To Invest or Not To Invest: An Economic Demonstration of Buridan's Ass

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Abstract

We analyze a model of irreversible investment with two sources of uncertainty. A risk-neutral decision maker has the choice between two mutually exclusive projects under input price and output price uncertainty. We propose a complete study of the shape of the rational investment region and we prove that it is never optimal to invest when the alternative investments generate the same payoff independently of their size. A key feature of this bidimensional degree of uncertainty is thus that the payoff generated by each project is not a sufficient statistic to make a rational investment. In this context, our analysis provides a new motive for waiting to invest: the benefits associated with the dominance of one project over the other. As an illustration, we apply our methodology to power generation under uncertainty.

1 Introduction

Irreversible investment decisions by a public authority are highly difficult to take. Indeed, such decisions usually generate very important cash-flows and their irreversibility makes the decision taker quite prudent. Furthermore, he is usually confronted to a wide choice of possibilities to undertake the investment. For instance, when a government wants to expand or to replace the electricity capacity, different solutions have to be taken into account. It may decide to invest in the nuclear technology or in the gas technology. The aim of this paper is to describe such decisions in the case where different technologies are available. Our main result is that the presence of such a choice makes the decision taker more reluctant to take a decision. He prefers to wait to invest later in the technology that turns out to be the most profitable. Like Buridan's ass that hesitates between drinking and eating, a decision taker hesitates between each technology that supplies electricity. But unlike Buridan's ass story, we will characterize the event on which a decision ends to be taken. Our model is quite general but has been more particulary designed for the electricity sector. Therefore, our results could be a way to explain the lack of electricity capacity investment. Indeed, according to the International Energy Agency, "electricity capacity

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reserve margins are declining in most OECD countries signalling the need for new investment. The supply disruptions in parts of North America and Europe in summer 2006 have raised again questions about the adequacy of generation margins and investment in network infrastructure."[1] Here our interpretation of this fact is that the simultaneous presence of different technologies (coal, gas and nuclear plant, renewable resources) in the electricity sector delays any investment decision.

This work is linked to the literature on investment under uncertainty that has developed very quickly since the early works by Arrow and Fisher [2] or by Henry [11]. When an investor has an investment opportunity, he faces a tradeoff: either he invests immediately or he waits to obtain more information about the quality of the investment project. The classic rule saying that investing is optimal as soon as the net present value is positive is not always valid since the option to wait in order to be better informed has to be taken into account. Therefore, investment under uncertainty creates what is commonly called a "time value". The existence of such an option value requires three features: first, the investment problem has to be dynamic, second, there must be some uncertainty concerning the cash flow that will be generated in the future, and finally, the investment decision has to be irreversible. McDonald and Siegel [16] were the first to give an expression to the option value. Moreover, they showed that when the underlying value of the investment project evolves as a geometric Brownian motion, the optimal strategy is usually a trigger strategy, that is, invest as soon as the investment value is greater than a threshold that can sometimes be explicitly computed using standard smooth-fit techniques (see Dixit and Pindyck [8]). Many authors extended the original model in different directions. Dixit [7], Kandel and Pearson [13] and Aguerrevere [1] studied how such an approach could be used by a firm to choose both an optimal capacity and an optimal timing. Other authors rather concentrated on a strategic viewpoint by considering not a monopolist but many firms and they tried to characterize the competitive equilibrium. Leahy [14] showed that "the interaction of competition does not affect the timing of irreversible investment decisions at all".

In this work, we thus revisit an old simple problem, namely, the choice between mutually exclusive investment projects by considering two sources of uncertainty. Despite its relative simplicity, this is still a timely question, as illustrated for instance by Dias' [5] recent survey of real options in the petroleum industry or by our previous discussion on electricity capacity needs. One of the major features of bidimensional investment problems is that the investment value is no longer a sufficient statistic to undertake optimally the project. Indeed, as we show, it may be optimal not to invest in any project even if their expected profit tends to infinity. This fact makes unexpected an explicit computation of the optimal time to invest and that is the main reason why the bidimensional investment models received little attention in the litterature. In this paper, the two mutually exclusive investment projects N and G are affected by output price uncertainty, but only the second one (project G) is affected by input price uncertainty. We

[1] See World Energy Outlook 2006 [12].

do not make any assumption on the output flow ranking of each project, but we suppose that the sunk capital cost of project N is higher than that of project G. This model is an extension of the one developed by Décamps et al. [4] in which they study the choice of two mutually exclusive investment projects under output price uncertainty: a project with high payoffs and high costs and a project with low payoffs and low costs. They find that the investment strategy is not a trigger strategy any more and that there exists an intermediate "inaction region": if the initial price had been lower, the investor would have invested in the small project, and had it been higher, he would have invested in the large project. But in this intermediate region, more information is needed to know in which direction the price will evolve and to be sure of the decision that will be taken.

The introduction of input price uncertainty in addition to the usual output price uncertainty makes the problem quite more complex from a mathematical viewpoint. Moreover, the presence of the two uncertainty sources reinforces the applicability of our model. One of our major features is that it is never optimal to invest when the two projects generate the same expected payoff whatever size it has. This result is very interesting since the size of the payoff is not the unique decision variable anymore. Moreover, this is quite new relative to Décamps et al. [4]: even if at the begining, the state variables are very low, we prove that there exists a path for which even if the expected payoff of each technology tends to infinity, no investment will be undertaken. Our results are related to the literature concerning American options on multiple assets. Broadie and Detemple [3] and Villeneuve [20] studied the exercise regions of such American options (they mostly focused on convex payoff options) and both showed that exercise regions may exhibit interesting shapes. In particular, in the case of an option on the maximum between two assets, when the underlying assets are equal, it is not optimal to invest in one of them even if the payoff process tends to infinity, but it is optimal to wait in order to collect information about the evolution of the state variables. From an empirical viewpoint, Dias et al. [6] worked on a model that is similar to the one of Décamps et al. [4]. They considered an investor who has the choice between three projects: each project is affected by output price uncertainty and has its own sunk capital cost. Although they did not present theoretical results, their simulations showed the existence of intermediate "inaction regions" as in Décamps et al. [4]. Moreover, they allowed the output to follow different processes: either a geometric Brownian motion or a mean reverting process. Geltner et al. [9] considered an investor who has the choice to invest in a land but for two different uses: if the first use is chosen, the value of the land follows a geometric Brownian motion, but if the second use is chosen, the value is a different state variable that also follows a geometric Brownian motion. The construction cost is assumed to be fixed and to be the same in the two cases. The investor chooses the use that yields the highest payoff. Geltner et al. studied the exercise region in this bidimensional setting and found that it can be decomposed into two symmetric disjoints regions (one for each use). When the value of each use generates the same profit, the investor prefers to wait than to invest in one of the two.

As already explained above, this paper focuses on a bidimensional setting. But in contrast to Geltner et al., the output process is the same for both projects and the second source of uncertainty comes from the input price. Moreover, unlike Broadie and Detemple [3], we do not consider an option on the maximum of two different assets, but on the maximum of two different linear combinations of assets. In our setting, we prove the existence of an "inaction region". When both projects have the same value or very similar values, it is optimal to wait rather than to invest in one of the two. In this bidimensional setting, the investment value is not a sufficient statistic to take a decision. Indeed, we prove that the investor might decide not to invest in any project even if each payoff tends to infinity. The shape of the exercise regions is quite different depending on the ranking of the output flow of project N, β_N, relative to the output flow of project G, β_G. This means that the investment decision not only depends on the level of the state variable but also on the output flow. It is interesting to note that if each project had been evaluated separately, exercise regions would have been quite different. Indeed we prove that the introduction of the choice modifies the exercise regions of each project taken separately: it can be optimal to delay investment whereas without this choice immediate investment would have been optimal. We thus introduce the concept of "choice value" between the two alternative projects. It is straightforward to extend these results to the case of n mutually exclusive projects.

Once the theoretical results have been presented, we turn to an application of our model to power generation under uncertainty. We assume technology N produces electricity from a nuclear power plant whereas technology G produces electricity from gas. Applying our results to this example, we find that the investment decision not only depends on the values taken by the state variables but also on the cash-flow generated by each technology (β_N or β_G). These coefficients are function of the construction time and the lifetime of each technology. Leaving aside social cost (CO_2, nuclear waste) and taking into account almost realistic parameters' values, our model suggests that investment in the nuclear technology is more likely to be optimal.

The next section of this paper describes the model and gives the first properties of the value function. In section 3, exercise regions are described for the different possible ranking of the output flows and their different properties are carefully stated. In section 4, we illustrate the theoretical model with power generation under uncertainty. Section 5 concludes.

2 The model

This is a model of choice between two technologies, technology N and technology G, both producing the same output by different means. Technology G has a stochastic input. Time is continuous and labeled by $t \geq 0$. There is a single risk-neutral investor who can engage in one of these two projects. To give a rigorous formulation to our model, we start with a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ representing the information available at time t. We consider a bidimensional $\mathcal{F}_t$-Brownian motion $\left(W_t^1, W_t^2\right)$. The output price $P = \{P_t; t \geq 0\}$ is a geometric Brownian motion with drift $(r - \delta_P)$, strictly positive convenience yield δ_P and volatility σ_P.

$$\frac{dP_t}{P_t} = (r - \delta_P)\, dt + \sigma_P dW_t^1. \tag{1}$$

Let P_t^p be the solution of (1) starting from $P_0^p = p$. The input price $X = \{X_t; t \geq 0\}$ is also a geometric Brownian motion with drift $(r - \delta_X)$, strictly positive convenience yield δ_X and volatility σ_X

$$\frac{dX_t}{X_t} = (r - \delta_X)\, dt + \sigma_X \left(\rho dW_t^1 + \sqrt{1-\rho^2} dW_t^2\right). \tag{2}$$

Let X_t^x be the solution of (2) starting from $X_0^x = x$. The correlation between P_t and X_t is equal to ρt. The instantaneous cash-flow generated by each project is equal to $\beta_i P_t, i = N, G$. We refer to β_i as the output flow or as the price sensibility of technology i. For a sake of completeness, we study the three different cases: $\beta_N > \beta_G$, $\beta_N < \beta_G$ and $\beta_N = \beta_G$. The sunk cost of project N, I_N, is greater than the sunk cost of project G, I_G. The second project is the only one to generate a strictly positive variable cost $\gamma_G X_t$. The net expected profits are thus equal to

$$\Psi_N(p) = \beta_N p - I_N \text{ for technology } N \text{ and,} \tag{3}$$

$$\Psi_G(p,x) = \beta_G p - \gamma_G x - I_G \text{ for technology } G. \tag{4}$$

Let $\mathcal{T}$ be the set of all stopping times adapted to $\mathcal{F}_t$. Because the investor has the opportunity to choose between the two projects, he shall invest in the project with the highest payoff. The value function associated to this investment problem can thus be formulated as an optimal stopping time problem

$$V(p,x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau} \max\left(\Psi_N\left(P_\tau^p\right), \Psi_G\left(P_\tau^p, X_\tau^x\right)\right)\right], \tag{5}$$

that is defined for $p \geq 0$ and $x \geq 0$. We define the investment region as

$$\mathcal{I} = \left\{(p,x) \in \mathbb{R}_+^2 \mid V(p,x) = \max\left(\Psi_N(p), \Psi_G(p,x)\right)\right\}. \tag{6}$$

The investment region is the set where the decision maker can invest optimally. Since the function $(p,x) \mapsto \max\left(\Psi_N\left(P_t^p\right), \Psi_G\left(P_t^p, X_t^x\right)\right)$ is continuous and since $e^{-rt}\left(\Psi_N\left(P_t^p\right), \Psi_G\left(P_t^p, X_t^x\right)\right)$ converges to 0 as $t \uparrow +\infty$, Theorem 10.1.9 by Øksendal [17] gives that τ_I, defined by $\tau_I = \inf\{t \geq 0 \mid (P_t, X_t) \in \mathcal{I}\}$, is an optimal stopping time. Analytically, this means that

$$V(p,x) = \mathbb{E}\left[e^{-r\tau_I}\left(\Psi_N\left(P_{\tau_I}^p\right), \Psi_G\left(P_{\tau_I}^p, X_{\tau_I}^x\right)\right)\right]. \tag{7}$$

We also define the indifference line as

$$\mathcal{D} = \left\{(p,x) \in \mathbb{R}_+^2 | \Psi_N(p) = \Psi_G(p,x)\right\}. \tag{8}$$

For a vector (p,x) of output/input values that belongs to $\mathcal{D}$, the two alternative technologies deliver the same payoff and a decision maker who would be forced to immediately invest would be indifferent between the two projects. If $(p,x) \in \mathcal{D}$ then the following relation holds

$$\beta_N p - I_N = \beta_G p - \gamma_G x - I_G,$$

or written in a different way

$$\begin{cases} p = \frac{1}{\beta_N - \beta_G} \left(I_N - I_G - \gamma_G x \right) & \text{if } \beta_N \neq \beta_G, \\ x = \frac{I_N - I_G}{\gamma_G} & \text{if } \beta_N = \beta_G. \end{cases}$$

We denote $\widetilde{p}$ the ratio $\dfrac{I_N - I_G}{\beta_N - \beta_G}$ which corresponds to the output value for which the payoff of the two projects are the same when the input price is equal to zero. We start our analysis with a proposition that summarizes the most intuitive properties of the value function V. To clarify the presentation of our results, all the proofs have been relegated to the Appendix.

Proposition 1 *We have the following properties concerning the value function V:*

1. $\forall (p, x) \in \mathbb{R}_+^2, V(p, x) < +\infty$,

2. $p \mapsto V(p, x)$ *is an increasing function,*

3. $x \mapsto V(p, x)$ *is a decreasing function,*

4. $(p, x) \mapsto V(p, x)$ *is a convex function.*

Results 2, 3 and 4 are intuitive. When the value of the output price increases, the investment opportunity becomes more valuable since the promised payoffs are higher. Furthermore, when the input price increases, the opportunity to invest becomes less valuable since technology G induces a higher production cost. Concerning the convexity result 4, the decision maker is ready to accept risky bets on the initial values for the output and input prices simultaneously.

3 Shape and properties of the investment region

We analyze in this section the properties of the investment region $\mathcal{I}$. We first try to elicit information from the one-dimensional setting. In the standard real option framework, an increase in the output price does not change the decision to invest when it has already crossed the investment threshold. If P_t lies in the investment region then it is also true for λP_t for any $\lambda > 1$. By analogy, it seems reasonable to claim that if (P_t, X_t) lies in $\mathcal{I}$, so lie $(\lambda P_t, X_t)$ and $\left(P_t, \frac{1}{\lambda} X_t\right)$. It seems also reasonable to claim that investment is optimal as soon as the payoff is sufficiently large. We will see that these two conjectures turn out to be false.

In order to describe the investment region $\mathcal{I}$, let us remind the investment thresholds corresponding to the two competitive projects taken separately.[2] If we only focus on an investment in technology N, we consider

$$V_N(p) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau} \left(\beta_N P_\tau - I_N \right) \right]. \tag{9}$$

The investment threshold corresponding to this project is equal to

$$p_N^* = \frac{\beta}{\beta - 1} \frac{I_N}{\beta_N}, \tag{10}$$

[2]These results can be found in Dixit and Pindyck, chapter V [8]

where β is the positive root of the usual characteristic equation $1/2\sigma_P^2\beta(\beta-1)+(r-\delta_P)\beta-r=0$. Similarly, if we only focus on an investment in technology G, we consider

$$V_G(p,x)=\sup_{\tau}\mathbb{E}\left[e^{-r\tau}\left(\beta_G P_\tau^p-\gamma_G X_\tau^x-I_G\right)\right]. \tag{11}$$

In the special case where $x=0$, we obtain

$$V_G(p)=\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}\left(\beta_G P_\tau-I_G\right)\right], \tag{12}$$

and the investment threshold that triggers investment is equal to

$$p_G^*=\frac{\beta}{\beta-1}\frac{I_G}{\beta_G}. \tag{13}$$

According to Loubergé et al. [15], the investment region corresponding to the general case $(x>0)$ takes the following form

$$\begin{aligned}\widetilde{\mathcal{I}}_G &= \left\{(x,p)\,|V_G(p,x)=\beta_G p-\gamma_G x-I_G\right\},\\ &= \left\{(x,p)\,|p\geq C_1x+p_G^*\right\}.\end{aligned}$$

A more involved problem we can focus on is the special case of our bidimensional problem when the input price is zero. In this case, the value function becomes

$$V(p)=\sup_{\tau\in\mathcal{T}}\mathbb{E}\left[e^{-r\tau}\max\left(\beta_N P_\tau-I_N,\beta_G P_\tau-I_G\right)\right]. \tag{14}$$

This problem has been deeply studied by Décamps et al. [4]. Under the assumption that $p_N^*<\widetilde{p}$, they find that there exist two thresholds p_1 and p_2 such that for every $p\in[p_G^*,p_1[$, it is optimal to invest in technology G, and that $\forall p\in]p_2,+\infty[$, it is optimal to invest in technology N. But for $p\in]p_1,p_2[$, it is not optimal to invest neither in technology N nor in technology G. They called the interval (p_1,p_2) the "inaction region".

Before giving further results in our setting, let us put some restrictions on the parameters' values. From now on, we assume that

$$A1: p_N^*<\widetilde{p}. \tag{15}$$

Assumption $A1$ means that there exists an inaction region in the one dimensional setting (see Décamps et al. [4] p.431). Since $\beta>1$, then $\frac{\beta}{\beta-1}>1$. Therefore, if $p_N^*<\widetilde{p}$, then $\frac{I_N}{\beta_N}<\frac{I_N-I_G}{\beta_N-\beta_G}$, implying that $p_G^*<p_N^*$. When the input price equals zero, another result allows to rank the different thresholds.

Proposition 2 *We have a lower bound for the threshold p_2 given by*

$$p_2>\frac{\beta}{\beta-1}\widetilde{p}.$$

We are now in a position to prove the existence of a similar inaction region in the bidimensional setting.

Theorem 1 *The indifference line $\mathcal{D}$ never belongs to the investment region. Analytically, we have $V(p,x) > \beta_N p - I_N$ for all $(p,x) \in \mathcal{D}$.*

This result extends the one dimensional result obtained by Décamps et al. [4]. The investor's preference to wait in order to collect more information about the dominance of one project over the other before investing creates an inaction region. In a two dimension space, the interpretation is the same. When the variables are on the indifference line, the investor prefers to wait in order to collect more information about the dominance of one project over the other. If the initial output price decreases, the investor may optimally content to invest in the technology with the lowest output flow rather than wait with the hope to reach the set of optimal investment in the technology with the highest output flow. On the contrary, if the output price increases relative to the input price, the investor has more chances to invest in the technology with the highest output flow. In fact, this kind of result has already been obtained by Broadie and Detemple [3] or Villeneuve [20] in the case of financial options. They show indeed that with an American option on the maximum of two assets, it is never optimal to exercise the option when the prices of underlying assets are equal. Our setting is close to this one, except that in our case, the underlying assets are more complicated since we have linear combinations of state variables. According to this result, we now have a more precise idea of the shape of the investment region. It can be decomposed into two disjoint sets $\mathcal{I} = \mathcal{I}_N \bigcup \mathcal{I}_G$. $\mathcal{I}_N$ is the investment region in which it is optimal to invest in technology N and $\mathcal{I}_G$ the one in which it is optimal to invest in technology G. $\mathcal{I}_N$ and $\mathcal{I}_G$ are defined by

$$\mathcal{I}_N = \left\{(p,x) \in \mathbb{R}_+^2 | V(p,x) = \beta_N p - I_N\right\} \text{ and,} \tag{16}$$

$$\mathcal{I}_G = \left\{(p,x) \in \mathbb{R}_+^2 | V(p,x) = \beta_G p - \gamma_G x - I_G\right\}. \tag{17}$$

We focus on the shape of the investment regions and we give some general properties.

Proposition 3 *Let $(p_0, x_0) \in \mathbb{R}_+^2$. The following properties hold*

1. *If $(p_0, x_0) \in \mathcal{I}_G$, then $\forall x \leq x_0, (p_0, x) \in \mathcal{I}_G$,*
2. *If $(p_0, x_0) \in \mathcal{I}_N$, then $\forall x \geq x_0, (p_0, x) \in \mathcal{I}_N$,*
3. *If $\beta_N \geq \beta_G$ and if $(p_0, x_0) \in \mathcal{I}_N$, then $\forall p \geq p_0, (p, x_0) \in \mathcal{I}_N$,*
4. *If $\beta_G \geq \beta_N$ and if $(p_0, x_0) \in \mathcal{I}_G$, then $\forall p \geq p_0, (p, x_0) \in \mathcal{I}_G$.*

These four results are quite intuitive. Result 1 states that if it is optimal to invest in technology G, it will remain so if the input price decreases. Indeed, its expected profit increases whereas the expected profit generated by technology N remains constant. On the contrary, when investment in technology N is optimal, it remains so if the input price increases (Result 2). Such an increase indeed has no effect on the expected payoff generated by technology N and at the same time it makes technology G less competitive. In the case where the price sensibility of technology N is higher than the one of technology G, if it is already optimal to invest in technology N for a given level of output price, it is all the more optimal to invest in technology N with a higher output price and hence a higher profit (Result 3). Result 4 tells the same story in the case where the price sensibility of technology N is lower than the one of technology G.

This proposition gives a first idea of the shape of the investment region. But a more precise study requires a separation of the different cases depending on the ranking of the output flows. Before going further, we present the graphs of the two investment regions in the three cases $\beta_N > \beta_G$, $\beta_G > \beta_N$ and $\beta_N = \beta_G$.

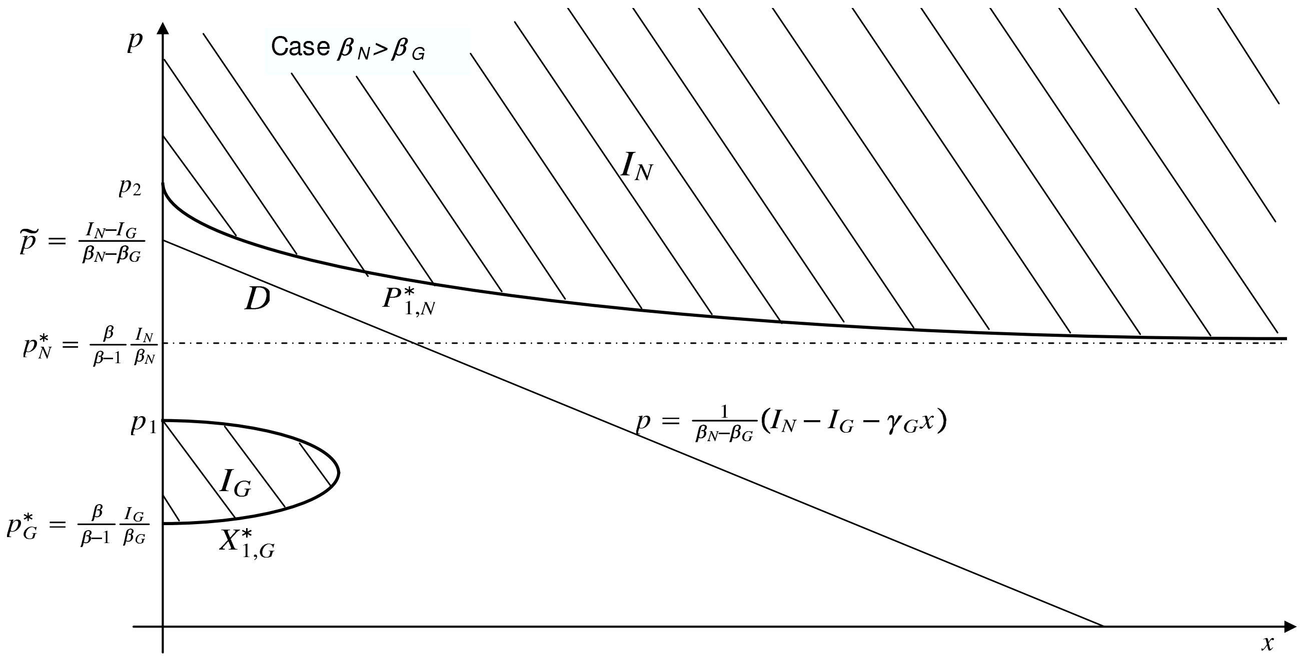

Figure 1: Shape of the investment regions when $\beta_N > \beta_G$

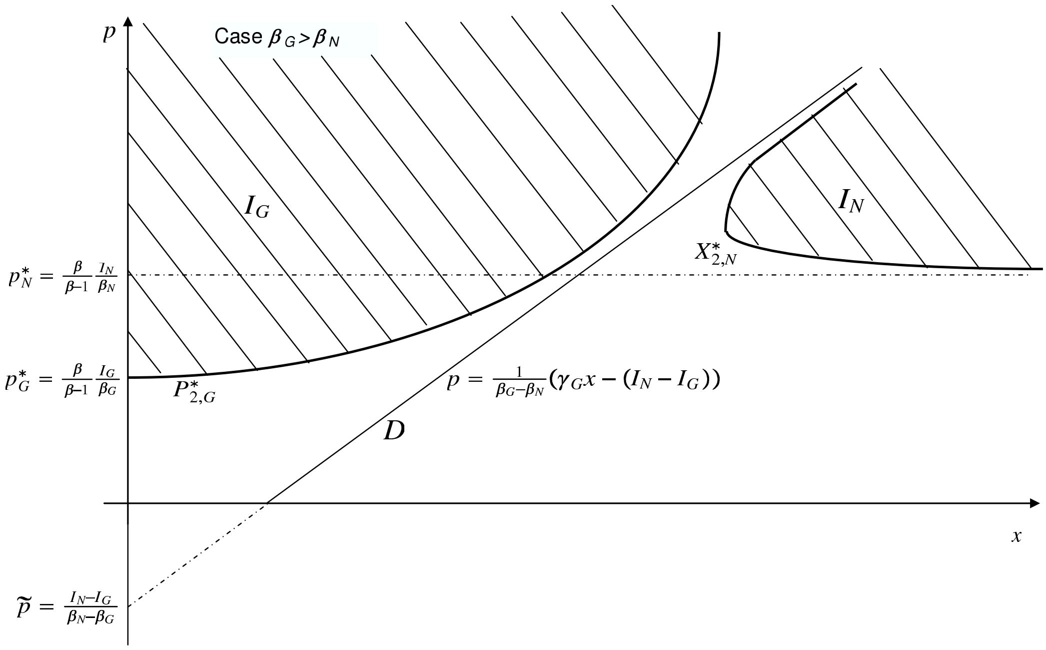

Figure 2: Shape of the investment regions when $\beta_N < \beta_G$

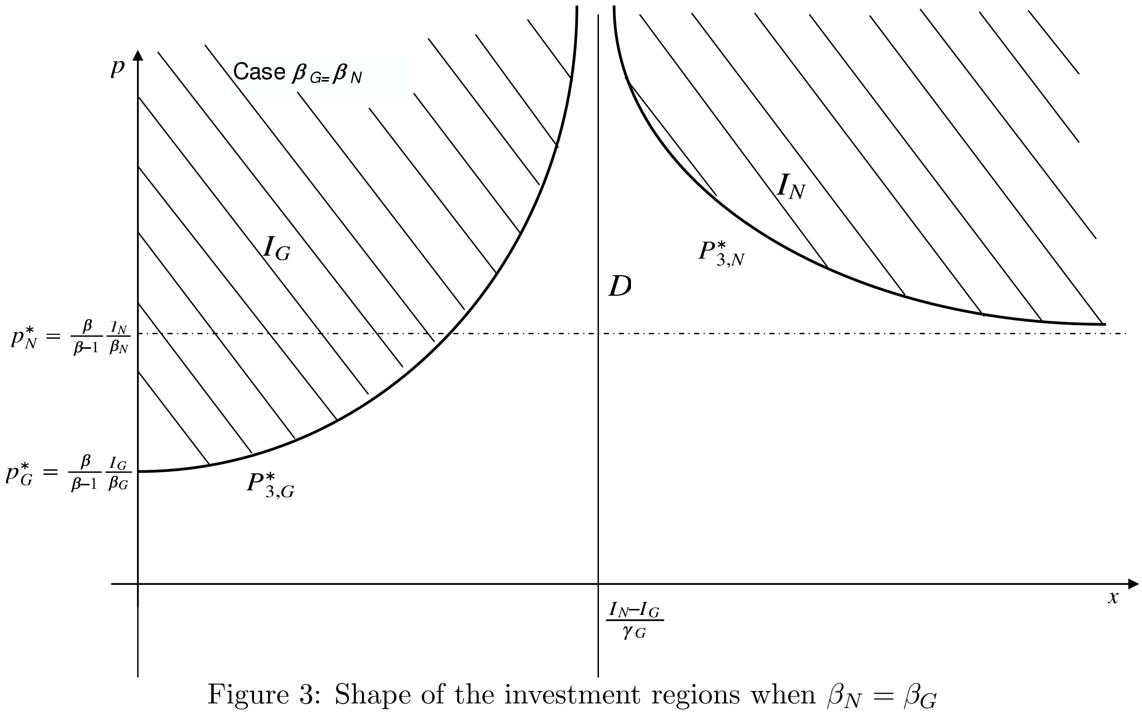

Figure 3: Shape of the investment regions when $\beta_N = \beta_G$

We begin by carefully examining the case $\beta_N > \beta_G$. As the remaining two cases will exhibit similar properties, developments will be shorter.

3.1 The output flow of technology N is higher than the output flow of technology G: $\beta_N > \beta_G$

In this paragraph, we describe the exercise region that corresponds to the investor's problem when the output flow generated by technology N is greater than that generated by technology G. First, we study the investment region for which it is optimal to invest in technology N, $\mathcal{I}_N$. Let us define

$$P_{1,N}^{*}(x) = \inf\{p \in \mathbb{R}_{+} | (x,p) \in \mathcal{I}_N\}, \tag{18}$$

which is the minimal level of output price for which it is optimal to invest in technology N given that the input price equals x. The next proposition gives the main features of this function $P_{1,N}^{*}$.

Proposition 4 *MacDonald and Siegel [16] or Dixit and Pindyck [8] We have the following properties concerning function* $P_{1,N}^{*}$*:*

1. $x \mapsto P_{1,N}^{*}(x)$ *is a decreasing function,*
2. $x \mapsto P_{1,N}^{*}(x)$ *is a convex function,*
3. $P_{1,N}^{*}(0) = p_1$,
4. $\lim_{x\to+\infty} P_{1,N}^{*}(x) = p_N^{*}$.

Proposition 4 describes the shape of the investment region $\mathcal{I}_N$. Result 1 states that when the input price increases, the threshold value of the output price for which it is optimal to invest in technology N decreases. Indeed, when the input price increases, the expected payoff generated by technology G decreases whereas the expected payoff generated by technology N remains constant. Knowing that technology G becomes less profitable, the investor chooses a threshold value of the output price for which it is optimal to invest in technology N that is decreasing with the input price. This effect decreases as the input price increases (Result 2): in this case, technology G is less competitive and plays almost no role in the decision any more. Ultimately, when the input price tends to infinity, project G totally disappears, coming back to the basic setting where there is only one project. It is thus optimal to invest as soon as the output price is greater than the usual threshold p_N^{*} (Result 4).

Concerning investment region $\mathcal{I}_G$, we have to prove first that it is nonempty under Assumption $A1$.

Proposition 5 *Under Assumption* $A1$*,* $\mathcal{I}_G$ *is nonempty.*

We define function

$$p \mapsto X_{1,G}^{*}(p) = \sup\{x \in \mathbb{R}_{+} | V(x,p) = \beta_G p - \gamma_G x - I_G\} \tag{19}$$

which has to be viewed as the maximal level of input price for which it is optimal to invest in technology G given that the output price equals p. Its main features are summarized in the following proposition.

Proposition 6 *We have the following properties concerning function $X_{1,G}^*$:*

1. $p \mapsto X_{1,G}^*(p)$ *is a concave function,*

2. $X_{1,G}^*(p_G^*) = 0$.

Concavity of function $X_{1,G}^*$ implies the existence of a maximum level of input cost above which it is never optimal to invest in technology G. As soon as the input cost increases, the set of output prices for which it is optimal to invest in technology G becomes smaller and tends to disappear. Moreover, the shape of the investment region gives some counterintuitive results. Let us imagine that the input/output prices are such that they are "just above" $\mathcal{I}_G$ so that it is not optimal to invest immediately. If the output price decreases and the input price increases in such a way that they fall into $\mathcal{I}_G$, it becomes optimal to invest in technology G though both the output and the input prices decreased. In this case, the investor is indeed sure that it will be too long and thus costly to reach $\mathcal{I}_N$. He thus accepts to invest in the project with the lowest price sensibility. When we consider the two projects simultaneously, the investment regions are quite different from the case where each project is taken separately. The presence of the two projects makes the investor more reluctant to invest in one of the two projects when the projects' profits are close and even if they are very high. He prefers to wait to obtain more information about the dominance of one project over the other: a choice value is created. Let us now turn to the case where the output flow from technology G is greater than the one of technology N.

3.2 The output flow of technology G is higher than the output flow of technology N: $\beta_G > \beta_N$

In order to study the shape of the investment regions, we define $P_{2,G}^*(x) = \inf\{p \in \mathbb{R}_+ |\, (x,p) \in \mathcal{I}_G\}$. To be more explicit, we have

$$P_{2,G}^*(x) = \inf\{p \in \mathbb{R}_+ | V(x,p) = \beta_G p - \gamma_G x - I_G\}. \tag{20}$$

$P_{2,G}^*$ has to be viewed as the minimal level of output price for which it is optimal to invest in technology G given that the input price equals x. We first focus on investment region $\mathcal{I}_G$ and on the general properties of function $P_{2,G}^*$.

Proposition 7 *We have the following properties concerning function $P_{2,G}^*$:*

1. $x \mapsto P_{2,G}^*(x)$ *is an increasing function,*

2. $x \mapsto P_{2,G}^*(x)$ *is a convex function,*

3. $P_{2,G}^*(0) = p_G^*$.

These results are very similar to the ones obtained in Proposition 4 when $\beta_N > \beta_G$. As the input price increases, the threshold value of the output for which it is optimal to invest in technology G increases. Indeed, for a given output price, when the input price increases, the expected payoff generated by technology G decreases. Therefore, in order technology G to remain the optimal choice, the optimal threshold has to increase (Result 1). Convexity of the optimal threshold (Result 2) shows that the choice value created by the competition between the two projects is all the more important as the input price increases. When the input price is equal to zero, technology G clearly dominates technology N since it has a higher payoff and a lower cost. It is indeed as if technology N did not exist any more and it is optimal to invest in technology G as soon as the output price exceeds the usual threshold p_G^* (Result 3). Next proposition states precisely the behavior of function $P_{2,G}^*$ for large input price and confirms that the choice value increases with the input price.

Proposition 8 *We have the following result concerning function* $P_{2,G}^*$

$$\lim_{x\to+\infty} \frac{P_{2,G}^*(x)}{x} \in \left] \frac{\gamma_G}{\beta_G-\beta_N}, \frac{\gamma_G}{\beta_G-\beta_N}\kappa^* \right[,$$

where κ^* *is defined by*

$$\kappa^* = \inf\left\{\kappa \geq 0 | \forall p \geq \kappa x, C_e(p,x) = p - x\right\},$$

and where $C_e(p,x)$ *is an exchange option defined by*

$$C_e(p,x) = \sup_\tau \mathbb{E}\left[e^{-r\tau}\left(X_\tau^x - P_\tau^p\right)\right].$$

The limit of $P_{2,G}^*$ is difficult to obtain. However, its asymptote lies in an interval that we determine. As the input price x increases, $P_{2,G}^*$ moves away the bisecting line. The choice value is thus unbounded for large values of x. Now, we focus on the other investment region $\mathcal{I}_N$ and on function

$$p \mapsto X_{2,N}^*(p) = \inf\left\{x \in \mathbb{R}_+ | V(p,x) = \beta_N p - I_N\right\}, \tag{21}$$

which is the minimal level of input price for which it is optimal to invest in technology N.

Proposition 9 *Function* $X_{2,N}^*$ *has the following properties:*

1. $p \mapsto X_{2,N}^*(p)$ *is a convex function,*

2. $\lim_{p\downarrow p_N^*} X_{2,N}^*(p) = +\infty$.

The findings concerning the investment regions are symmetric with the case $\beta_N > \beta_G$. Along the indifference line and despite the fact that the profit is unbounded, it is not optimal to invest in any project due to the choice value generated by the competition between the two projects. Moreover, there is a minimum level of input price that makes investment in technology N optimal. For a given input price that is very low, investment can only occur in technology G. On the contrary, for a given input price that is high enough, investment can occur in the two technologies depending on the level of the output price. We study the last case where the two technologies exactly generate the same output flow.

3.3 The output flow of both technologies are equal: $\beta_N = \beta_G$

In this section, the two technologies present the same price sensibility. The indifference line is then equal to $x = \frac{I_N - I_G}{\gamma_G}$. It is interesting to note that the three different cases are described by a rotation of the indifference line. Here, we are in the extreme case where the indifference line is vertical. As in the first two cases, we define

$$P_{3,G}^{*}(x) = \inf\{p \in \mathbb{R}_{+} | V(p,x) = \beta_G p - \gamma_G x - I_G\} \text{ and} \tag{22}$$

$$P_{3,N}^{*}(x) = \inf\{p \in \mathbb{R}_{+} | V(p,x) = \beta_N p - I_N\}. \tag{23}$$

Proposition 10 *We have the following properties concerning functions $P_{3,G}^{*}$ and $P_{3,N}^{*}$:*

1. *$x \mapsto P_{3,G}^{*}(x)$ is an increasing and convex function,*
2. *$x \mapsto P_{3,N}^{*}(x)$ is a decreasing and convex function,*
3. *$P_{3,G}^{*}(0) = p_G^{*}$ and $\lim_{x \uparrow \frac{I_N - I_G}{\gamma_G}} P_{3,G}^{*}(x) = +\infty$,*
4. *$\lim_{x \downarrow \frac{I_N - I_G}{\gamma_G}} P_{3,N}^{*}(x) = +\infty$ and $\lim_{x \to +\infty} P_{3,N}^{*}(x) = p_N^{*}$.*

When the input price is equal to zero, this is as if technology G were unique and investment in technology G is thus optimal as soon as the output price is greater than the usual threshold p_G^{*}. On the contrary, when the input price tends to $+\infty$, this is as if there were only technology N and investment is optimal as soon as the output price is greater than p_N^{*}. Here the two investment regions are clearly separated by a vertical line that corresponds to the indifference line. When the input price is lower than $\frac{I_N - I_G}{\gamma_G}$, any potential investment would only occur in technology G, whereas when the input price is greater than $\frac{I_N - I_G}{\gamma_G}$, it would only occur in technology N. That $P_{3,G}^{*}$ is increasing and $P_{3,N}^{*}$ is decreasing illustrates the interaction between the two technologies. This effect is at its height when the input price exactly equals $\frac{I_N - I_G}{\gamma_G}$, since the investor will never invest in any of the two projects even if the common profit tends to infinity. Indeed, his indifference makes him wait to choose the most favorable technology. With this extreme case, we see that the level of the future cash-flow is not a sufficient statistic to take any decision in this bidimensional setting. We now have a precise idea of the shape of the investment regions for different values taken by the pair input/output prices and by β_N and β_G. We can go to the next section that proposes an application of this model to power generation under uncertainty.

4 Application: power generation under uncertainty

As pointed out in the introduction, the multiple technologies to produce electricity makes any investment decision difficult. For instance, what should an investor choose between a technology with high sunk costs and a high price sensibility and a technology that is more flexible but that presents a lower price sensibility at the same time? Moreover, how to take into account the characteristics of the electricity market? This is this kind of questions we are trying to answer in this section. We consider two ways to produce electricity:

- technology N produces electricity with a nuclear power plant,
- technology G produces electricity with a gas power plant.

In this particular case, P can be considered as the electricity price, whereas X can be viewed as the cost of supplying gas. More precisely, we have the following features concerning each technology.

-Technology *N* T_N years are needed to build the production unit. This means that once the investment decision has been taken, the investor does not get any profit immediately: there is a lag between the time at which investment is decided and the time at which power generation starts. We assume moreover that the production unit lasts L_N years, implying that the profit flow only exists on the time period $[T_N, T_N + L_N]$. Sunk capital cost is denoted I_N. The profit is thus given by the following function:

$$\Psi_N(p) = e^{-rT_N}\mathbb{E}\left[\int_{T_N}^{T_N+L_N} P(t)\, e^{-rt}dt|P(0) = p\right] - I_N.$$

According to the dynamic of P_t, we have for $t \geq T_N$,

$$P_t = P_{T_N} \exp\left\{\left(r - \delta_P - \frac{1}{2}\sigma_P^2\right)(t - T_N) + \sigma_P\left(W_t^1 - W_{T_N}^1\right)\right\}.$$

Therefore

$$\begin{aligned}\mathbb{E}\left[\int_{T_N}^{T_N+L_N} P(t)\, e^{-rt}dt|P(0) = p\right] &= \frac{1}{\delta_P}\left(1 - e^{-\delta_P L_N}\right)\mathbb{E}\left[e^{-rT_N}P_{T_N}|P(0) = p\right], \\ &= \frac{p}{\delta_P}e^{-\delta_P T_N}\left(1 - e^{-\delta_P L_N}\right).\end{aligned}$$

We finally have that

$$\Psi_N(p) = \beta_N p - I_N \tag{24}$$

with $\beta_N = \frac{1}{\delta_P}e^{-(r+\delta_P)T_N}\left(1 - e^{-\delta_P L_N}\right)$. We recover the expression of the initial model with the output flow β_N. β_N that can also be seen as the price sensibility of the technology is an increasing function of L_N, and a decreasing function of T_N. The more important is the time lag between the decision and the effective electricity generation, the less profit the investor gets.

-Technology *G* T_G years are needed to build the production unit that lasts L_G years. Sunk capital cost is denoted I_G. The amount of gas required to generate one electricity unit is equal to l_G. The profit is given by the following function:

$$\begin{aligned}\Psi_G(p, x) &= e^{-rT_G}\mathbb{E}\left[\int_{T_G}^{T_G+L_G} P(t)\, e^{-rt}dt|P(0) = p\right] \\ &\quad - e^{-rT_G}\mathbb{E}\left[\int_{T_G}^{T_G+L_G} l_G X(t)\, e^{-rt}dt|X(0) = x\right] - I_G.\end{aligned}$$

By making similar computations than in the case of the nuclear technology, we easily obtain that

$$\Psi_G(p,x) = \beta_G p - \gamma_G x - I_G, \tag{25}$$

with $\beta_G = \frac{1}{\delta_P} e^{-(r+\delta_P)T_G}\left(1 - e^{-\delta_P L_G}\right)$ and $\gamma_G = \frac{l_G}{\delta_X} e^{-(r+\delta_X)T_G}\left(1 - e^{-\delta_X L_G}\right)$.

As for technology *N*, β_G is increasing in L_G and decreasing in T_G. But the effect on the profit function is not clear, because here we have to take into account the variable cost and γ_G is increasing in L_G and decreasing in T_G. Therefore, the total effect of L_G and T_G on $\Psi_G(p,x)$ is not determined.

We make the following assumptions on the parameters' values:

- $I_N > I_G$,
- $l_G > 0$ and so $\gamma_G > 0$.

Thanks to these assumptions and depending on the values taken by T_i and L_i, for $i = N, G$, the three cases concerning the ranking of the price sensibility β_i may arise. We suppose that the length of life of a nuclear power plant is twice longer than that of a thermal power plant and that there exists $\xi > 1$ such that $T_N = \xi T_G$. The three following cases arise:

$$-\text{If } \xi \in \left]1, 1 + \frac{1}{(r+\delta_P)T_G} \ln\left(\frac{1-e^{-2\delta_P L_G}}{1-e^{-\delta_P L_G}}\right)\right[, \text{then } \beta_N > \beta_G,$$

$$-\text{If } \xi \in \left]1 + \frac{1}{(r+\delta_P)T_G} \ln\left(\frac{1-e^{-2\delta_P L_G}}{1-e^{-\delta_P L_G}}\right), +\infty\right[, \text{then } \beta_G > \beta_N,$$

$$-\text{If } \xi = 1 + \frac{1}{(r+\delta_P)T_G} \ln\left(\frac{1-e^{-2\delta_P L_G}}{1-e^{-\delta_P L_G}}\right), \text{then } \beta_N = \beta_G.$$

With the assumption on the length of life, the output flow generated by the nuclear power plant is greater than the one generated by the thermal power plant if the construction time of a nuclear power plant is not too long relative to the thermal power plant. We recover the characteristics of the investment regions obtained in the previous part. In the case where the output flow of the nuclear power plant is greater than the one of the gas power plant, it can be optimal to invest in a gas power plant after a fall in the electricity price and an increase in the gas price. Indeed, the investor prefers to be sure he does not lose an opportunity to invest in the nuclear power plant, therefore he waits until it becomes too costly to invest in the nuclear technology.

When the price sensibility of the gas power plant is higher than the one of the nuclear power plant, the optimal choice goes from the gas power plant to the nuclear technology as the cost of supplying gas increases. If gas is not too costly, it is preferred because the technology is more flexible than a nuclear power plant.

Another surprising result occurs when $\beta_N = \beta_G$. Indeed, when $x = \frac{I_N - I_G}{\gamma_G}$, we are on the indifference line and even if the output price tends to infinity, the investor is indifferent between the two projects. Although it is possible for him to obtain an infinite profit, he prefers to wait to know in which direction the state variables are going to evolve and which technology to select. We recover the fact that the profit level is not a decision variable any more.

In this application, depending on the values given to the construction lag and to the lifetime of the production unit, we may obtain very different results as far as the position of β_N with respect to β_G may change. The results are thus highly dependent on the characteristics of each power plant.

5 Concluding remarks

This paper studies the choice by an investor between two mutually exclusive projects under both output price and input price uncertainty. In this bidimensional setting, the main difficulty is to determine the set of values for which it is optimal to invest. Our main finding is that it is never optimal to invest when the competitive projects yield the same profit, that is when the investor is indifferent between the two. The interpretation is that the investor prefers to wait in order to collect information rather than to invest too fast in a project that turns out to be unprofitable.

The study of the different possible investment regions shows us that they are quite different depending on the ranking of the price sensibilities. When $\beta_G \geq \beta_N$, for low values of the input price, optimal investment may only occur in technology *G*, and for high values of the input price, optimal investment occurs more likely in technology *N*. When $\beta_N > \beta_G$, for high values of the input price, optimal investment can only occur in technology *N*, and for low values of the input price, optimal investment may occur in both technology. The shape of the exercise regions is very different than if each project were taken separately: the interaction between the two projects creates what we shall call a "choice value". It has to be added to the "time value" that corresponds to the optimal moment to invest and that has been demonstrated by McDonald and Siegel [16] or by Henry [11]. A natural extension could be to consider such a technology choice in a competitive setting. Do firms still take the time to ensure their investment decision? The fear of being preempted will certainly decrease the choice value but to which extent?

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6 Appendix

6.1 Proof of Proposition 1

To prove Result 1, we fix a stopping time τ and we have

$$\begin{aligned}\mathbb{E}\left[e^{-r\tau}\max\left(\Psi_N\left(P_\tau^p\right),\Psi_G\left(P_\tau^p,X_\tau^x\right)\right)\right] &\leq \mathbb{E}\left[e^{-r\tau}\left(\Psi_N\left(P_\tau^p\right)\right)\right]+\mathbb{E}\left[e^{-r\tau}\left(\Psi_G\left(P_\tau^p,X_\tau^p\right)\right)\right],\\ &= \mathbb{E}\left[e^{-r\tau}\left(\beta_N P_\tau^p-I_N\right)\right]+\mathbb{E}\left[e^{-r\tau}\left(\beta_G P_\tau^p-\gamma_G X_\tau^x-I_G\right)\right],\\ &\leq \mathbb{E}\left[e^{-r\tau}\left(\beta_N P_\tau^p-I_N\right)\right]+\mathbb{E}\left[e^{-r\tau}\left(\beta_G P_\tau^p-I_G\right)\right],\\ &\leq \sup_\tau \mathbb{E}\left[e^{-r\tau}\left(\beta_N P_\tau^p-I_N\right)\right]+\sup_\tau \mathbb{E}\left[e^{-r\tau}\left(\beta_G P_\tau^p-I_G\right)\right].\end{aligned}$$

This implies that $V(p,x)\leq V_N(p)+V_G(p,0)$. Because $\delta_P>0$ and $\delta_X>0$, $V_N(p)$ and $V_G(p,0)$ are explicit functions (see MacDonald and Siegel [16] or Dixit and Pindyck [8]) that are finite. The value function V is thus finite.
Results 2 and 3 immediately follow from a composition of monotonic functions.
Concerning Result 4, we have to show that

$$V\left(\lambda p_0+(1-\lambda)p_1,\lambda x_0+(1-\lambda)x_1\right)\leq\lambda V\left(p_0,x_0\right)+(1-\lambda)V\left(p_1,x_1\right),$$

for any (p_0,x_0), (p_1,x_1) and $\lambda\in[0,1]$.
By definition, putting $p(\lambda)=\lambda p_0+(1-\lambda)p_1$ and $x(\lambda)=\lambda x_0+(1-\lambda)x_1$ we have

$$V\left(p(\lambda),x(\lambda)\right)=\sup_{\tau\in\mathcal{T}^P}\mathbb{E}\left[e^{-r\tau}\max\left(\Psi_N\left(P_\tau^{p(\lambda)}\right),\Psi_G\left(P_\tau^{p(\lambda)},X_\tau^{x(\lambda)}\right)\right)\right].$$

Focusing on the left hand side, we have

$$\begin{aligned}&\mathbb{E}\left[e^{-r\tau}\max\left(\beta_N P_\tau^{\lambda p_0+(1-\lambda)p_1}-I_N,\beta_G P_\tau^{\lambda p_0+(1-\lambda)p_1}-\gamma_G X_\tau^{\lambda x_0+(1-\lambda)x_1}-I_G\right)\right]\\ =\ &\mathbb{E}\left[e^{-r\tau}\max\left\{\beta_N\left(\lambda P_\tau^{p_0}+(1-\lambda)P_\tau^{p_1}\right)-I_N,\beta_G\left(\lambda P_\tau^{p_0}+(1-\lambda)P_\tau^{p_1}\right)-\gamma_G\left(\lambda X_\tau^{x_0}+(1-\lambda)X_\tau^{x_1}\right)-I_G\right\}\right]\\ \leq\ &\lambda\mathbb{E}\left[e^{-r\tau}\max\left(\beta_N P_\tau^{p_0}-I_N,\beta_G P_\tau^{p_0}-\gamma_G X_\tau^{x_0}-I_G\right)\right]\\ +\ &(1-\lambda)\mathbb{E}\left[e^{-r\tau}\max\left\{\beta_N P_\tau^{p_1}-I_N,\beta_G P_\tau^{p_1}-\gamma_G X_\tau^{x_1}-I_G\right\}\right].\end{aligned}$$

Because this inequality is true for every stopping times τ, it follows that

$$V\left(\lambda p_0+(1-\lambda)p_1,\lambda x_0+(1-\lambda)x_1\right)\leq\lambda V\left(p_0,x_0\right)+(1-\lambda)V\left(p_1,x_1\right),$$

what concludes the proof. □

6.2 Proof of Proposition 2

Let us introduce the value function

$$C(p) = \sup_{\tau} \mathbb{E}\left[e^{-r\tau}\left(\left(\beta_N - \beta_G\right) P_\tau - \left(I_N - I_G\right)\right)_+\right].$$

Using that $\max(x, y) = (x - y)_+ + y$, we have

$$\mathbb{E}\left[e^{-rt}\max\left(\beta_N P_\tau - I_N, \beta_G P_\tau - I_G\right)\right] = \mathbb{E}\left[e^{-rt}\left(\left(\beta_N - \beta_G\right) P_\tau - \left(I_N - I_G\right)\right)_+\right] + \mathbb{E}\left[e^{-rt}\left(\beta_G P_\tau - I_G\right)\right].$$

Taking the supremum over the stopping times τ gives the inequality $V(p) \leq C(p) + V_G(p)$. According to MacDonald and Siegel [16] or Dixit and Pindyck [8], the optimal threshold above which the value function $C(p)$ has to be exercised is given by

$$\frac{\beta}{\beta - 1}\frac{I_N - I_G}{\beta_N - \beta_G} = \frac{\beta}{\beta - 1}\widetilde{p}.$$

Therefore, for any $p \geq \frac{\beta}{\beta-1}\widetilde{p}$, we have $V(p) \leq \beta_N p - I_N$. It follows that $V(p) = \beta_N p - I_N$ and $p_2 \geq \frac{\beta}{\beta-1}\widetilde{p}$. □

6.3 Proof of Theorem 1

For every $t \geq 0$, we have by definition of the value function

$$\begin{aligned}
V(p,x) &\geq \mathbb{E}\Bigg[e^{-rt}\max\Bigg(\beta_N p e^{\left(r-\delta_P-\frac{\sigma_P^2}{2}\right)t+\sigma_P W_t^1} - I_N, \beta_G p e^{\left(r-\delta_P-\frac{\sigma_P^2}{2}\right)t+\sigma_P W_t^1} \\
&\quad -\gamma_G x e^{\left(r-\delta_X-\frac{\sigma_X^2}{2}\right)t+\sigma_X\left(\rho W_t^1+\sqrt{1-\rho^2}W_t^2\right)} - I_G\Bigg)\Bigg], \\
&\geq \mathbb{E}\Bigg[\max\Bigg(\beta_N p e^{-\delta_P t+\sigma_P W_t^1-\frac{\sigma_P^2}{2}t} - I_N, \beta_G p e^{-\delta_P t+\sigma_P W_t^1-\frac{\sigma_P^2}{2}t} \\
&\quad -\gamma_G x e^{-\delta_X t+\sigma_X^2\left(\rho W_t^1+\sqrt{1-\rho^2}W_t^2\right)-\frac{\sigma_X^2}{2}t} - I_G\Bigg)\Bigg], \\
&\geq \mathbb{E}\Big[\max\left(\beta_N p\left(1+\sigma_P W_t^1\right) - I_N, \beta_G p\left(1+\sigma_P W_t^1\right)\right. \\
&- \left.\gamma_G x\left(1+\sigma_X\left(\rho W_t^1+\sqrt{1-\rho^2}W_t^2\right)\right) - I_G\right)\Big] + \mathbb{E}\left[f\left(t, W_1^t, W_2^t\right)\right], \\
&= \beta_N p - I_N + \mathbb{E}\left[\max\left(\beta_N p\sigma_P W_t^1, \beta_G p\sigma_P W_t^1 - \gamma_G x\sigma_X\left(\rho W_t^1+\sqrt{1-\rho^2}W_t^2\right)\right)\right] \\
&+ \mathbb{E}\left[f\left(t, W_1^t, W_2^t\right)\right],
\end{aligned}$$

where the last equality comes from the fact that (p, x) belongs to the indifference line and where the function $f(.)$ is defined as

$$\begin{aligned}
f(t, y_1, y_2) &= \max(\beta_N p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1} - I_N, \beta_G p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1} - \gamma_G x e^{-(\delta_X+\frac{\sigma_X^2}{2})t+\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)} \\
&- I_G) \\
&- \max\left(\beta_N p - I_N + \beta_N p\sigma_P y_1, \beta_G p - \gamma_G x - I_G + \beta_G p\sigma_P y_1 - \gamma_G x\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)\right).
\end{aligned}$$

Now, we are going to show that for t small enough $\mathbb{E}|f\left(t,W_t^1,W_t^2\right)|\leq ct$ where c is a constant. Using that $\max(a,b)-\max(c,d)\leq\max(a-c,b-d)$, we have:

$$\begin{aligned} f\left(t,y_1,y_2\right) &\leq \max\Big[\beta_N p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_N p\left(1+\sigma_P y_1\right), \beta_G p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_G p\left(1+\sigma_P y_1\right) \\ &\quad -\left(\gamma_G x e^{-(\delta_X+\frac{\sigma_X^2}{2})t+\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)}-\gamma_G x\left(1+\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)\right)\right)\Big], \\ &\leq \Big|\beta_N p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_N p\left(1+\sigma_P y_1\right)|+|\beta_G p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_G p\left(1+\sigma_P y_1\right)\Big| \\ &\quad +\left|\left(\gamma_G x e^{-(\delta_X+\frac{\sigma_X^2}{2})t+\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)}-\gamma_G x\left(1+\sigma_X\left(\rho y_1+\sqrt{1-\rho^2}y_2\right)\right)\right)\right|. \end{aligned}$$

For each of the three terms of the right hand side, we use the following inequality: $|e^y-1-y|\leq\frac{y^2}{2}e^{|y|}$. So, for the first term, we obtain for t small enough,

$$\begin{aligned} \left|\beta_N p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_N p\left(1+\sigma_P y_1\right)\right| &= \Big|\beta_N p e^{-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1}-\beta_N p\left(1-(\delta_P+\frac{\sigma_P^2}{2})t+\sigma_P y_1\right) \\ &- \beta_N p\delta_P t\Big|, \\ &\leq \beta_N p\frac{\left(\sigma_P y_1-(\delta_P+\frac{\sigma_P^2}{2})t\right)^2}{2}e^{|\sigma_P y_1-(\delta_P+\frac{\sigma_P^2}{2})t|}+\beta_N p(\delta_P+\frac{\sigma_P^2}{2})t, \\ &\leq c_1 t. \end{aligned}$$

Hence, by repeating this operation twice, we obtain that $\mathbb{E}|f\left(t,W_t^1,W_t^2\right)|\leq(c_1+c_2+c_3)t=ct$. It follows that

$$\begin{aligned} V\left(p,x\right) &\geq \beta_N p-I_N \\ &+ \mathbb{E}\left[\max\left(\beta_N p\sigma_P W_t^1,\beta_G p\sigma_P W_t^1-\gamma_G x\sigma_X\left(\rho W_t^1+\sqrt{1-\rho^2}W_t^2\right)\right)\right]+o(t). \end{aligned}$$

Because W_t^1 (resp. W_t^2) has the same law as $\sqrt{t}g_1$ (resp. $\sqrt{t}g_2$), where the g_i are Gaussian random variables, we have

$$V\left(x,p\right)\geq\beta_N p-I_N+\sqrt{t}\mathbb{E}\left[\max\left(h_1,h_2\right)\right]+o(t),$$

where $h_1=\beta_N p\sigma_P g_1$ and $h_2=(\beta_G p\sigma_P-\gamma_G x\sigma_X)g_1-\sqrt{1-\rho^2}\gamma_G x\sigma_X g_2$ are also Gaussian random variables. According to the standard property:

$$\textit{If } \mathbb{E}g=\mathbb{E}h=0 \textit{ and } \mathbb{P}\left(h\neq g\right)>0, \textit{ then } \mathbb{E}\max\left(g,h\right)>0,$$

we have that $V\left(x,p\right)>\beta_N p-I_N$ which concludes the proof.

6.4 Proof of Proposition 3

Case 1: $\forall x\leq x_0$,

$$0 \leq V\left(p_0,x\right)-V\left(p_0,x_0\right)$$

$$
\begin{aligned}
&= \sup_{\tau\in\mathcal{T}^P} \mathbb{E}\left[e^{-r\tau}\max\left(\Psi_N\left(P_\tau\right),\Psi_G\left(X_\tau^x,P_\tau\right)\right)\right] \\
&- \sup_{\tau\in\mathcal{T}^P} \mathbb{E}\left[e^{-r\tau}\max\left(\Psi_N\left(P_\tau\right),\Psi_G\left(X_\tau^{x_0},P_\tau\right)\right)\right] \\
&\leq \sup_{\tau\in\mathcal{T}^P} \mathbb{E}\left[e^{-r\tau}\left(\max\left(\Psi_N\left(P_\tau\right),\Psi_G\left(X_\tau^x,P_\tau\right)\right)-\max\left(\Psi_N\left(P_\tau\right),\Psi_G\left(X_\tau^{x_0},P_\tau\right)\right)\right)\right], \\
&= \sup_{\tau\in\mathcal{T}^P} \mathbb{E}\left[e^{-r\tau}\left(\max\left(0,\gamma_G\left(X_\tau^{x_0}-X_\tau^x\right)\right)\right)\right], \\
&\leq \gamma_G\left(x_0-x\right).
\end{aligned}
$$

Because $V(p_0,x_0) \leq \beta_G p_0 - \gamma_G x_0 - I_G$, we get $V(p_0,x) \leq \beta_G p_0 - \gamma_G x - I_G$, and thus (p_0,x) belongs to the investment region.

Case 2: $\forall x \geq x_0$, $V(p_0,x) \leq V(p_0,x_0)$. So, we have $V(p_0,x) \leq \beta_N p_0 - I_N$.

It follows that $V(p_0,x) = \beta_N p_0 - I_N$, which ends the proof.

Case 3: In this case, $\beta_N \geq \beta_G$ and we take $p \geq p_0$:

$$
V(p,x_0)-V(p_0,x_0) \leq (p-p_0)\sup_{\tau\in\mathcal{T}^P}\mathbb{E}\left[e^{-r\tau}\max\left(\beta_N e^{\left(r-\delta_N-\frac{1}{2}\sigma_N^2\right)\tau+\sigma_N W_\tau^1},\beta_G e^{\left(r-\delta_N-\frac{1}{2}\sigma_N^2\right)\tau+\sigma_N W_\tau^1}\right)\right].
$$

As $\beta_N \geq \beta_G$, it follows that $V(p_0,x)-V(p_0,x_0) \leq (p-p_0)\beta_N$. Because we assume $V(p_0,x_0) = \beta_N p_0 - I_N$, we have: $V(p,x_0) \leq \beta_N p - I_N$ and the result follows.

Case 4: As $p \geq p_0$, we have the same inequality than above:

$$
V(p,x_0)-V(p_0,x_0) \leq (p-p_0)\sup_{\tau\in\mathcal{T}^P}\mathbb{E}\left[e^{-r\tau}\max\left(\beta_N e^{\left(r-\delta_N-\frac{1}{2}\sigma_N^2\right)\tau+\sigma_N W_\tau^1},\beta_G e^{\left(r-\delta_N-\frac{1}{2}\sigma_N^2\right)\tau+\sigma_N W_\tau^1}\right)\right].
$$

But, now, $\beta_G \geq \beta_N$ and consequently, $V(p_0,x) - V(p_0,x_0) \leq (p-p_0)\beta_G$. An analogous argument as above with $V(p_0,x_0) = \beta_G p_0 - \gamma_G - I_G$ leads to the result. □

6.5 Proof of Proposition 4

Result 1: Let $(x_0,x_1) \in \mathbb{R}^2$ such that $x_0 < x_1$. By definition of $P_{1,N}^*$, $\left(x_0, P_{1,N}^*(x_0)\right) \in \mathcal{I}_N$. According to the previous proposition, $\left(x_1, P_{1,N}^*(x_0)\right) \in \mathcal{I}_N$. By definition of $P_{1,N}^*$, $P_{1,N}^*(x_1) \leq P_{1,N}^*(x_0)$. It follows that $P_{1,N}^*(.)$ is a decreasing function.

Result 2: In order to show that $P_{1,N}^*$ is a convex function, we are going to proceed in several steps. The first step consists in proving that $\mathcal{I}_N$ is a convex set. We want to show that if (x_0,p_0) and $(x_1,p_1) \in (\mathcal{I}_N)^2$, then $(\lambda x_0 + (1-\lambda)x_1, \lambda p_0 + (1-\lambda)p_1) \in \mathcal{I}_N$.

$$
\begin{aligned}
V\left(\lambda p_0+(1-\lambda)p_1,\lambda x_0+(1-\lambda)x_1\right) &\leq \lambda V(p_0,x_0)+(1-\lambda)V(p_1,x_1), \\
&= \lambda\left(\beta_N p_0-I_N\right)+(1-\lambda)\left(\beta_N p_1-I_N\right), \\
&= \beta_N\left(\lambda p_0+(1-\lambda)p_1\right)-I_N.
\end{aligned}
$$

But, knowing that

$$V\left(\lambda p_0+(1-\lambda)\,p_1,\lambda x_0+(1-\lambda)\,x_1\right)\geq \max\left\{\beta_N\left(\lambda p_0+(1-\lambda)\,p_1\right)-I_N,\right.$$
$$\left.\beta_G\left(\lambda p_0+(1-\lambda)\,p_1\right)-\gamma_G\left(\lambda x_0+(1-\lambda)\,x_1\right)-I_G\right\}.$$

This implies that $V\left(\lambda x_0+(1-\lambda)\,x_1,\lambda p_0+(1-\lambda)\,p_1\right)=\beta_N\left(\lambda p_0+(1-\lambda)\,p_1\right)-I_N$ and thus $\left(\lambda x_0+(1-\lambda)\,x_1,\lambda p_0+(1-\lambda)\,p_1\right)\in\mathcal{I}_N$.
The second step consists in showing that $P^*_{1,N}$ is effectively a convex function. As $\mathcal{I}_N$ is a convex set,

$$\left(\lambda x_0+(1-\lambda)\,x_1,\lambda P^*_{1,N}\left(x_0\right)+(1-\lambda)\,P^*_{1,N}\left(x_1\right)\right)\in\mathcal{I}_N.$$

By definition, we have the following inequality:

$$P^*_{1,N}\left(\lambda x_0+(1-\lambda)\,x_1\right)\leq\lambda P^*_{1,N}\left(x_0\right)+(1-\lambda)\,P^*_{1,N}\left(x_1\right),$$

and it follows that $P^*_{1,N}$ is a convex function.
Note that we have also proven that $(p,x)\mapsto V(p,x)$ is a convex function.
Result 3: This result has been shown by Décamps, Mariotti and Villeneuve [4].
Result 4: Once again, we are going to demonstrate this result using several steps.
Note that $V_N(p)\leq V(p,x)$. Let us define the set $\mathcal{N}=\left\{\left(\S,\surd\right)\in\mathbb{R}^{\in}_{+}|\beta_{\mathcal{N}}\surd-\mathcal{I}_{\mathcal{N}}>\beta_{\mathcal{G}}\surd-\gamma_{\mathcal{G}}\S-\mathcal{I}_{\mathcal{G}}\right\}$ and take $(p,x)\in\mathcal{N}$ with $p<p^*_N$. We have the following inequalities:

$$\begin{aligned} V(p,x) &\geq V_N(p),\\ &> \beta_N p-I_N.\end{aligned}$$

It follows that (p,x) does not belong to $\mathcal{I}_N$. Moreover, we have $\lim\limits_{x\to+\infty}V\left(p^*_N,x\right)\geq V_N\left(p^*_N\right)$.
The next step consists in proving that $\lim\limits_{x\to+\infty}V\left(p^*_N,x\right)=V_\infty\left(p^*_N\right)$.
We take $\left(x_n\right)_{n\geq0}$ that tends to $+\infty$. If n is high enough, $\left(p^*_N,x_n\right)\in N$.

$$\begin{aligned} 0 &\leq V\left(p^*_N,x_n\right)-V_N\left(p^*_N\right),\\ &\leq \mathbb{E}\left[e^{-r\tau_n}\max\left(\beta_N P^{p^*_N}_{\tau_n}-I_N,\beta_G P^{p^*_N}_{\tau_n}-\gamma_G X^{x_n}_{\tau_n}-I_G\right)\right]-\mathbb{E}\left[e^{-r\tau_n}\left(\beta_N P^{p^*_N}_{\tau_n}-I_N\right)\right],\end{aligned}$$

with $\tau_n=\inf\left\{t\geq0|\left(P^{p^*_N}_t,X^{x_n}_t\right)\in\mathcal{I}\right\}$. It follows that:

$$\begin{aligned} 0 &\leq V\left(p^*_N,x_n\right)-V_N\left(p^*_N\right),\\ &\leq \mathbb{E}\left[e^{-r\tau_n}\left(\left(\beta_G-\beta_N\right)P^{p^*_N}_{\tau_n}-\gamma_G X^{x_n}_{\tau_n}-\left(I_G-I_N\right)\right)_+\right],\\ &\leq \mathbb{E}\left[e^{-r\tau_n}\left(I_N-I_G-\gamma_G X^{x_n}_{\tau_n}\right)_+\right],\\ &\leq \sup_\tau\mathbb{E}\left[e^{-r\tau}\left(I_N-I_G-\gamma_G X^{x_n}_{\tau}\right)_+\right],\\ &= P\left(\gamma_G x_n\right),\end{aligned}$$

where $P\left(\gamma_G x_n\right)$ is the price of a put option with a strike equal to I_N-I_G. But, we know that $\lim\limits_{n\to+\infty}P\left(\gamma_G x_n\right)=0$.

It follows that $\lim_{n\to+\infty} V\left(p_N^*, x_n\right) = V_N\left(p_N^*\right)$. Note that we have also proven that

$$\lim_{n\to+\infty} \mathbb{E}\left[e^{-r\tau_n}\left(\beta_N P_{\tau_n}^{p_N^*} - I_N\right)\right] = \beta_N p_N^* - I_N. \tag{26}$$

The last step consists in proving that $(\tau_n)_n$ tends in probability to 0. We have the following inequalities:

$$\begin{aligned}
\mathbb{E}\left[e^{-r\tau_n}\left(\beta_N P_{\tau_n}^{p_N^*} - I_N\right)\right] &\leq \mathbb{E}\left[e^{-r\tau_n} V_N\left(P_{\tau_n}^{p_N^*}\right)\right], \\
&= \beta_N p_N^* - I_N + \mathbb{E}\int_0^{\tau_n} e^{-ru}\left(rI_N - \delta_P P_u^{p_N^*}\right)\mathbb{I}_{P_u^{p_N^*}\geq p_N^*}\,du, \\
&\leq \beta_N p_N^* - I_N + \left(rI_N - \delta_P p_N^*\right)\mathbb{E}\int_0^{\tau_n} e^{-ru}\mathbb{I}_{P_u^{p_N^*}\geq p_N^*}\,du.
\end{aligned}$$

Since $rI_N - \delta_P p_N^*$ is nonpositive, (26) gives

$$\begin{aligned}
\lim_{n\to+\infty}\mathbb{E}\int_0^{\tau_n} e^{-ru}\mathbb{1}_{\{P_u^{p_N^*}\geq p_N^*\}}\,du &= \lim_{n\to+\infty}\mathbb{E}\int_0^{\tau_n} e^{-ru}\mathbb{1}_{\{\sigma_P W_u + \left(r-\delta_P-\frac{1}{2}\sigma_P^2\right)u\geq 0\}}du, \\
&= 0,
\end{aligned}$$

which implies that $(\tau_n)_n$ tends in probability to 0.
Finally, suppose $\lim_{x\to+\infty} P_{1,N}^*(x) = l > p_N^*$ and let ε be such that $\varepsilon < l - p_N^*$. Let us define $M = \frac{I_N - I_G}{\gamma_G}$ and the stopping times $\tau_M^n = \inf\left\{t\geq 0 | X_t^{x_n} \leq M\right\}$, and $\tau_\epsilon = \inf\left\{t \geq 0 | P_t^{p_N^*} \leq p_N^* + \epsilon\right\}$. We have $\tau_n \geq \tau_M^n \wedge \tau_\epsilon$ and $\lim_{n\to+\infty} \tau_M^n \wedge \tau_\epsilon = \tau_\epsilon$, what leads to a contradiction
All these steps allow us to conclude that $\lim_{x\to+\infty} P_{1,N}^*\left(x\right) = p_N^*$. □

6.6 Proof of Proposition 5

We will make a proof by contradiction assuming that $\mathcal{I}_G$ is empty. As a consequence, optimal stopping theory (see Theorems 10.1.9 and 10.1.12 in Øksendal [17]) gives

$$\begin{aligned}
V(p,x) &= \mathbb{E}\left[e^{-r\tau_I}\max(\Psi_N(P_{\tau_I}), \Psi_G(P_{\tau_I}, X_{\tau_I}))\right], \\
&= \mathbb{E}\left[e^{-r\tau_I}(\beta_N P_{\tau_I} - I_N)\right], \\
&\leq V_N(p).
\end{aligned}$$

Therefore, we have $V(p,x) = V_N(p)$. But, for $x < \frac{\beta_N - \beta_G}{\gamma_G}(\tilde{p} - p_N^*)$, we get

$$\begin{aligned}
\beta_G p_N^* - \gamma_G x - I_G &\leq V(p_N^*, x), \\
&= V_N(p_N^*), \\
&= \beta_N p_N^* - I_N, \\
&< \beta_G p_N^* - \gamma_G x - I_G,
\end{aligned}$$

which yields to a contradiction. □

6.7 Proof of Proposition 6

Concerning Result 1, we are going to use the same steps as the ones used to prove the convexity of the function $P_{1,N}^*$. $\mathcal{I}_G$ is a convex set which implies that

$$\left(\lambda X_{1,G}^*\left(p_0\right)+\left(1-\lambda\right)X_{1,G}^*\left(p_1\right),\lambda p_0+\left(1-\lambda\right)p_1\right)\in\mathcal{I}_G.$$

But, we also have that

$$\left(X_{1,G}^*\left(\lambda p_0+\left(1-\lambda\right)p_1\right),\lambda p_0+\left(1-\lambda\right)p_1\right)\in\mathcal{I}_G.$$

Therefore by definition of $X_{1,G}^*$,

$$X_{1,G}^*\left(\lambda p_0+\left(1-\lambda\right)p_1\right)\geq\lambda X_{1,G}^*\left(p_0\right)+\left(1-\lambda\right)X_{1,G}^*\left(p_1\right).$$

Concerning Result 2, recall that function V_G defined in the previous section is such that $V_G\left(p,x\right)\leq V\left(p,x\right)$. It follows that $X_{1,G}^*\left(p\right)\in\widetilde{\mathcal{I}}_G$. Therefore

$$0\leq X_{1,G}^*\left(p\right)\leq\frac{p-p_G^*}{C_1}.$$

By letting p tend to p_G^*, we conclude that $X_{1,G}^*\left(p_G^*\right)=0$. □

6.8 Proof of Proposition 8

Let us define the exchange option by:

$$C_e\left(p,x\right)=\sup_\tau\mathbb{E}\left[e^{-r\tau}\left(xe^{\left(r-\delta_X-\frac{1}{2}\sigma_X^2\right)\tau+\sigma_X\left(\rho W_\tau^1+\sqrt{1-\rho^2}W_\tau^2\right)}-pe^{\left(r-\delta_P\frac{1}{2}\sigma_P^2\right)\tau+\sigma_P W_\tau^1}\right)\right].$$

Using

$$\begin{aligned}\max\left(\beta_N p-I_N,\beta_G p-\gamma_G x-I_G\right)&=\max\left(\beta_N p-I_N+I_G,\beta_G p-\gamma_G x\right)-I_G,\\&=\max\left(-\left(I_N-I_G\right),\left(\beta_G-\beta_N\right)p-\gamma_G x\right)-I_G+\beta_N p,\\&\leq\max\left(0,\left(\beta_G-\beta_N\right)p-\gamma_G x\right)-I_G+\beta_N p,\end{aligned}$$

we obtain

$$V\left(x,p\right)\leq C_e\left(\left(\beta_G-\beta_N\right)p,\gamma_G x\right)+C\left(\beta_N,p,I_G\right),$$

where $C\left(\beta_N,p,I_N\right)$ is defined by

$$C\left(\beta_N,p,I_G\right)=\sup_\tau\mathbb{E}\left[e^{-r\tau}\left(\beta_N P_\tau^p-I_G\right)\right].$$

With $\kappa^*=\inf\left\{\kappa\geq0|\forall p\geq\kappa x,C_e\left(p,x\right)=p-x\right\}$, if we take $p>\max\left\{\frac{\gamma_G}{\beta_G-\beta_N}\kappa^*x,\frac{\beta}{\beta-1}\frac{I_G}{\beta_N}\right\}$, we have

$$\begin{aligned}V\left(p,x\right)&\leq\left(\beta_G-\beta_N\right)p-\gamma_G x-I_G+\beta_N p,\\&=\beta_G p-\gamma_G x-I_G.\end{aligned}$$

In fact, the condition $p > \max\left\{\frac{\gamma_G}{\beta_G-\beta_N}\kappa^* x, \frac{\beta}{\beta-1}\frac{I_G}{\beta_N}\right\}$ becomes $p > \frac{\gamma_G}{\beta_G-\beta_N}\kappa^* x$ for x high enough. And for such an x, we finally have: $V(p,x) = \beta_G p - \gamma_G x - I_G$.
It follows that

$$\lim_{x\to+\infty} \frac{P^*_{2,G}(x)}{x} \in \left]\frac{\gamma_G}{\beta_G-\beta_N}, \frac{\gamma_G}{\beta_G-\beta_N}\kappa^*\right[,$$

what concludes the proof. □

6.9 Proof of Proposition 10

The proof of the first two results comes directly from the properties of functions $x \mapsto P^*_{2,G}(x)$ and $x \mapsto P^*_{1,N}(x)$.

Concerning Result 3, as $x \mapsto P^*_{3,G}(x)$ is an increasing and convex function, and as the indifference line does not belong to the stopping region, $\lim_{x\uparrow\frac{I_N-I_G}{\gamma_G}} P^*_{3,G}(x) = +\infty$.

A similar arguments holds for Result 4, and the limit when $x \to +\infty$ comes from the limit of $x \mapsto P^*_{1,N}(x)$. □