

On the Supremum of γ -reflected Processes with Fractional Brownian Motion as Input

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Abstract: Let $\{X_H(t), t \geq 0\}$ be a fractional Brownian motion with Hurst index $H \in (0, 1]$ and define a γ -reflected process $W_\gamma(t) = X_H(t) - ct - \gamma \inf_{s \in [0, t]} (X_H(s) - cs)$, $t \geq 0$ with $c > 0, \gamma \in [0, 1]$ two given constants. In this paper we establish the exact tail asymptotic behaviour of $M_\gamma(T) = \sup_{t \in [0, T]} W_\gamma(t)$ for any $T \in (0, \infty]$. Furthermore, we derive the exact tail asymptotic behaviour of the supremum of certain non-homogeneous mean-zero Gaussian random fields.

Key Words: γ -reflected process; fractional Brownian motion; supremum; exact asymptotics; ruin probability;

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1 Introduction

Let $\{X_H(t), t \geq 0\}$ be a standard fractional Brownian motion (fBm) with Hurst index $H \in (0, 1]$, i.e., X_H is a H -self-similar Gaussian process with stationary increments, and covariance function

$$\text{Cov}(X_H(t), X_H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

For two given constants $c > 0, \gamma \in [0, 1]$ define a new process $\{W_\gamma(t), t \geq 0\}$ by

$$W_\gamma(t) = X_H(t) - ct - \gamma \inf_{s \in [0, t]} (X_H(s) - cs), \quad t \geq 0. \quad (1)$$

Throughout this paper $\{W_\gamma(t), t \geq 0\}$ is referred to as a γ -reflected process with fBm as input since it reflects at rate γ when reaching its minimum.

In queuing theory W_1 is the so called *workload process* (or *queue length process*) see e.g., Harrison (1985), Zeevi and Glynn (2000), Whitt (2002) and Awad and Glynn (2009); alternatively one can refer to W_γ as a generalized workload process with fBm as input. In risk theory W_γ can be interpreted as a *claim surplus process* since the surplus process of an insurance portfolio can be defined by

$$U_\gamma(t) = u + ct - X_H(t) - \gamma \sup_{s \in [0, t]} (cs - X_H(s)) = u - W_\gamma(t), \quad t \geq 0$$

for any nonnegative initial reserve u . In the literature, see e.g., Asmussen and Albrecher (2010) the process $\{U_\gamma(t), t \geq 0\}$ is referred to as the risk process with tax payments of a *loss-carry-forward* type.

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This contribution is concerned with the tail asymptotic behaviour of the supremum $M_\gamma(T) = \sup_{t \in [0, T]} W_\gamma(t)$ for $T \in (0, \infty]$, i.e., we shall investigate the rate of convergence to 0 of $\psi_{\gamma, T}(u) := \mathbb{P}(M_\gamma(T) > u)$ as $u \rightarrow \infty$. The exact tail asymptotic behaviour of $M_\gamma(T)$ is known only for $\gamma = 0$. The case $T = \infty$ is already dealt with in Hüsler and Piterbarg (1999), whereas the case $T \in (0, \infty)$ has been investigated in Dębicki and Rolski (2002) and Dębicki and Sikora (2011), see our Theorem 4.1 in Appendix. Note in passing that $M_1(t) \rightarrow \infty$ almost surely as $t \rightarrow \infty$ (e.g., Duncan and Jin (2008)), therefore we shall assume below that $\gamma \in (0, 1)$ when $T = \infty$.

The principal result of this paper is Theorem 1.1 below, which establishes a unique asymptotic relationship between $\psi_{\gamma, T}(u)$ and $\psi_{0, T}(u)$ as $u \rightarrow \infty$ for any $T \in (0, \infty]$. Surprisingly, the following positive constant

$$\mathcal{P}_\alpha^a := \lim_{S \rightarrow \infty} \mathcal{P}_\alpha^a[0, S] = \lim_{S \rightarrow \infty} \mathcal{P}_\alpha^a[-S, 0], \quad \alpha \in (0, 2], \quad a > 0,$$

where

$$\mathcal{P}_\alpha^a[-S_1, S_2] = \mathbb{E} \left(\exp \left(\sup_{t \in [-S_1, S_2]} \left(\sqrt{2} B_\alpha(t) - (1+a)|t|^\alpha \right) \right) \right) \in (0, \infty), \quad 0 \leq S_1, S_2 < \infty, \quad (2)$$

with $\{B_\alpha(t), t \in \mathbb{R}\}$ a fBm defined on \mathbb{R} with Hurst index $\alpha/2 \in (0, 1]$, determines the ratio

$$R_{\gamma, T}(u) := \frac{\psi_{\gamma, T}(u)}{\psi_{0, T}(u)}$$

for all u large. Specifically, we have:

Theorem 1.1. *For any $H, \gamma \in (0, 1)$ and any $T \in (0, \infty]$*

$$\lim_{u \rightarrow \infty} R_{\gamma, T}(u) = \mathcal{M}_{H, \gamma, T}, \quad (3)$$

where $\mathcal{M}_{H, \gamma, T} = \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}$ if $T = \infty$, and for $T < \infty$

$$\mathcal{M}_{H, \gamma, T} = \begin{cases} \mathcal{P}_{2H}^{\frac{1-\gamma}{\gamma}}, & \text{if } H < 1/2, \\ \mathcal{P}_1^{\frac{2-\gamma}{\gamma}}, & \text{if } H = 1/2, \\ 1 & \text{if } H > 1/2. \end{cases}$$

The exact values of \mathcal{P}_α^a are known only for $\alpha = 1$ or 2 , namely,

$$\mathcal{P}_1^a = 1 + \frac{1}{a} \quad \text{and} \quad \mathcal{P}_2^a = \frac{1}{2} \left(1 + \sqrt{1 + \frac{1}{a}} \right) \quad (4)$$

see e.g., Piterbarg (1996) or Dębicki and Mandjes (2003). For general $\alpha \in (0, 2)$, bounds for \mathcal{P}_α^a are derived in Dębicki and Tabiś (2013).

The asymptotic relation described by (3) is of relevance for theoretical models in queuing theory and insurance mathematics. Moreover, a strong merit of (3) is that its proof for the case $H > 1/2$ and $T = \infty$ is closely related with the

exact tail asymptotics of the supremum of certain non-homogeneous Gaussian random fields, a result which has not been known in the literature so far. Given the importance of that result for extremes of Gaussian random fields, in the next section we present first an asymptotic expansion of the tail of supremum of certain non-homogeneous Gaussian random fields. We proceed then with the asymptotic formulas of $\psi_{\gamma,T}(u)$ for both cases $T = \infty$ and $T \in (0, \infty)$. All proofs are relegated to Section 3 followed by some technical results displayed in Appendix.

2 Main Results

We prefer to state first our new result on the tail asymptotic behaviour of the supremum of certain non-homogeneous Gaussian fields, since it is of theoretical importance going beyond the scope of queuing and risk theory. We need to introduce some more notation starting with the well-known Pickands constant \mathcal{H}_α defined by

$$\mathcal{H}_\alpha = \lim_{T \rightarrow \infty} \frac{1}{T} \mathcal{H}_\alpha[0, T], \quad \alpha \in (0, 2],$$

with

$$\mathcal{H}_\alpha[0, T] = \mathbb{E} \left(\exp \left(\sup_{t \in [0, T]} \left(\sqrt{2} B_\alpha(t) - t^\alpha \right) \right) \right) \in (0, \infty), \quad T \in (0, \infty).$$

It is known that $\mathcal{H}_1 = 1$ and $\mathcal{H}_2 = 1/\sqrt{\pi}$, see Pickands (1969), Berman (1992), Piterbarg (1996), Dębicki (2002), Mandjes (2007), Dębicki and Mandjes (2011), Dębicki and Dieker and Yakir (2012) for various properties of Pickands constant and its generalizations.

Throughout this paper, $x_+ = \max(0, x)$ for any $x \in \mathbb{R}$, and $\Psi(u), u \in \mathbb{R}$ denotes the survival function of the standard normal distribution $N(0, 1)$. Furthermore, we introduce the following constant

$$\tilde{\mathcal{P}}_\alpha^a = \lim_{S \rightarrow \infty} \mathcal{P}_\alpha^a[-S, S], \quad \alpha \in (0, 2], \quad a > 0,$$

where $\mathcal{P}_\alpha^a[-S, S]$ is given as in (2).

We state next our first result.

Theorem 2.1. *Let S, T be two positive constants, and let $\{X(s, t), (s, t) \in [0, S] \times [0, T]\}$ be a zero-mean Gaussian random field, with standard deviation function $\sigma(\cdot, \cdot)$ and correlation function $r(\cdot, \cdot, \cdot, \cdot)$. Assume that $\sigma(\cdot, \cdot)$ attains its unique maximum on $[0, S] \times [0, T]$ at (s_0, t_0) , and further*

$$\sigma(s, t) = 1 - b_1 |s - s_0|^\beta (1 + o(1)) - b_2 (t - t_0)^2 (1 + o(1)) - b_3 |(t - t_0)(s - s_0)| (1 + o(1)) \quad \text{as } (s, t) \rightarrow (s_0, t_0) \quad (5)$$

for some positive constants $b_i, i = 1, 2, 3$ and $1 < \beta < 2$. Suppose further that

$$r(s, s', t, t') = 1 - (a_1 |s - s'|^\beta + a_2 |t - t'|^\beta) (1 + o(1)) \quad \text{as } (s, t), (s', t') \rightarrow (s_0, t_0) \quad (6)$$

for some positive constants $a_i, i = 1, 2$. If there exist two positive constants G, μ with $\mu \in (0, 2]$ such that

$$\mathbb{E} \left((X(s, t) - X(s', t'))^2 \right) \leq G(|s - s'|^\mu + |t - t'|^\mu) \quad (7)$$

for any $(s, t), (s', t') \in [0, S] \times [0, T]$, then

$$\mathbb{P} \left(\sup_{(s, t) \in [0, S] \times [0, T]} X(s, t) > u \right) = \widehat{\mathcal{P}}_\beta^{b_1/a_1} \widehat{I}_2 \mathcal{H}_\beta \frac{\sqrt{\pi} a_2^{1/\beta}}{2\sqrt{b_2}} u^{2/\beta-1} \Psi(u) (1 + o(1)) \quad \text{as } u \rightarrow \infty, \quad (8)$$

where the constant $\widehat{\mathcal{P}}_\beta^{b_1/a_1}$ is equal to $\widetilde{\mathcal{P}}_\beta^{b_1/a_1}$ if $s_0 \in (0, S)$ and equal to $\mathcal{P}_\beta^{b_1/a_1}$ if $s_0 = 0$ or S , and \widehat{I}_2 is equal to 2 if $t_0 \in (0, T)$ and equal to 1 if $t_0 = 0$ or T .

Remark 2.2. The claim of Theorem 2.1 still holds for $b_3 < 0$ satisfying $b_2 + b_3/2 > 0$ which can be shown by utilising Theorem 2.1 and noting that

$$1 - \sigma(s, t) \geq \frac{b_1}{2} |s - s_0|^\beta (1 + o(1)) + \left(b_2 + \frac{b_3}{2} \right) (t - t_0)^2 (1 + o(1))$$

as $(s, t) \rightarrow (s_0, t_0)$. Note that (8) does not depend on the value of the constant b_3 .

Next, we return to our principal problem deriving below the exact asymptotic behaviour of $\psi_{\gamma, T}(u)$ as $u \rightarrow \infty$. Although the limit of the ratio $R_{\gamma, T}(u)$ as $u \rightarrow \infty$ remains constant, both cases $T = \infty$ and $T \in (0, \infty)$ are very different and will therefore be dealt with separately. We shall analyse first the case $T = \infty$.

Below, we set $Y_u(s, t) := \frac{X_H(ut) - \gamma X_H(us)}{(1 + ct - c\gamma s)u^H}$, and then write

$$\begin{aligned} \psi_{\gamma, \infty}(u) &= \mathbb{P} \left(\sup_{t \geq 0} \left(X_H(t) - ct - \gamma \inf_{s \in [0, t]} (X_H(s) - cs) \right) > u \right) \\ &= \mathbb{P} \left(\sup_{0 \leq s \leq t < \infty} Y_u(s, t) > u^{1-H} \right). \end{aligned} \quad (9)$$

The above alternative formula for $\psi_{\gamma, \infty}(u)$ together with Theorem 2.1 and Lemma 3.1 is crucial for the derivation of the tail asymptotic behaviour of $M_\gamma(\infty)$.

Theorem 2.3. We have, for $H, \gamma \in (0, 1)$

$$\psi_{\gamma, \infty}(u) = \mathcal{W}_H(u) \Psi \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right) (1 + o(1)) \quad \text{as } u \rightarrow \infty, \quad (10)$$

where

$$\mathcal{W}_H(u) = 2^{\frac{1}{2} - \frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \mathcal{H}_{2H} \mathcal{P}_{2H}^{\frac{1-\gamma}{H}} \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right)^{1/H-1}.$$

Remark 2.4. If $H = 1$, then $X_H(t) = \mathcal{N}t$ with \mathcal{N} a standard normal random variable (i.e., \mathcal{N} is $N(0, 1)$ distributed).

Consequently, for any $c > 0$ and $\gamma \in (0, 1)$

$$\psi_{\gamma, \infty}(u) = \mathbb{P} \left(\sup_{0 \leq s \leq t < \infty} \frac{(t - \gamma s)\mathcal{N}}{1 + c(t - \gamma s)} > 1 \right) = \Psi(c) = \mathbb{P} \left(\sup_{t \geq 0} (\mathcal{N}t - ct) > u \right) = \psi_{0, \infty}(u)$$

holds for all $u \geq 0$.

Example: Consider the case of the γ -reflected process with Brownian motion as input, i.e., $H = 1/2$. It is well-known that

$$\psi_{0,\infty}(u) = \mathbb{P}\left(\sup_{t \in [0,\infty)} (B_1(t) - ct) > u\right) = e^{-2cu}, \quad u \geq 0.$$

Further, for this case Theorem 2.3 together with (4) imply

$$\psi_{\gamma,\infty}(u) = \frac{2\sqrt{2\pi}c^{1/2}}{1-\gamma} u^{1/2} \Psi\left(2c^{1/2}u^{1/2}\right) (1 + o(1))$$

as $u \rightarrow \infty$. Therefore

$$\psi_{\gamma,\infty}(u) = \frac{1}{1-\gamma} \psi_{0,\infty}(u) (1 + o(1)) \quad \text{as } u \rightarrow \infty,$$

which also follows from the following tax identity (see e.g., Asmussen and Albrecher (2010), Albrecher et al. (2013))

$$\psi_{\gamma,\infty}(u) + (1 - \psi_{0,\infty}(u))^{1-\gamma} = 1, \quad \forall u \geq 0.$$

We conclude this section with an explicit asymptotic expansion for $\psi_{\gamma,T}(u)$ with $T \in (0, \infty)$. For any $u \geq 0$

$$\begin{aligned} \psi_{\gamma,T}(u) &= \mathbb{P}\left(\sup_{t \in [0,T]} \left(X_H(t) - ct - \gamma \inf_{s \in [0,t]} (X_H(s) - cs)\right) > u\right) \\ &= \mathbb{P}\left(\sup_{0 \leq s \leq t \leq T} (Z(s,t) - (ct - c\gamma s)) > u\right), \end{aligned}$$

where $Z(s,t) := X_H(t) - \gamma X_H(s)$. It follows that the variance function of $Z(s,t)$ is given by

$$V_Z^2(s,t) = \mathbb{E}((X_H(t) - \gamma X_H(s))^2) = (1-\gamma)t^{2H} + (\gamma^2 - \gamma)s^{2H} + \gamma(t-s)^{2H}.$$

Clearly, $V_Z(s,t)$ attains its unique maximum on the set $A := \{(s,t) : 0 \leq s \leq t \leq T\}$ at $(s_0, t_0) = (0, T)$. This fact is crucial for our last result stated below.

Theorem 2.5. *For any $T \in (0, \infty)$ and $H, \gamma \in (0, 1]$, we have*

$$\psi_{\gamma,T}(u) = \mathcal{D}_{H,\gamma} \left(\frac{u + cT}{T^H}\right)^{\frac{(1-2H)_+}{H}} \Psi\left(\frac{u + cT}{T^H}\right) (1 + o(1)) \quad \text{as } u \rightarrow \infty, \quad (11)$$

where

$$\mathcal{D}_{H,\gamma} = \begin{cases} 2^{-\frac{1}{2H}} H^{-1} \mathcal{H}_{2H} \mathcal{P}_{2H}^{\frac{1-\gamma}{2}}, & \text{if } H < 1/2, \\ \frac{4}{2-\gamma}, & \text{if } H = 1/2, \\ 1 & \text{if } H > 1/2, \end{cases} \quad \gamma \in (0, 1), \quad \mathcal{D}_{H,1} = \begin{cases} 2^{-\frac{1}{H}} H^{-2} (\mathcal{H}_{2H})^2, & \text{if } H < 1/2, \\ 4, & \text{if } H = 1/2, \\ 1 & \text{if } H > 1/2. \end{cases}$$

3 Proofs

In this section, we give proofs of all the results. Hereafter the positive constant \mathbb{C} may be different from line to line. Furthermore, a mean-zero Gaussian process (or a random Gaussian field) $\{\xi(t), t \geq 0\}$ with a bar denotes the corresponding standardized process (or random field), i.e., $\bar{\xi}(t) = \xi(t)/\sqrt{\mathbb{E}(\xi(t))^2}$.

PROOF OF THEOREM 1.1 In view of (32) in Theorem 4.1 in Appendix the claim for the case $T = \infty$ follows immediately from Theorem 2.3. Further, by combining the result of Theorem 2.5 with that of (33) in Theorem 4.1 we establish the claim for the case $T \in (0, \infty)$. \square

PROOF OF THEOREM 2.1 Set $\eta(s, t) = X(s + s_0, t + t_0)$, $(s, t) \in [-s_0, S - s_0] \times [-t_0, T - t_0]$. It follows that the standard deviation function $\sigma_\eta(s, t)$ of $\eta(s, t)$ attains its unique maximum equal to 1 on $[-s_0, S - s_0] \times [-t_0, T - t_0]$ at $(0, 0)$. Further (5) and (6) are valid for the standard deviation function σ_η and the correlation function r_η with (s_0, t_0) replaced by $(0, 0)$. Moreover, (7) is established for the random field η over $[-s_0, S - s_0] \times [-t_0, T - t_0]$. There are nine cases to be considered depending on whether 0 is an inner point or a boundary point of $[-s_0, S - s_0]$ or $[-t_0, T - t_0]$. We investigate next on the case that $(s_0, t_0) = (0, 0)$, and thus $\eta = X$. The other cases can be analysed with the same argumentations. In the light of Theorem 8.1 in Piterbarg (1996) (or Theorem 8.1 in Piterbarg (2001)) for u sufficiently large (set $\delta(u) = \ln u/u$, $\tilde{\Delta}_u = [0, \delta(u)] \times [0, \delta(u)]$)

$$\mathbb{P} \left(\sup_{(s,t) \in [0,S] \times [0,T] / \tilde{\Delta}_u} X(s, t) > u \right) \leq \mathbb{C} u^{4/\mu} \exp \left(-\frac{u^2}{2 - C\delta(u)^2} \right) \quad (12)$$

holds for some positive constant \mathbb{C} not depending on u . Next we analyse

$$\mathbb{P} \left(\sup_{(s,t) \in \tilde{\Delta}_u} X(s, t) > u \right)$$

as $u \rightarrow \infty$, which has the same asymptotic behaviour as (set $\tilde{\xi}(s, t) = \frac{\xi(s, t)}{(1+b_1 s^\beta)(1+b_2 t^2 + b_3 ts)}$)

$$\pi(u) = \mathbb{P} \left(\sup_{(s,t) \in \tilde{\Delta}_u} \tilde{\xi}(s, t) > u \right) \quad \text{as } u \rightarrow \infty,$$

where $\{\xi(s, t), s, t \geq 0\}$ is a mean-zero Gaussian random field with covariance function given by

$$r_\xi(s, t) = \exp(-a_1 s^\beta - a_2 t^\beta), \quad s, t \geq 0.$$

For simplicity, we shall assume that $a_1 = a_2 = 1$. The general case can be analysed by rescaling the time. It follows from Lemma 6.1 in Piterbarg (1996) that

$$\mathbb{P} \left(\sup_{(s,t) \in [0, u^{-2/\beta} S] \times [0, u^{-2/\beta} T]} \frac{\xi(s, t)}{1 + b_1 s^\beta} > u \right) = \mathcal{P}_\beta^{b_1} [0, S] \mathcal{H}_\beta [0, T] \Psi(u) (1 + o(1)) \quad \text{as } u \rightarrow \infty. \quad (13)$$

Since $\beta \in (1, 2)$, for any positive constant S_1 , we can divide the interval $[0, \delta(u)]$ into several sub-intervals of length $S_1 u^{-2/\beta}$. Specifically, let for $S_1, S_2 > 0$

$$\Delta_0^i = u^{-2/\beta} [0, S_i], \quad \Delta_k^i = u^{-2/\beta} [kS_i, (k+1)S_i], \quad k \in \mathbb{N}, \quad i = 1, 2$$

and let further

$$h_i(u) = \lfloor S_i^{-1} u^{\frac{2}{\beta}-1} \ln u \rfloor + 1, \quad i = 1, 2, \quad u > 0.$$

Bonferroni inequality yields

$$\begin{aligned}\pi(u) &\leq \sum_{k_2=0}^{h_2(u)} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_{k_2}^2} \tilde{\xi}(s,t) > u \right) + \sum_{k_1=1}^{h_1(u)} \sum_{k_2=0}^{h_2(u)} \mathbb{P} \left(\sup_{(s,t) \in \Delta_{k_1}^1 \times \Delta_{k_2}^2} \tilde{\xi}(s,t) > u \right) \\ &=: I_1(u) + I_2(u)\end{aligned}$$

and

$$\begin{aligned}\pi(u) &\geq \sum_{k_2=0}^{h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_{k_2}^2} \tilde{\xi}(s,t) > u \right) \\ &\quad - \sum_{0 \leq i < j \leq h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_i^2} \tilde{\xi}(s,t) > u, \sup_{(s,t) \in \Delta_0^1 \times \Delta_j^2} \tilde{\xi}(s,t) > u \right) \\ &=: J_1(u) - J_2(u).\end{aligned}$$

Next we calculate the required asymptotic bounds for $I_1(u)$ and $J_1(u)$ and show that

$$I_2(u) = J_2(u)(1 + o(1)) = o(I_1(u)) \quad \text{as } u \rightarrow \infty, S_i \rightarrow \infty, i = 1, 2. \quad (14)$$

We derive that

$$\begin{aligned}J_1(u) &= \sum_{k_2=0}^{h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_{k_2}^2} \tilde{\xi}(s,t) > u \right) \\ &\geq \sum_{k_2=0}^{h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_{k_2}^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > u(1 + b_2((k_2 + 1)S_2 u^{-2/\beta})^2 + b_3((k_2 + 1)S_2 u^{-2/\beta})(S_1 u^{-2/\beta})) \right).\end{aligned}$$

In view of (13)

$$\begin{aligned}J_1(u) &\geq \mathcal{P}_\beta^{b_1}[0, S_1] \mathcal{H}_\beta[0, S_2] \frac{1}{\sqrt{2\pi}} \sum_{k_2=0}^{h_2(u)-1} \frac{1}{u(1 + b_2((k_2 + 1)S_2 u^{-2/\beta})^2 + b_3((k_2 + 1)S_2 u^{-2/\beta})(S_1 u^{-2/\beta}))} \\ &\quad \exp \left(-\frac{u^2(1 + b_2((k_2 + 1)S_2 u^{-2/\beta})^2 + b_3((k_2 + 1)S_2 u^{-2/\beta})(S_1 u^{-2/\beta}))^2}{2} \right) (1 + o(1)) \\ &= \mathcal{P}_\beta^{b_1}[0, S_1] \frac{\mathcal{H}_\beta[0, S_2]}{S_2} \Psi(u) u^{2/\beta-1} \int_0^\infty e^{-b_2 x^2} dx (1 + o(1))\end{aligned} \quad (15)$$

as $u \rightarrow \infty$, where in the last equation we used the facts that, as $u \rightarrow \infty$

$$h_2(u) \rightarrow \infty, \quad h_2(u)u^{1-2/\beta} \rightarrow \infty, \quad h_2(u)u^{2-4/\beta} \rightarrow 0.$$

Similarly

$$I_1(u) \leq \mathcal{P}_\beta^{b_1}[0, S_1] \frac{\mathcal{H}_\beta[0, S_2]}{S_2} \Psi(u) u^{2/\beta-1} \int_0^\infty e^{-b_2 x^2} dx (1 + o(1)) \quad (16)$$

as $u \rightarrow \infty$. Moreover, (14) can be shown as in Piterbarg (1996). Specifically

$$I_2(u) = \sum_{k_1=1}^{h_1(u)} \sum_{k_2=0}^{h_2(u)} \mathbb{P} \left(\sup_{(s,t) \in \Delta_{k_1}^1 \times \Delta_{k_2}^2} \tilde{\xi}(s,t) > u \right)$$

$$\leq \sum_{k_1=1}^{h_1(u)} \sum_{k_2=0}^{h_2(u)} \mathbb{P} \left(\sup_{(s,t) \in \Delta_{k_1}^1 \times \Delta_{k_2}^2} \xi(s,t) > u(1 + b_1(k_1 S_1 u^{-2/\beta})^\beta + b_2(k_2 S_2 u^{-2/\beta})^2) \right).$$

Similar argumentations as in (15) yield

$$I_2(u) \leq \mathcal{H}_\beta[0, S_1] \mathcal{H}_\beta[0, S_2] \Psi(u) (S_2^{-1} u^{2/\beta-1}) \int_0^\infty e^{-b_2 x^2} dx \sum_{k_1=1}^{h_1(u)} \exp(-b_1(k_1 S_1)^\beta) (1 + o(1))$$

as $u \rightarrow \infty$. Further

$$J_2(u) = \sum_{0 \leq i < j \leq h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_i^2} \tilde{\xi}(s,t) > u, \sup_{(s,t) \in \Delta_0^1 \times \Delta_j^2} \tilde{\xi}(s,t) > u \right) =: \Sigma_1(u) + \Sigma_2(u),$$

where $\Sigma_1(u)$ is the sum over indices $j = i + 1$, and similarly $\Sigma_2(u)$ is the sum over indices $j > i + 1$. Let

$$B(i, S_2, u) = u(1 + b_2(i S_2 u^{-2/\beta})^2), \quad i \in \mathbb{N}, \quad S_2 > 0, \quad u > 0.$$

It follows that

$$\Sigma_1(u) \leq \sum_{0 \leq i \leq h_2(u)-1} \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_i^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u), \sup_{(s,t) \in \Delta_0^1 \times \Delta_{i+1}^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u) \right)$$

and

$$\begin{aligned} & \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_i^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u), \sup_{(s,t) \in \Delta_0^1 \times \Delta_{i+1}^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u) \right) \\ &= \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_0^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u) \right) + \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times \Delta_1^2} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u) \right) \\ & - \mathbb{P} \left(\sup_{(s,t) \in \Delta_0^1 \times (\Delta_0^2 \cup \Delta_1^2)} \frac{\xi(s,t)}{1 + b_1 s^\beta} > B(i, S_2, u) \right). \end{aligned}$$

Therefore, using the same reasoning as (15), we conclude that

$$\limsup_{u \rightarrow \infty} \frac{\Sigma_1(u)}{\Psi(u) u^{2/\beta-1}} \leq \mathcal{P}_\beta^{b_1}[0, S_1] \frac{2\mathcal{H}_\beta[0, S_2] - \mathcal{H}_\beta[0, 2S_2]}{S_2} \int_0^\infty e^{-b_2 x^2} dx. \quad (17)$$

Further

$$\Sigma_2(u) \leq \sum_{i=0}^{h_2(u)-1} \sum_{j \geq 2} \mathbb{P} \left(\sup_{\substack{(s,t) \in \Delta_0^1 \times \Delta_0^2 \\ (s',t') \in \Delta_0^1 \times \Delta_j^2}} \zeta(s,t,s',t') > 2B(i, S_2, u) \right),$$

where

$$\zeta(s,t,s',t') = \xi(s,t) + \xi(s',t'), \quad s, s', t, t' \geq 0.$$

Now, for u sufficiently large

$$2 \leq \mathbb{E}((\zeta(s,t,s',t'))^2) = 4 - 2(1 - r(|s-s'|, |t-t'|)) \leq 4 - ((j-1)S_2)^\beta u^{-2}$$

for any $(s, t) \in \Delta_0^1 \times \Delta_0^2$, $(s', t') \in \Delta_0^1 \times \Delta_0^2$. Thus, using similar argumentations as in Lemma 6.3 of Piterbarg (1996), we conclude that

$$\limsup_{u \rightarrow \infty} \frac{\Sigma_2(u)}{\Psi(u)u^{2/\beta-1}} \leq \mathbb{C} (\mathcal{H}_\beta[0, S_1])^2 S_2 \sum_{j \geq 1} \exp\left(-\frac{1}{8}(jS_2)^\beta\right). \quad (18)$$

Consequently, the claim follows from (12) and (14–16) by letting $S_2, S_1 \rightarrow \infty$. \square

Before proceeding with the proof of Theorem 2.3 observe first that the variance function of $Y_u(s, t)$ is given by

$$V_Y^2(s, t) = \frac{(1-\gamma)t^{2H} + (\gamma^2 - \gamma)s^{2H} + \gamma(t-s)^{2H}}{(1+ct-c\gamma s)^2}.$$

In fact, the distribution function of Y_u does not depend on u , so in the following we deal with $Y(s, t) := \frac{X_H(t) - \gamma X_H(s)}{1+ct-c\gamma s}$ instead of $Y_u(s, t)$. The next lemma will be used in the proof of Theorem 2.3.

Lemma 3.1. *The variance function $V_Y^2(s, t)$ attains its unique global maximum over set $B := \{(s, t) : 0 \leq s \leq t < \infty\}$ at $(\tilde{s}_0, \tilde{t}_0)$, with $\tilde{s}_0 = 0$ and $\tilde{t}_0 = \frac{H}{c(1-H)}$. Further*

$$V_Y(0, \tilde{t}_0) = \frac{H^H(1-H)^{1-H}}{c^H}.$$

PROOF OF THEOREM 2.3 The theorem will be proved in the following two steps.

Step 1. Let $K > \tilde{t}_0$ be a sufficiently large integer. We first derive the asymptotics of

$$\mathbb{P}\left(\sup_{0 \leq s \leq t < K} Y(s, t) > u^{1-H}\right) \text{ as } u \rightarrow \infty.$$

Define $\tilde{B}_\delta := \{(s, t) : s \in (0, \delta), t \in (\tilde{t}_0 - \delta, \tilde{t}_0 + \delta)\}$, for $\delta > 0$ sufficiently small, and let $B_K := \{(s, t) : 0 \leq s \leq t < K\}$.

We write

$$\pi(u) := \mathbb{P}\left(\sup_{(s,t) \in \tilde{B}_\delta} \tilde{Y}(s, t) > \frac{u^{1-H}}{V_Y(0, \tilde{t}_0)}\right) \text{ with } \tilde{Y}(s, t) := \bar{Y}(s, t) \frac{V_Y(s, t)}{V_Y(0, \tilde{t}_0)}.$$

Clearly

$$\pi(u) \leq \mathbb{P}\left(\sup_{0 \leq s \leq t < K} Y(s, t) > u^{1-H}\right) \leq \pi(u) + \mathbb{P}\left(\sup_{(s,t) \in B_K / \tilde{B}_\delta} \tilde{Y}(s, t) > \frac{u^{1-H}}{V_Y(0, \tilde{t}_0)}\right).$$

Therefore, we can conclude that

$$\mathbb{P}\left(\sup_{0 \leq s \leq t < K} Y(s, t) > u^{1-H}\right) = \pi(u)(1 + o(1)) \text{ as } u \rightarrow \infty,$$

if

$$\mathbb{P}\left(\sup_{(s,t) \in B_K / \tilde{B}_\delta} \tilde{Y}(s, t) > \frac{u^{1-H}}{V_Y(0, \tilde{t}_0)}\right) = o(\pi(u)) \text{ as } u \rightarrow \infty. \quad (19)$$

We calculate next the asymptotics of $\pi(u)$ and show that (19) holds. In view of Lemma 3.1 the standard deviation function of \tilde{Y} given by

$$\sigma_{\tilde{Y}}(s, t) := \frac{V_Y(s, t)}{V_Y(0, \tilde{t}_0)}, \quad (s, t) \in B,$$

attains its unique maximum over $\tilde{B}_{\tilde{\delta}}$ at the point $(0, \tilde{t}_0)$, and $\sigma_{\tilde{Y}}(0, \tilde{t}_0) = 1$. Straightforward calculations yield

$$1 - \sigma_{\tilde{Y}}(s, t) = \begin{cases} \frac{c^2(1-H)^3}{2H}(\tilde{t}_0 - t)^2(1 + o(1)) + \frac{(\gamma - \gamma^2)(1-H)^{2H}c^{2H}}{2H^{2H}}s^{2H}(1 + o(1)), & H \leq 1/2, \\ \frac{c^2(1-H)^3}{2H}(\tilde{t}_0 - t + \gamma s)^2(1 + o(1)) + \frac{(\gamma - \gamma^2)(1-H)^{2H}c^{2H}}{2H^{2H}}s^{2H}(1 + o(1)), & H > 1/2 \end{cases} \quad (20)$$

as $(s, t) \rightarrow (0, \tilde{t}_0)$. Additionally

$$1 - \text{Cov}(\bar{Y}(s, t), \bar{Y}(s', t')) = \frac{1}{2\tilde{t}_0^{2H}} (|t - t'|^{2H} + \gamma^2 |s - s'|^{2H}) (1 + o(1)) \quad (21)$$

as $(s, t), (s', t') \rightarrow (0, \tilde{t}_0)$ and for any $s, t, s', t' \in \tilde{B}_{\tilde{\delta}}$

$$\mathbb{E} \left(\tilde{Y}(s, t) - \tilde{Y}(s', t') \right)^2 \leq \mathbb{C}(|t - t'|^{2H} + |s - s'|^{2H}).$$

Using Theorem 4.2 for $H \leq 1/2$ and Theorem 2.1 with Remark 2.2 for $H > 1/2$, we conclude that

$$\pi(u) = \mathcal{W}_H(u) \Psi \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right) (1 + o(1)), \quad (22)$$

where

$$\mathcal{W}_H(u) = 2^{\frac{1}{2} - \frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \mathcal{H}_{2H} \mathcal{P}_{2H}^{\frac{1-\gamma}{2}} \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right)^{(1/H-1)}.$$

Next we give the proof of (19). Since $\sigma_{\tilde{Y}}(s, t)$ is continuous, there exists some positive constant ρ such that

$$\sup_{(s, t) \in B_K / \tilde{B}_{\tilde{\delta}}} \sigma_{\tilde{Y}}(s, t) < \rho < 1$$

for the chosen small $\tilde{\delta}$. Therefore, in view of Borell-TIS inequality (e.g., Adler and Taylor (2007)), for u sufficiently large

$$\mathbb{P} \left(\sup_{(s, t) \in B_K / \tilde{B}_{\tilde{\delta}}} \tilde{Y}(s, t) > \frac{u^{1-H}}{V_Y(0, \tilde{t}_0)} \right) \leq \exp \left(-\frac{(u^{1-H} - a)^2}{2V_Y^2(0, \tilde{t}_0)\rho^2} \right)$$

for some constant $a > 0$. Consequently, Eq. (19) is established by comparing the last inequality with (22).

Step 2. We show that, for the chosen large enough integer $K > \tilde{t}_0$

$$\mathbb{P} \left(\sup_{s, K \leq t} Y(s, t) > u^{1-H} \right) = o \left(\mathbb{P} \left(\sup_{0 \leq s \leq t < K} Y(s, t) > u^{1-H} \right) \right) \quad \text{as } u \rightarrow \infty.$$

For any $u > 0$ we have (set $I_n = [n, n+1), n \in \mathbb{N}$)

$$\begin{aligned} \mathbb{P} \left(\sup_{s, K \leq t} Y(s, t) > u^{1-H} \right) &\leq \mathbb{P} \left(\sup_{s, K \leq t} \frac{X_H(t)}{1 + ct - c\gamma s} + \sup_{s, K \leq t} \frac{-\gamma X_H(s)}{1 + ct - c\gamma s} > u^{1-H} \right) \\ &\leq \mathbb{P} \left(\sup_{s, K \leq t} \frac{X_H(t)}{t^H} \frac{t^H}{1 + ct - c\gamma s} > \frac{u^{1-H}}{2} \right) + \mathbb{P} \left(\sup_{K \leq s \leq t} \frac{-\gamma X_H(s)}{s^H} \frac{s^H}{1 + ct - c\gamma s} > \frac{u^{1-H}}{2} \right) \end{aligned}$$

$$\begin{aligned}
& +\mathbb{P}\left(\sup_{s \leq K \leq t} \frac{-\gamma X_H(s)}{(1+ct-c\gamma s)} > \frac{u^{1-H}}{2}\right) \\
& \leq 2J_1(u) + J_2(u),
\end{aligned}$$

where

$$J_1(u) := \sum_{i \geq K} \mathbb{P}\left(\sup_{s \in I_i} \frac{X_H(s)}{s^H} > \frac{1+c(1-\gamma)i}{2i^H} u^{1-H}\right), \quad J_2(u) := \mathbb{P}\left(\sup_{s \leq K} X_H(s) > \frac{1+c(1-\gamma)K}{2\gamma} u^{1-H}\right).$$

Furthermore, it follows that, for any $s, t \in I_i, i \geq K$

$$\mathbb{E}\left(\left(\frac{X_H(t)}{t^H} - \frac{X_H(s)}{s^H}\right)^2\right) = \frac{2s^H t^H - 2\mathbb{E}(X_H(t)X_H(s))}{s^H t^H} \leq \frac{|t-s|^{2H}}{i^{2H}} \leq |t-s|^{2H}.$$

Using Fernique's Lemma (e.g., Leadbetter et al. (1983)) for some absolute positive constants C_1, C_2

$$\mathbb{P}\left(\sup_{t \in I_i} \frac{X_H(t)}{t^H} > \frac{1+c(1-\gamma)i}{2i^H} u^{1-H}\right) \leq C_1 \exp\left(-C_2 i^{2(1-H)} u^{2(1-H)}\right)$$

from which we conclude that, for K sufficiently large

$$J_1(u) \leq \sum_{i \geq K} C_1 \exp\left(-C_2 i^{2(1-H)} u^{2(1-H)}\right).$$

In the light of (33) of Theorem 4.1 we see that

$$J_2(u) = \mathcal{D}_H \left(\frac{1+c(1-\gamma)K}{2\gamma K^H} u^{1-H}\right)^{\frac{(1-2H)_+}{H}} \Psi\left(\frac{1+c(1-\gamma)K}{2\gamma K^H} u^{1-H}\right) (1+o(1)) \quad \text{as } u \rightarrow \infty.$$

Consequently, for sufficiently large K

$$\mathbb{P}\left(\sup_{s, K \leq t} Y(s, t) > u^{1-H}\right) \leq 2J_1(u) + J_2(u) = o\left(\mathbb{P}\left(\sup_{0 \leq s \leq t < K} Y(s, t) > u^{1-H}\right)\right)$$

as $u \rightarrow \infty$, hence the proof is complete. \square

PROOF OF THEOREM 2.5 Without loss of generality, we give only the proof of the case $\gamma \in (0, 1)$. Firstly, we give the asymptotic expansion of the standard deviation function $V_Z(s, t)$ at the point $(0, T)$. It follows that

$$V_Z(s, t) = \begin{cases} T^H (1 - HT^{-1}(T-t) - H\gamma T^{-1}s) + o((T-t) + s), & H > 1/2, \\ T^{1/2} \left(1 - \frac{1}{2}T^{-1}(T-t) - \left(\frac{1}{2}\gamma T^{-1} + \frac{\gamma-\gamma^2}{2}T^{-1}\right)s\right) + o((T-t) + s), & H = 1/2, \\ T^H \left(1 - HT^{-1}(T-t) - \frac{\gamma-\gamma^2}{2}T^{-2H}s^{2H}\right) + o((T-t) + s^{2H}), & H < 1/2, \end{cases} \quad (23)$$

as $(s, t) \rightarrow (0, T)$, hence there exists a positive constant $\delta > 0$ such that

$$|t - T - \gamma s| \leq \mathbb{C}(V_Z(0, T) - V_Z(s, t)) \quad (24)$$

uniformly in $A_\delta := \{(s, t) : (s, t) \in [0, \delta] \times [T - \delta, T]\}$. Next, we study the asymptotics of the supremum of the Gaussian random field defined on A_δ . Set below

$$\nu_u(s, t) = \frac{u + ct - c\gamma s}{V_Z(s, t)} \quad \text{and} \quad \Pi(u) = \mathbb{P}\left(\sup_{(s, t) \in A_\delta} \frac{\bar{Z}(s, t) \nu_u(0, T)}{\nu_u(s, t)} > \nu_u(0, T)\right).$$

For any $u > 0$

$$\Pi(u) \leq \mathbb{P} \left(\sup_{(s,t) \in A} (Z(s,t) - (ct - c\gamma s)) > u \right) \leq \Pi(u) + \mathbb{P} \left(\sup_{(s,t) \in A/A_\delta} \bar{Z}(s,t) \frac{\nu_u(0,T)}{\nu_u(s,t)} > \nu_u(0,T) \right). \quad (25)$$

Since

$$\frac{\nu_u(0,T)}{\nu_u(s,t)} = 1 - \frac{V_Z(0,T) - V_Z(s,t)}{V_Z(0,T)} - \frac{(c(t-T) - c\gamma s)V_Z(s,t)}{(u + ct - c\gamma s)V_Z(0,T)}, \quad (26)$$

we have, in view of (24), for any $\varepsilon \in (0,1)$, and sufficiently large u

$$1 - \frac{V_Z(0,T) - V_Z(s,t)}{V_Z(0,T)} \leq \frac{\nu_u(0,T)}{\nu_u(s,t)} \leq 1 - (1-\varepsilon) \frac{V_Z(0,T) - V_Z(s,t)}{V_Z(0,T)}$$

uniformly in $(s,t) \in A_\delta$. Consequently

$$\mathbb{P} \left(\sup_{(s,t) \in A_\delta} Z_0(s,t) > \nu_u(0,T) \right) \leq \Pi(u) \leq \mathbb{P} \left(\sup_{(s,t) \in A_\delta} Z_\varepsilon(s,t) > \nu_u(0,T) \right), \quad (27)$$

where the random field $\{Z_\varepsilon(s,t), s, t \geq 0\}$ is defined as

$$Z_\varepsilon(s,t) := \bar{Z}(s,t) \left(1 - (1-\varepsilon) \frac{V_Z(0,T) - V_Z(s,t)}{V_Z(0,T)} \right), \quad \varepsilon \in [0,1).$$

Direct calculations show that the standard deviation function $\sigma_{Z_\varepsilon}(s,t) := \sqrt{\mathbb{E}((Z_\varepsilon(s,t))^2)}$ attains its unique maximum over A_δ at $(0,T)$ with $\sigma_{Z_\varepsilon}(0,T) = 1$. Thus, in the light of (23), we have

$$\sigma_{Z_\varepsilon}(s,t) = \begin{cases} 1 - (1-\varepsilon) (HT^{-1}(T-t) + H\gamma T^{-1}s) (1 + o(1)), & H > 1/2, \\ 1 - (1-\varepsilon) \left(\frac{1}{2}T^{-1}(T-t) + \left(\frac{1}{2}\gamma T^{-1} + \frac{\gamma-\gamma^2}{2}T^{-1} \right) s \right) (1 + o(1)), & H = 1/2, \\ 1 - (1-\varepsilon) \left(HT^{-1}(T-t) + \frac{\gamma-\gamma^2}{2}T^{-2H}s^{2H} \right) (1 + o(1)), & H < 1/2 \end{cases} \quad (28)$$

as $(s,t) \rightarrow (0,T)$. Furthermore, it follows that

$$1 - Cov(Z_\varepsilon(s,t), Z_\varepsilon(s',t')) = \frac{1}{2T^{2H}} (|t-t'|^{2H} + \gamma^2 |s-s'|^{2H}) (1 + o(1)) \quad (29)$$

as $(s,t), (s',t') \rightarrow (0,T)$. In addition, we obtain

$$\mathbb{E}((Z_\varepsilon(s,t) - Z_\varepsilon(s',t'))^2) \leq \mathbb{C}(2|t-t'|^{2H} + 2\gamma^2|s-s'|^{2H})$$

for $(s,t), (s',t') \in A_\delta$, consequently, by Theorem 4.2

$$\mathbb{P} \left(\sup_{(s,t) \in A_\delta} Z_\varepsilon(s,t) > \nu_u(0,T) \right) = \mathcal{D}_{H,\gamma,\varepsilon} \left(\frac{u + cT}{T^H} \right)^{\left(\frac{1}{H}-2\right)_+} \Psi \left(\frac{u + cT}{T^H} \right) (1 + o(1)) \quad (30)$$

as $u \rightarrow \infty$, where

$$\mathcal{D}_{H,\gamma,\varepsilon} = \begin{cases} (1-\varepsilon)^{-1} 2^{-\frac{1}{2H}} H^{-1} \mathcal{H}_{2H} \mathcal{P}_{2H}^{(1-\varepsilon)\frac{1}{2H} \frac{1-\gamma}{\gamma}}, & \text{if } H < 1/2, \\ \mathcal{P}_1^{(1-\varepsilon)\frac{2-\gamma}{\gamma}} \times \mathcal{P}_1^{(1-\varepsilon)}[-\infty, 0], & \text{if } H = 1/2, \\ 1 & \text{if } H > 1/2 \end{cases}$$

and thus letting $\varepsilon \rightarrow 0$, we obtain the asymptotic upper bound for $\Pi(u)$ on the set A_δ . The asymptotic lower bound can be derived using the same arguments. In order to complete the proof we need to show further that

$$\mathbb{P} \left(\sup_{(s,t) \in A/A_\delta} \bar{Z}(s,t) \frac{\nu_u(0,T)}{\nu_u(s,t)} > \nu_u(0,T) \right) = o(\Pi(u)) \quad \text{as } u \rightarrow \infty. \quad (31)$$

In the light of (26) for all u sufficiently large

$$\sup_{(s,t) \in A/A_\delta} \text{Var} \left(\bar{Z}(s,t) \frac{\nu_u(0,T)}{\nu_u(s,t)} \right) \leq (\rho(\delta))^2 < 1,$$

where $\rho(\delta)$ is a positive function in δ which exists due to the continuity of $V_Z(s,t)$ in A . Additionally, by the almost surely continuity of the random field, we have, for some constant $a > 0$

$$\mathbb{P} \left(\sup_{(s,t) \in A/A_\delta} \bar{Z}(s,t) \frac{\nu_u(0,T)}{\nu_u(s,t)} > a \right) \leq \mathbb{P} \left(\sup_{(s,t) \in A/A_\delta} \bar{Z}(s,t) \left(1 - \frac{V_Z(0,T) - V_Z(s,t)}{2V_Z(0,T)} \right) > a \right) \leq 1/2.$$

Therefore, a direct application of the Borell inequality (e.g., Theorem D.1 of Piterbarg (1996)) implies

$$\mathbb{P} \left(\sup_{(s,t) \in A/A_\delta} \bar{Z}(s,t) \frac{\nu_u(0,T)}{\nu_u(s,t)} > \nu_u(0,T) \right) \leq 2\Psi \left(\frac{\nu_u(0,T) - a}{\rho(\delta)} \right) = o(\Pi(u)) \quad \text{as } u \rightarrow \infty.$$

Consequently, Eq. (31) is established, and thus the proof is complete. \square

4 Appendix

The next theorem consists of two known results given in Hüsler and Piterbarg (1999) for the case $T = \infty$ and in Dębicki and Rolski (2002) when $T \in (0, \infty)$.

Theorem 4.1. *If $\{X_H(t), t \geq 0\}$ is a fBm with Hurst index $H \in (0, 1]$, then for any $H \in (0, 1)$*

$$\psi_{0,\infty}(u) = 2^{\frac{1}{2} - \frac{1}{2H}} \frac{\sqrt{\pi}}{\sqrt{H(1-H)}} \mathcal{H}_{2H} \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right)^{1/H-1} \Psi \left(\frac{c^H u^{1-H}}{H^H (1-H)^{1-H}} \right) (1 + o(1)) \quad (32)$$

holds as $u \rightarrow \infty$, and for any $H \in (0, 1]$ and $T \in (0, \infty)$

$$\psi_{0,T}(u) = \mathcal{D}_H \left(\frac{u + cT}{T^H} \right)^{\frac{(1-2H)_+}{H}} \Psi \left(\frac{u + cT}{T^H} \right) (1 + o(1)) \quad (33)$$

holds as $u \rightarrow \infty$, where \mathcal{D}_H is equal to $H^{-1}2^{-1/(2H)}\mathcal{H}_{2H}$ if $H < 1/2$, 2 if $H = 1/2$, and 1 if $H > 1/2$.

In the following theorem we present some results used in the proof of our main theorems; denote the Euler Gamma function by $\Gamma(\cdot)$.

Theorem 4.2. *Let S, T be two positive constants, and let $\{X(s,t), (s,t) \in [0, S] \times [0, T]\}$ be a zero-mean Gaussian random field with standard deviation function $\sigma(\cdot, \cdot)$ and correlation function $r(\cdot, \cdot, \cdot, \cdot)$. Assume that $\sigma(\cdot, \cdot)$ attains its unique maximum on $[0, S] \times [0, T]$ at (s_0, t_0) , and further*

$$\sigma(s, t) = 1 - b_1 |s - s_0|^{\beta_1} (1 + o(1)) - b_2 |t - t_0|^{\beta_2} (1 + o(1)), \quad \text{as } (s, t) \rightarrow (s_0, t_0) \quad (34)$$

for some positive constants $b_i, \beta_i, i = 1, 2$. Let, moreover

$$r(s, s', t, t') = 1 - (a_1|s - s'|^{\alpha_1} + a_2|t - t'|^{\alpha_2})(1 + o(1)) \quad \text{as } (s, t), (s', t') \rightarrow (s_0, t_0)$$

for some positive constants $a_i, i = 1, 2$ and $\alpha_i \in (0, 2], i = 1, 2$. In addition, there exist two positive constants G, μ with $\mu \in (0, 2]$ such that

$$\mathbb{E}((X(s, t) - X(s', t'))^2) \leq G(|s - s'|^\mu + |t - t'|^\mu)$$

for any $(s, t), (s', t') \in [0, S] \times [0, T]$. Then as $u \rightarrow \infty$

i) if $\alpha_1 < \beta_1$ and $\alpha_2 < \beta_2$

$$\mathbb{P}\left(\sup_{(s,t) \in [0,S] \times [0,T]} X(s, t) > u\right) = \prod_{i=1}^2 \left(\mathcal{H}_{\alpha_i} a_i^{1/\alpha_i} b_i^{-1/\beta_i} \widehat{I}_i \Gamma\left(\frac{1}{\beta_i} + 1\right) u^{2/\alpha_i - 2/\beta_i} \right) \Psi(u)(1 + o(1));$$

ii) if $\alpha_1 < \beta_1$ and $\alpha_2 = \beta_2$

$$\mathbb{P}\left(\sup_{(s,t) \in [0,S] \times [0,T]} X(s, t) > u\right) = \mathcal{H}_{\alpha_1} a_1^{1/\alpha_1} b_1^{-1/\beta_1} \widehat{I}_1 \Gamma\left(\frac{1}{\beta_1} + 1\right) \widehat{\mathcal{P}}_{\alpha_2}^{b_2/a_2} u^{2/\alpha_1 - 2/\beta_1} \Psi(u)(1 + o(1));$$

iii) if $\alpha_1 = \beta_1$ and $\alpha_2 = \beta_2$

$$\mathbb{P}\left(\sup_{(s,t) \in [0,S] \times [0,T]} X(s, t) > u\right) = \widehat{\mathcal{P}}_{\alpha_1}^{b_1/a_1} \widehat{\mathcal{P}}_{\alpha_2}^{b_2/a_2} \Psi(u)(1 + o(1));$$

iv) if $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$

$$\mathbb{P}\left(\sup_{(s,t) \in [0,S] \times [0,T]} X(s, t) > u\right) = \Psi(u)(1 + o(1)),$$

where \widehat{I}_2 is the same as in (8) and

$$\widehat{\mathcal{P}}_{\alpha_1}^{b_1/a_1} := \begin{cases} \widetilde{\mathcal{P}}_{\alpha_1}^{b_1/a_1}, & \text{if } s_0 \in (0, S), \\ \mathcal{P}_{\alpha_1}^{b_1/a_1}, & \text{if } s_0 = 0 \text{ or } S, \end{cases} \quad \widehat{\mathcal{P}}_{\alpha_2}^{b_2/a_2} := \begin{cases} \widetilde{\mathcal{P}}_{\alpha_2}^{b_2/a_2}, & \text{if } t_0 \in (0, T), \\ \mathcal{P}_{\alpha_2}^{b_2/a_2}, & \text{if } t_0 = 0 \text{ or } T, \end{cases} \quad \widehat{I}_1 := \begin{cases} 2, & \text{if } s_0 \in (0, S), \\ 1, & \text{if } s_0 = 0 \text{ or } S. \end{cases}$$

PROOF OF THEOREM 4.2 In the context of Piterbarg (1996) and Fatalov (1992) condition (34) is formulated as

$$\sigma(s, t) = 1 - (b_1|s - s_0|^{\beta_1} + b_2|t - t_0|^{\beta_2})(1 + o(1)), \quad \text{as } (s, t) \rightarrow (s_0, t_0). \quad (35)$$

In fact, conditions (34) and (35) play the same roles in the proof, since only bounds of the form

$$(b_1|s - s_0|^{\beta_1} + b_2|t - t_0|^{\beta_2})(1 - \epsilon) \leq 1 - \sigma(s, t) \leq (b_1|s - s_0|^{\beta_1} + b_2|t - t_0|^{\beta_2})(1 + \epsilon)$$

for any $\epsilon > 0$, as $(s, t) \rightarrow (s_0, t_0)$, are needed. Therefore, the claims follow by similar argumentations as in Piterbarg (1996) and Fatalov (1992). \square

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References

- [1] Adler, R.J., Taylor, J.E., 2007. *Random Fields and Geometry*. Springer.
- [2] Albrecher, H., Avram, F., Constantinescu, C., Ivanovs, J., (2013). The tax identity for Markov additive risk processes. *Methodology and Computing in Applied Probability*, in press.
- [3] Asmussen, S., Albrecher, H., 2010. *Ruin probabilities* (Second Edition). World Scientific, New Jersey.
- [4] Awad, H., Glynn, P., 2009. Conditional limit theorem for regulated fractional Brownian motion. *The Annals of Applied Probability* 19, 2102-2136.
- [5] Berman, M.S., 1992. *Sojourns and extremes of stochastic processes*, Wadsworth & Brooks/ Cole, Boston.
- [6] Dębicki, K., 2002. Ruin probability for Gaussian integrated processes. *Stochastic Processes and their Applications* 98, 151-174.
- [7] Dębicki, K., Mandjes, M., 2003. Exact overflow asymptotics for queues with many Gaussian inputs. *Journal of Applied Probability* 40, 704-720.
- [8] Dębicki, K., Mandjes, M., 2011. Open problems in Gaussian fluid queueing theory. *Queueing Systems Theory Appl.* 68, 267-273.
- [9] Dębicki, K., Rolski, T., 2002. A note on transient Gaussian fluid models. *Queueing Systems Theory Appl.* 42, 321-342.
- [10] Dębicki, K., Sikora, G., 2011. Infinite time asymptotics of fluid and ruin models: multiplexed fractional Brownian motions case. *Applicationes Mathematicae* 38, 107-116.
- [11] Dieker, A.B., Yakir, B., 2012. On asymptotic constants in the theory of Gaussian processes. *Bernoulli*, in press.
- [12] Duncan, T.E., Jin, Y., 2008. Maximum Queue Length of a Fluid Model with an aggregated fractional Brownian input. *IMS Collections Markov Processes and Related Topics: A Festschrift for Thomas G. Kurtz Vol. 4*, 235-251.
- [13] Fatalov, V.R., 1992. Asymptotics of large deviation probabilities for Gaussian fields. I, *J. Contemp. Math. Analysis* (Armenian Acad. Sci.) 27, 48-70.
- [14] Harrison, M.J., 1985. *Brownian motion and stochastic flow system*. Wiley, New York.
- [15] Hüsler, J., Piterbarg, V.I., 1999. Extremes of a certain class of Gaussian processes. *Stochastic Processes and their Applications* 83, 257-271.

- [16] Leadbetter, M., Lindgren, G., Rootzén, H., 1983. *Extremes and related properties of random sequences and processes*. Springer Series in Statistics. Springer, Berlin.
- [17] Mandjes, M., 2007. *Large deviations of Gaussian queues*. Wiley, Chichester, UK.
- [18] Pickands, J. III., 1969. Asymptotic properties of the maximum in a stationary Gaussian process. *Transactions of the American Mathematical Society* 145, 75-86.
- [19] Piterbarg, V.I., 2001. Large deviations of a storage process with fractional Brownian motion as input. *Extremes* 4, 147-164.
- [20] Piterbarg, V.I., 1996. *Asymptotic methods in the theory of Gaussian processes and fields*. In: *Transl. Math. Monographs*, vol. 148. AMS, Providence, RI.
- [21] Dębicki, K., Tabiś, K., 2013. Constants in the asymptotics of suprema of Gaussian processes. Preprint.
- [22] Whitt, W., 2002. *Stochastic-process limits. An introduction to stochastic-process limits and their application to queues*. Springer.
- [23] Zeevi, A.J., Glynn, P.W., 2000. On the maximum workload of a queue fed by fractional Brownian motion. *The Annals of Applied Probability* 10, 1084-1099.