Maxima of Skew Elliptical Triangular Arrays

Enkelejd Hashorva and Chengxiu Ling

University of Lausanne, 1015 Lausanne, Switzerland

December 8, 2013

Abstract: In this paper we investigate the asymptotic behaviour of the componentwise maxima for two bivariate skew elliptical triangular arrays with components given in terms of skew transformations of bivariate spherical random vectors. We find the weak limit of the normalized maxima for both cases that the random radius pertaining to the elliptical random vectors is either in the Gumbel or in the Weibull max-domain of attractions.

Key words and phrases: maxima of triangular arrays; skew elliptical distributions; Gumbel max-domain attraction; Weibull max-domain attraction; Hüsler-Reiss distribution; max-infinitely divisible distribution.

1 Introduction

In the seminal paper [22], Hüsler and Reiss showed that the maxima of dependent Gaussian triangular arrays converge in distribution (after normalization) to a random vector with max-stable distribution function (df) referred to as Hüsler-Reiss df. Specifically, if $(X_{jn}, Z_{jn}), j \le n, n \ge 1$ is a triangular array of independent bivariate Gaussian random vectors such that $(X_{jn}, Z_{jn}) \stackrel{d}{=} (X, \rho_n X + \sqrt{1-\rho_n^2} Y)$ for $\rho_n \in (-1,1)$ where X, Y are independent $N(0,1)$ random variables with df Φ and $\stackrel{d}{=}$ stands for equality of dfs, then the convergence in distribution

$$\left(\frac{\max_{1\le j\le n} X_{jn} - b_n}{a_n}, \frac{\max_{1\le j\le n} Z_{jn} - b_n}{a_n} \right) \stackrel{d}{\to} (\mathcal{M}_1, \mathcal{M}_2), \quad n \to \infty \tag{1.1}$$

holds with $a_n = 1/\sqrt{2\ln n}, b_n = \Phi^{-1}(1-1/n)$ and $(\mathcal{M}_1, \mathcal{M}_2)$ having the Hüsler-Reiss df H_λ given by

$$H_\lambda(x,y) = \exp\left(-\Phi\left(\lambda + \frac{x-y}{2\lambda}\right) e^{-y} - \Phi\left(\lambda + \frac{y-x}{2\lambda}\right) e^{-x} \right), \quad \lambda \in [0,\infty),$$

provided that the Hüsler-Reiss condition

$$\lim_{n\to\infty} (1-\rho_n) \ln n \quad = \quad \lambda^2 \tag{1.2}$$

holds. The marginal dfs of H_λ are the unit Gumbel df $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$.

A popular extension of the normal distribution is the skew normal one, see e.g., [1,25]; a natural question is whether skewing has an effect on the asymptotic behaviour of the normalized maxima. Specifically, define a triangular array

$(X_{jn}, Z_{jn}), j \le n, n \ge 1$ of bivariate skew normal random vectors as

$$(X_{jn}, Z_{jn}) \stackrel{d}{=} (|X|, \rho_n |X| + \sqrt{1-\rho_n^2} Y), \quad \rho_n \in (-1,1), \quad j \le n. \tag{1.3}$$

If the Hüsler-Reiss condition (1.2) holds, then by conditioning it follows that (1.1) is still valid where instead of $b_n = \Phi^{-1}(1-1/n)$, we put $b_n = \Phi^{-1}(1-1/n) + a_n \ln 2$; in view of the findings of [4,19] the above result is expected. However, for the half-skew model, surprisingly the limiting df is not the Hüsler-Reiss df any more. Indeed, if we suppose that for all $j \le n$

$$(X_{jn}, Z_{jn}) \stackrel{d}{=} (X, \rho_n |X| + \sqrt{1-\rho_n^2} Y), \quad \rho_n \in (-1,1), \tag{1.4}$$

then again under the Hüsler-Reiss condition (1.2) we obtain the joint convergence in distribution

$$\left(\frac{\max_{1 \le j \le n} X_{jn} - b_n}{a_n}, \frac{\max_{1 \le j \le n} Z_{jn} - (b_n + a_n \ln 2)}{a_n} \right) \stackrel{d}{\to} (\widetilde{\mathcal{M}_1}, \widetilde{\mathcal{M}_2}), \quad n \to \infty, \tag{1.5}$$

where $(\widetilde{\mathcal{M}_1}, \widetilde{\mathcal{M}_2})$ has df $\widetilde{H_\lambda}$ given by

$$\widetilde{H_\lambda}(x,y) = H_\lambda(x, y + \ln 2) \Lambda(y + \ln 2), \quad x, y \in \mathbb{R}. \tag{1.6}$$

As shown for instance in [18] the Hüsler-Reiss df appears due to the fact that Gaussian random vectors belong to the larger class of elliptically symmetrical (elliptical for shorthand) random vectors with pertaining random radius which has df in the Gumbel max-domain of attraction (MDA), see definition below.
In this paper we are concerned with the behaviour of maxima of the larger classes of skew and half-skew elliptically symmetric distributions. For simplicity, we shall deal with the bivariate setup assuming that (X,Y) has stochastic representation

$$(X, Y) \stackrel{d}{=} R(\cos \Theta, \sin \Theta), \tag{1.7}$$

where the random angle Θ is independent of $R > 0$ and follows the uniform distribution on $(-\pi, \pi)$ (abbreviate this by $\Theta \sim U(-\pi, \pi)$). In the special case that R^2 is chi-square distributed with 2 degrees of freedom then X and Y are independent with df Φ.

Our investigation will be concerned with two different cases: a) we consider the case that the random radius R has df in the Gumbel MDA which includes the Gaussian one mentioned above, and b) we shall assume that R has df in the Weibull MDA, see definition below. The motivation for the latter assumption comes from [14, 18]. Interestingly, as shown in the aforementioned papers the case of the Weibull MDA leads to a limiting distribution which is different from the Hüsler-Reiss distribution, see also [18]. This will be confirmed in this paper also for the skew elliptical and half-skew elliptical models.

Our main findings are presented in Theorem 2.1 and Theorem 2.3 below. Therein we show that maxima of triangular arrays of half-skew elliptically symmetric distributions is different as those of elliptically symmetric and skew elliptically symmetric distributions. Since for the skew elliptical case the limiting distributions are new our findings are of some theoretical and applied interest; in a forthcoming paper we shall discuss aspects of statistical modelling using skew and half-skew elliptically symmetric distributions.

The organisation of the paper is as follows: In Section 2 we present our main results. Two illustrating examples are given in Section 3. The proofs of the results are relegated to Section 4.

2 Main Results

In this section we shall investigate the asymptotic behavior of maxima of triangular arrays defined via (1.4) with (X, Y) specified by (1.7). The main assumption imposed on the random radius R is that it has df F in the Gumbel or in the Weibull MDA. There is no loss of generality if we fix the right endpoint x_F of F. Therefore, hereafter we shall assume that $x_F \in \{1, \infty\}$.

2.1 Gumbel MDA

Next we deal with the case that F is in the Gumbel MDA with some positive scaling function w, i.e., for any $s \in \mathbb{R}$

$$\lim_{u \uparrow x_F} \frac{\bar{F}(u + s/w(u))}{\bar{F}(u)} = \exp(-s), \quad \bar{F} := 1 - F. \tag{2.1}$$

The scaling function w can be defined asymptotically by (cf. [5, 9])

$$w(t) = (1 + o(1)) \frac{\bar{F}(t)}{\int_t^{x_F} \bar{F}(s)\, ds}, \quad t \to x_F.$$

For $X = R\cos\Theta$ as defined in (1.7) the assumption (2.1) implies that the df G of X is also in the Gumbel MDA with the same scaling function w as for F. In fact also the converse result holds, see Theorem 4.1 in [20]. Furthermore, from the aforementioned theorem or Theorem 12.3.1 in [2] (see also Theorem 3 in [15], Proposition A.3 in [16] and Theorem 1 in [17] for more general results) condition (2.1) implies with $v(u) = \sqrt{uw(u)}$

$$\int_0^1 \bar{F}\left(\frac{u}{\sqrt{1-y}}\right) dB(y; a, b) = \frac{2^a \Gamma(a+b)}{\Gamma(b)} (v(u))^{-2a} \bar{F}(u)(1 + o(1)), \quad u \uparrow x_F,$$

where $B(y; a, b)$ is the beta df with positive parameters a, b which has density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}, \quad x \in (0, 1).$$

Here $\Gamma(\cdot)$ denotes the Euler Gamma function. Since $(\cos\Theta)^2$ has beta df $B(y;1/2,1/2)$ we have thus

$$2\mathbb{P}\{X>u\}=\mathbb{P}\{R|\cos\Theta|>u\}=\sqrt{\frac{2}{\pi}}\frac{\bar{F}(u)}{\nu(u)}(1+o(1)),\quad u\uparrow x_F.$$

This fact can be also formulated in the classical framework of extreme value theory as

$$\lim_{n\to\infty}\sup_{x\in\mathbb{R}}\Big|G^n(a_nx+b_n)-\Lambda(x)\Big|=0,$$

with

$$b_n=G^{\leftarrow}(1-1/n)=\inf\{x:G(x)>1-1/n\},\quad a_n=\frac{1}{w(b_n)}. \tag{2.2}$$

Next we state our first result.

Theorem 2.1 *Let (X,Y) be given by (1.7) with F the df of the random radius R which satisfies (2.1) with some positive scaling function w. Let $\rho_n\in(0,1]$ and let a_n,b_n be given by (2.2). If*

$$\lim_{n\to\infty}\frac{(1-\rho_n)b_n}{a_n}=2\lambda^2\in[0,\infty) \tag{2.3}$$

and $(X_{jn},Z_{jn}),j\le n,n\ge 1$ is given by (1.4), then

$$\frac{\max_{1\le j\le n}Z_{jn}-(b_n+a_n\ln 2)}{a_n}\overset{d}{\to}\mathcal{M},\quad n\to\infty, \tag{2.4}$$

where $\mathcal{M}$ has unit Gumbel df Λ. Furthermore, the joint convergence in (1.5) holds.

Remark 2.2 *a) In the case that $\rho_n=\rho\in(0,1)$ for all large n, then it follows that Theorem 2.1 holds with $\lambda=\infty$ and $\widetilde{H_\infty}(x,y)=\Lambda(x)\Lambda(y)$.*
b) Let $(\widetilde{M_{n1}},\widetilde{M_{n2}})=(\max_{1\le j\le n}Z^{(1)}_{jn},\max_{1\le j\le n}Z^{(2)}_{jn})$ with

$$(Z^{(1)}_{jn},Z^{(2)}_{jn})\overset{d}{=}(\rho_{1n}|X|+\sqrt{1-\rho_{1n}^2}Y,\rho_{2n}|X|+\sqrt{1-\rho_{2n}^2}Y). \tag{2.5}$$

Suppose that ρ_{in} satisfies condition (2.3) with $\lambda_i\in[0,\infty),i=1,2$. Using Lemma 4.1 and following the arguments of the proof of Theorem 2.1 we obtain

$$\left(\frac{\widetilde{M_{n1}}-(b_n+a_n\ln 2)}{a_n},\frac{\widetilde{M_{n2}}-(b_n+a_n\ln 2)}{a_n}\right)\overset{d}{\to}(\mathcal{M}_1,\mathcal{M}_2),\quad n\to\infty,$$

where $(\mathcal{M}_1,\mathcal{M}_2)$ has Hüsler-Reiss df H_λ with $\lambda=|\lambda_1-\lambda_2|$, see (1.2).
c) The bivariate Hüsler-Reiss df appeared initially in [3]; see also the recent articles [6, 24, 27]. Related results for more general triangular arrays can be found in [7, 10, 11, 21, 23, 26]; see also [8] for novel statistical applications.

2.2 Weibull case

The main assumption on F in this section is that it belongs to the Weibull MDA, hence x_F is necessarily finite. We assume again that $x_F = 1$. Specifically, we shall suppose that F is in the Weibull MDA with index $\alpha > 0$, i.e., (cf. [5])

$$\lim_{t \downarrow 0} \frac{\bar{F}(1-tx)}{\bar{F}(1-t)} = x^{\alpha}, \quad x > 0. \tag{2.6}$$

In view of Theorem 4.5 in [20] the df G of X is in the Weibull MDA, and the converse result is also valid. For notational simplicity define below a positive constant $\mathcal{I}_\alpha$ and a df $\Upsilon_\alpha(\cdot)$ by

$$\mathcal{I}_\alpha = \frac{\sqrt{2}}{\pi} \int_0^1 (1-s^2)^\alpha \, ds = \frac{1}{\sqrt{2\pi}} \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+3/2)}, \quad \Upsilon_\alpha(t) = \frac{\int_{-1}^t (1-s^2)^\alpha \, ds}{\int_{-1}^1 (1-s^2)^\alpha \, ds}, \quad t \in [-1,1]. \tag{2.7}$$

Theorem 2.3 *Let (X,Y) be given by (1.7) with F the df of the random radius R and $\rho_n \in (0,1]$. Suppose that F is in the Weibull MDA satisfying (2.6) for some $\alpha > 0$.*

a) If for some $u_n \downarrow 0$

$$\lim_{n\to\infty} \frac{1-\rho_n}{u_n} = 2\lambda^2 \in [0,\infty), \tag{2.8}$$

then

$$\lim_{n\to\infty} \frac{\mathbb{P}\left\{\rho_n |X| + \sqrt{1-\rho_n^2} Y > 1 + u_n x\right\}}{u_n^{1/2} \bar{F}(1-u_n)} = 2\mathcal{I}_\alpha |x|^{1/2+\alpha}, \quad x < 0. \tag{2.9}$$

b) Let $(X_{jn}, Z_{jn}), j \le n, n \ge 1$ given by (1.4). If (2.8) holds for u_n such that $u_n^{1/2}\bar{F}(1-u_n) = (1+o(1))/n$, then

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left\{\frac{\max_{1\le j\le n} X_{jn} - 1}{u_n} \le x, \frac{\max_{1\le j\le n} Z_{jn} - 1}{u_n} \le y\right\} \\
&= \exp\left(-\mathcal{I}_\alpha \left(|x|^{1/2+\alpha}\Upsilon_\alpha\left(\frac{\lambda + (y-x)/(2\lambda)}{\sqrt{2|x|}}\right) + |y|^{1/2+\alpha}\left(1 + \Upsilon_\alpha\left(\frac{\lambda+(x-y)/(2\lambda)}{\sqrt{2|y|}}\right)\right)\right)\right)
\end{aligned} \tag{2.10}$$

holds for all $x, y < 0$.

Remark 2.4 *a) The right-hand side of (2.10) is a bivariate df with dependent Weibull marginal dfs.*

b) If $\rho_n = \rho \in (0,1)$ for all large n, then $\lambda = \infty$ and the triangular array $(X_{jn}, Z_{jn}), j \le n, n \ge 1$ given by (1.4) is asymptotically tail independent.

c) Let $(\widetilde{M_{n1}}, \widetilde{M_{n2}})$ be given as in Remark 2.2 b). Suppose that ρ_{in} satisfies condition (2.8) with $\lambda_i \in [0,\infty), i = 1,2$ and u_n such that $u_n^{1/2}\bar{F}(1-u_n) = (1+o(1))/n$. Using Lemma 4.1 and following the arguments of the proof of Theorem 2.3 we obtain that for all $x, y < 0$

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{P}\left\{\frac{\widetilde{M_{n1}} - 1}{u_n} \le x, \frac{\widetilde{M_{n2}} - 1}{u_n} \le y\right\} \\
&= \exp\left(-2\mathcal{I}_\alpha\left(|x|^{1/2+\alpha}\Upsilon_\alpha\left(\frac{\lambda+(y-x)/(2\lambda)}{\sqrt{2|x|}}\right) + |y|^{1/2+\alpha}\Upsilon_\alpha\left(\frac{\lambda+(x-y)/(2\lambda)}{\sqrt{2|y|}}\right)\right)\right)
\end{aligned}$$

holds with $\lambda = |\lambda_1 - \lambda_2|$. Note that the limit joint df above is the same as that without skew transformation, see (2.8) therein in Theorem 2.1 in [14].

d) For both cases that F is in the Gumbel or in the Weibull MDA utilising the arguments in [14] the more general cases that $(\sin\Theta)^2$ has regularly varying tail at $\pi/2$ can be dealt with following the ideas of our proofs here; we omit those results.

3 Examples

In this section, we give two illustrating examples.

Example 3.1 *(Kotz Type I)* Consider a triangular array as in Theorem 2.1 with almost surely positive random radius R and df F such that as $x \to \infty$

$$\bar{F}(x) = (1 + o(1))Kx^{\varsigma} \exp\left(-cx^{\tau}\right), \quad \varsigma \in \mathbb{R}, K, c, \tau > 0.$$

It follows that F is in the Gumbel MDA with auxiliary function $w(x) = c\tau x^{\tau-1}$ and further by (2.2)

$$\bar{G}(x) = (1 + o(1))\frac{K}{\sqrt{2\pi c\tau}}x^{\varsigma-\tau/2} \exp\left(-cx^{\tau}\right) =: (1 + o(1))\widetilde{K}x^{\widetilde{\varsigma}} \exp\left(-cx^{\tau}\right), \quad x \to \infty,$$

which implies that (see e.g., [5])

$$b_n = G^{\leftarrow}(1 - 1/n) = (1 + o(1))(c^{-1}\ln n)^{1/\tau}, \quad n \to \infty$$

and thus $b_n/a_n = (1 + o(1))c\tau(b_n)^{\tau} = (1 + o(1))\tau \ln n$. Therefore, condition (2.3) can be written as

$$\lim_{n\to\infty} (1 - \rho_n) \ln n = \frac{2\lambda^2}{\tau}.$$

Note in passing that for the Gaussian case, which corresponds to $\tau = 2$, the above asymptotic condition reduces to the Hüsler-Reiss condition (1.2).

Example 3.2 *(Weibull case)* Consider a triangular array as in Theorem 2.3 with almost surely positive random radius R being beta distributed with parameters $a, b > 0$, thus

$$\bar{F}(1 - x) = \frac{\Gamma(a + b)}{b\Gamma(a)\Gamma(b)}x^b \left(1 - \frac{b(a - 1)x}{(b + 1)}(1 + o(1))\right), \quad x \downarrow 0.$$

Consequently, condition (2.6) holds with $\alpha = b$ and thus (2.8) is satisfied if further

$$\lim_{n\to\infty} (1 - \rho_n) \left(\frac{b\Gamma(a)\Gamma(b)}{n\Gamma(a + b)}\right)^{-1/(1/2+b)} = 2\lambda^2$$

is valid.

4 Proofs

We present first a lemma. Its proof is deferred to Appendix.

For any $x \neq 0, y \in \mathbb{R}$ and $\rho \in (0,1]$ denote

$$\beta = \beta(x,y,\rho) = \arctan\left(\frac{y/x-\rho}{\sqrt{1-\rho^2}}\right), \quad \psi = \arccos\rho \in [0,\pi/2). \tag{4.1}$$

In the following $c_n \sim d_n$ means $\lim_{n\to\infty} c_n/d_n = 1$ for $c_n, d_n, n \geq 1$ given positive constants, and instead of $B(y;a,b)$ we shall write $B(y)$ simply for the beta distribution with parameters $1/2, 1/2$.

Lemma 4.1 *Let $(X,Y) \stackrel{d}{=} R(\cos\Theta, \sin\Theta)$ with independent random variables $R = \sqrt{X^2+Y^2} > 0$ almost surely and $\Theta \in (-\pi,\pi)$. Then for $x,y>0$ and ψ,β given by (4.1), we have*

$$\mathbb{P}\left\{X > x, \rho|X| + \sqrt{1-\rho^2}Y > y\right\} = \mathbb{P}\left\{R > \frac{x}{\cos\Theta}, \Theta \in (\beta, \frac{\pi}{2})\right\} + \mathbb{P}\left\{R > \frac{y}{\cos(\Theta-\psi)}, \Theta \in (-\frac{\pi}{2}+\psi, \beta)\right\}$$

and

$$\begin{aligned}\mathbb{P}\left\{\rho|X| + \sqrt{1-\rho^2}Y > x\right\} &= \mathbb{P}\left\{R > \frac{x}{\cos(\Theta-\psi)}, \Theta-\psi \in (-\frac{\pi}{2}, \frac{\pi}{2}-\psi)\right\} \\ &\quad + \mathbb{P}\left\{R > -\frac{x}{\cos(\Theta+\psi)}, \Theta \in (\frac{\pi}{2},\pi) \cup (-\pi, -\frac{\pi}{2}-\psi)\right\}.\end{aligned}$$

Proof of Theorem 2.1: By Lemma 4.1 putting $\psi_n = \arccos\rho_n$ and $u_n(y) = a_n y + b_n$ we obtain

$$\begin{aligned}&n\mathbb{P}\left\{\rho_n|X| + \sqrt{1-\rho_n^2}Y > u_n(y)\right\} \\ &= \frac{n}{2\pi}\left(\int_{-\pi/2}^{0} \mathbb{P}\{R\cos\theta > u_n(y)\}\, d\theta + \int_0^{\pi/2-\psi_n} \mathbb{P}\{R\cos\theta > u_n(y)\}\, d\theta \right. \\ &\quad \left. + \int_{\psi_n+\pi/2}^{\pi} \mathbb{P}\{R|\cos\theta| > u_n(y)\}\, d\theta + \int_{\psi_n-\pi}^{-\pi/2} \mathbb{P}\{R|\cos\theta| > u_n(y)\}\, d\theta + \int_{\pi}^{\pi+\psi_n} \mathbb{P}\{R|\cos\theta| > u_n(y)\}\, d\theta\right) \\ &= \frac{2n}{2\pi}\int_0^{\pi/2} \mathbb{P}\{R\cos\theta > u_n(y)\}\, d\theta + \frac{2n}{2\pi}\int_0^{\pi/2-\psi_n} \mathbb{P}\{R\cos\theta > u_n(y)\}\, d\theta =: A_n + B_n.\end{aligned}$$

It follows from (2.2) that

$$A_n = \frac{n}{2}\int_0^1 \mathbb{P}\left\{R > \frac{u_n(y)}{\sqrt{1-s}}\right\} dB(s) = n\bar{G}(u_n(y)) \to e^{-y}, \quad n \to \infty.$$

Further, since $\rho_n \leq 1$ and $\rho_n > \epsilon_0$ for sufficiently large n and any given $\epsilon_0 \in (0,1)$,

$$\begin{aligned}\int_0^1 \mathbb{P}\left\{R > \frac{u_n(y)}{\sqrt{1-s}}\right\} dB(s) &\geq \int_0^{\rho_n^2} \mathbb{P}\left\{R > \frac{u_n(y)}{\sqrt{1-s}}\right\} dB(s) = \frac{2B_n}{n} \\ &\geq \int_0^{\epsilon_0^2} \mathbb{P}\left\{R > \frac{u_n(y)}{\sqrt{1-s}}\right\} dB(s) \sim \int_0^1 \mathbb{P}\left\{R > \frac{u_n(y)}{\sqrt{1-s}}\right\} dB(s), \quad n \to \infty,\end{aligned}$$

where the last step above follows by Proposition 12.2.1 in [2]. Consequently,

$$\lim_{n\to\infty} n\mathbb{P}\left\{\rho_n|X| + \sqrt{1-\rho_n^2}Y > u_n(y)\right\} = 2e^{-y}.$$

and thus (2.4) follows.

Next we assume first that $x_F = \infty$, which implies that both $u_n(x)$ and $u_n(y)$ tend to infinity as $n \to \infty$. By Lemma 4.1

$$\begin{aligned} n\mathbb{P}\left\{X > u_n(x), \rho_n|X| + \sqrt{1-\rho_n^2}Y > u_n(y)\right\} &= \frac{n}{2\pi}\int_{\beta_n}^{\pi/2} \bar{F}\left(\frac{u_n(x)}{\cos\theta}\right) d\theta + \frac{n}{2\pi}\int_{-\pi/2}^{\beta_n-\psi_n} \bar{F}\left(\frac{u_n(y)}{\cos\theta}\right) d\theta \\ &=: I_n + J_n, \end{aligned}$$

where

$$\beta_n = \arctan\left(\frac{u_n(y)/u_n(x) - \rho_n}{\sqrt{1-\rho_n^2}}\right), \quad \psi_n = \arccos\rho_n \in [0, \pi/2),$$

with

$$\tan(\beta_n) = \frac{u_n(y)/u_n(x) - \rho_n}{\sqrt{1-\rho_n^2}} = \left(\frac{y - x + (1-\rho_n)x}{(1-\rho_n)b_n w(b_n)} + 1\right)\sqrt{\frac{1-\rho_n}{1+\rho_n}}\left(1 + \frac{x}{b_n w(b_n)}\right)^{-1}.$$

The asymptotic behaviours of integrals similar to I_n and J_n are derived in several contributions, see e.g., [2,12–16]. The idea is to transform both the integrand and the distribution function (in our case $B(y)$) so that the integrand converges locally uniformly and so does the sequence of distribution functions. Below we shall treat separately two cases (i) $\lambda \in (0, \infty)$ and (ii) $\lambda = 0$. Denote $\nu_n = \nu(b_n) = \sqrt{b_n w(b_n)}$.

(i) If $\lambda_{x,y} := \lambda + (y-x)/(2\lambda) > 0$, then $\beta_n > 0$ holds for sufficiently large n. Thus, for any given $\epsilon \in (0,1)$ and all n large

$$\begin{aligned} I_n &= \frac{n}{4}\int_{(\sin\beta_n)^2}^{1} \bar{F}\left(\frac{u_n(x)}{\sqrt{1-s}}\right) dB(s) \\ &\sim \frac{n}{4\nu_n}\int_{(\sin\beta_n)^2\nu_n^2}^{\epsilon^2\nu_n^2} \bar{F}\left(\frac{u_n(x)}{\sqrt{1-s/\nu_n^2}}\right) d\nu_n B(s/\nu_n^2) \\ &\sim (1 - \Phi(\lambda_{x,y}))\, e^{-x} \end{aligned} \tag{4.2}$$

since $w(u_n(x)) \sim w(b_n), b_n w(b_n) \to \infty$ as $n \to \infty$ and

$$(\sin\beta_n)^2\nu_n^2 \sim \beta_n^2\nu_n^2 \sim \left(\frac{y - x + (1-\rho_n)x}{(1-\rho_n)b_n w(b_n)} + 1\right)^2 \frac{1-\rho_n}{1+\rho_n} b_n w(b_n) \sim (\lambda_{x,y})^2.$$

Similarly, (4.2) holds for $\lambda_{x,y} \le 0$. Further, $\psi_n \sim \sqrt{2(1-\rho_n)} \to 0$ as $n \to \infty$ and

$$\begin{aligned} (\sin(\beta_n - \psi_n))^2\nu_n^2 &\sim (\beta_n - \psi_n)^2 b_n w(b_n) \sim b_n w(b_n)\left(\frac{u_n(y)/u_n(x) - \rho_n}{\sqrt{1-\rho_n^2}} - \sqrt{2(1-\rho_n)}\right)^2 \\ &\sim (\lambda_{y,x})^2. \end{aligned}$$

Hence, it follows that for both two cases $\lambda_{x,y} > 2\lambda$ and $\lambda_{x,y} \le 2\lambda$ we have

$$\lim_{n\to\infty} J_n = (1 - \Phi(\lambda_{y,x}))\, e^{-y}, \tag{4.3}$$

which implies that for $(M_{n1}, M_{n2}) := (\max_{1\le j\le n} X_{jn}, \max_{1\le j\le n} Z_{jn})$ and $(X_{jn}, Z_{jn}), j \le n$ given as in (1.4)

$$-\lim_{n\to\infty} n \ln \mathbb{P}\left\{M_{n1} \le u_n(x), M_{n2} \le u_n(y)\right\} = e^{-y} + \Phi\left(\lambda_{x,y}\right) e^{-x} + \Phi\left(\lambda_{y,x}\right) e^{-y}.$$

(ii) For $\lambda = 0$, note that as $n \to \infty$

$$\begin{aligned}
(\sin\beta_n)\nu_n &= \frac{(u_n(y) - \rho_n u_n(x))\sqrt{b_n w(b_n)}}{\sqrt{(1-\rho_n^2)(u_n(x))^2 + (u_n(y) - \rho_n u_n(x))^2}} \\
&\sim \frac{(y - \rho_n x)\sqrt{b_n w(b_n)}}{\sqrt{2(1-\rho_n)(b_n w(b_n) + x)^2 + (y - \rho_n x)^2}} \to \pm\infty, \\
(\sin(\beta_n - \psi_n))\nu_n &= \frac{(1-\rho_n) b_n w(b_n) + (y - \rho_n x) - (1-\rho_n^2)(b_n w(b_n) + x)}{\sqrt{(1-\rho_n^2)(b_n w(b_n) + x)^2 + ((1-\rho_n) b_n w(b_n) + (y - \rho_n x))^2}} \sqrt{b_n w(b_n)} \\
&\to \pm\infty,
\end{aligned}$$

where the sign above depends on $y > x$ and $y \le x$, respectively. Thus using similar arguments as above for (i) it follows that both (4.2) and (4.3) hold also for $\lambda = 0$.

If the upper endpoint $x_F \in (0, \infty)$, then $\bar{F}(x/\cos\theta) = 0$ for $x/\cos\theta > x_F$, thus one can substitute the upper limits of the integrals above accordingly and obtain the results. Hence the proof is complete. $\square$

Proof of Theorem 2.3: $a)$ First recall that

$$z_n = \arccos\rho_n \in [0, \pi/2), \quad \psi_n = \arccos(1 - u_n) \sim \sqrt{2u_n} \to 0, \quad n \to \infty.$$

By Lemma 4.1 and the fact that $x_F = 1$ we have

$$\begin{aligned}
\mathbb{P}\left\{\rho_n |X| + \sqrt{1-\rho_n^2} Y > 1 - u_n\right\} &= \mathbb{P}\left\{R\cos(\Theta - z_n) > 1 - u_n, \Theta - z_n \in (-\psi_n, \psi_n)\right\} \\
&\quad + \mathbb{P}\left\{R\cos(\Theta + z_n) < -(1 - u_n), \Theta + z_n \in (\pi - \psi_n, \pi + \min(z_n, \psi_n))\right\} \\
&\quad + \mathbb{P}\left\{R\cos(\Theta + z_n) < -(1 - u_n), \Theta + z_n \in (z_n - \pi, \psi_n - \pi)\right\} \\
&=: I_n + J_n + K_n.
\end{aligned}$$

Next, for all large n

$$\begin{aligned}
I_n &= \frac{2}{2\pi} \int_0^{\psi_n} \bar{F}\left(\frac{1 - u_n}{\cos\theta}\right) d\theta = \frac{1}{2} \int_0^{(\sin\psi_n)^2} \bar{F}\left(\frac{1 - u_n}{\sqrt{1 - s}}\right) dB(s) \\
&= \frac{1}{2} \int_0^{(\sin\psi_n)^2/(2u_n)} \bar{F}\left(\frac{1 - u_n}{\sqrt{1 - 2u_n s}}\right) dB(2u_n s) \\
&= \frac{\sqrt{2u_n}\bar{F}(1 - u_n)}{2} \int_0^{(\sin\psi_n)^2/(2u_n)} \frac{\bar{F}((1 - u_n)/\sqrt{1 - 2u_n s})}{\bar{F}(1 - u_n)} d\frac{B(2u_n s)}{\sqrt{2u_n}}.
\end{aligned}$$

By Theorem 12.3.3 in [2] (see also Theorem 2.1 in [14])

$$I_n \sim \mathcal{I}_\alpha u_n^{1/2} \bar{F}(1 - u_n), \quad n \to \infty,$$

with $\mathcal{I}_\alpha$ given by (2.7). For J_n and K_n, we consider first that $2\lambda^2 > 1$, then $z_n > \psi_n$ for sufficiently large n. Thus $K_n = 0$ and

$$\begin{aligned} J_n &= \mathbb{P}\{R\cos(\Theta + z_n) < -(1-u_n), \Theta + z_n \in (\pi - \psi_n, \pi + \psi_n)\} \\ &= \mathbb{P}\{R\cos(\Theta + z_n - \pi) > 1 - u_n, \Theta + z_n - \pi \in (-\psi_n, \psi_n)\} = I_n. \end{aligned}$$

Similarly for $2\lambda^2 < 1$ and $2\lambda^2 = 1$ it follows that $J_n + K_n = I_n$. Hence the proof of a) follows by utilising further (2.6).

b) For any x, y negative the case proved in a) above implies that

$$\lim_{n\to\infty} n\mathbb{P}\left\{\rho_n|X| + \sqrt{1-\rho_n^2}Y > 1 + u_n y\right\} = 2\mathcal{I}_\alpha |y|^{1/2+\alpha}$$

and

$$\lim_{n\to\infty} n\mathbb{P}\{X > 1 + u_n x\} = \lim_{n\to\infty} \frac{n}{2}\mathbb{P}\{|X| > 1 + u_n x\} = \mathcal{I}_\alpha |x|^{1/2+\alpha}.$$

It thus remains to deal with

$$A_n := \mathbb{P}\left\{X > 1 + u_n x, \rho_n|X| + \sqrt{1-\rho_n^2}Y > 1 + u_n y\right\}.$$

By Lemma 4.1 we have

$$\begin{aligned} A_n &= \mathbb{P}\{R\cos\Theta > 1 + u_n x, \Theta \in (\beta_n, \pi/2)\} + \mathbb{P}\{R\cos(\Theta - z_n) > 1 + u_n y, \Theta - z_n \in (-\pi/2, \beta_n - z_n)\} \\ &= \mathbb{P}\left\{R > \frac{1+u_n x}{\cos\Theta}, \Theta \in (\max(-\psi_{1n}, \beta_n), \psi_{1n})\right\} + \mathbb{P}\left\{R > \frac{1+u_n y}{\cos(\Theta - z_n)}, \Theta - z_n \in (-\psi_{2n}, \min(\beta_n - z_n, \psi_{2n}))\right\} \\ &=: A_{1n} + A_{2n}, \end{aligned}$$

where $z_n = \arccos \rho_n \sim 2\lambda\sqrt{u_n}$ as $n \to \infty$ and

$$\begin{aligned} &\psi_{1n} = \arccos(1 + u_n x) \sim \sqrt{2u_n|x|}, \quad \psi_{2n} = \arccos(1 + u_n y) \sim \sqrt{2u_n|y|}, \\ &\beta_n = \arctan\left(\frac{(1+u_n y)/(1+u_n x) - \rho_n}{\sqrt{1-\rho_n^2}}\right) \sim \left(\lambda + \frac{y-x}{2\lambda}\right)\sqrt{u_n}. \end{aligned}$$

If $0 < \lambda_{x,y} := \lambda + \frac{y-x}{2\lambda} < \sqrt{2|x|}$, then $0 < \beta_n < \psi_{1n}$ holds for sufficiently large n. Hence

$$\begin{aligned} nA_{1n} &= \frac{n}{2\pi}\int_{\beta_n}^{\psi_{1n}} \bar{F}\left(\frac{1+u_n x}{\cos\theta}\right) d\theta = \frac{n}{4}\int_{(\sin\beta_n)^2}^{(\sin\psi_{1n})^2} \bar{F}\left(\frac{1+u_n x}{\sqrt{1-s}}\right) dB(s) \\ &= \frac{n}{4}\int_{(\sin\beta_n)^2/(2u_n|x|)}^{(\sin\psi_{1n})^2/(2u_n|x|)} \bar{F}\left(\frac{1+u_n x}{\sqrt{1+2u_n x s}}\right) dB(2u_n|x|s) \\ &\sim |x|^{1/2+\alpha}(nu_n^{1/2}\bar{F}(1-u_n))\frac{\sqrt{2}}{2\pi}\int_{\lambda_{x,y}/\sqrt{2|x|}}^{1} (1-s^2)^\alpha\, ds \\ &\sim |x|^{1/2+\alpha}\frac{\sqrt{2}}{2\pi}\int_{\lambda_{x,y}/\sqrt{2|x|}}^{1} (1-s^2)^\alpha\, ds = |x|^{1/2+\alpha}\mathcal{I}_\alpha\left(1 - \Upsilon_\alpha\left(\frac{\lambda_{x,y}}{\sqrt{2|x|}}\right)\right), \end{aligned} \tag{4.4}$$

where $\mathcal{I}_\alpha$ and $\Upsilon_\alpha(\cdot)$ are given by (2.7).

If $-\sqrt{2|x|} < \lambda_{x,y} < 0$, then $-\psi_{1n} < \beta_n \le 0$ holds for all large n. Hence

$$\begin{aligned}
nA_{1n} &= \frac{n}{2\pi}\left(\int_0^{-\beta_n} \bar{F}\left(\frac{1+u_n x}{\cos\theta}\right) d\theta + \int_0^{\psi_{1n}} \bar{F}\left(\frac{1+u_n x}{\cos\theta}\right) d\theta\right)\\
&\sim |x|^{1/2+\alpha}(n u_n^{1/2}\bar{F}(1-u_n))\frac{\sqrt{2}}{2\pi}\left(\int_0^{-\lambda_{x,y}/\sqrt{2|x|}} (1-s^2)^\alpha\, ds + \int_0^1 (1-s^2)^\alpha\, ds\right)\\
&\sim |x|^{1/2+\alpha}\frac{\sqrt{2}}{2\pi}\int_{\lambda_{x,y}/\sqrt{2|x|}}^1 (1-s^2)^\alpha\, ds = |x|^{1/2+\alpha}\mathcal{I}_\alpha\left(1-\Upsilon_\alpha\left(\frac{\lambda_{x,y}}{\sqrt{2|x|}}\right)\right).
\end{aligned}$$

Similarly, (4.4) holds for the other three cases: $\lambda_{x,y} = -\sqrt{2|x|}$, $\lambda_{x,y} = 0$ and $\lambda_{x,y} \ge \sqrt{2|x|}$, respectively.

For A_{2n}, if $0 < \lambda_{y,x} := \lambda + \frac{x-y}{2\lambda} < \sqrt{2|y|}$, then $-\psi_{2n} < \beta_n - z_n < 0$ holds for sufficiently large n. Hence

$$\begin{aligned}
nA_{2n} &= \frac{n}{2\pi}\int_{-\psi_{2n}}^{\beta_n - z_n} \bar{F}\left(\frac{1+u_n y}{\cos\theta}\right) d\theta = \frac{n}{4}\int_{(\sin(z_n-\beta_n))^2}^{(\sin\psi_{2n})^2} \bar{F}\left(\frac{1+u_n y}{\sqrt{1-s}}\right) dB(s)\\
&= \frac{n}{4}\int_{(\sin(z_n-\beta_n))^2/(2u_n|y|)}^{(\sin\psi_{2n})^2/(2u_n|y|)} \bar{F}\left(\frac{1+u_n y}{\sqrt{1+2u_n ys}}\right) dB(2u_n|y|s)\\
&\sim |y|^{1/2+\alpha}(n u_n^{1/2}\bar{F}(1-u_n))\frac{\sqrt{2}}{2\pi}\int_{\lambda_{y,x}/\sqrt{2|y|}}^1 (1-s^2)^\alpha\, ds\\
&\sim |y|^{1/2+\alpha}\frac{\sqrt{2}}{2\pi}\int_{\lambda_{y,x}/\sqrt{2|y|}}^1 (1-s^2)^\alpha\, ds = |y|^{1/2+\alpha}\mathcal{I}_\alpha\left(1-\Upsilon_\alpha\left(\frac{\lambda_{y,x}}{\sqrt{2|y|}}\right)\right).
\end{aligned}$$

The other three cases $\lambda_{y,x} \le -\sqrt{2|y|}, -\sqrt{2|y|} < \lambda_{y,x} \le 0$ and $\lambda_{y,x} \ge \sqrt{2|y|}$ follow with similar arguments as above establishing thus the claim in *b*). Consequently, the proof is complete. □

Acknowledgments. We would like to thank the referees for several comments and suggestions. C. Ling kindly acknowledges support from the Swiss National Science Foundation Grant 200021-140633/1. Both authors have been partially supported by the project RARE -318984 (an FP7 Marie Curie IRSES Fellowship).

Appendix

The Appendix contains the proof of Lemma 4.1 and the direct verification of (1.5) for the half-skew Gaussian case.

Proof of Lemma 4.1: Recall that $\psi = \arccos\rho \in [0, \pi/2)$ and the representation given by (1.7) implies

$$(X, \rho|X| + \sqrt{1-\rho^2}Y) \stackrel{d}{=} R(\cos\Theta, |\cos\Theta|\cos\psi + \sin\Theta\sin\psi),$$

where $\Theta \in (-\pi, \pi)$, independent of the random radius R. Hence,

$$\Theta_\rho := \rho|\cos\Theta| + \sqrt{1-\rho^2}\sin\Theta = \begin{cases} \cos(\Theta-\psi), & \Theta \in (-\pi/2, \pi/2),\\ -\cos(\Theta+\psi), & \Theta \in (-\pi, -\pi/2) \cup (\pi/2, \pi). \end{cases}$$

Further, $\Theta_\rho > 0$ if and only if

$$\Theta_\rho = \begin{cases} \cos(\Theta - \psi), & \Theta - \psi \in (-\pi/2, \pi/2 - \psi), \\ -\cos(\Theta + \psi), & \Theta + \psi \in (\psi - \pi, -\pi/2) \cup (\psi + \pi/2, \psi + \pi). \end{cases}$$

Therefore, for $x, y > 0$

$$\begin{aligned} \mathbb{P}\left\{\rho|X| + \sqrt{1-\rho^2}Y > x\right\} &= \mathbb{P}\left\{R > \frac{x}{\cos(\Theta - \psi)}, \Theta - \psi \in (-\frac{\pi}{2}, \frac{\pi}{2} - \psi)\right\} \\ &\quad + \mathbb{P}\left\{R > -\frac{x}{\cos(\Theta + \psi)}, \Theta \in (\frac{\pi}{2}, \pi) \cup (-\pi, -\frac{\pi}{2} - \psi)\right\} \end{aligned}$$

and

$$\begin{aligned} \mathbb{P}\left\{X > x, \rho|X| + \sqrt{1-\rho^2}Y > y\right\} &= \mathbb{P}\left\{R\cos\Theta > x, R\cos(\Theta - \psi) > y, \cos\Theta > 0\right\} \\ &= \mathbb{P}\left\{R > \max\left(\frac{x}{\cos\Theta}, \frac{y}{\cos(\Theta - \psi)}\right), \Theta \in (\psi - \frac{\pi}{2}, \frac{\pi}{2})\right\} \\ &= \mathbb{P}\left\{R > \frac{x}{\cos\Theta}, \Theta \in (\beta, \frac{\pi}{2})\right\} + \mathbb{P}\left\{R > \frac{y}{\cos(\Theta - \psi)}, \Theta \in (-\frac{\pi}{2} + \psi, \beta)\right\}, \end{aligned}$$

where β is the solution of $\cos(\theta - \psi)/\cos\theta = y/x$ with respect to $\theta \in (-\pi/2, \pi/2)$, i.e.,

$$\beta := \arctan\left(\frac{y/x - \rho}{\sqrt{1-\rho^2}}\right).$$

$\square$

Proof of the claim in (1.5).

Recall that X, Y are independent standard normal distributed and $b_n = G^{\leftarrow}(1 - 1/n) \sim \sqrt{2\ln n}, a_n = 1/b_n$, thus $\varphi(b_n)/b_n \sim 1/n$ with $\varphi(\cdot)$ the density of an $N(0,1)$ df. Further (set $u_n(y) = y/b_n + b_n$)

$$\begin{aligned} & n\mathbb{P}\left\{\rho_n|X| + \sqrt{1-\rho_n^2}Y > u_n(y)\right\} = 2n\int_0^\infty \mathbb{P}\left\{Y > \frac{u_n(y) - \rho_n x}{\sqrt{1-\rho_n^2}}\right\} d\Phi(x) \\ =\ & 2\int_{-b_n^2}^\infty \frac{n\varphi(u_n(x))}{b_n}\mathbb{P}\left\{Y > \frac{u_n(y) - \rho_n u_n(x)}{\sqrt{1-\rho_n^2}}\right\} dx \\ \sim\ & 2\int_{-b_n^2}^\infty \exp\left(-\left(x + \frac{x^2}{2b_n^2}\right)\right)\left(1 - \Phi\left(\frac{y-x}{\sqrt{1-\rho_n^2}b_n} + \frac{(1-\rho_n)x}{\sqrt{1-\rho_n^2}b_n} + \frac{1-\rho_n}{\sqrt{1-\rho_n^2}}b_n\right)\right) dx \\ =\ & 2\int_0^{b_n^2} \exp\left(x - \frac{x^2}{2b_n^2}\right)\left(1 - \Phi\left(\frac{y+x}{\sqrt{1-\rho_n^2}b_n} + \frac{(1-\rho_n)(b_n^2 - x)}{\sqrt{1-\rho_n^2}b_n}\right)\right) dx \\ & +2\int_0^\infty \exp\left(-\left(x + \frac{x^2}{2b_n^2}\right)\right)\left(1 - \Phi\left(\frac{y-x}{\sqrt{1-\rho_n^2}b_n} + \frac{(1-\rho_n)x}{\sqrt{1-\rho_n^2}b_n} + \frac{1-\rho_n}{\sqrt{1-\rho_n^2}}b_n\right)\right) dx \\ =:\ & 2(I_n + J_n). \end{aligned}$$

Note that

$$I_n \le \int_0^{b_n^2} e^x\left(1 - \Phi\left(\frac{y+x}{\sqrt{1-\rho_n^2}b_n}\right)\right) dx \le \int_0^\infty e^x\left(1 - \Phi\left(\frac{y+x}{\sqrt{1-\rho_n^2}b_n}\right)\right) dx < \infty,$$

where the last inequality holds by the dominated convergence theorem. In fact $\sqrt{1-\rho_n^2}b_n < 2\lambda+1, y+x > 2\lambda+1$ hold for sufficiently large n and x. Further since $1-\Phi(x) < \varphi(x)/x, x>0$ we obtain

$$e^x\left(1-\Phi\left(\frac{y+x}{\sqrt{1-\rho_n^2}b_n}\right)\right) < \frac{\varphi((y+x)/(2\lambda+1))}{(y+x)/(2\lambda+1)}e^x \leq \varphi\left(\frac{y+x}{2\lambda+1}\right)e^x.$$

Consequently, as $n\to\infty$

$$I_n \sim 2\int_0^\infty \left(1-\Phi\left(\frac{y+x}{2\lambda}+\lambda\right)\right)e^x\,dx =: 2I, \quad J_n \sim 2\int_0^\infty \left(1-\Phi\left(\frac{y-x}{2\lambda}+\lambda\right)\right)e^{-x}\,dx =: 2J.$$

Partial integration for I and J yields that (cf. [22])

$$I = -\left(1-\Phi\left(\frac{y}{2\lambda}+\lambda\right)\right)+\frac{e^{-y}}{2\lambda}\int_0^\infty \varphi\left(\frac{y+x}{2\lambda}-\lambda\right)dx = -\left(1-\Phi\left(\frac{y}{2\lambda}+\lambda\right)\right)+e^{-y}\Phi\left(\lambda-\frac{y}{2\lambda}\right)$$
$$J = 1-\int_0^\infty \Phi\left(\frac{y-x}{2\lambda}+\lambda\right)e^{-x}\,dx = 1-\Phi\left(\frac{y}{2\lambda}+\lambda\right)+e^{-y}\left(1-\Phi\left(\lambda-\frac{y}{2\lambda}\right)\right).$$

Hence,

$$\lim_{n\to\infty} n\mathbb{P}\left\{\rho_n|X|+\sqrt{1-\rho_n^2}Y > u_n(y)\right\} = 2e^{-y},$$

i.e.,

$$\lim_{n\to\infty} n\mathbb{P}\left\{\rho_n|X|+\sqrt{1-\rho_n^2}Y > a_n y+(b_n+a_n\ln 2)\right\} = e^{-y}.$$

Next, note that $u_n(x)>0$ for all $x\in\mathbb{R}$ and sufficiently large n

$$\begin{aligned} n\mathbb{P}\left\{X>u_n(x), \rho_n|X|+\sqrt{1-\rho_n^2}Y>u_n(y)\right\} &= n\int_{u_n(x)}^\infty \mathbb{P}\left\{Y>\frac{u_n(y)-\rho_n t}{\sqrt{1-\rho_n^2}}\right\}d\Phi(t) \\ &= \int_x^\infty \frac{n\varphi(u_n(t))}{b_n}\left(1-\Phi\left(\frac{u_n(y)-\rho_n u_n(t)}{\sqrt{1-\rho_n^2}}\right)\right)dt \\ &\sim \int_x^\infty \frac{\varphi(u_n(t))}{\varphi(b_n)}\left(1-\Phi\left(\frac{(y-t)+(1-\rho_n)t}{\sqrt{1-\rho_n^2}b_n}+\sqrt{\frac{1-\rho_n}{1+\rho_n}}b_n\right)\right)dt \\ &\sim \int_x^\infty\left(1-\Phi\left(\frac{y-t}{2\lambda}+\lambda\right)\right)e^{-t}\,dt \end{aligned} \tag{4.5}$$

by the dominated convergence theorem. It follows from [22] that the left-hand side of (4.5) is asymptotically equal to

$$e^{-x}+e^{-y}-\left(\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\right),$$

establishing the claim. $\square$

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