On Laplace asymptotic method, with application to random chaos

Enkelejd Hashorva*, Dmitry Korshunov† and Vladimir I. Piterbarg‡

June 10, 2015

Abstract: We investigate finite-dimensional Laplace integrals with phase minimum set of arbitrary dimension and then present applications to the extremal behaviour of Gaussian random chaos.

Key words: Laplace asymptotic method; Gelfand–Leray differential form; Random chaos

Subject Classification: 41A60, 60J57, 60E99

1 Introduction

Many problems of asymptotic behaviour of tail distributions may be reduced to asymptotic analysis of the Laplace integral

$$I_\lambda = \int_U a(\boldsymbol{u}) e^{-\lambda f(\boldsymbol{u})} d\boldsymbol{u}, \tag{1}$$

where U is a compact set in $\mathbb{R}^d$. Here a is called the amplitude and f the phase. Both a and f are supposed to be smooth enough. We shall assume below that $\min_{\boldsymbol{u}\in U} f(\boldsymbol{u}) = 0$ and that the set

$$\mathcal{M} = \{\boldsymbol{u} \in U : \ f(\boldsymbol{u}) = 0\}$$

is an m-dimensional smooth manifold, $0 \le m \le d-1$.

Textbooks and monographs on Laplace asymptotic method usually consider the case $m = 0$, that is, $f(\boldsymbol{u})$ has isolated points of maximum, in a pinch $m = d - 1$, whereas in applications one often meets intermediate cases. There are several approaches to study the case of an arbitrary m. For instance, an interesting approach is to integrate over the level set $f(\boldsymbol{u}) = c$ and then with respect to $c \ge 0$. Then the

*University of Lausanne, UNIL-Dorigny 1015 Lausanne, Switzerland, enkelejd.hashorva@unil.ch

†Sobolev Institute of Mathematics, 630090 Novosibirsk, Russia, korshunov@math.nsc.ru

‡Lomonosov Moscow State University, 1 Leninskie Gory, 119234 Moscow, Russia, piter@mech.math.msu.su

integral becomes a usual Laplace integral, with the differential form $df(\boldsymbol{u})\wedge\omega_f(\boldsymbol{u}) = dx_1\wedge\cdots\wedge dx_d$, where the $(d-1)$-form $\omega_f(\boldsymbol{u})$ is called the Gelfand–Leray differential form. For those $\boldsymbol{u}_0$ where the gradient $\nabla f(\boldsymbol{u}_0)$ is non-zero, the form $\omega_f(\boldsymbol{u})$ exists in a neighborhood of $\boldsymbol{u}_0$; its restriction to the level manifold $\{\boldsymbol{u} : f(\boldsymbol{u}) = c\}$ is uniquely defined and

$$\omega_f(\boldsymbol{u}) = \frac{1}{|\nabla f(\boldsymbol{u})|^2}\sum_{j=1}^{d}(-1)^{j-1}\frac{\partial f(\boldsymbol{u})}{\partial u_j}du_1\wedge\cdots\wedge\widehat{du_j}\wedge\cdots\wedge du_d, \tag{2}$$

where $\widehat{du_j}$ means its absence, see e.g. Arnold et al. [2, Chap. 7]. Therefore, we have

$$\int_U a(\boldsymbol{u})e^{-\lambda f(\boldsymbol{u})}d\boldsymbol{u} = \int_0^\infty e^{-\lambda c}\left(\int_{f(\boldsymbol{u})=c} a(\boldsymbol{u})\omega_f(\boldsymbol{u})\right)dc. \tag{3}$$

In this way the problem has been reduced to the asymptotic expansion of the following function at zero

$$F(c) = \int_{f(\boldsymbol{u})=c} a(\boldsymbol{u})\omega_f(\boldsymbol{u}).$$

Assume for a moment that $F(c)$ can be expanded as

$$F(c) = F_0c^{\rho_0} + F_1c^{\rho_1} + \cdots \quad \text{as } c\downarrow 0, \tag{4}$$

for some $-1 < \rho_0 < \rho_1 < \ldots$ and non-zero F_j's; the main difficulty to follow this approach is how to deduce such an expansion especially for the case of an arbitrary dimension m of the manifold $\mathcal{M}$. However, assuming that (4) holds, then the integration with respect to c finally leads to the expansion

$$I_\lambda = F_0\Gamma(\rho_0+1)\lambda^{-\rho_0-1} + F_1\Gamma(\rho_1+1)\lambda^{-\rho_1-1} + \cdots .$$

This approach was discussed by Combet [7, Chap. 2] for the case of a single minimum, i.e., $m = 0$.

In this paper we shall follow another, also direct approach based on integrating along $\mathcal{M}$, using Fubini's Theorem, and then using uniform asymptotic single-point-minimum results. So we start with the classical Laplace asymptotic method result which deals with the case where the minimum is attained at a single point, so that $m = 0$.

2 Standard Laplace asymptotic method in case $m = 0$

We assume $f\in C^2(U)$; denote by $f''(\boldsymbol{u}) := \left[f''_{ij}, i,j = 1,\ldots,d\right]$ the Hessian matrix of the amplitude f at point $\boldsymbol{u}$, where f''_{ij} is the partial derivative of f with respect to u_i and u_j.

Theorem 1. *Let a function $f(\boldsymbol{u}) \geq 0$ have the unique inner point of minimum in U, say $\boldsymbol{u}_0$,*

$$f(\boldsymbol{u}_0) = 0, \qquad \inf_{\boldsymbol{u}:\|\boldsymbol{u}-\boldsymbol{u}_0\|\geq\varepsilon} f(\boldsymbol{u}) > 0 \quad \textit{for every } \varepsilon > 0. \tag{5}$$

Suppose that the minimum of f is non-degenerate, i.e.,

$$\det f''(\boldsymbol{u}_0) > 0. \tag{6}$$

If, in addition, $a \in C^{2r}(U)$ and $f \in C^{2r+2}(U)$ for some $r \in \mathbb{Z}^+$, then the following decomposition holds:

$$\int_U a(\boldsymbol{u}) e^{-\lambda f(\boldsymbol{u})} d\boldsymbol{u} = \lambda^{-d/2}\left[c_0 + \sum_{i=1}^{r} c_i \lambda^{-i} + o(\lambda^{-r})\right] \quad \textit{as } \lambda \to \infty, \tag{7}$$

where

$$c_0 = a(\boldsymbol{u}_0)\frac{(2\pi)^{d/2}}{\sqrt{\det f''(\boldsymbol{u}_0)}}. \tag{8}$$

The intuition behind the fact that so many derivatives are necessary for obtaining the decomposition (7) with $r+1$ terms may be found, e.g. in Bender and Orszag [4, Sect. 6.4], where the decomposition with two terms, $r = 1$, was considered in the univariate case $d = 1$, see also Fedoryuk [9, p. 69] and Trofimov and Friezen [14]. We should also mention asymptotic expansions, $r = \infty$, by Combet [7, Chap. 1] and by Wong [17, Chap. IX, Sect. 5]; and, for a boundary point $\boldsymbol{u}_0$ and finite r, by Bleistein and Handelsman [5, Sect. 8.3]. Notice that Fulks and Sather [10] derived a similar asymptotic expansion of the form $\sum_{i=0}^{r} c_i \lambda^{-(d+i)/2}$ assuming asymptotic expansions of a and f along every radius instead of differentiability; our assumptions on differentiability of a and f cancel the terms $\lambda^{-(d+i)/2}$ with i odd.

Various approaches for calculation of the coefficients c_j for $j \geq 1$ in the one-dimensional case $d = 1$ are discussed, in particular, in Wojdylo [15] and López et al. [12].

Proof. In order to make formulas shorter, let us suppose that $\mathbf{0} \in U$ and $\boldsymbol{u}_0 = \mathbf{0}$. Since the point of minimum is non-degenerate and the function is everywhere positive except $\mathbf{0}$ (the conditions (6) and (5)), there exists a $\delta > 0$ such that $f(\boldsymbol{u}) \geq \delta\|\boldsymbol{u}\|^2$ for all $\boldsymbol{u} \in U$. Therefore,

$$\begin{aligned}\int_{\|\boldsymbol{u}\|\geq\frac{\log\lambda}{\sqrt{\lambda}}} |a(\boldsymbol{u})| e^{-\lambda f(\boldsymbol{u})} d\boldsymbol{u} &\leq \|a(\boldsymbol{u})\|_{C(U)} \int_{\|\boldsymbol{u}\|\geq\frac{\log\lambda}{\sqrt{\lambda}}} e^{-\lambda\delta\|\boldsymbol{u}\|^2} d\boldsymbol{u} \\ &= o(\lambda^{-r}) \quad \text{as } \lambda \to \infty. \end{aligned} \tag{9}$$

Since $a(\boldsymbol{u}) \in C^{2r}(U)$, Taylor's theorem for multivariate functions justifies the following decomposition:

$$a(\boldsymbol{u}) = a(\mathbf{0}) + \sum_{1\leq|\boldsymbol{\gamma}|\leq 2r} \frac{D^{\boldsymbol{\gamma}} a(\mathbf{0})}{\boldsymbol{\gamma}!} \boldsymbol{u}^{\boldsymbol{\gamma}} + o(\|\boldsymbol{u}\|^{2r}) \quad \text{as } \boldsymbol{u} \to \mathbf{0},$$

where, for $\boldsymbol{\gamma} \in (\mathbb{Z}^+)^d$ and $\boldsymbol{u} \in \mathbb{R}^d$, we follow the multi-index notation $|\boldsymbol{\gamma}| = \gamma_1 + \cdots + \gamma_d$, $\boldsymbol{\gamma}! = \gamma_1! \cdots \gamma_d!$, $\boldsymbol{u}^{\boldsymbol{\gamma}} = u_1^{\gamma_1} \cdots u_d^{\gamma_d}$ and $D^{\boldsymbol{\gamma}} a = \frac{\partial^{|\gamma|} a}{\partial u_1^{\gamma_1} \cdots \partial u_d^{\gamma_d}}$. Then

$$a(\boldsymbol{u}/\sqrt{\lambda}) = a(\boldsymbol{0}) + \sum_{i=1}^{2r} A_i(\boldsymbol{u})\lambda^{-i/2} + o(\|\boldsymbol{u}\|^{2r}\lambda^{-r}) \tag{10}$$

as $\lambda \to \infty$ uniformly in $\|\boldsymbol{u}\| \leq \log\lambda$, where $A_i(\boldsymbol{u})$ is a homogeneous polynomial of degree i.

Similarly, since $f(\boldsymbol{u}) \in C^{2r+2}(U)$, $f(\boldsymbol{0}) = 0$ and $\partial f(\boldsymbol{0})/\partial u_i = 0$,

$$f(\boldsymbol{u}) = \frac{1}{2}(f''(\boldsymbol{0})\boldsymbol{u}, \boldsymbol{u}) + R(\boldsymbol{u}),$$

where

$$R(\boldsymbol{u}) := \sum_{3 \leq |\boldsymbol{\gamma}| \leq 2r+2} \frac{D^{\boldsymbol{\gamma}} f(\boldsymbol{0})}{\boldsymbol{\gamma}!}\boldsymbol{u}^{\boldsymbol{\gamma}} + o(\|\boldsymbol{u}\|^{2r+2}) \quad \text{as } \boldsymbol{u} \to \boldsymbol{0}.$$

In particular,

$$\sup_{\|\boldsymbol{u}\| \leq \frac{\log\lambda}{\sqrt{\lambda}}} |\lambda R(\boldsymbol{u})| \leq \text{const}\frac{\log^3\lambda}{\sqrt{\lambda}} \to 0 \quad \text{as } \lambda \to \infty.$$

Therefore, Taylor's expansion for the exponent is applicable:

$$e^{-\lambda R(\boldsymbol{u})} = 1 + \sum_{i=1}^{2r} \lambda^i R^i(\boldsymbol{u})\frac{1}{i!} + O((\lambda R(\boldsymbol{u}))^{2r+1})$$

as $\lambda \to \infty$ uniformly in $\|\boldsymbol{u}\| \leq \frac{\log\lambda}{\sqrt{\lambda}}$. Hence, as $\lambda \to \infty$ uniformly in $\|\boldsymbol{u}\| \leq \log\lambda$,

$$e^{-\lambda R(\boldsymbol{u}/\sqrt{\lambda})} = 1 + \sum_{i=1}^{2r} Q_i(\boldsymbol{u})\lambda^{-i/2} + o(\lambda^{-r}), \tag{11}$$

where $Q_{2i}(\boldsymbol{u})$ is a polynomial consisting of terms of even degree and $Q_{2i+1}(\boldsymbol{u})$ is a polynomial consisting of terms of odd degree.

Combining (10) and (11), we deduce that

$$a(\boldsymbol{u}/\sqrt{\lambda})e^{-\lambda f(\boldsymbol{u}/\sqrt{\lambda})} = e^{-(f''(\boldsymbol{0})\boldsymbol{u},\boldsymbol{u})/2}\left[a(\boldsymbol{0}) + \sum_{i=1}^{2r} T_i(\boldsymbol{u})\lambda^{-i/2} + o(\lambda^{-r})\right], \tag{12}$$

where $T_{2i}(\boldsymbol{u})$ is a polynomial consisting of terms of even degree and $T_{2i+1}(\boldsymbol{u})$ is a polynomial consisting of terms of odd degree. For i odd,

$$\int_{\mathbb{R}^d} e^{-(f''(\boldsymbol{0})\boldsymbol{u},\boldsymbol{u})/2} T_i(\boldsymbol{u})d\boldsymbol{u} = 0.$$

The latter integral over the set $\|\boldsymbol{u}\| \geq \log \lambda$ is of order $o(\lambda^{-r})$, so that

$$\int_{\|\boldsymbol{u}\|<\log \lambda} e^{-(f''(\mathbf{0})\boldsymbol{u},\boldsymbol{u})/2} T_i(\boldsymbol{u}) d\boldsymbol{u} = O(\lambda^{-r}) \quad \text{as } \lambda \to \infty.$$

Thus, integrating (12) over $\|\boldsymbol{u}\| < \log \lambda$ we should exclude all polynomials T_i of odd degrees and then, as $\lambda \to \infty$,

$$\int_{\|\boldsymbol{u}\|<\log \lambda} a(\boldsymbol{u}/\sqrt{\lambda}) e^{-\lambda f(\boldsymbol{u}/\sqrt{\lambda})} d\boldsymbol{u} = a(\mathbf{0}) \int_{\|\boldsymbol{u}\|<\log \lambda} e^{-(f''(\mathbf{0})\boldsymbol{u},\boldsymbol{u})/2} d\boldsymbol{u}$$
$$+ \sum_{i=1}^{r} \lambda^{-i} \int_{\|\boldsymbol{u}\|<\log \lambda} T_{2i}(\boldsymbol{u}) e^{-(f''(\mathbf{0})\boldsymbol{u},\boldsymbol{u})/2} d\boldsymbol{u} + o(\lambda^{-r}),$$

which together with (9) is equivalent to the theorem conclusion. □

We also need a generalization of this theorem to functions a and f depending on some parameter. In [8] there is a corresponding theorem for the case $d = 1$. We formulate a version of its multidimensional generalisation. Consider the integral

$$I_{\lambda,\theta} = \int_U a(\boldsymbol{u}, \theta) e^{\lambda f(\boldsymbol{u},\theta)} d\boldsymbol{u}, \tag{13}$$

where $\theta \in \Theta$ is a parameter. We assume that $f \in C^2(U)$ for every $\theta \in \Theta$.

Theorem 2. *Let a function $f(\boldsymbol{u}, \theta) \geq 0$ have the unique inner point of minimum in U, say $\boldsymbol{u}_0(\theta)$,*

$$f(\boldsymbol{u}_0(\theta), \theta) = 0, \qquad \inf_{\theta\in\Theta, \boldsymbol{u}: \|\boldsymbol{u}-\boldsymbol{u}_0(\theta)\| \geq \varepsilon} f(\boldsymbol{u}, \theta) > 0 \quad \textit{for every } \varepsilon > 0.$$

Suppose that, for every $\theta \in \Theta$, the minimum of f is non-degenerate, i.e.,

$$\det f''(\boldsymbol{u}_0(\theta), \theta) > 0.$$

Suppose that, for some $r \in \mathbb{Z}^+$, we have $a \in C^{2r}(U)$ and $f \in C^{2r+2}(U)$ in $\boldsymbol{u} \in U$ for every $\theta \in \Theta$. If all partial derivatives of $a(\boldsymbol{u}, \theta)$ of order $2r$ and of $f(\boldsymbol{u}, \theta)$ of order $2r + 2$ are uniformly continuous in $\theta \in \Theta$, then the following decomposition holds:

$$\int_U a(\boldsymbol{u}, \theta) e^{-\lambda f(\boldsymbol{u},\theta)} d\boldsymbol{u} = \lambda^{-d/2} \left[c_0(\theta) + \sum_{i=1}^{r} c_i(\theta) \lambda^{-i} + \psi(\lambda, \theta) \right], \tag{14}$$

where $\psi(\lambda, \theta)\lambda^r \to 0$ as $\lambda \to \infty$ uniformly in $\theta \in \Theta$.

Proof. Since all partial derivatives of $a(\boldsymbol{u}, \theta)$ of order $2r$ and of $f(\boldsymbol{u}, \theta)$ of order $2r+2$ are uniformly continuous in $\theta \in \Theta$, we may apply Taylor's decomposition as in the previous proof with the remainder terms uniform in θ. This allows to follow the same calculations as above. □

Notice that, for every j, the coefficient $c_i(\theta)$ is a smooth function of the partial derivatives of $a(\boldsymbol{u}, \theta)$ at point $\boldsymbol{u}_0(\theta)$ of order not greater than $2i$ and of partial derivatives of $f(\boldsymbol{u}, \theta)$ at point $\boldsymbol{u}_0(\theta)$ of order not greater than $2i + 2$.

Notice also that due to the uniformity of the decomposition it can be integrated in θ in the case where Θ is a manifold, say, in $\mathbb{R}^{d_1}$.

3 Laplace asymptotic method in case $1 \le m \le d-1$

In this section we consider the case where $\mathcal{M}$—the set of minimum of the phase $f(\boldsymbol{u})$—is a m-dimensional manifold without boundary, $1 \le m \le d-1$, of finite and positive volume.

We assume that all the points of $\mathcal{M}$ are points of non-degenerate maximum in the sense that for any $\boldsymbol{v} \in \mathcal{M}$, the rank of $f''(\boldsymbol{v})$ is equal to $d-m$. Denote by $\det f''_{d-m}(\boldsymbol{v})$ any non-zero $(d-m)$-minor of the matrix $f''(\boldsymbol{v})$; notice that all $(d-m)$-minors are equal one to another, by using orthogonal transform.

Fix some $r \in \mathbb{Z}^+$. We assume that the manifold $\mathcal{M}$ is C^{2r+2}-smooth. Moreover, we assume that U may be partitioned into a finite number of disjoint sets $U_1, \ldots, U_n$ such that, for every $1 \le j \le n$, the manifold $\mathcal{M} \cap U_j$ is elementary, that is, there exists a bijection $h_j : [0,2]^d \to \mathrm{cl}\,(U_j)$ (the closure of U_j) which is $2r+2$ times differentiable, non-degenerate and such that

$$h_j([0,2]^m \times \{1\}^{d-m}) = \mathcal{M} \cap \mathrm{cl}\,(U_j).$$

It is non-degenerate in a sense that its Jacobian $J_j(\boldsymbol{z}) := \det h'_j(\boldsymbol{z})$ is non-zero at every point $\boldsymbol{z} \in [0,2]^d$.

For every $\boldsymbol{u} \in U$, denote by $\rho(\boldsymbol{u}, \mathcal{M}) := \inf_{\boldsymbol{v}\in\mathcal{M}} \|\boldsymbol{u} - \boldsymbol{v}\|$ the distance from $\boldsymbol{u}$ to the manifold $\mathcal{M}$.

Theorem 3. *Suppose that, for every $\varepsilon > 0$,*

$$\inf_{\boldsymbol{u}\in U:\rho(\boldsymbol{u},\mathcal{M})\| \ge \varepsilon} f(\boldsymbol{u}) > 0. \tag{15}$$

Suppose that

$$\inf_{\boldsymbol{v}\in\mathcal{M}} \det f''_{d-m}(\boldsymbol{v}) > 0. \tag{16}$$

If the above conditions on $\mathcal{M}$ are fulfilled and if $a(\boldsymbol{u}) \in C^{2r}$, $f(\boldsymbol{u}) \in C^{2r+2}$, then the following asymptotical expansion takes place:

$$I_\lambda = \lambda^{-\frac{d-m}{2}} \left(c_0 + \sum_{i=1}^{r} c_i \lambda^{-i} + o(\lambda^{-r}) \right) \quad \text{as } \lambda \to \infty, \tag{17}$$

where

$$c_0 \;\; := \;\; (2\pi)^{\frac{d-m}{2}} \int_{\mathcal{M}} \frac{a(\boldsymbol{v})}{\sqrt{\det f''_{d-m}(\boldsymbol{v})}} dV,$$

where dV is the m-dimensional volume element of $\mathcal{M}$ and $c_1, \ldots, c_r \in \mathbb{R}$.

Similar integrals over the manifold—where the function attains its minimum—appear in different sources, for instance, in an asymptotic equivalence proven by Barbe [3, Theorem 7.1] for the case where $\mathbf{0} \notin U$ and f is a strictly convex, α-positively homogeneous function, so that $\mathcal{M}$ is a boundary set of U; by Breitung [6, Theorem 50].

Proof. Consider the following decomposition:

$$I_\lambda = \sum_{j=1}^{n} \int_{U_j} a(\boldsymbol{u}) e^{-\lambda f(\boldsymbol{u})} d\boldsymbol{u} =: \sum_{j=1}^{n} I_{\lambda,j}$$

and compute the asymptotic behaviour of the jth integral.

Since $h_j([0,2]^m \times \{1\}^{d-m}) = \mathcal{M} \cap \mathrm{cl}\,(U_j)$, we have $f(h_j(\boldsymbol{s},\boldsymbol{1})) = 0$ for every point $\boldsymbol{s} \in [0,2]^m$. For every $\boldsymbol{s}$, the function $f(h_j(\boldsymbol{s},\cdot,\ldots,\cdot))$ of $d-m$ arguments is $2r+2$ times differentiable while the function $a(h_j(\boldsymbol{s},\cdot,\ldots,\cdot))$ of $d-m$ arguments is $2r$ times differentiable. Applying Theorem 2 to functions $a \circ h_j$ and $f \circ h_j$ and parameter $\theta = \boldsymbol{s} \in [0,2]^m$, we obtain

$$\int_{\boldsymbol{t}\in[0,2]^{d-m}} a(h_j(\boldsymbol{s},\boldsymbol{t})) e^{\lambda f(h_j(\boldsymbol{s},\boldsymbol{t}))} |\det J_j(\boldsymbol{s},\boldsymbol{t})| d\boldsymbol{t} = \lambda^{-\frac{d-m}{2}} \left(c_{0j}(\boldsymbol{s}) + \sum_{i=1}^{r} c_{ij}(\boldsymbol{s}) \lambda^{-i} + o(\lambda^{-r}) \right)$$

as $\lambda \to \infty$ uniformly in $\boldsymbol{s}$, with

$$c_{j0}(\boldsymbol{s}) = (2\pi)^{\frac{d-m}{2}} \frac{a(h_j(\boldsymbol{s},\boldsymbol{1}))|\det J_j(\boldsymbol{s},\boldsymbol{1})|}{\sqrt{|\det(f\circ h_j)''_{d-m}(\boldsymbol{s},\boldsymbol{1})|}},$$

where the Hessian of $g \circ h_j$ is taken with respect to the last $d-m$ arguments. Integration over $\boldsymbol{s} \in [0,2]^m$ finally implies that

$$\begin{aligned} I_{\lambda,j} &= \int_{(\boldsymbol{s},\boldsymbol{t})\in[0,2]^d} a(h_j(\boldsymbol{s},\boldsymbol{t})) e^{\lambda f(h_j(\boldsymbol{s},\boldsymbol{t}))} |\det J_j(\boldsymbol{s},\boldsymbol{t})| d\boldsymbol{t} d\boldsymbol{s} \\ &= \lambda^{-\frac{d-m}{2}} \left(c_{0j} + \sum_{i=1}^{r} c_{ij} \lambda^{-i} + o(\lambda^{-r}) \right) \end{aligned}$$

as $\lambda \to \infty$ where

$$\begin{aligned} c_{0j} &= (2\pi)^{\frac{d-m}{2}} \int_{[0,2]^m} \frac{a(h_j(\boldsymbol{s},\boldsymbol{1}))|\det J_j(\boldsymbol{s},\boldsymbol{1})|}{\sqrt{|\det(f\circ h_j)''_{d-m}(\boldsymbol{s},\boldsymbol{1})|}} d\boldsymbol{s} \\ &= (2\pi)^{\frac{d-m}{2}} \int_{\mathcal{M}\cap U_j} \frac{a(\boldsymbol{v})}{\sqrt{|\det f''_{d-m}(\boldsymbol{v})|}} dV. \end{aligned}$$

Summation over $j \le n$ completes the proof. □

4 Weibullian type random chaos

In this section we present a family of Gaussian and non-Gaussian random chaoses such that their tail asymptotics may be calculated via the Laplace asymptotic method.

Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$ be a random vector in $\mathbb{R}^d$, $d \geq 2$, with the standard normal distribution. Let $g : \mathbb{R}^d \to \mathbb{R}$ be a continuous homogeneous function of order $\alpha > 0$, that is, $g(x\boldsymbol{t}) = x^\alpha g(\boldsymbol{t})$ for all $x > 0$ and $\boldsymbol{t} = (t_1, \dots, t_d) \in \mathbb{R}^d$. We say that the random variable $g(\boldsymbol{\eta})$ is a *Gaussian chaos of order* α. In the literature, the term Gaussian chaos of order $\alpha \in \mathbb{N}$ is traditionally reserved for the case where g is a homogeneous polynomial of degree α; this case goes back to Wiener [16] where polynomial chaos processes were first time introduced. Here we follow the extended version of the term Gaussian chaos. Examples of the Gaussian chaoses include quadratic forms of components of Gaussian vector, other homogeneous polynomials of the components (for example some Hoeffding symmetric statistics), random determinants, products of degrees of Gaussian variables.

Gaussian chaos may be considered as a particular example of more general Weibullian type random chaos. We say that a random vector in $\mathbb{R}^d$, $d \geq 2$, say $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$, has density function of Weibullian type, if its density function may be represented as

$$p_{\boldsymbol{\eta}}(\boldsymbol{v}) \quad = \quad a\Big(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_\beta}\Big)\|\boldsymbol{v}\|_\beta^{\beta_a} e^{-f(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_\beta})\|\boldsymbol{v}\|_\beta^\beta}, \quad \boldsymbol{v} \in \mathbb{R}^d, \tag{18}$$

where both a and f are nonnegative functions on the unit sphere $\mathbb{S}_{d-1,\beta}$ in L_β; the function $a(\cdot)$ is homogeneous of order β_a, while the function $f(\cdot)$ is homogeneous of order β. Hereinafter $\|\boldsymbol{v}\|_\beta$ stands for the L_β-norm of the vector $\boldsymbol{v} \in \mathbb{R}^d$, that is, for $(v_1^\beta + \dots + v_d^\beta)^{1/\beta}$. As above, $\|\boldsymbol{v}\| := \|\boldsymbol{v}\|_2$.

Equivalently, the density (18) may be rewritten in terms of the L_2-norm in the following way:

$$p_{\boldsymbol{\eta}}(\boldsymbol{v}) \quad = \quad \widetilde{a}\Big(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\Big)\|\boldsymbol{v}\|^{\beta_a} e^{-\widetilde{f}(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|})\|\boldsymbol{v}\|^\beta},$$

where the functions $\widetilde{a}$ and $\widetilde{f}$ are defined on the unit sphere $\mathbb{S}_{d-1}$ in L_2 as follows: for $\boldsymbol{u} \in \mathbb{S}_{d-1}$,

$$\widetilde{a}(\boldsymbol{u}) = a\Big(\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_\beta}\Big)\Big\|\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_\beta}\Big\|^{-\beta_a} \text{ and } \widetilde{f}(\boldsymbol{u}) = f\Big(\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_\beta}\Big)\Big\|\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_\beta}\Big\|^{-\beta_a}, \quad \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|_\beta} \in \mathbb{S}_{d-1,\beta}.$$

Now let us show how the Laplace asymptotic method helps to derive the asymptotic behaviour of the tail distribution of the Weibullian chaos $g(\boldsymbol{\eta})$. We suppose that g is not negative, that is, for some x, $g(x) > 0$, otherwise our problem is trivial.

We start with the equality

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \quad = \quad \int_{\{\boldsymbol{v}\in\mathbb{R}^d : g(\boldsymbol{v})>x\}} p_{\boldsymbol{\eta}}(\boldsymbol{v}) d\boldsymbol{v}.$$

By homogeneity of g, the domain of integration is determined by the inequality $\|\boldsymbol{v}\|_\beta^\alpha g(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta) > x$, so that

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \int_{\{\boldsymbol{v}:\ \|\boldsymbol{v}\|_\beta > \frac{x^{1/\alpha}}{g^{1/\alpha}(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta)},\ g(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta)>0\}} a\Big(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_\beta}\Big)\|\boldsymbol{v}\|_\beta^{\beta_a} e^{-f(\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_\beta})\|\boldsymbol{v}\|_\beta^\beta} d\boldsymbol{v}.$$

Now it is natural to introduce new integrating variables $\boldsymbol{v} = (r, \boldsymbol{\ell})$, where $r = \|\boldsymbol{v}\|_\beta \geq 0$ and $\boldsymbol{\ell} = \frac{\boldsymbol{v}}{\|\boldsymbol{v}\|_\beta} \in \mathbb{S}_{d-1,\beta}$. The volume of $d\boldsymbol{v}$ is equal to $r^{d-1}J(1, \boldsymbol{\ell})d\boldsymbol{\ell}dr$. Changing in such a way variables, we have (we set $g(\boldsymbol{\ell}) = g(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta)$, $a(\boldsymbol{\ell}) = a(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta)$ and $f(\boldsymbol{\ell}) = f(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta))$

$$\begin{aligned}\mathbb{P}\{g(\boldsymbol{\eta}) > x\} &= \int_{\{r > \frac{x^{1/\alpha}}{g^{1/\alpha}(\boldsymbol{\ell})},\ g(\boldsymbol{\ell})>0\}} a(\boldsymbol{\ell}) r^{d-1+\beta_a} e^{-f(\boldsymbol{\ell})r^\beta} dr d\boldsymbol{\ell} \\ &= \frac{1}{\beta} \int_{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta}:\ g(\boldsymbol{\ell})>0} \frac{a(\boldsymbol{\ell})}{f^{\frac{d+\beta_a}{\beta}}(\boldsymbol{\ell})} \left[\int_{f(\boldsymbol{\ell})\frac{x^{\beta/\alpha}}{g^{\beta/\alpha}(\boldsymbol{\ell})}}^\infty s^{\frac{d+\beta_a}{\beta}-1} e^{-s} ds\right] d\boldsymbol{\ell}, \qquad (19)\end{aligned}$$

where we use Fubini's theorem and put $s = f(\boldsymbol{\ell})r^\beta$.

The inner integral is just the incomplete Gamma function which may be approximated in the following way (see, e.g., Abramowitz and Stegun [1, 6.5.32]):

$$\int_y^\infty s^\beta e^{-s} ds \quad = \quad y^\beta e^{-y}\left[1 + \sum_{k=1}^{n-1} \beta \cdots (\beta + 1 - k) y^{-k} + R_n(y)\right], \qquad (20)$$

where $R_n(y) = O(y^{-n})$ as $y \to \infty$ for every fixed n and, moreover,

$$|R_n(y)| \leq |\beta \cdots (\beta + 1 - n)| y^{-n} \quad \text{for } n > \beta.$$

Notice that for $\beta \in \mathbb{N}$ the sum is finite, up to $\beta + 1$. Therefore,

$$\begin{aligned}\int_{f(\boldsymbol{\ell})\frac{x^{\beta/\alpha}}{g^{\beta/\alpha}(\boldsymbol{\ell})}}^\infty s^{\frac{d+\beta_a}{\beta}-1} e^{-s} ds \quad &= \quad f^{\frac{d+\beta_a}{\beta}-1}(\boldsymbol{\ell}) \frac{x^{\frac{d+\beta_a-\beta}{\alpha}}}{g^{\frac{d+\beta_a-\beta}{\alpha}}(\boldsymbol{\ell})} e^{-f(\boldsymbol{\ell})\frac{x^{\beta/\alpha}}{g^{\beta/\alpha}(\boldsymbol{\ell})}} \\ &\times \quad \left[1 + \sum_{k=1}^\infty \Big(\frac{d+\beta_a}{\beta} - 1\Big) \cdots \Big(\frac{d+\beta_a}{\beta} - k\Big) \Big(f(\boldsymbol{\ell}) \frac{x^{\beta/\alpha}}{g^{\beta/\alpha}(\boldsymbol{\ell})}\Big)^{-k}\right],\end{aligned}$$

this asymptotic expansion holds uniformly in $\boldsymbol{\ell} \in \mathbb{S}_{d-1,\beta}$ because both $f(\boldsymbol{\ell})$ and $g(\boldsymbol{\ell})$ are continuous on $\mathbb{S}_{d-1,\beta}$ which implies that $f(\boldsymbol{\ell})$ is bounded away from zero and $g(\boldsymbol{\ell})$ from infinity.

Inputting this into (19) we get

$$\begin{aligned}\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \quad &= \quad \frac{x^{\frac{d+\beta_a-\beta}{\alpha}}}{\beta} \int_{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta}:g(\boldsymbol{\ell})>0} a_0(\boldsymbol{\ell}) e^{-f_0(\boldsymbol{\ell})x^{\beta/\alpha}} d\boldsymbol{\ell} \\ &\times\left[1 + \sum_{k=1}^\infty \Big(\frac{d+\beta_a}{\beta} - 1\Big) \cdots \Big(\frac{d+\beta_a}{\beta} - k\Big) \Big(f(\boldsymbol{\ell}) \frac{x^{\beta/\alpha}}{g^{\beta/\alpha}(\boldsymbol{\ell})}\Big)^{-k}\right],\end{aligned}$$

where

$$a_0(\boldsymbol{\ell}) := \frac{a(\boldsymbol{\ell})}{f(\boldsymbol{\ell}) g^{\frac{d+\beta_a-\beta}{\alpha}}(\boldsymbol{\ell})} \quad \text{and} \quad f_0(\boldsymbol{\ell}) := \frac{f(\boldsymbol{\ell})}{g^{\beta/\alpha}(\boldsymbol{\ell})}. \qquad (21)$$

Therefore,

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = \frac{x^{\frac{d+\beta_a-\beta}{\alpha}}}{\beta}\Big[I_0(x) + \sum_{k=1}^{\infty}\Big(\frac{d+\beta_a}{\beta}-1\Big)\cdots\Big(\frac{d+\beta_a}{\beta}-k\Big)x^{-k\frac{\beta}{\alpha}}I_k(x)\Big], \tag{22}$$

where

$$I_k(x) := \int_{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta}:\ g(\boldsymbol{\ell})>0} a_k(\boldsymbol{\ell})e^{-f_0(\boldsymbol{\ell})x^{\beta/\alpha}}d\boldsymbol{\ell}$$

and

$$a_k(\boldsymbol{\ell}) := a_0(\boldsymbol{\ell})\Big(\frac{g^{\beta/\alpha}(\boldsymbol{\ell})}{f(\boldsymbol{\ell})}\Big)^k = \frac{a(\boldsymbol{\ell})}{f^{k+1}(\boldsymbol{\ell})g^{\frac{d+\beta_a-(k+1)\beta}{\alpha}}(\boldsymbol{\ell})}.$$

We see that our problem has been reduced to the problem of finding the asymptotic behaviour of the integral $I_k(x)$ for $k \geq 0$ as $x \to \infty$. In order to apply Laplace method, we need to introduce some parametrisation on the unit sphere $\mathbb{S}_{d-1,\beta}$ in L_β. We pass to the hyperspherical coordinates, $\boldsymbol{\ell}(\boldsymbol{\varphi})$, that is, for $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_d) \in \mathbb{S}_{d-1,\beta}$,

$$\begin{aligned}
&\ell_1 = \|\boldsymbol{\ell}\|\cos\varphi_1\\
&\ell_2 = \|\boldsymbol{\ell}\|\sin\varphi_1\cos\varphi_2\\
&\ldots\\
&\ell_{d-1} = \|\boldsymbol{\ell}\|\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{d-2}\cos\varphi_{d-1}\\
&\ell_d = \|\boldsymbol{\ell}\|\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{d-2}\sin\varphi_{d-1},
\end{aligned} \tag{23}$$

where $\boldsymbol{\varphi} = (\varphi_1, \ldots, \varphi_{d-1}) \in \Pi_{d-1} := [0,\pi)^{d-2}\times[0,2\pi)$ are the angular coordinates of $\boldsymbol{\ell} \in \mathbb{S}_{d-1,\beta}$; clearly, $\|\boldsymbol{\ell}\|$ depends on $\boldsymbol{\varphi}$. (One may also pass to the so-called generalised spherical coordinates, which are more adjusted for L_β, see [13].) Its Jacobian is equal to

$$\begin{aligned}
\det J(\boldsymbol{\varphi}) &:= \sin^{d-2}\varphi_1\cdots\sin\varphi_{d-2}\frac{\|\nabla(\ell_1^\beta+\cdots+\ell_d^\beta)\|\|\boldsymbol{\ell}\|}{(\nabla(\ell_1^\beta+\cdots+\ell_d^\beta),\boldsymbol{\ell})}\\
&= \frac{\sin^{d-2}\varphi_1\cdots\sin\varphi_{d-2}}{\sqrt{\ell_1^{2(\beta-1)}+\cdots+\ell_d^{2(\beta-1)}}\|\boldsymbol{\ell}\|}.
\end{aligned} \tag{24}$$

Changing in such a way variables, we have (we set $g(\boldsymbol{\varphi}) = g(\boldsymbol{\ell})$, $a_k(\boldsymbol{\varphi}) = a_k(\boldsymbol{\ell})$ and $f_0(\boldsymbol{\varphi}) = f_0(\boldsymbol{\ell})$)

$$I_k(x) = \int_{\boldsymbol{\varphi}\in\Pi_{d-1}:g(\boldsymbol{\varphi})>0} a_k(\boldsymbol{\varphi})e^{-f_0(\boldsymbol{\varphi})x^{\beta/\alpha}}|\det J(\boldsymbol{\varphi})|d\boldsymbol{\varphi}. \tag{25}$$

In the light of the Laplace methodology we are interested in the set of minimum points of $f_0(\boldsymbol{\ell})$, $\boldsymbol{\ell} \in \mathbb{S}_{d-1,\beta}$. Denote

$$\hat{f}_0 := \min_{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta}} f_0(\boldsymbol{\ell}) = \min_{\boldsymbol{\varphi}\in\Pi_{d-1}} f_0(\boldsymbol{\varphi})$$

and

$$\mathcal{M} := \{\boldsymbol{\ell} \in \mathbb{S}_{d-1,\beta} : f_0(\boldsymbol{\ell}) = \hat{f}_0\}, \qquad \mathcal{M}_\varphi := \{\boldsymbol{\varphi} \in \Pi_{d-1} : f_0(\boldsymbol{\varphi}) = \hat{f}_0\}.$$

We consider two different cases of the structure of the set $\mathcal{M}$:

(i) $\mathcal{M}$ consists of a finite number of isolated points.

(ii) $\mathcal{M}$ is a smooth m-dimensional manifold without boundary, $1 \le m \le d-2$, on the unit sphere $\mathbb{S}_{d-1,\beta}$ in L_β.

In fact, the first case is a particular case of the second one, the dimension of the manifold equals zero, nevertheless we consider it separately, because of our considerations in this case are elementary applications of the classical multivariate Laplace asymptotic method.

4.1 The case of finite $\mathcal{M}$

Here we consider a homogeneous continuous function $g : \mathbb{R}^d \to \mathbb{R}$ of order $\alpha > 0$ and a function f such that $\mathcal{M}$ consists of a finite number of points, say $\mathcal{M} = \{\boldsymbol{\ell}^{(1)}, \dots, \boldsymbol{\ell}^{(k)}\}$; equivalently, $\mathcal{M}_\varphi = \{\boldsymbol{\varphi}^{(1)}, \dots, \boldsymbol{\varphi}^{(k)}\}$.

Let $g(\boldsymbol{\varphi})$, $f(\boldsymbol{\varphi}) \in C^2(\Pi_{d-1})$. Assume that $\det f_0''(\boldsymbol{\varphi}^{(j)}) > 0$ for every $j = 1, \dots, k$, where

$$f_0''(\boldsymbol{\varphi}) := \left[\frac{\partial^2 f_0(\boldsymbol{\varphi})}{\partial\varphi_i\partial\varphi_l}\right]_{i,l=1,\dots,d-1}$$

is the Hessian matrix of $f_0(\boldsymbol{\varphi})$. Applying Theorem 1 to the integrals $I_k(x)$ and substituting the resulting asymptotics into (22), we deduce the following asymptotic expansion for the Weibullian chaos.

Theorem 4. *Let $a(\boldsymbol{\varphi}) \in C^{2r}(\Pi_{d-1})$ and $g(\boldsymbol{\varphi})$, $f(\boldsymbol{\varphi}) \in C^{2r+2}(\Pi_{d-1})$ for some $r \ge 0$. Then the following asymptotic expansion holds:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} = x^{\frac{2d+2\beta_a-(d+1)\beta}{2\alpha}} e^{-\hat{f}_0 x^{\beta/\alpha}} \left(h_0 + \sum_{i=1}^r h_i x^{-i\beta/\alpha} + o(x^{-r\beta/\alpha})\right)$$

as $x \to \infty$, where

$$h_0 := \frac{1}{\beta}(2\pi)^{\frac{d-1}{2}} \sum_{j=1}^k \frac{a_0(\boldsymbol{\varphi}^{(j)})|\det J(\boldsymbol{\varphi}^{(j)})|}{\sqrt{\det f_0''(\boldsymbol{\varphi}^{(j)})}}$$

and $h_1, \dots, h_r \in \mathbb{R}$.

4.2 The case of a manifold

Now consider the case where $\mathcal{M}_\varphi$ is a m-dimensional manifold without boundary, $1 \le m \le d-2$, of finite volume.

We assume that the rank of $f_0''(\boldsymbol{\varphi})$ is equal to $d-1-m$ for every $\boldsymbol{\varphi} \in \mathcal{M}_\varphi$. Denote by $\det f''_{0,d-1-m}(\boldsymbol{\varphi})$ any non-zero $(d-1-m)$-minor of the matrix $f_0''(\boldsymbol{\varphi})$; notice that all such minors are equal one to another. Denote

$$h_0 := \frac{1}{\beta}(2\pi)^{\frac{d-1-m}{2}} \int_{\mathcal{M}_\varphi} \frac{a_0(\boldsymbol{\varphi})|\det J(\boldsymbol{\varphi})|}{\sqrt{\det f''_{0,d-1-m}(\boldsymbol{\varphi})}} dV_\varphi,$$

where dV_φ is the volume element of $\mathcal{M}_\varphi \subset \Pi_{d-1}$. Applying now Theorem 3 to the integrals $I_k(x)$ we deduce the following result.

Theorem 5. *Let the manifold $\mathcal{M}_\varphi$ is C^{2r+2}-smooth for some $r \ge 0$. Assume also that $a(\boldsymbol{\varphi}) \in C^{2r}(\Pi_{d-1})$ and $f(\boldsymbol{\varphi})$, $g(\boldsymbol{\varphi}) \in C^{2r+2}(\Pi_{d-1})$. Then the following asymptotic expansion holds:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \quad = \quad x^{\frac{2d+2\beta_a-(d+1-m)\beta}{2\alpha}} e^{-\hat{f}_0 x^{\beta/\alpha}} \left(h_0 + \sum_{i=1}^{r} h_i x^{-i\beta/\alpha} + o(x^{-r\beta/\alpha}) \right)$$

as $x \to \infty$, where $h_1, \ldots, h_r \in \mathbb{R}$.

This result generalises asymptotics given in Barbe [3, Theorem 7.1], where it was additionally assumed that (i) $a(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta) \equiv \text{const}$, (ii) $\beta_a = 0$, (iii) the function $f(\boldsymbol{v}/\|\boldsymbol{v}\|_\beta)\|\boldsymbol{v}\|_\beta^\beta$ is strictly convex (in particular, $\beta > 1$).

5 Random chaos with independent coordinates

Now consider a special case of the Weibullian chaos where the coordinates of the random vector $\boldsymbol{\eta}$ in $\mathbb{R}^d$ are independent with the following marginal density function:

$$p_{\eta_k}(v) = c_1 e^{-c_2|v|^\beta}, \qquad k = 1, \ldots, d,$$

where $c_2 > 0$, $\beta > 0$ and c_1 is the normalising constant, so that

$$p_{\boldsymbol{\eta}}(\boldsymbol{v}) = c_1^d e^{-c_2(|v_1|^\beta + \cdots + |v_d|^\beta)} = c_2^d e^{-c_2\|\boldsymbol{v}\|_\beta^\beta}. \tag{26}$$

It is a particular case of (18) with $a(\boldsymbol{v}) \equiv c_1^d$, $\beta_a = 0$ and $f(\boldsymbol{v}) \equiv c_2$.

Let us apply Theorem 5 in order to understand the tail behaviour of $g(\boldsymbol{\eta})$ in this case. Then the functions defined in (21) are equal to

$$a_0(\boldsymbol{\ell}) = \frac{c_1^d}{c_2 g^{\frac{d-\beta}{\alpha}}(\boldsymbol{\ell})} \quad \text{and} \quad f_0(\boldsymbol{\ell}) = \frac{c_2}{g^{\beta/\alpha}(\boldsymbol{\ell})}.$$

We have

$$\hat{f}_0 = \min_{\boldsymbol{\varphi}\in\Pi_{d-1}} f_0(\boldsymbol{\varphi}) = \frac{c_2}{\hat{g}^{\beta/\alpha}}$$

where

$$\hat{g} := \max_{\boldsymbol{\varphi}\in\Pi_{d-1}} g(\boldsymbol{\varphi}) = \max_{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta}} g(\boldsymbol{\ell}).$$

Then

$$\mathcal{M} := \{\boldsymbol{\ell}\in\mathbb{S}_{d-1,\beta} : g(\boldsymbol{\ell}) = \hat{g}\}, \qquad \mathcal{M}_\varphi := \{\boldsymbol{\varphi}\in\Pi_{d-1} : g(\boldsymbol{\varphi}) = \hat{g}\}.$$

The Hessian of f_0 at the point of its minimum is equal to

$$f_0''(\boldsymbol{\varphi}) = -\frac{c_2\beta}{\alpha\hat{g}^{\frac{\beta+\alpha}{\alpha}}} g''(\boldsymbol{\varphi}),$$

so that, for a non-zero $(d-1-m)$-minor of the matrix $f_0''(\boldsymbol{\varphi})$, we have

$$\det f_{0,d-1-m}''(\boldsymbol{\varphi}) = \left(\frac{c_2\beta}{\alpha\hat{g}^{\frac{\beta+\alpha}{\alpha}}}\right)^{d-1-m} |\det g_{d-1-m}''(\boldsymbol{\varphi})|.$$

Therefore, in the case of Weibullian radius (26), Theorem 5 reads as follows.

Theorem 6. *Let the manifold $\mathcal{M}_\varphi$ is C^{2r+2}-smooth for some $r\geq 0$. Assume also that $g(\boldsymbol{\varphi})\in C^{2r+2}(\Pi_{d-1})$. Then the following asymptotic expansion holds:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \;=\; (x/\hat{g})^{\frac{2d-(d+1-m)\beta}{2\alpha}} e^{-c_2(x/\hat{g})^{\beta/\alpha}} \left(h_0 + \sum_{i=1}^{r} h_i x^{-i\beta/\alpha} + o(x^{-r\beta/\alpha})\right)$$

as $x\to\infty$, where

$$h_0 := \left(\frac{2\pi}{c_2\beta}\right)^{\frac{d-1-m}{2}} \frac{c_1^d}{\beta c_2} (\alpha\hat{g})^{\frac{d-1-m}{2}} \int_{\mathcal{M}_\varphi} \frac{|\det J(\boldsymbol{\varphi})|}{\sqrt{|\det g_{d-1-m}''(\boldsymbol{\varphi})|}} dV_\varphi,$$

and $h_1, \ldots, h_r\in\mathbb{R}$.

6 Gaussian random chaos

Here we deduce a corollary of Theorem 6 for the case of the Gaussian chaos which was defined at the beginning of Section 4. So, now $\boldsymbol{\eta}$ is a random vector in $\mathbb{R}^d$ with the standard normal distribution which means that, in the representation (18), $a(\boldsymbol{v})\equiv c_1^d = (2\pi)^{-d/2}$, $\beta_a = 0$ and $f(\boldsymbol{v})\equiv c_2 = 1/2$, $\beta = 2$. Then we have the following corollary of Theorem 6 for the case of Gaussian chaos.

Corollary 7. *Let the manifold* $\mathcal{M}_\varphi$ *is* C^{2r+2}*-smooth for some* $r \geq 0$*. Assume also that* $g(\boldsymbol{\varphi}) \in C^{2r+2}(\Pi_{d-1})$*. Then the following asymptotic expansion holds:*

$$\mathbb{P}\{g(\boldsymbol{\eta}) > x\} \;=\; (x/\hat{g})^{\frac{m-1}{\alpha}} e^{-(x/\hat{g})^{2/\alpha}/2}\left(h_0 + \sum_{i=1}^{r} h_i x^{-2i/\alpha} + o(x^{-2r/\alpha})\right) \quad (27)$$

as $x \to \infty$*, where*

$$h_0 := \frac{1}{(2\pi)^{\frac{1+m}{2}}}(\alpha\hat{g})^{\frac{d-1-m}{2}} \int_{\mathcal{M}_\varphi} \frac{|\det J(\boldsymbol{\varphi})|}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}} dV_\varphi,$$

and $h_1, \ldots, h_r \in \mathbb{R}$.

Corollary (27) was first proved in [11] by a direct probabilistic method. For references to the corresponding literature on the topic of various Gaussian models the interested reader is referred to the aforementioned reference.

Acknowledgment

E. Hashorva and D. Korshunov kindly acknowledge partial support from the Swiss National Science Foundation Grants 200021-140633/1 and 200021-134785. The research of V.I. Piterbarg is supported by the Russian Foundation for Basic Research, Project 11-01-00050-a. The authors were partially supported also by the project RARE-318984 (an FP7 Marie Curie IRSES Fellowship).

References

[1] Abramowitz, M., Stegun, I. A., eds. (1972) *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. New York: Dover Publications.

[2] Arnold, V. I. Varchenko, A. N., and Gusein-Zade, S. M. (1987) *Singularities of Differentiable Maps*. Vol. 2: Monodromy and asymptotic integrals (Monographs in Mathematics), Birkhäuser.

[3] Barbe, Ph. (2003) *Approximation of Integrals over Asymptotic Sets with Applications to Probability and Statistics*. arXiv:math/0312132.

[4] Bender, C. M., and Orszag, S. A. (1999) *Advanced Mathematical Methods for Scientists and Engineers. I. Asymptotic Methods and Perturbation Theory.* Reprint of the 1978 original. Springer-Verlag, New York.

[5] Bleistein, N., and Handelsman, R. A. (1975) *Asymptotic Expansions of Integrals*. Holt, Rinehart and Winston, New York.

[6] Breitung, K. W. (1994) *Asymptotic Approximations for Probability Integrals.* Lecture Notes in Mathematics, 1592, Springer-Verlag, Berlin.

[7] Combet, E. (1982) *Intégrales Exponentielles. Développements Asymptotiques, Propriétés Lagrangiennes*. Lecture Notes in Mathematics, 937. Springer-Verlag, Berlin-New York.

[8] Fedoryuk, M. V. (1977) *Saddle Point Method*. Moscow, Nauka (in Russian).

[9] Fedoryuk, M. V. (1987) *Asymptotics: Integrals and Series*. Nauka, Moscow (in Russian).

[10] Fulks, W., and Sather, J. O. (1961) Asymptotics II: Laplace's method for multiple integrals. *Pacific J. Math.* **11**, 185-192.

[11] Hashorva, E., Korshunov, D., and Piterbarg, V. I. (2013) Extremal behaviour of Gaussian chaos. *Submitted* , –.

[12] López, J. L., Pagola, P., and Sinusía, E. P. (2009) A simplification of Laplaces method: Applications to the Gamma function and Gauss hypergeometric function. *J. Approx. Theory* **161**, 280-291.

[13] Richter, W. D. (2007) Generalized spherical and simplicial coordinates. *J. Math. Anal. Appl.* **336**, 1187-1202.

[14] Trofimov, O. E. abd Friezen, D. G. (1981) Coefficients of asymptotic decompositions of integrals by Laplace method. *Avtometria* **2**, 94 (in Russian).

[15] Wojdylo, J. (2006) Computing the coefficients in Laplace's method. *SIAM Rev.* **48**, 76–96.

[16] Wiener, N. (1938) The homogeneous chaos. *Amer. J. Math.* **60**, 897–936.

[17] Wong, R. (1989) *Asymptotic Approximations of Integrals.* Computer Science and Scientific Computing. Academic Press, Inc., Boston, MA.