$, hence by Theorem 3 we find that, as $x \to \infty$,

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \sim h_0(xw(x/\hat{g}))^{\frac{1-d}{2}}\mathbb{P}\{\chi^\alpha > x/\hat{g}\}$$

as $x \uparrow \hat{g}x_+$, where

$$h_0 = \frac{2^{3d/2-3/2}\Gamma(d/2)\hat{g}^{\frac{d-1}{2}}}{\sqrt{\pi}(\alpha(2-\alpha))^{\frac{d-1}{2}}}.$$

In the case $\alpha > 2$, then $\hat{g} = 1$ and $m = 0$ as in Example 1 and we find that

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \sim \left(\frac{2}{\alpha}\right)^{\frac{d-1}{2}} \frac{d\Gamma(d/2)}{\sqrt{\pi}}\big(xw(x)\big)^{\frac{1-d}{2}}\mathbb{P}\{\chi^\alpha > x\} \quad \text{as } x \to \infty.$$

We note in passing that the above results agree with the direct calculations in [16]; the case $\alpha = 2$ is discussed in [18].

Example 11. (Determinant of a random spherical matrix) Let $A = [A_{ij}]_{i,j=1}^n$ be a random square matrix of order n and let $\boldsymbol{A}^* = (\boldsymbol{a}_1,\dots,\boldsymbol{a}_n)$ be the $n^2 \times 1$ vector obtained pasting the rows of A, i.e., $\boldsymbol{a}_i = (A_{i1},\dots,A_{in})$ is the ith row of A. Suppose that $\boldsymbol{A}^*$ is a spherically symmetric random vector meaning that

$$\boldsymbol{A}^* \stackrel{d}{=} \chi\boldsymbol{\zeta}$$

with $\chi > 0$ being independent of $\boldsymbol{\zeta}$ which is uniformly distributed on $\mathbb{S}_{n^2-1}$. We consider again $g(A) = \det A$ which is a continuous homogeneous function of order $\alpha = n$. Note that if χ^2 has a chi-square distribution with n^2 degrees of freedom, then A is the matrix in Example 7. Hence for this case, if $\chi^\alpha \in GMDA(w, x_+)$ with $x_+ = \infty$, say, then since by Example 7 we have $m = (n^2-n)/2$ and $\hat{g} = n^{-n/2}$, $d = n^2$, then Theorem 3 entails

$$\mathbb{P}\{\det A > x\} \sim c^*\big(xw(xn^{n/2})\big)^{-\frac{n^2+n-2}{4}}\mathbb{P}\{\chi^n > xn^{n/2}\} \tag{28}$$

as $x \to \infty$ for some constant c^*, which can be calculated iteratively by applying Lemma 3.2 in [1] as mentioned in Example 7.

3 Proof of Theorem 1

The proof is based on the polar representation (3) for a d-dimensional centered Gaussian random vector $\boldsymbol{\eta}$ with identity covariance matrix, $\boldsymbol{\eta} \stackrel{d}{=} \chi\boldsymbol{\zeta}$, where χ and $\boldsymbol{\zeta}$ are independent, $\chi^2 = \sum_{i=1}^d \eta_i^2$ has χ^2-distribution with d degrees of freedom and $\boldsymbol{\zeta}$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1} \subset \mathbb{R}^d$. The tail distribution of the random variable

$$g(\boldsymbol{\eta}) \stackrel{d}{=} g(\chi\boldsymbol{\zeta}) = \chi^\alpha g(\boldsymbol{\zeta})$$

is equal to

$$\begin{aligned}\mathbb{P}\{g(\boldsymbol{\eta}) > x\} &= \int_{x/\hat{g}}^\infty p_{\chi^\alpha}(y)\mathbb{P}\{g(\boldsymbol{\zeta}) > x/y\}dy \\ &= \frac{1}{\alpha 2^{d/2-1}\Gamma(d/2)} \int_{x/\hat{g}}^\infty y^{d/\alpha-1}e^{-y^{2/\alpha}/2}\mathbb{P}\{g(\boldsymbol{\zeta}) > x/y\}\mathrm{d}y \qquad (29)\end{aligned}$$

by the equality (5) and boundedness $g(\boldsymbol{\zeta}) \le \hat{g}$. In order to compute the asymptotics for the latter integral, we first need to estimate the probability $\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\}$ for small positive values of t. Hereinafter $\mathrm{Vol}\,\mathbb{B}_{d-1}$ stands for the volume of the unit ball $\mathbb{B}_{d-1}$ in $\mathbb{R}^{d-1}$.

Lemma 1 *Under the conditions of Theorem 1*

$$\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\} = \sum_{i=0}^r g_i t^{\frac{d-1}{2}+i} + o(t^{\frac{d-1}{2}+r}) \quad \text{as } t \downarrow 0,$$

where

$$g_0 = 2^{\frac{d-1}{2}} \frac{\mathrm{Vol}\,\mathbb{B}_{d-1}}{\mathrm{mes}\,\mathbb{S}_{d-1}} \sum_{j=1}^k \big|\det\big(g''_{d-1}(\boldsymbol{v}_j) - (\alpha\hat{g})I_{d-1}\big)\big|^{-1/2},$$

and where further coefficients $g_1, \ldots, g_r$ only depend on α, $\hat{g}$, and derivatives of $g(\boldsymbol{\varphi})$ at points $\boldsymbol{\varphi}_j$.

Since $\mathrm{Vol}\,\mathbb{B}_{d-1} = \frac{\pi^{(d-1)/2}}{\Gamma((d+1)/2)}$ and $\mathrm{mes}\,\mathbb{S}_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, the expression for the coefficient g_0 may be rewritten in the hyperspherical coordinates as follows (see (10)):

$$g_0 = \frac{2^{d/2-1}}{\sqrt{2\pi}} \frac{\Gamma(d/2)}{\Gamma((d+1)/2)} \sum_{j=1}^k \frac{|\det J(1, \boldsymbol{\varphi}_j)|}{\sqrt{|\det g''(\boldsymbol{\varphi}_j)|}}.$$

Proof Without loss of generality we consider the case where $\mathcal{M}$ consists of a single point $\boldsymbol{v}_1$. First prove that

$$\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\} \sim g_0 t^{\frac{d-1}{2}} \quad \text{as } t \downarrow 0.$$

Introduce the hyperspherical coordinates of the random vector $\boldsymbol{\zeta} = (\zeta_1, \ldots, \zeta_d) \in \mathbb{S}_{d-1}$ as $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{d-1}) \in \Pi_{d-1}$. Since $\boldsymbol{\zeta}$ is uniformly distributed on the unit sphere $\mathbb{S}_{d-1}$ in $\mathbb{R}^d$, the density function of the random vector $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_{d-1}) \in \Pi_{d-1}$ equals $\frac{|\det J(1,\boldsymbol{\varphi})|}{\mathrm{mes}\,\mathbb{S}_{d-1}}$, $\boldsymbol{\varphi} \in \Pi_{d-1}$, which implies

$$\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\} = \frac{1}{\mathrm{mes}\,\mathbb{S}_{d-1}} \int_{\boldsymbol{\varphi} \in \Pi_{d-1} : g(\boldsymbol{\varphi}) > \hat{g} - t} |\det J(1, \boldsymbol{\varphi})| d\boldsymbol{\varphi}. \qquad (30)$$

Since g is at least twice differentiable and attains its maximum at point $\boldsymbol{\varphi}_1$,

$$g(\boldsymbol{\varphi}) = \hat{g} + \frac{1}{2}\Big((g''(\boldsymbol{\varphi}_1) + A(\boldsymbol{\varphi}))(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1), \boldsymbol{\varphi} - \boldsymbol{\varphi}_1\Big),$$

where all the coefficients of the matrix $A(\boldsymbol{\varphi})$ go to 0 as $\boldsymbol{\varphi} \to \boldsymbol{\varphi}_1$. Therefore, the inequality $g(\boldsymbol{\varphi}) > \hat{g} - t$ is equivalent to

$$-\Big((g''(\boldsymbol{\varphi}_1) + A(\boldsymbol{\varphi}))(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1), \boldsymbol{\varphi} - \boldsymbol{\varphi}_1\Big) \le 2t.$$

Fix $\varepsilon > 0$. There exists $\delta > 0$ such that

$$-\varepsilon I_{d-1} \le A(\boldsymbol{\varphi}) \le \varepsilon I_{d-1} \quad \text{for all } \boldsymbol{\varphi} \text{ such that } \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_1\| \le \delta. \qquad (31)$$

Then, for all sufficiently small $t > 0$, the set $\{\boldsymbol{\varphi} : g(\boldsymbol{\varphi}) > \hat{g} - t\}$ is contained in the $(d-1)$-dimensional ellipsoid

$$\Big((-g''(\boldsymbol{\varphi}_1) - \varepsilon I_{d-1})(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1), \boldsymbol{\varphi} - \boldsymbol{\varphi}_1\Big) \le 2t,$$

whose volume is

$$\frac{\mathrm{Vol}\,\mathbb{B}_{d-1}}{\sqrt{|\det(g''(\boldsymbol{\varphi}_1) + \varepsilon I_{d-1})|}}(2t)^{\frac{d-1}{2}}.$$

On the other hand, (31) implies that, for all sufficiently small $t > 0$, the set $\{\boldsymbol{\varphi} : g(\boldsymbol{\varphi}) > \hat{g} - t\}$ contains the $(d-1)$-dimensional ellipsoid

$$\Big((-g''(\boldsymbol{\varphi}_1) + \varepsilon I_{d-1})(\boldsymbol{\varphi} - \boldsymbol{\varphi}_1), \boldsymbol{\varphi} - \boldsymbol{\varphi}_1\Big) \le 2t,$$

whose volume is

$$\frac{\operatorname{Vol} \mathbb{B}_{d-1}}{\sqrt{|\det(g''(\boldsymbol{\varphi}_1) - \varepsilon I_{d-1})|}}(2t)^{\frac{d-1}{2}}.$$

Since $\varepsilon > 0$ may be chosen as small as we please, the above arguments yield that the volume of the set $\{\boldsymbol{\varphi} : g(\boldsymbol{\varphi}) > \hat{g} - t\}$ is proportional to

$$\frac{\operatorname{Vol} \mathbb{B}_{d-1}}{\sqrt{|\det g''(\boldsymbol{\varphi}_1)|}}(2t)^{\frac{d-1}{2}} \quad \text{as } t \downarrow 0.$$

Together with (30) this proves the required asymptotic behavior of the probability $\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\}$.

Next, given that g is differentiable sufficiently many times, the probability $\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\}$ clearly possesses the decomposition with terms $t^{\frac{d-1+i}{2}}$. It turns out that in reality all terms with i odd have zero coefficients. So, it remains to prove that the asymptotic expansion of the integral (30) only contains the terms $t^{\frac{d-1}{2}+i}$ and does not contain terms of order $\frac{d-1}{2} + \frac{1}{2} + i$. It is done in Lemmas 2 and 3 below and the proof of Lemma 1 follows.

Lemma 2 *Let a function $g(u) : [-1, 1] \to \mathbb{R}^+$ possess an asymptotic expansion*

$$g(u) = \sum_{i=2}^{2r+2} g_i u^i + o(u^{2r+2}) \quad \text{as } u \to 0,$$

where $g_2 > 0$. Let $g(u)$ be strictly decreasing for $u \in [-1, 0]$ and strictly increasing for $u \in [0, 1]$. For $t \in (0, g(-1) \wedge g(1))$, denote by $u^+(t)$ the unique positive value of $g^{-1}(t)$ and by $u^-(t)$ the negative one. Let a function $w(u)$ possess an asymptotic expansion

$$w(u) = \sum_{i=0}^{2r} w_i u^i + o(u^{2r}) \quad \text{as } u \to 0.$$

Then

$$\int_{u^-(t)}^{u^+(t)} w(u')du' = \frac{2w_0}{\sqrt{g_2}}\sqrt{t} + \sum_{i=1}^{r} u_i t^{1/2+i} + o(t^{1/2+r}) \quad \text{as } t \downarrow 0 \tag{32}$$

and

$$\int_{u^-(t)}^{u^+(t)} |u'|w(u')du' = \frac{w_0}{g_2}t + \sum_{i=2}^{r+1} \widetilde{u}_i t^i + o(t^{r+1}) \quad \text{as } t \downarrow 0, \tag{33}$$

where coefficients $u_1, \ldots, u_r$ and $\widetilde{u}_2, \ldots, \widetilde{u}_{r+1}$ only depend on g_i's and w_i's.

Moreover, let Θ be a parameter set and let, for every fixed $\theta \in \Theta$, the functions $g(u, \theta)$ and $w(u, \theta)$ satisfy the conditions stated above. Suppose that

$$\inf_{\theta \in \Theta} g_2(\theta) > 0$$

and that all coefficients are uniformly bounded on Θ,

$$\sup_{\theta \in \Theta} |g_i(\theta)| < \infty, \quad i \in \{1, 2, \ldots, 2r+2\},$$
$$\sup_{\theta \in \Theta} |w_i(\theta)| < \infty, \quad i \in \{0, 1, \ldots, 2r\},$$

and the remainder terms are uniform on Θ:

$$\sup_{\theta \in \Theta} \Big| g(u, \theta) - \sum_{i=2}^{2r+2} g_i(\theta) u^i \Big| = o(u^{2r+2}),$$
$$\sup_{\theta \in \Theta} \Big| w(u, \theta) - \sum_{i=0}^{2r} w_i(\theta) u^i \Big| = o(u^{2r})$$

as $u \to 0$. Then, for

$$t \in \Big(0, \min_{\theta \in \Theta} g(-1, \theta) \wedge \min_{\theta \in \Theta} g(1, \theta)\Big),$$

the asymptotic expansion

$$\int_{u^-(t,\theta)}^{u^+(t,\theta)} w(u', \theta)du' = \frac{2w_0(\theta)}{\sqrt{g_2(\theta)}}\sqrt{t} + \sum_{i=1}^{r} u_i(\theta) t^{1/2+i} + o(t^{1/2+r}) \tag{34}$$

holds as $t \downarrow 0$ uniformly on Θ and

$$\int_{u^-(t,\theta)}^{u^+(t,\theta)} |u'|w(u', \theta)du' = \frac{w_0(\theta)}{g_2(\theta)}t + \sum_{i=2}^{r+1} \widetilde{u}_i(\theta) t^i + o(t^{r+1}). \tag{35}$$

Conditions of Lemma 2 almost immediately imply that

$$u^{\pm}(t) = \sum_{i=0}^{2r} u_i^{\pm} t^{1/2+i/2} + o(t^{1/2+r}) \quad \text{as } t \downarrow 0,$$

so that the asymptotic expansion (32), with $w(u) = u$, is equivalent to the nontrivial property that $u_i^+ = u_i^-$ for odd i. It is unclear how it may be proven directly, so our proof of (32) is based on a different approach.

Proof The function

$$f(u) := u\sqrt{g(u)/u^2}, \quad u \in [-1, 1],$$

is invertible. Here the function $g(u)/u^2$ possesses the asymptotic expansion

$$g(u)/u^2 = g_2 + \sum_{i=1}^{2r} g_{i+2}u^i + o(u^{2r}) \quad \text{as } u \to 0.$$

Therefore,

$$f(u) = \sqrt{g_2}\bigg(u + \sum_{i=2}^{2r+1} f_i u^i + o(u^{2r+1})\bigg) \quad \text{as } u \to 0,$$

where f_i is a polynomial of $g_3/g_2, \ldots, g_{i+1}/g_2$. Let $f^{-1}(t)$ be f inverse. It follows that the inverse function possesses an asymptotic expansion at zero up to order $2r+1$:

$$f^{-1}(t) = t/\sqrt{g_2} + \sum_{i=2}^{2r+1} c_i (t/\sqrt{g_2})^i + o(t^{2r+1}) \quad \text{as } t \to 0; \tag{36}$$

here c_i is a polynomial of $g_3/g_2, \ldots, g_{i+1}/g_2$; the remainder term $o(t^{2r+1})$ may be bounded via the remainder term in the asymptotic expansion of g and the coefficients in it. Since the function $W(u) := \int_0^u w(u')du'$ possesses an asymptotic expansion at the origin up to order $2r+1$,

$$W(f^{-1}(t)) = w_0 t/\sqrt{g_2} + \sum_{i=2}^{2r+1} \widetilde{c}_i (t/\sqrt{g_2})^i + o(t^r) \quad \text{as } t \to 0,$$

where $\widetilde{c}_i$ is a polynomial of the coefficients $g_3/g_2, \ldots, g_{i+1}/g_2, w_0, \ldots, w_{i-1}$; here the remainder term $o(t^{2r+1})$ may be bounded via the remainder term in the asymptotic expansions of g and W and the coefficients there. Therefore, as $t \to 0$,

$$W(f^{-1}(t)) - W(f^{-1}(-t)) = 2w_0 t/\sqrt{g_2} + \sum_{i=1}^{r} 2\widetilde{c}_{2i+1} (t/\sqrt{g_2})^{2i+1} + o(t^{2r+1}).$$

Taking into account that $u^+(t) = f^{-1}(\sqrt{t})$ and $u^-(t) = f^{-1}(-\sqrt{t})$ we conclude the desired asymptotic expansion (32). The uniform version of it—(34)—follows by noting that the coefficient $g_2(\theta)$ is bounded away from zero and that all the coefficients $\widetilde{c}_i(\theta)$ are polynomials of bounded coefficients $g_3(\theta)/g_2(\theta), \ldots, g_{2r+2}(\theta)/g_2(\theta), w_0(\theta), \ldots, w_{2r}(\theta)$.

Concerning (33), denote

$$\widetilde{W}(u) = \int_0^u u'w(u')du',$$

then

$$\begin{aligned}\int_{u^-(t)}^{u^+(t)} |u'|w(u')du' &= \int_0^{u^+(t)} u'w(u')du' - \int_{u^-(t)}^0 u'w(u')du' \\ &= \widetilde{W}(u^+(t)) + \widetilde{W}(u^-(t)).\end{aligned}$$

Since the function $\widetilde{W}(u)$ possesses the asymptotic expansion

$$\widetilde{W}(u) = \frac{w_0}{2}u^2 + \sum_{i=1}^{2r} \frac{w_i}{i+2}u^{i+2} + o(u^{2r+2}) \quad \text{as } u \to 0,$$

we conclude from (36) that

$$\widetilde{W}(f^{-1}(t)) = \frac{w_0}{2g_2}t^2 + \sum_{i=3}^{2r+2} \widetilde{\widetilde{c}}_i (t/\sqrt{g_2})^i + o(t^{2r+2}).$$

Therefore, as $t \to 0$,

$$\widetilde{W}(f^{-1}(t)) + \widetilde{W}(f^{-1}(-t)) = \frac{w_0}{g_2}t^2 + \sum_{i=2}^{r+1} 2\widetilde{\widetilde{c}}_{2i} (t/\sqrt{g_2})^{2i} + o(t^{2r+2}).$$

Substituting the equalities $u^+(t) = f^{-1}(\sqrt{t})$ and $u^-(t) = f^{-1}(-\sqrt{t})$, we deduce the desired asymptotic expansion (33).

Lemma 3 *Let a function $g(\boldsymbol{u}) : \mathbb{B}_d \to \mathbb{R}^+$ possess an asymptotic expansion*

$$g(\boldsymbol{u}) = (G_2\boldsymbol{u}, \boldsymbol{u}) + \sum_{i=3}^{2r+2} g_i(\boldsymbol{u}) + o(\|\boldsymbol{u}\|^{2r+2}) \quad \text{as } \boldsymbol{u} \to \boldsymbol{0},$$

where G_2 is a positive definite matrix and $g_i(\boldsymbol{u})$ is a homogeneous polynomial of degree i. For $t > 0$, denote by $B(t)$ the set of all $\boldsymbol{u}$ such that $g(\boldsymbol{u}) \le t$. Let a function $w(\boldsymbol{u})$ possess an asymptotic expansion

$$w(\boldsymbol{u}) = \sum_{i=0}^{2r} w_i(\boldsymbol{u}) + o(\|\boldsymbol{u}\|^{2r}) \quad \text{as } \boldsymbol{u} \to \boldsymbol{0},$$

where $w_i(\boldsymbol{u})$ is a homogeneous polynomial of degree i. Then

$$\int_{B(t)} w(\boldsymbol{u})d\boldsymbol{u} = \frac{w_0 \operatorname{Vol}\mathbb{B}_d}{\sqrt{\det G_2}} t^{d/2} + \sum_{i=1}^{r} u_i t^{d/2+i} + o(t^{d/2+r}) \quad \text{as } t \downarrow 0, \tag{37}$$

where coefficients $u_1, \ldots, u_r$ only depend on coefficients of the polynomials g_i's and w_i's.

It is questionable how to extend the previous proof for multidimensional case $d \ge 2$. For example, if $g(\boldsymbol{u}) = \|\boldsymbol{u}\|^2 + u_1^3$, then one may think of considering the invertible function

$$f(\boldsymbol{u}) = \boldsymbol{u}\sqrt{g(\boldsymbol{u})/\|\boldsymbol{u}\|^2} = \boldsymbol{u}\sqrt{1 + u_1^3/\|\boldsymbol{u}\|^2}.$$

Clearly, the function $u_1^3/\|\boldsymbol{u}\|^2$ doesn't possesses an asymptotic expansion with respect to $\boldsymbol{u}$ and this observation blocks the proof available in dimension 1. By this reason we proceed in a different way, by passing to hyperspherical coordinates which allows to reduce the problem to the case $d = 1$.

Proof For $d = 1$, $\operatorname{Vol}\mathbb{B}_1 = 2$ and the assertion is proven in Lemma 2.

Consider the case $d = 2$. For $\theta \in [0, \pi)$, let $l(\theta)$ be the line passing through the points $(0, 0)$ and $(\cos\theta, \sin\theta)$. Since $G_2 > 0$, there exists a $t_0 > 0$ such that, for all $t \le t_0$ and $\theta \in [0, \pi)$, the set $B(t) \cap l(\theta)$ represents a segment, say $[b^-(\theta, t), b^+(\theta, t)]$ where $b^+(\theta, t) \in \mathbb{R} \times \mathbb{R}^+$. Denote $u^\pm(\theta, t) := \|b^\pm(\theta, t)\|$. Then passing to the spherical coordinates (θ, u) we deduce the equality

$$\int_{B(t)} w(\boldsymbol{u})d\boldsymbol{u} = \int_0^\pi d\theta \int_{u^-(\theta,t)}^{u^+(\theta,t)} |u| w(u\cos\theta, u\sin\theta)du. \tag{38}$$

We have

$$w(u\cos\theta, u\sin\theta) = w_0 + \sum_{i=1}^{2r} w_i(\cos\theta, \sin\theta)u^i + o(u^{2r}),$$

so that Lemma 2 is applicable and we conclude that

$$\int_{u^-(\theta,t)}^{u^+(\theta,t)} |u| w(u\cos\theta, u\sin\theta)du = \frac{w_0}{g_2(\theta)} t + \sum_{i=2}^{r+1} u_i(\theta)t^i + o(t^{r+1})$$

as $t \downarrow 0$ uniformly on $[0, \pi)$ where

$$g_2(\theta) = (G_2(\cos\theta, \sin\theta), (\cos\theta, \sin\theta)).$$

Taking into account that

$$\int_0^\pi \frac{d\theta}{(G_2(\cos\theta, \sin\theta), (\cos\theta, \sin\theta))} = \frac{\pi}{\sqrt{\det G_2}}$$

and $\operatorname{Vol}\mathbb{B}_2 = \pi$, we finally come to (37) for $d = 2$.

In the same way we may proceed with an arbitrary $d \ge 3$. First we pass to the hyperspherical coordinates

$$\int_{B(t)} w(\boldsymbol{u})d\boldsymbol{u} = \int_{[0,\pi)^{d-1}} \det J(1, \boldsymbol{\theta})d\boldsymbol{\theta} \int_{u^-(\boldsymbol{\theta},t)}^{u^+(\boldsymbol{\theta},t)} |u|^{d-1} w(u, \boldsymbol{\theta})du,$$

then integration along the radius which is covered by Lemma 2. Final integration with respect to angles completes the proof of the asymptotic expansion and the lemma follows.

We also need the following version of Watson's lemma.

Lemma 4 *Fix $\gamma \in \mathbb{R}$ and positive y_0, β, c and δ. Then, for any $r > 0$, the integral*

$$I(x) := \int_{x/y_0}^\infty y^\gamma e^{-cy^\beta}(y_0 - x/y)^\delta \mathrm{d}y$$

possesses the expansion

$$I(x) = \Big(\frac{x}{y_0}\Big)^{1+\gamma-(1+\delta)\beta} e^{-c(x/y_0)^\beta} \Big(\sum_{i=0}^{r} I_i x^{-\beta i} + O(x^{-\beta(r+1)})\Big) \quad \text{as } x \to \infty,$$

where $I_0 = \Gamma(1+\delta)y_0^\delta/(c\beta)^{1+\delta}$.

Proof Denote $\lambda := c(x/y_0)^\beta$. Changing variable $y := (x/y_0)z^{1/\beta}$ we find that

$$I(x) = \Big(\frac{x}{y_0}\Big)^{1+\gamma}\frac{y_0^\delta}{\beta}\int_1^\infty z^{\frac{\gamma-\delta+1}{\beta}-1}(z^{1/\beta}-1)^\delta e^{-\lambda z}\mathrm{d}z$$
$$= \Big(\frac{x}{y_0}\Big)^{1+\gamma}\frac{y_0^\delta}{\beta}e^{-\lambda}\int_0^\infty (1+u)^{\frac{\gamma+1}{\beta}-1}(1-(1+u)^{-1/\beta})^\delta e^{-\lambda u}\mathrm{d}u.$$

If $r > \frac{\gamma+1}{\beta} - 2 - \delta$ then, for all $u > 0$,

$$\bigg|(1+u)^{\frac{\gamma+1}{\beta}-1}\Big(\frac{1-(1+u)^{-1/\beta}}{u}\Big)^\delta - \frac{1}{\beta^\delta} + \sum_{i=1}^r c'_i u^i\bigg| \le c' u^{r+1}$$

for some c'_i and $c' < \infty$. Hence,

$$\bigg|I(x) - \Big(\frac{x}{y_0}\Big)^{1+\gamma}e^{-\lambda}\sum_{i=0}^r I'_i\int_0^\infty u^{\delta+i}e^{-\lambda u}\mathrm{d}u\bigg| \le I' x^{1+\gamma}e^{-\lambda}\int_0^\infty u^{\delta+r+1}e^{-\lambda u}\mathrm{d}u,$$

where $I'_0 = y_0^\delta/\beta^{1+\delta}$ and $I'_i = c'_i y_0^\delta/\beta$ for $i \ge 1$. In its turn,

$$\int_0^\infty u^{\delta+i}e^{-\lambda u}\mathrm{d}u = \frac{\Gamma(\delta+i+1)}{\lambda^{1+\delta+i}} = \frac{\Gamma(\delta+i+1)}{c^{1+\delta+i}}\Big(\frac{x}{y_0}\Big)^{-\beta(1+\delta+i)},$$

which completes the proof.

Let us proceed with the proof of Theorem 1. Substituting the result of Lemma 1 into (29) we deduce that, as $x \to \infty$

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \sim \frac{1}{\sqrt{2\pi}\alpha\Gamma((d+1)/2)}\sum_{j=1}^k \frac{|\det J(1,\boldsymbol{\varphi}_j)|}{\sqrt{|\det g''(\boldsymbol{\varphi}_j)|}}$$
$$\times \int_{x/\hat{g}}^\infty y^{d/\alpha-1}e^{-y^{2/\alpha}/2}(\hat{g}-x/y)^{(d-1)/2}\mathrm{d}y.$$

Now we apply Lemma 4 with $y_0 := \hat{g}$, $\gamma := d/\alpha - 1$, $\beta := 2/\alpha$, $c := 1/2$ and $\delta := (d-1)/2$ we deduce from that the following asymptotics, as $x \to \infty$:

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \sim \frac{(\alpha\hat{g})^{\frac{d-1}{2}}}{\sqrt{2\pi}}\sum_{j=1}^k \frac{|\det J(1,\boldsymbol{\varphi}_j)|}{\sqrt{|\det g''(\boldsymbol{\varphi}_j)|}}\Big(\frac{x}{\hat{g}}\Big)^{-1/\alpha}e^{-(x/\hat{g})^{2/\alpha}/2},$$

which completes the proof of the tail asymptotics.

Next, we prove the claim in (2) which shows a tractable expression of the density of $g(\boldsymbol{\eta})$ in terms of tail characteristics of $g(\boldsymbol{\eta})$.

Lemma 5 *The density function $p_{g(\eta)}$ of distribution of $g(\boldsymbol{\eta})$ restricted to $\mathbb{R}\setminus\{0\}$ exists and possesses the representation (2). Moreover,*

$$p_{g(\boldsymbol{\eta})}(x) = \frac{1}{\alpha x}\bigg(\frac{1}{\alpha 2^{d/2-1}\Gamma(d/2)}\int_{x/\hat{g}}^\infty y^{(d+2)/\alpha-1}e^{-y^{2/\alpha}/2}\mathbb{P}\{g(\boldsymbol{\zeta}) > x/y\}\mathrm{d}y$$
$$-d\cdot\mathbb{P}\{g(\boldsymbol{\eta}) > x\}\bigg), \tag{39}$$

where $\boldsymbol{\zeta}$ is uniformly distributed on $\mathbb{S}_{d-1}$.

Proof Since $\boldsymbol{\eta}$ is a standard Gaussian random vector,

$$p_{g(\boldsymbol{\eta})}(x) = -\frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\{g(x^{-1/\alpha}\boldsymbol{\eta}) > 1\}$$
$$= -\frac{\mathrm{d}}{\mathrm{d}x}\bigg(\frac{x^{d/\alpha}}{(2\pi)^{d/2}}\int_{\{\boldsymbol{v}\in\mathbb{R}^d : g(\boldsymbol{v})>1\}} e^{-x^{2/\alpha}\|\boldsymbol{v}\|^2/2}\mathrm{d}\boldsymbol{v}\bigg)$$
$$= -\frac{d}{\alpha x}\frac{x^{d/\alpha}}{(2\pi)^{d/2}}\int_{\{\boldsymbol{v}\in\mathbb{R}^d : g(\boldsymbol{v})>1\}} e^{-x^{2/\alpha}\|\boldsymbol{v}\|^2/2}\mathrm{d}\boldsymbol{v}$$
$$+\frac{x^{2/\alpha-1}}{\alpha}\frac{x^{d/\alpha}}{(2\pi)^{d/2}}\int_{\{\boldsymbol{v}\in\mathbb{R}^d : g(\boldsymbol{v})>1\}} |\boldsymbol{v}|^2 e^{-x^{2/\alpha}\|\boldsymbol{v}\|^2/2}\mathrm{d}\boldsymbol{v}.$$

Therefore,

$$p_{g(\boldsymbol{\eta})}(x) = -\frac{d}{\alpha x}\mathbb{P}\{g(x^{-1/\alpha}\boldsymbol{\eta}) > 1\} + \frac{x^{2/\alpha-1}}{\alpha}\mathbb{E}\{\|x^{-1/\alpha}\boldsymbol{\eta}\|^2; g(x^{-1/\alpha}\boldsymbol{\eta}) > 1\},$$

which implies (2).

Similarly to (29), we derive from (5) the equality

$$\begin{aligned}\mathbb{E}\{\|\boldsymbol{\eta}\|^2; g(\boldsymbol{\eta}) > x\} &= \mathbb{E}\big\{\mathbb{E}\big\{(\chi^\alpha)^{2/\alpha}\mathbb{I}\{g(\boldsymbol{\zeta}) > x/\chi^\alpha\}\big\} \mid \chi\big\} \\ &= \int_{x/\hat{g}}^{\infty} y^{2/\alpha} p_{\chi^\alpha}(y)\mathbb{P}\{g(\boldsymbol{\zeta}) > x/y\}\mathrm{d}y \\ &= \frac{1}{\alpha 2^{d/2-1}\Gamma(d/2)} \int_{x/\hat{g}}^{\infty} y^{(d+2)/\alpha-1} e^{-y^{2/\alpha}/2}\mathbb{P}\{g(\boldsymbol{\zeta}) > x/y\}\mathrm{d}y\end{aligned}$$

establishing thus the proof.

Further application of Lemmas 1 and 4 completes the proof of the density function asymptotic expansion.

In the Gaussian case, yet another approach for estimating the tail of Gaussian chaos seems to be applicable. Consider n independent copies $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_n$ of $\boldsymbol{\eta}$, then

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \mathbb{P}\Big\{g\Big(\frac{\boldsymbol{\eta}_1 + \ldots + \boldsymbol{\eta}_n}{\sqrt{n}}\Big) > x\Big\} = \mathbb{P}\Big\{g\Big(\frac{\boldsymbol{\eta}_1 + \ldots + \boldsymbol{\eta}_n}{n}\Big) > \frac{x}{n^{\alpha/2}}\Big\}.$$

Therefore, considering $x = tn^{\alpha/2}$, we have

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \mathbb{P}\Big\{g\Big(\frac{\boldsymbol{\eta}_1 + \ldots + \boldsymbol{\eta}_n}{n}\Big) > t\Big\}.$$

Hence, this reduces the problem of the tail behavior as $x \to \infty$ to that of large deviation as $n \to \infty$. Then one may try to apply some results on asymptotic expansions in large deviations, for the distribution as well as for the density, see e.g., Borovkov and Rogozin [8]. We just mention that it is easily seen that the integration over a domain of the asymptotic expansion for the large deviation probabilities is not simpler than our integration related to a chi-squared distribution.

We conclude this section with the proof of the equality (10) which follows from the following result.

Lemma 6 *Under the conditions of Theorem 1 at every point* $\boldsymbol{v}_j = h_j(\boldsymbol{z}_j) \in \mathcal{M}$, $\boldsymbol{z}_j \in (0,2)^{d-1} \times \{1\}$

$$(J_j^{-1}(\boldsymbol{z}_j))^{\mathrm{T}}(g \circ h_j)''_{d-1}(\boldsymbol{z}_j)J_j^{-1}(\boldsymbol{z}_j) = g''_{d-1}(\boldsymbol{v}_j) - \hat{g}\alpha I_{d-1}.$$

Proof Indeed, since $\boldsymbol{v}_j$ is the point of the maximum of the function g, Taylor's expansion at this points reads as follows: with $\boldsymbol{v} = h_j(\boldsymbol{z})$,

$$\begin{aligned}g(\boldsymbol{v}) &= \hat{g} + \frac{1}{2}\big((g \circ h_j)''_{d-1}(\boldsymbol{z}_j)(\boldsymbol{z} - \boldsymbol{z}_j), \boldsymbol{z} - \boldsymbol{z}_j\big) + o(\|\boldsymbol{v} - \boldsymbol{v}_j\|^2) \\ &= \hat{g} + \frac{1}{2}\big((g \circ h_j)''_{d-1}(\boldsymbol{z}_j)(h_j^{-1}(\boldsymbol{v}) - h_j^{-1}(\boldsymbol{v}_j)), h_j^{-1}(\boldsymbol{v}) - h_j^{-1}(\boldsymbol{v}_j)\big) + o(\|\boldsymbol{v} - \boldsymbol{v}_j\|^2)\end{aligned}$$

as $\boldsymbol{v} \to \boldsymbol{v}_j$. Since

$$h_j^{-1}(\boldsymbol{v}) - h_j^{-1}(\boldsymbol{v}_j) = J_j^{-1}(\boldsymbol{z}_j)(\boldsymbol{v} - \boldsymbol{v}_j) + o(\|\boldsymbol{v} - \boldsymbol{v}_j\|),$$

we obtain that

$$g(\boldsymbol{v}) = \hat{g} + \frac{1}{2}\big((g \circ h_j)''(\boldsymbol{z}_j)J_j^{-1}(\boldsymbol{z}_j)(\boldsymbol{v} - \boldsymbol{v}_j), J_j^{-1}(\boldsymbol{z}_j)(\boldsymbol{v} - \boldsymbol{v}_j)\big) + o(\|\boldsymbol{v} - \boldsymbol{v}_j\|^2) \tag{40}$$

as $\boldsymbol{v} \to \boldsymbol{v}_j$. Consider the projection $\boldsymbol{u}$ of the point $\boldsymbol{v}$ onto the hyperplane $(u_1 - (\boldsymbol{v}_j)_1)(\boldsymbol{v}_j)_1 + \ldots + (u_d - (\boldsymbol{v}_j)_d)(\boldsymbol{v}_j)_d = 0$. The equalities

$$\begin{aligned}\|\boldsymbol{v} - \boldsymbol{u}\| &= 1 - (\boldsymbol{v}, \boldsymbol{v}_j) = (\boldsymbol{v}, \boldsymbol{v} - \boldsymbol{v}_j) \\ &= (\boldsymbol{v} - \boldsymbol{v}_j + \boldsymbol{v}_j, \boldsymbol{v} - \boldsymbol{v}_j) \\ &= \|\boldsymbol{v} - \boldsymbol{v}_j\|^2 + (\boldsymbol{v}_j, \boldsymbol{v}) - 1\end{aligned}$$

yield that

$$\|\boldsymbol{v} - \boldsymbol{u}\| = \|\boldsymbol{v} - \boldsymbol{v}_j\|^2/2 = \|\boldsymbol{u} - \boldsymbol{v}_j\|^2/2 + o(\|\boldsymbol{u} - \boldsymbol{v}_j\|^4). \tag{41}$$

Applying this in (40) we deduce that

$$g(\boldsymbol{v}) = \hat{g} + \frac{1}{2}\big((J_j^{-1}(\boldsymbol{z}_j))^{\mathrm{T}}(g \circ h_j)''_{d-1}(\boldsymbol{z}_j)J_j^{-1}(\boldsymbol{z}_j)(\boldsymbol{u} - \boldsymbol{v}_j), \boldsymbol{u} - \boldsymbol{v}_j\big) + o(\|\boldsymbol{u} - \boldsymbol{v}_j\|^2). \tag{42}$$

On the other hand, again by Taylor's expansion

$$g(\boldsymbol{u}) = \hat{g} + \frac{1}{2}(g''_{d-1}(\boldsymbol{v}_j)(\boldsymbol{u} - \boldsymbol{v}_j), \boldsymbol{u} - \boldsymbol{v}_j) + o(\|\boldsymbol{u} - \boldsymbol{v}_j\|^2) \quad \text{as } \boldsymbol{u} \to \boldsymbol{v}_j. \tag{43}$$

In addition

$$\begin{aligned}g(\boldsymbol{v}) &= g(\boldsymbol{u}) + (\nabla g(\boldsymbol{u}), \boldsymbol{v} - \boldsymbol{u}) + O(\|\boldsymbol{v} - \boldsymbol{u}\|^2) \\ &= g(\boldsymbol{u}) + (\nabla g(\boldsymbol{v}_j), \boldsymbol{v} - \boldsymbol{u}) + O(\|\boldsymbol{u} - \boldsymbol{v}_j\|^2)\end{aligned} \tag{44}$$

because $\nabla g(\boldsymbol{u}) \to \nabla g(\boldsymbol{v}_j)$ and $\boldsymbol{v}-\boldsymbol{u} = O(\|\boldsymbol{u}-\boldsymbol{v}_j\|^2)$. Since $\boldsymbol{v}_j$ and $\boldsymbol{v}-\boldsymbol{u}$ are collinear,

$$(\nabla g(\boldsymbol{v}_j), \boldsymbol{v}-\boldsymbol{u}) = -\|\boldsymbol{v}-\boldsymbol{u}\| \lim_{\varepsilon\to 0} \frac{g(\boldsymbol{v}_j+\varepsilon\boldsymbol{v}_j)-g(\boldsymbol{v}_j)}{\varepsilon}.$$

By the homogeneity of the function g,

$$\lim_{\varepsilon\to 0} \frac{g(\boldsymbol{v}_j+\varepsilon\boldsymbol{v}_j)-g(\boldsymbol{v}_j)}{\varepsilon} = g(\boldsymbol{v}_j)\lim_{\varepsilon\to 0}\frac{(1+\varepsilon)^\alpha-1}{\varepsilon} = -\hat{g}\alpha,$$

so that we have

$$\begin{aligned}(\nabla g(\boldsymbol{v}_j), \boldsymbol{v}-\boldsymbol{u}) &= -\hat{g}\alpha\|\boldsymbol{v}-\boldsymbol{u}\| \\ &= -\frac{1}{2}\hat{g}\alpha(\boldsymbol{u}-\boldsymbol{v}_j, \boldsymbol{u}-\boldsymbol{v}_j) + o(\|\boldsymbol{u}-\boldsymbol{v}_j\|^4),\end{aligned}$$

by the equality (41). Combining (42), (43) and (44) we conclude the desired equality of the matrices.

4 Proof of Theorem 2

Since $\mathcal{M}$ is C^{2r+2}-smooth manifold in $\mathbb{S}_{d-1}$, there exists some neighborhood U of $\mathcal{M}$ in $\mathbb{R}^d$ such that it may be partitioned into a finite number of disjoint sets $U_1, \ldots, U_n$ such that, for every $1 \le j \le n$, the manifold $\mathcal{M}\cap U_j$ is elementary, that is, there exists some bijection $h_j : [0,2]^d \to \mathrm{cl}\,(U_j)$ (the closure of U_j) which is $2r+2$ times differentiable, non-degenerate and such that

$$h_j([0,2]^{d-1}\times\{1\}) = \mathbb{S}_{d-1}\cap \mathrm{cl}\,(U_j) \quad \text{and} \quad h_j([0,2]^m\times\{1\}^{d-m}) = \mathcal{M}\cap\mathrm{cl}\,(U_j).$$

It is non-degenerate in the sense that its Hessian is non-zero at every point $\boldsymbol{z}\in[0,2]^d$.

The proof of Theorem 2 follows the lines of the previous one. The main difference consists in the estimation of the probability $\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g}-t\}$ for small values of $t>0$. Because of this, we only need to show the following result.

Lemma 7 *Under the conditions of Theorem 2*

$$\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g}-t\} = \sum_{i=0}^{r} g_i t^{\frac{d-1-m}{2}+i} + o(t^{\frac{d-1-m}{2}+r}) \quad \textit{as } t\downarrow 0,$$

where

$$g_0 = 2^{\frac{d-1-m}{2}}\frac{\mathrm{Vol}\,\mathbb{B}_{d-1-m}}{\mathrm{mes}\,\mathbb{S}_{d-1}}\int_{\mathcal{M}} \left|\det(g''_{d-1-m}(\boldsymbol{v}) - (\alpha\hat{g})I_{d-1-m})\right|^{-1/2}\mathrm{d}V.$$

Since $\mathrm{Vol}\,\mathbb{B}_{d-1-m} = \frac{\pi^{(d-1-m)/2}}{\Gamma((d+1-m)/2)}$ and $\mathrm{mes}\,\mathbb{S}_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$, we have the following alternative representation for the constant g_0, in terms of the hyperspherical coordinates:

$$g_0 = \frac{2^{d/2-1}}{(2\pi)^{(1+m)/2}}\frac{\Gamma(d/2)}{\Gamma((d+1-m)/2)}\int_{\mathcal{M}_\varphi}\frac{|\det J(1,\boldsymbol{\varphi})|}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}}\mathrm{d}V_{\boldsymbol{\varphi}}.$$

Proof For every $j\le n$, consider a random vector $\boldsymbol{\nu}_j$ valued in $[0,2]^{d-1}$ with density function $\frac{|\det J_j(\boldsymbol{z})|}{\mathrm{mes}\,(\mathbb{S}_{d-1}\cap U_j)}$, $\boldsymbol{z}\in[0,2]^{d-1}$, where $J_j(\boldsymbol{z})$ is the Jacobian matrix of the function h_j restricted to the first $d-1$ coordinates. Then $h_j(\boldsymbol{\nu}_j,1)$ has the uniform distribution on the set $\mathbb{S}_{d-1}\cap U_j$.

Let $t>0$ be so small that the t-neighborhood of $\mathcal{M}$ is contained in the set U. Consider the following decomposition:

$$\begin{aligned}\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g}-t\} &= \sum_{j=1}^{n}\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g}-t, \boldsymbol{\zeta}\in\mathbb{S}_{d-1}\cap U_j\} \\ &= \sum_{j=1}^{n}\mathbb{P}\{g(h_j(\boldsymbol{\nu}_j,1)) > \hat{g}-t\}\frac{\mathrm{mes}\,(\mathbb{S}_{d-1}\cap U_j)}{\mathrm{mes}\,\mathbb{S}_{d-1}}\end{aligned}$$

and compute the asymptotic behaviour of the jth term on the right. Since $h_j([0,2]^m\times\{1\}^{d-m}) = \mathcal{M}\cap\mathrm{cl}\,(U_j)$, we have $g(h_j(\boldsymbol{s},1,\ldots,1)) = \hat{g}$ for every point $\boldsymbol{s}\in[0,2]^m$. The function $g(h_j(\boldsymbol{s},\cdot,\ldots,\cdot,1))$ of $d-1-m$ arguments is $2r+2$ times differentiable. Then the same arguments as in Lemma 1 yield the decomposition

$$\mathbb{P}\{g(h_j(\boldsymbol{\nu}_j,1)) > \hat{g}-t \mid \boldsymbol{\nu}_j\in\{\boldsymbol{s}\}\times[0,2]^{d-1-m}\} = \sum_{i=0}^{r} g_{ji}(\boldsymbol{s})t^{\frac{d-1-m}{2}+i} + o(t^{\frac{d-1-m}{2}+r})$$

as $t\downarrow 0$ where

$$g_{j0}(\boldsymbol{s}) = 2^{\frac{d-1-m}{2}}\frac{\mathrm{Vol}\,\mathbb{B}_{d-1-m}}{\mathrm{mes}\,(\mathbb{S}_{d-1}\cap U_j)}\frac{|\det J_j(\boldsymbol{s},1,\ldots,1)|}{\sqrt{|\det(g\circ h_j)''(\boldsymbol{s},1,\ldots,1)|}},$$

where the Hessian of $g \circ h_j$ is taken with respect to the last $d-1-m$ arguments. Integration over $\boldsymbol{s} \in [0,2]^m$ finally implies that

$$\mathbb{P}\{g(h_j(\boldsymbol{\nu}_j, 1)) > \hat{g} - t\} = \sum_{i=0}^{r} g_{ji} t^{\frac{d-1-m}{2}+i} + o(t^{\frac{d-1-m}{2}+r})$$

as $t \downarrow 0$ where

$$g_{j0} = 2^{\frac{d-1-m}{2}} \frac{\mathrm{Vol}\,\mathbb{B}_{d-1-m}}{\mathrm{mes}\,(\mathbb{S}_{d-1} \cap U_j)} \int_{[0,2]^m} \frac{|\det J_j(\boldsymbol{s}, 1, \ldots, 1)|}{\sqrt{|\det(g \circ h_j)''(\boldsymbol{s}, 1, \ldots, 1)|}} \mathrm{d}\boldsymbol{s},$$

which proves the lemma.

5 Proof of Theorem 3

The crucial step of the proof is again to find the tail asymptotics of $g(\boldsymbol{\zeta})$. As in the proof of Lemma 7 we have

$$\mathbb{P}\{g(\boldsymbol{\zeta}) > \hat{g} - t\} \sim g_0 t^{\frac{d-1-m}{2}} \quad \text{as } t \downarrow 0,$$

where

$$g_0 = \frac{(2\pi)^{\frac{d-1-m}{2}}}{\Gamma((d+1-m)/2)} \int_{\mathcal{M}_\varphi} \frac{p_{\boldsymbol{\nu}}(\boldsymbol{\varphi})}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}} \mathrm{d}V_{\boldsymbol{\varphi}}. \tag{45}$$

Next—here we follow a simplified version compared to Hashorva (2012)—

$$\begin{aligned}
\mathbb{P}\{g(\boldsymbol{\eta}) > x\} &= \mathbb{P}\{\chi^\alpha g(\boldsymbol{\zeta}) > x\} \\
&= \int_0^\infty \mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \frac{x}{x/\hat{g} + y}\Big\} \mathbb{P}\{\chi^\alpha \in x/\hat{g} + \mathrm{d}y\} \\
&= \int_0^\infty \mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \frac{x}{x/\hat{g} + y/w(x/\hat{g})}\Big\} \mathbb{P}\Big\{\chi^\alpha \in \frac{x}{\hat{g}} + \frac{\mathrm{d}y}{w(x/\hat{g})}\Big\} \\
&= \mathbb{P}\Big\{\chi^\alpha > \frac{x}{\hat{g}}\Big\} \int_0^\infty \mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \hat{g} - \frac{\hat{g}^2}{xw(x/\hat{g})} \frac{y}{1 + y\hat{g}/xw(x/\hat{g})}\Big\} \\
&\qquad \mathbb{P}\Big\{\chi^\alpha \in \frac{x}{\hat{g}} + \frac{\mathrm{d}y}{w(x/\hat{g})}\Big| \chi^\alpha > \frac{x}{\hat{g}}\Big\}.
\end{aligned} \tag{46}$$

Since $xw(x) \to \infty$ as $x \to \infty$,

$$\mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \hat{g} - \frac{\hat{g}^2}{xw(x/\hat{g})} \frac{y}{1 + y\hat{g}/xw(x/\hat{g})}\Big\} \sim g_0 \Big(\frac{\hat{g}^2 y}{xw(x/\hat{g})}\Big)^{\frac{d-1-m}{2}}$$

as $x \to \infty$ uniformly on any y-compact set. In addition,

$$\begin{aligned}
\mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \hat{g} - \frac{\hat{g}^2}{xw(x/\hat{g})} \frac{y}{1 + y\hat{g}/xw(x/\hat{g})}\Big\} &\le \mathbb{P}\Big\{g(\boldsymbol{\zeta}) > \hat{g} - \frac{y\hat{g}^2}{xw(x/\hat{g})}\Big\} \\
&\le c\Big(\frac{y}{xw(x/\hat{g})}\Big)^{\frac{d-1-m}{2}},
\end{aligned}$$

for some $c < \infty$. The latter asymptotics and upper bound allow to apply Lebesgue's dominated convergence theorem in (46) and to conclude that, as $x \to \infty$,

$$\begin{aligned}
&\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \\
&\sim g_0 \Big(\frac{\hat{g}^2}{xw(x/\hat{g})}\Big)^{\frac{d-1-m}{2}} \mathbb{P}\Big\{\chi^\alpha > \frac{x}{\hat{g}}\Big\} \int_0^\infty y^{\frac{d-1-m}{2}} \mathbb{P}\Big\{\chi^\alpha \in \frac{x}{\hat{g}} + \frac{\mathrm{d}y}{w(x/\hat{g})}\Big| \chi^\alpha > \frac{x}{\hat{g}}\Big\}.
\end{aligned}$$

It follows from the Davis–Resnick tail property—see [11, Proposition 1.1]—that, for any fixed $\gamma > 0$, there exists a $c_1 < \infty$ such that for all u, $v > 0$

$$\mathbb{P}\{\chi^\alpha > u + v/w(u) \mid \chi^\alpha > u\} \le c_1/v^\gamma.$$

This ensures the following convergence of moments

$$\begin{aligned}
\int_0^\infty y^{\frac{d-1-m}{2}} \mathbb{P}\Big\{\chi^\alpha \in \frac{x}{\hat{g}} + \frac{\mathrm{d}y}{w(x/\hat{g})}\Big| \chi^\alpha > \frac{x}{\hat{g}}\Big\} &\to \int_0^\infty y^{\frac{d-1-m}{2}} e^{-y} \mathrm{d}y \\
&= \Gamma\Big(\frac{d+1-m}{2}\Big),
\end{aligned}$$

and hence the first claim follows.

Since further the scaling function $w(\cdot)$ is self-neglecting (see e.g., Resnick (1987)) i.e.,

$$w(t + s/w(t)) \sim w(t) \quad \text{as } t \uparrow x_+$$

locally uniformly in s, then $g(\boldsymbol{\eta})$ is also in the Gumbel max-domain of attraction with the same scaling function w as χ^α. Thus the second claim follows.

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