Approximation of a random process with variable smoothness

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Abstract We consider the rate of piecewise constant approximation to a locally stationary process $X(t), t \in [0,1]$, having a variable smoothness index $\alpha(t)$. Assuming that $\alpha(\cdot)$ attains its unique minimum at zero and satisfies

$$\alpha(t) = \alpha_0 + bt^\gamma + \mathrm{o}(t^\gamma) \quad \text{as } t \to 0,$$

we propose a method for construction of observation points (composite dilated design) such that the integrated mean square error

$$\int_0^1 \mathbb{E}\{(X(t) - X_n(t))^2\}dt \sim \frac{K}{n^{\alpha_0}(\log n)^{(\alpha_0+1)/\gamma}} \qquad \text{as } n \to \infty,$$

where a piecewise constant approximation X_n is based on $N(n) \sim n$ observations of X. Further, we prove that the suggested approximation rate is optimal, and then show how to find an optimal constant K.

1 Introduction

Probabilistic models based on the locally stationary processes with variable smoothness became recently an object of interest for applications in various areas (e.g., In-

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ternet traffic, financial records, natural landscapes) due to their flexibility for matching local regularity properties (see, e.g., [15, 17] and references therein). The most known representative random process of this class is a multifractional Brownian motion (mBm) independently introduced in [6] and [24]. We refer to [3] for a survey and to [4, 5, 19, 29] for studies of particular aspects of mBm.

A more general class of $\alpha(\cdot)$-locally stationary Gaussian processes with a variable smoothness index $\alpha(t), t \in [0,1]$, was elaborated in [16]. This class generalizes also the class of locally stationary Gaussian processes with index α (introduced by Berman, [8]). It is worthwhile, however, to notice that another approach to "local stationarity" is possible whenever the processes with time varying parameters are considered. This direction leads to interesting models and applications in Statistics, long memory theory, etc, see [15] for more information. The two approaches are technically different but describe, in our opinion, the same phenomena. In this paper we stick to $\alpha(\cdot)$-local stationarity, as defined in (1) below. Whenever we need to model such processes with a given accuracy, the approximation (time discretization) accuracy has to be evaluated.

More specifically, consider a random process $X(t), t \in [0,1]$, with finite second moment and variable quadratic mean smoothness (see precise definition (1) below). The process X is observed at $N = N(n)$ points and a piecewise constant approximation X_n is built upon these observations. The approximation performance on the entire interval is measured by the integrated mean square error (IMSE) $\int_0^1 \mathbb{E}\{(X(t) - X_n(t))^2\}dt$. We construct a sequence of sampling designs (i.e., sets of observation points) taking into account the varying smoothness of X such that on a class of processes, the IMSE decreases faster when compared to conventional regular sampling designs (see, e.g., [28]) or to quasi-regular designs, [2], used for approximation of locally stationary random processes and random processes with an isolated singularity point, respectively.

The approximation results obtained in this paper can be used in various problems in signal processing, e.g., in optimization of compressing digitized signals, (see, e.g., [12]), in numerical analysis of random functions (see, e.g., [7, 13, 14]), in simulation studies with controlled accuracy for functionals on realizations of random processes (see, e.g., [1, 18]). It is known that a piecewise constant approximation gives an optimal rate for certain class of continuous random processes satisfying a Hölder condition (see, e.g., [11, 28]). In this paper we develop a technique improving this rate for a certain class of locally stationary processes with variable smoothness. The developed technique can be generalized for more advanced approximation methods (e.g., Hermite splines) and various classes of random processes and fields. Some related approximation results for continuous and smooth random functions can be found in [20, 21, 27]. The book [25] contains a very detailed survey of various random function approximation problems.

The paper is organized as follows. In Section 2 we specify the problem setting. We recall a notion of a locally stationary process with variable smoothness, introduce a class of piecewise constant approximation processes, and define integrated mean square error (IMSE) as a measure of approximation accuracy. Furthermore, we introduce a special method of *composite dilated sampling designs* that suggests how to distribute the observation points sufficiently densely located near the point of the lowest smoothness. The implementation of this design depends on some functional and numerical parameters, and we set up a certain number of mild assumptions about these parameters. In Section 3, our main results are stated. Namely, for a locally stationary process with known smoothness, we consider the piecewise constant interpolation related to dilated sampling designs (adjusted to smoothness parameters) and find the asymptotic behavior of its approximation error. In the second part of that section, the approximation for conventional regular and some quasi-regular sampling designs are studied. In Section 4, the results and conjectures related to the optimality of our bounds are discussed. Section 5 contains the proofs of the statements from Section 3.

2 Variable smoothness random processes and approximation methods. Basic notation

2.1 Approximation problem setting

Let $X = X(t), t \in [0,1]$, be an $\alpha(\cdot)$-locally stationary random process, i.e., $\mathbb{E}\{X(t)^2\} < \infty$ and

$$\lim_{s\to 0} \frac{||X(t+s)-X(t)||^2}{|s|^{\alpha(t)}} = c(t) \quad \text{uniformly in } t \in [0,1], \tag{1}$$

where $||Y|| := (\mathbb{E}Y^2)^{1/2}$, $\alpha(\cdot), c(\cdot) \in C([0,1])$ and $2 \geq \alpha(t) > 0$, $c(t) > 0$.

We assume that the following conditions hold for the function $\alpha(\cdot)$ describing the smoothness of X:

(C1) $\alpha(\cdot)$ attains its global minimum $\alpha_0 := \alpha(0)$ at the unique point $t_0 = 0$.

(C2) there exist $b, \gamma > 0$ such that

$$\alpha(t) = \alpha_0 + bt^\gamma + \mathrm{o}(t^\gamma) \quad \text{as } t \to 0.$$

The choice $t_0 = 0$ in $(C1)$ is made only for notational convenience. The results are essentially the same for any location of the unique minimum of $\alpha(\cdot)$.

Let X be sampled at the distinct design points $T_n = (t_0(n), \ldots, t_N(n))$ (also referred to as knots), where $0 = t_0(n) < t_1(n) < \cdots < t_N(n) = 1$, $N = N(n)$. We suppress the auxiliary integer argument n for design points $t_j = t_j(n)$ and for the number of points $N = N(n)$ when doing so causes no confusion. The corresponding piecewise constant approximation is defined by

$$X_n(t) := X(t_{j-1}), \qquad t_{j-1} \leq t < t_j,\ j = 1, \ldots, N.$$

In this article, we consider the accuracy of the approximation to X by X_n with respect to the integrated mean square error (IMSE)

$$e_n^2 = ||X - X_n||_2^2 := \int_0^1 ||X(t) - X_n(t)||^2 dt.$$

We shall describe below a construction of sampling designs $\{T_n\}$ providing the fastest decay of e_n^2.

2.2 Sampling design construction

The construction idea is as follows. In order to achieve a rate-optimal approximation of X by X_n, we introduce a sequence of *dilated* sampling designs $\{T_n\}$.

Recall first that any probability density $f(t), t \in [0,1]$, generates a sequence of associated conventional sampling designs, (cf., e.g., [26], [7], [28]) defined by

$$\int_0^{t_j} f(t)dt = \frac{j}{n}, \qquad j = 0, \dots, n, \tag{2}$$

i.e., the corresponding sampling points are (j/n)-percentiles of the distribution having density $f(\cdot)$. We call $f(\cdot)$ a *sampling density*.

Let $p(\cdot)$ be a probability density on $\mathbb{R}_+ := [0,\infty)$; we shall refer to it as the *design density*. In our problem, it turns out to be useful to *dilate* the design density $p(\cdot)$ by replacing it with a *dilated sampling density*

$$p_n(t) := d_n\, p(d_n t), \qquad t \in [0,1], \tag{3}$$

where $d_n \nearrow \infty$ is a *dilation coefficient*. Note, that formally $p_n(\cdot)$ is not a probability density, but

$$\int_0^1 p_n(t)dt = \int_0^{d_n} p(u)du \to 1 \text{ as } n \to \infty.$$

The idea of dilation is obvious: we wish to put more knots near the point of the worst smoothness. The delation coefficient should be chosen according to the smoothness behavior at this critical point. In our case, $(C2)$ requires the choice

$$d_n := (\log n)^{1/\gamma}$$

that will be maintained in the sequel. As in (2), we define the knots by

$$\int_0^{t_j} p_n(t)dt = \frac{j}{n}. \tag{4}$$

Further optimization of the approximation accuracy bound requires one more adjustment: it turns out to be useful to choose the knots t_j as in (4) using different densities in a neighborhood of the critical point and outside of it. We call *composite* such sampling design constructions operating differently on two disjoint domains.

Now we pass to the rigorous description of our sampling designs. Let $p(u)$ and $\widetilde{p}(u), u \in [0,\infty)$, be two probability densities. Let the dilated sampling densities $p_n(\cdot)$ be defined as in (3). Similarly, $\widetilde{p}_n(t) := d_n\, \widetilde{p}(d_n t)$.

For $0 < \rho < 1$, we define the *composite dilated* $(p,\rho,\widetilde{p})$*-designs* T_n by choosing t_j according to (4) for

$$0 \le j \le J(p,\rho,n) := n\int_0^{\rho} p_n(t)dt = n\int_0^{\rho d_n} p(u)du \le n.$$

Notice that for these knots, we have $0 \le t_j \le \rho$. Furthermore, we fill the interval $(\rho,1]$ with analogous knots t_i using the probability density $\widetilde{p}(\cdot)$,

$$\int_0^{t_i} \widetilde{p}_n(t)dt = \frac{j}{n}, \tag{5}$$

where

$$J(\widetilde{p},\rho,n) < j \le J(\widetilde{p},1,n),$$
$$i = j + J(p,\rho,n) - J(\widetilde{p},\rho,n).$$

For these knots we clearly have $\rho < t_i \le 1$. Note, it follows by definition that

$$J(p,\rho,n) = n\int_0^{\rho d_n} p(u)du \sim n \qquad \text{as } n \to \infty,$$

and similarly, in the interval $[\rho,1]$, the number of points does not exceed

$$n - J(\widetilde{p},\rho,n) = n\int_{\rho d_n}^{\infty} \widetilde{p}(u)du = \mathrm{o}(n) \qquad \text{as } n \to \infty,$$

that is the total number of sampling points satisfies

$$N(n) \sim J(p,\rho,n) \sim n \text{ as } n \to \infty. \tag{6}$$

In the sequel, we will use $(p,\rho,\widetilde{p})$-designs satisfying the following additional assumptions on $p(\cdot)$, ρ, and $\widetilde{p}(\cdot)$:

(A1) The design density $p(\cdot)$ is bounded, non-increasing, and

$$p(u) \ge q_1 \exp\{-q_2 u^{\gamma}\}, \qquad u \ge 0,\ q_1 > 0,\ \tfrac{b}{\alpha_0} > q_2 > 0. \tag{7}$$

(A2) We assume that $\widetilde{p}$ is *regularly varying* at infinity with some index $r \le -1$. This means that for all $\lambda > 0$,

$$\frac{\widetilde{p}(\lambda u)}{\widetilde{p}(u)} \to \lambda^r \quad \text{ as } u \to \infty. \tag{8}$$

(A3) Finally, we assume that the parameter ρ is small enough. Namely, applying $q_2 < b/\alpha_0$ and using $(C2)$ we may choose ρ satisfying

$$q_2 \sup_{0\le t\le\rho} \alpha(t) < \inf_{0\le t\le\rho} \frac{\alpha(t)-\alpha(0)}{t^\gamma} \tag{9}$$

and

$$q_2\rho^\gamma < 1. \tag{10}$$

For example, let $\alpha(t) = 1+t^\gamma$. Then $(C1)$, $(C2)$ hold and (A3) corresponds to $\rho < (1/q_2-1)^{1/\gamma}$, where $0 < q_2 < 1$.

Regularly varying probability densities satisfy (7) for large u, thus we could simplify the design construction by letting $p = \widetilde{p}$. For example, the choice

$$p(u) = \widetilde{p}(u) := (1+u)^{-2}$$

agrees with (A1) and (A2).

Moreover, in this case the knots may be easily calculated explicitely, as $t_j = \frac{j}{d_n(n-j)}$. However, this kind of the simplified choice does not provide an optimal constant K in the main approximation error asymptotics (11) below.

3 Main results

3.1 Dilated approximation designs

In the following theorem, we give the principal result of the paper and consider IMSE e_n^2 of approximation to X by X_n for the proposed sequence of composite dilated sampling designs $T_n, n\ge 1$. It follows from (A1) that the following constant is finite,

$$K = K(c,\alpha,(p,\rho,\widetilde{p})) := \frac{c_0}{\alpha_0+1}\int_0^\infty p(u)^{-\alpha_0} e^{-bu^\gamma} du < \infty,$$

where $c_0 := c(0)$.

Theorem 1. *Let $X(t), t\in[0,1]$, be an $\alpha(\cdot)$-locally stationary random process such that assumptions $(C1)$, $(C2)$ hold. Let X_n be the piecewise constant approximations corresponding to composite dilated $(p,\rho,\widetilde{p})$-designs $\{T_n\}$ satisfying $(A1)$-$(A3)$. Then $N(n)\sim n$ and*

$$||X-X_n||_2^2 \sim \frac{K}{n^{\alpha_0}(\log n)^{(\alpha_0+1)/\gamma}} \sim \frac{K}{N^{\alpha_0}(\log N)^{(\alpha_0+1)/\gamma}} \qquad \text{as } n\to\infty. \tag{11}$$

Remark 1. Among the assumptions of Theorem 1, the monotonicity of $p(\cdot)$ is worth of a discussion. Of course, it agrees with the heuristics to put more knots at places where the smoothness of the process is worse. However, this assumption may be easily replaced by some mild regularity assumptions on $p(\cdot)$.

Remark 2. The following probability density $p^*(\cdot)$

$$p^*(u) = Ce^{-bu^\gamma/(\alpha_0+1)}, \quad C = \frac{b^{1/\gamma}}{(\alpha_0+1)^{1/\gamma}\Gamma(1/\gamma+1)}$$

minimizes the constant K in Theorem 1 and generates the asymptotically optimal sequence of designs T_n^*. For the optimal T_n^*,

$$K^* := \frac{c_0}{\alpha_0+1}\left(\int_0^\infty e^{-bu^\gamma/(\alpha_0+1)}du\right)^{\alpha_0+1} = \frac{c_0}{\alpha_0+1}\left(\frac{(\alpha_0+1)^{1/\gamma}\Gamma(1/\gamma+1)}{b^{1/\gamma}}\right)^{\alpha_0+1},$$

see, e.g., [28]. We emphasize that $p^*(\cdot)$ satisfies assumption (7) but it is not regularly varying. In other words, a simple design based on $\widetilde{p} = p = p^*$ does not fit in theorem's assumptions.

Remark 3. The idea of considering composite designs might seem to be overcomplicated at first glance. However, in some sense it can not be avoided. The previous remark shows that if we want to get the optimal constant K, we must handle the exponentially decreasing densities. Assume that

$$p(u) \le q_1 \exp\{-q_2 u^\gamma\}. \tag{12}$$

If we would simplify the design by defining $t_j(n)$ as in (4) for the entire interval, i.e., with $\rho = 1$, then we would have

$$\int_0^{t_j} p_n(t)dt = \frac{j}{n},$$

hence,

$$\begin{aligned}\frac{1}{n} &= \int_{t_j}^{t_{j+1}} p_n(t)dt = \int_{t_j}^{t_{j+1}} d_n p(d_n t)dt \le d_n q_1 \int_{t_j}^{t_{j+1}} \exp\{-q_2(d_n t)^\gamma\}dt \\ &\le d_n q_1 (t_{j+1} - t_j)\exp\{-q_2(d_n t_j)^\gamma\}.\end{aligned}$$

Let $a \in (0,1)$ and $t_j \in [1-a, 1]$. Then for the length of the corresponding intervals, we have

$$t_{j+1} - t_j \ge \frac{\exp\{q_2(d_n t_j)^\gamma\}}{n d_n q_1} \ge \frac{\exp\{q_2 \log n (1-a)^\gamma\}}{n d_n q_1}.$$

If $q_2 > 1$ and a is so small that $q_2(1-a)^\gamma > 1$, we readily obtain $t_{j+1} - t_j > a$ for large n which is impossible. Therefore, for $q_2 > 1$ there are no sampling points t_j in $[1-a, 1]$, i.e., clearly $e_n^2 \ge C > 0$ for any n, i.e., IMSE does not tend to zero at all.

The confusion described above may really appear in practice because $q_2 > 1$ is compatible with the assumption $q_2 < b/\alpha_0$ from (7) whenever $b > \alpha_0$.

Theorem 1 shows that for the design densities with regularly varying tails, we may define all knots by (4) without leaving empty intervals as above. However, we can not achieve the optimal constant K on this simpler way.

Remark 4. Actually, the choice of knots outside of $[0,\rho]$ is not relevant for the approximation rate. One can replace the knots from (5) with a uniform grid of knots $t_i = in^{-\mu}$ with appropriate $\mu < 1$.

3.2 Regular sampling designs

The approximation algorithm investigated in Theorem 1 is based upon the assumption that we know the point where $\alpha(\cdot)$ attains its minimum, as well as the index γ in $(C2)$. If for the same process neither the critical point nor the index γ are known, a conventional *regular design* can be used.

Let $X(t), t \in [0,1]$, be an $\alpha(\cdot)$-locally stationary random process, i.e., (1) holds. Consider now sampling designs $T_n = \{t_j(n),\ j = 0,1,\dots,n\}$ generated by a regular positive continuous density $p(t), t \in [0,1]$, (see, e.g., [26], [28]) through (13), i.e.,

$$\int_0^{t_j} p(t)dt = \frac{j}{n}\,, \qquad 0 \le j \le n. \tag{13}$$

Let the constant

$$K_1 := \frac{c_0}{\alpha_0+1}\,\frac{\Gamma(1/\gamma+1)}{p_0^{\alpha_0} b^{1/\gamma}}\,, \quad p_0 := p(0).$$

Theorem 2. *Let $X(t), t \in [0,1]$, be an $\alpha(\cdot)$-locally stationary random process such that $(C1)$, $(C2)$ hold. Let X_n be the piecewise constant approximations corresponding to the (regular) sampling designs $\{T_n\}$ generated by $p(\cdot)$. Then*

$$||X - X_n||_2^2 \sim \frac{K_1}{n^{\alpha_0}(\log n)^{1/\gamma}} \qquad \text{as } n \to \infty.$$

Remark 5. If the point where $\alpha(\cdot)$ attains its minimum, is known but γ is unknown, we may build the designs without dilating the design density. Instead, one could use quasi-regular sampling designs generated by a possibly unbounded design density $p(t), t \in (0,1]$, at the singularity point $t_0 = 0$ (cf., [2]). For example, if $p(\cdot)$ is a probability density on $(0,1]$ such that

$$p(t) \sim A\,t^{-\kappa} \qquad \text{as } t \searrow 0, \qquad 0 < \kappa < 1,$$

and $t_j(n)$ are chosen through (13), then for an $\alpha(\cdot)$-locally stationary random process X satisfying $(C1)$ and $(C2)$, it is possible to show a slightly weaker asymptotics than that of Theorem 1, namely,

$$e_n^2 \sim \frac{K_2}{n^{\alpha_0}(\log n)^{(1+\kappa\alpha_0)/\gamma}} \qquad \text{as } n \to \infty,$$

with $K_2 := c_0 A^{-\alpha_0}\Gamma(1/\gamma+1)/((\alpha_0+1)b^{1/\gamma})$.

Of course, all above mentioned asymptotics differ only by a degree of logarithm while the polynomial rate is determined by the minimal regularity index α_0. But for some large scale approximation problems and certain regularity properties (C1), (C2) of $\alpha(t)$ this gain could be significant.

Remark 6. As an anonymous referee pointed out to us, it would be interesting to extend the results to the case of more general behavior of the function $\alpha(\cdot)$ at the critical point by replacing (C2) with

$$\alpha(t) = \alpha_0 + g(t) + \mathrm{o}(g(t)) \qquad \text{as } t \to 0,$$

with a given function $g(\cdot)$ from an appropriate class. For example, it is clear that the results will be pretty much the same if we allow g to be γ-regularly varying at zero. Yet we preferred to consider here only the simplest (and arguably the most important) polynomial case and leave the general case for further research.

4 Optimality

4.1 Optimality of the rate for piecewise constant approximations

We explain here that the approximation rate $l_n^{-1}, l_n := n^{\alpha_0} d_n^{(\alpha_0+1)}$, achieved in Theorem 1 is optimal in the class of piecewise constant approximations for every $\alpha(\cdot)$-locally stationary random process satisfying $(C1)$ and $(C2)$. For a sampling design T_n, let the mesh size $|T_n| := \max\{(t_j - t_{j-1}), j = 1, \ldots, n\}$.

Proposition 1. *Let X_n be piecewise constant approximations to an $\alpha(\cdot)$-locally stationary random process X satisfying $(C1)$ and $(C2)$ constructed according to designs $\{T_n\}$ such that $N_n \sim n$ and $|T_n| \to 0$ as $n \to \infty$. Then*

$$\liminf_{n\to\infty} l_n e_n^2 > 0. \tag{14}$$

Proof. Let $r_n := d_n^{-1} = (\log n)^{-1/\gamma}$ and $J_n := \inf\{j : t_j = t_j(n) \geq r_n\}$. Then (19) entails

$$e_n^2 \geq \sum_{j=1}^{J_n} e_{n,j}^2 = \sum_{j=1}^{J_n} B_{j-1} w_j^{\alpha(t_{j-1})+1}(1+\mathrm{o}(1)) = B \sum_{j=1}^{J_n} w_j^{a_n+1}(1+\mathrm{o}(1)),$$

where $a_n := \sup_{0\leq t\leq r_n} \alpha(t)$ and $w_j = t_j - t_{j-1}$. By using the convexity of the power function $w \to w^{a_n+1}$, we obtain

$$\frac{1}{J_n}\sum_{j=1}^{J_n} w_j^{a_n+1} \geq \left(\frac{1}{J_n}\sum_{j=1}^{J_n} w_j\right)^{a_n+1} \geq \left(\frac{r_n}{J_n}\right)^{a_n+1},$$

hence,

$$\sum_{j=1}^{J_n} w_j^{a_n+1} \geq \frac{r_n^{a_n+1}}{J_n^{a_n}} \geq \frac{r_n^{a_n+1}}{N_n^{a_n}} ,$$

whereas

$$\begin{aligned} e_n^2 &\geq B\, \frac{r_n^{a_n+1}}{N_n^{a_n}}\, (1+\mathrm{o}(1)) = B\, \frac{1}{d_n^{a_n+1}}\, \frac{1}{N_n^{a_n}}\, (1+\mathrm{o}(1)) \\ &= B\, \frac{1}{d_n^{\alpha_0+1} n^{\alpha_0}} \left(\frac{1}{d_n n}\right)^{a_n-\alpha_0} \left(\frac{n}{N_n}\right)^{a_n} (1+\mathrm{o}(1)) \\ &= B\, l_n^{-1} \left(\frac{1}{d_n n}\right)^{a_n-\alpha_0} (1+\mathrm{o}(1)). \end{aligned}$$

Recall that by $(C2)$, $a_n - \alpha_0 = O(r_n^\gamma) = O((\log n)^{-1})$ and thus (14) follows. □

4.2 Optimality of the rate in a class of linear methods

We explain here that the approximation rate l_n^{-1} achieved in Theorem 1 is optimal not only in the class of piecewise constant approximations but in a much wider class of linear methods, – at least for some $\alpha(\cdot)$-locally stationary random processes satisfying $(C1)$ and $(C2)$. The corresponding setting is based on the notion of Gaussian approximation numbers, or ℓ-numbers, that we recall here.

Gaussian approximation numbers of a Gaussian random vector X taking values in a normed space $\mathscr{X}$ are defined by

$$\ell_n(X;\mathscr{X})^2 = \inf_{\substack{x_1,\dots,x_{n-1}\\ \xi_1,\dots,\xi_{n-1}}} \mathbb{E}\left\{ \left\| X - \sum_{j=1}^{n-1} \xi_j x_j \right\|_{\mathscr{X}}^2 \right\}, \tag{15}$$

where infimum is taken over all $x_j \in \mathscr{X}$ and all Gaussian vectors $\xi = (\xi_1,\dots,\xi_{n-1}) \in \mathbb{R}^{n-1}, n \geq 2$, see [22, 23]. If $\mathscr{X}$ is a Hilbert space, then

$$\ell_n(X;\mathscr{X})^2 = \sum_{j=n}^{\infty} \lambda_j,$$

where λ_j is a decreasing sequence of eigenvalues of the covariance operator of X.

Recall that a multifractional Brownian motion (mBm) with a variable smoothness index (or fractality function) $\alpha(\cdot) \in (0,2)$ introduced in [6, 24] and studied in [3, 4, 5] is a Gaussian process defined through its white noise representation

$$X(t) = \int_{-\infty}^{\infty} \frac{e^{itu}-1}{|u|^{(\alpha(t)+1)/2}}\, dW(u),$$

where $W(t), t \in \mathbb{R}$, is a standard Brownian motion. Notice that mBm is a typical example of a locally stationary process whenever $\alpha(\cdot)$ is a continuous function.

In the particular case of the constant fractality $\alpha(t) \equiv \alpha$, we obtain an ordinary fractional Brownian motion B^α, $\alpha \in (0,2)$. For $X = B^\alpha$ considered as an element of $\mathscr{X} = L_2[0,1]$, the behavior of its eigenvalues λ_j is well known, cf. [9]. Namely,

$$\lambda_j \sim c_\alpha j^{-\alpha-1} \qquad \text{as } j \to \infty,$$

with some $c_\alpha > 0$ continuously depending on $\alpha \in (0,2)$. It follows that

$$\ell_n(B^\alpha; L_2[0,1])^2 \sim \alpha^{-1} c_\alpha n^{-\alpha} \qquad \text{as } n \to \infty.$$

Hence, for all $n \ge 1$,

$$\ell_n(B^\alpha; L_2[0,1])^2 \ge C_\alpha n^{-\alpha}, \quad C_\alpha > 0.$$

Furthermore, since B^α is a self-similar process, we can scale this estimate from $\mathscr{X} = L_2[0,1]$ to $\mathscr{X} = L_2[0,r]$ with arbitrary $r > 0$. An easy computation shows that

$$\ell_n(B^\alpha; L_2[0,r])^2 = r^{\alpha+1} \ell_n(B^\alpha; L_2[0,1])^2 \ge C_\alpha r^{\alpha+1} n^{-\alpha}.$$

Let us now consider a multifractional Brownian motion X parameterized by a fractality function $\alpha(\cdot)$ satisfying $(C2)$. For example, let

$$\alpha(t) := \alpha_0 + b t^\gamma, \qquad 0 \le t \le 1, \tag{16}$$

with $\alpha_0, b > 0$ chosen so small that $\alpha_0 + b < 2$. This choice secures the necessary condition $0 < \alpha(t) < 2, \quad 0 \le t \le 1$. Then, letting $r = r_n := d_n^{-1}$, we have

$$\begin{aligned} &\ell_n(X; L_2[0,1])^2 \ge \ell_n(X; L_2[0,r_n])^2 \ge M \ell_n(B^{\alpha(r_n)}; L_2[0,r_n])^2 \\ &\ge M C_{\alpha(r_n)} r_n^{\alpha(r_n)+1} n^{-\alpha(r_n)} = M C_{\alpha(r_n)} d_n^{-\alpha(r_n)-1} n^{-\alpha(r_n)} \\ &\ge C l_n^{-1} (d_n n)^{\alpha_0 - \alpha(r_n)} = C l_n^{-1} (d_n n)^{-b r_n^\gamma} = C l_n^{-1} (d_n n)^{-b(\log n)^{-1}} \ge \widetilde{C} l_n^{-1}, \end{aligned}$$

for some positive $M, C_{\alpha(r_n)}, C, \widetilde{C}$. All bounds here are obvious except for the second inequality comparing approximation rate of multifractional Brownian motion with that of a standard fractional Brownian motion. We state this fact as a separate result.

Proposition 2. *Let $X(t), a \le t \le b$, be a multifractional Brownian motion corresponding to a continuous fractality function $\alpha : [a,b] \to (0,2)$. Let B^β be a fractional Brownian motion such that $\inf_{a \le t \le b} \alpha(t) \le \beta < 2$. Then there exists $M = M(\alpha(\cdot), \beta) > 0$ such that*

$$\ell_n(X; L_2[a,b]) \ge M \ell_n(B^\beta, L_2[a,b]), \qquad n \ge 1.$$

The proof of this proposition requires different methods from those used in this article. We relegate it to another publication.

Our conclusion is that a multifractional Brownian motion with fractality function (16) provides an example of an $\alpha(\cdot)$-locally stationary random process satisfying assumptions $(C1)$ and $(C2)$ such that no linear approximation method provides a better approximation rate than l_n^{-1}.

5 Proofs

Proof of Theorem 1:
We represent the IMSE $e_n^2 = ||X(t) - X_n(t)||_2^2$ as the following sum

$$e_n^2 = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} ||X(t) - X_n(t)||^2 dt = \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} ||X(t) - X(t_{j-1})||^2 dt =: \sum_{j=1}^{N} e_{n,j}^2. \quad (17)$$

Next, for a large $U > 0$, let

$$e_n^2 = \sum_{j=1}^{N} e_{n,j}^2 = S_1 + S_2 + S_3,$$

where the sums S_1, S_2, S_3 include the terms $e_{n,j}^2$ such that $[t_{j-1}, t_j]$ belongs to $[0, U/d_n]$, $[U/d_n, \rho]$, and $[\rho, 1]$, respectively. Let J_1 and J_2 denote the corresponding boundaries for the index j. Recall that $l_n = n^{\alpha_0} d_n^{\alpha_0+1} = n^{\alpha_0} (\log n)^{(\alpha_0+1)/\gamma}$ and l_n^{-1} is the approximation rate announced in the theorem. We show that only S_1 is relevant to the asymptotics of e_n^2, namely, that $l_n S_3 = \mathrm{o}(1)$ as $n \to \infty$, while

$$\limsup_{n\to\infty} \, l_n S_2 = \mathrm{o}(1) \qquad \text{as } U \to \infty. \quad (18)$$

Let $w_j := t_j - t_{j-1}$, $u_j := d_n t_j$ be the normalized knots and denote by $v_j := u_j - u_{j-1} = d_n w_j$ the corresponding dilated interval lengths. It follows by the definition of $\alpha(\cdot)$-local stationarity (1) that for large n,

$$\begin{aligned} e_{n,j}^2 &= c(t_{j-1}) \int_{t_{j-1}}^{t_j} (t - t_{j-1})^{\alpha(t_{j-1})} dt \; (1 + r_{n,j}) \\ &= B_{j-1} \, (t_j - t_{j-1})^{\alpha(t_{j-1})+1} \, (1 + r_{n,j}) \\ &= B_{j-1} \, (v_j/d_n)^{\alpha(t_{j-1})+1} \, (1 + r_{n,j}), \end{aligned} \quad (19)$$

where $|T_n| = \max_j w_j = \mathrm{o}(1)$ and $\max_j r_{n,j} = \mathrm{o}(1)$ as $n \to \infty$ and

$$B_j := \frac{c(t_j)}{\alpha(t_j) + 1}, \qquad j = 1, \ldots, N.$$

First, we evaluate S_3. Recall that for $j > J_2$ we have $\rho d_n \le u_{j-1} < u_j \le d_n$. We use now the following property of regularly varying functions (see, e.g., [10]): convergence in (8) is uniform for all intervals $0 < a \le \lambda \le b < \infty$. Using this uniformity we obtain, for some $C_1 > 0$,

$$\inf_{u_{j-1}\le u\le u_j} \widetilde{p}(u) \ge \inf_{\rho d_n\le u\le d_n} \widetilde{p}(u) \ge C_1\, \widetilde{p}(d_n).$$

It follows by (5) that

$$\int_{u_{j-1}}^{u_j} \widetilde{p}(u)du = \int_{t_{j-1}}^{t_j} \widetilde{p}_n(t)dt = \frac{1}{n}\,.$$

Hence, for some $C_2 > 0$,

$$\begin{aligned} v_j &\le \left(\inf_{u_{j-1}\le u\le u_j} \widetilde{p}(u)\right)^{-1} \int_{u_{j-1}}^{u_j} \widetilde{p}(u)du \le \frac{1}{n}\left(\inf_{u_{j-1}\le u\le u_j} \widetilde{p}(u)\right)^{-1} \\ &\le \frac{1}{C_1\, n\widetilde{p}(d_n)} \le C_2 \frac{d_n^{|r|+1}}{n}, \quad j = J_2+1,\dots,N, \end{aligned} \tag{20}$$

and $\max_{j>J_2} w_j = d_n^{|r|}/n$. Recall that by assumption $(C1)$,

$$\alpha_1 := \inf_{t\in[\rho,1]} \alpha(t) > \alpha_0.$$

Therefore, for large n, we get by (19) and (20), $C_3, C_4 > 0$,

$$S_3 \le n \max_{j>J_2} e_{n,j}^2 \le nC_3(v_j/d_n)^{\alpha_1+1} \le C_4 \frac{d_n^{|r|(\alpha_1+1)}}{n^{\alpha_1}} = \mathrm{o}(l_n^{-1}) \text{ as } n\to\infty. \tag{21}$$

Now consider the first two zones corresponding to S_1, S_2. We have by definition

$$\int_0^{u_j} p(u)du = \frac{j}{n}\,, \qquad 0 \le j < J.$$

Since the function $p_n(t), t \in [0,1]$, is non-increasing, the sequence $\{v_j\}$ is non-decreasing. In fact,

$$\frac{1}{n} = \int_{u_{j-1}}^{u_j} p(u)du \in [p(u_j)v_j, p(u_{j-1})v_j],$$

and therefore,

$$\frac{1}{np(u_{j-1})} \le v_j \le \frac{1}{np(u_j)} \le v_{j+1} \tag{22}$$

and it follows by (A1) that $\max_{j\le J_2} w_j = \mathrm{o}(1)$ as $n\to\infty$. For $j \le J_2$, the bounds (19) and (22) yield for n large,

$$\begin{aligned}
e_{n,j}^2 &= B_{j-1}(v_j/d_n)^{\alpha(t_{j-1})}\frac{v_j}{d_n}(1+\mathrm{o}(1))\\
&\le B_{j-1}(nd_n p(u_j))^{-\alpha(t_{j-1})}\ \frac{v_j}{d_n}(1+\mathrm{o}(1))\\
&\le B_{j-1}(np(u_j))^{-\alpha(t_{j-1})}d_n^{-\alpha_0-1}v_j(1+\mathrm{o}(1))\\
&= B_{j-1}l_n^{-1}n^{-(\alpha(t_{j-1})-\alpha_0)}p(u_j)^{-\alpha(t_{j-1})}v_j(1+\mathrm{o}(1)). \qquad (23)
\end{aligned}$$

From now on, we proceed differently in the first and in the second zone.

For the second zone, $J_1 \le j \le J_2$, we do not care about the constant by using

$$B_j \le B_* := \max_{0\le t\le 1}\ \frac{c(t)}{\alpha(t)+1}\ . \qquad (24)$$

Next, (7) and (9) give

$$p(u_j)^{-\alpha(t_{j-1})} \le C\exp\{q_2\alpha(t_{j-1})u_j^\gamma\} \le C\exp\{\beta_1 u_j^\gamma\}, \quad C>0, \qquad (25)$$

where $\beta_1 := q_2\sup_{0\le t\le\rho}\alpha(t)$. On the other hand, we infer from (9) that

$$n^{-(\alpha(t_{j-1})-\alpha_0)} = n^{-\frac{\alpha(t_{j-1})-\alpha_0}{t_{j-1}^\gamma}t_{j-1}^\gamma} \le n^{-\beta_2\frac{u_{j-1}^\gamma}{\log n}} = \exp\{-\beta_2 u_{j-1}^\gamma\}, \qquad (26)$$

where $\beta_2 := \inf_{0\le t\le\rho}(\alpha(t)-\alpha_0)/t^\gamma > \beta_1$ by (9).

Recall that by (10), we have $1-q_2\rho^\gamma > 0$. Moreover, for $U \le u_j \le \rho d_n$, we derive from (7) and (22)

$$v_j \le n^{-1}p(\rho d_n)^{-1} \le Cn^{-1}\exp\{q_2(\rho d_n)^\gamma\} = Cn^{-(1-q_2\rho^\gamma)}, \quad C>0,$$

and it follows

$$u_{j+1}^\gamma - u_{j-1}^\gamma = u_{j-1}^\gamma\left(\left(\frac{u_{j+1}}{u_{j-1}}\right)^\gamma - 1\right) = O\left(d_n^\gamma n^{-(1-q_2\rho^\gamma)}\right) = \mathrm{o}(1) \text{ as } n\to\infty \qquad (27)$$

uniformly in $J_1 \le j \le J_2$.

Since $\{v_j\}$ is non-decreasing, (27) implies an integral bound

$$\begin{aligned}
&\exp\{-\beta_2 u_{j-1}^\gamma\}\exp\{\beta_1 u_j^\gamma\}v_j\\
&= \exp\{\beta_2[u_{j+1}^\gamma - u_{j-1}^\gamma]\}\exp\{\beta_1 u_j^\gamma - \beta_2 u_{j+1}^\gamma\}v_j\\
&\le C\inf_{u_j\le u\le u_{j+1}}\exp\{\beta_1 u^\gamma - \beta_2 u^\gamma\}v_{j+1}\\
&\le C\int_{u_j}^{u_{j+1}} e^{-(\beta_2-\beta_1)u^\gamma}du, \quad C>0. \qquad (28)
\end{aligned}$$

By plugging (24), (26), and (28) into (23), and summing up the resulting bounds over $J_1 < j \le J_2$, we obtain

$$S_2 \le B_* l_n^{-1} \int_U^\infty e^{-(\beta_2-\beta_1)u^\gamma} du\,(1+\mathrm{o}(1)) \qquad \text{as } n\to\infty. \tag{29}$$

Therefore, (18) is valid.

In the first zone, $j \le J_1, t_j \le U/d_n$, the knots are uniformly small. Hence, B_{j-1} are uniformly close to B due to the continuity of the functions $\alpha(\cdot)$ and $c(\cdot)$. Moreover, by $(C2)$ for any $\varepsilon > 0$, we have for all n large enough

$$\alpha_0 + (b-\varepsilon)t_{j-1}^\gamma \le \alpha(t_{j-1}) \le \alpha_0 + (b+\varepsilon)t_{j-1}^\gamma, \qquad j \le J_1. \tag{30}$$

Hence (23) yields

$$\begin{aligned} e_{n,j}^2 &\le (B+\varepsilon) l_n^{-1} n^{-(b-\varepsilon)t_{j-1}^\gamma}\ p(u_j)^{-\alpha_0} p(u_j)^{-(\alpha(t_{j-1})-\alpha_0)}\ v_j \\ &= (B+\varepsilon) l_n^{-1}\ n^{-(b-\varepsilon)(u_{j-1}/d_n)^\gamma}\ p(u_j)^{-\alpha_0} p(u_j)^{-(\alpha(t_{j-1})-\alpha_0)}\ v_j. \end{aligned} \tag{31}$$

Recall that by the definition of d_n, we have

$$n^{-(b-\varepsilon)(u_{j-1}/d_n)^\gamma} = n^{-(b-\varepsilon)u_{j-1}^\gamma/\log n} = \exp\{-(b-\varepsilon)u_{j-1}^\gamma\}.$$

Since $p(\cdot)$ in non-increasing and $\{v_j\}$ is non-decreasing, we also have an integral bound

$$\begin{aligned} &\exp\{-(b-\varepsilon)u_{j-1}^\gamma\}\ p(u_j)^{-\alpha_0} v_j \\ &= \exp\{(b-\varepsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\}\ \ p(u_j)^{-\alpha_0} \exp\{-(b-\varepsilon)u_{j+1}^\gamma\} v_j \\ &\le \exp\{(b-\varepsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \inf_{u_j \le u \le u_{j+1}} (p(u)^{-\alpha_0} e^{-(b-\varepsilon)u^\gamma)}\ v_j \\ &\le \exp\{(b-\varepsilon)[u_{j+1}^\gamma - u_{j-1}^\gamma]\} \int_{u_j}^{u_{j+1}} p(u)^{-\alpha_0} e^{-(b-\varepsilon)u^\gamma} du. \end{aligned} \tag{32}$$

Moreover, for $u_j \le U$, we derive from (A1) and (22)

$$v_j \le n^{-1} p(U)^{-1}.$$

By using the convexity and the concavity of the power function for $\gamma \ge 1$ and $\gamma \le 1$, respectively, we get

$$\begin{aligned} u_{j+1}^\gamma - u_{j-1}^\gamma &\le \gamma U^{\gamma-1}(u_{j+1}-u_j) = \gamma U^{\gamma-1}(v_j + v_{j+1}) \\ &\le 2\gamma U^{\gamma-1} v_{j+1} = \mathrm{o}(1) \qquad \text{as } n\to\infty \qquad (\gamma \ge 1); \\ u_{j+1}^\gamma - u_{j-1}^\gamma &\le (u_{j+1}-u_{j-1})^\gamma \\ &= (v_j + v_{j+1})^\gamma = \mathrm{o}(1) \qquad \text{as } n\to\infty \qquad (\gamma \le 1). \end{aligned}$$

Therefore, the exponential factor in (32) turns out to be negligible.

Finally, for $u_j \le U$, the property $d_n \to \infty$ yields

$$p(u_j)^{-(\alpha(t_{j-1})-\alpha_0)} \le \max\{1, p(U)^{-\max_{0\le t\le U/d_n}(\alpha(t)-\alpha_0)}\} = 1 + \mathrm{o}(1). \tag{33}$$

By plugging (32) and (33) into (31), and summing up the resulting bounds over $j \le J_1$, we obtain

$$S_1 \le (B+2\varepsilon) l_n^{-1} \int_0^\infty p(u)^{-\alpha_0} e^{-(b-\varepsilon)u^\gamma} du \qquad \text{as } n \to \infty.$$

Since ε can be chosen arbitrarily small, we arrive at

$$\limsup_{n\to\infty} \, l_n S_1 \le B \int_0^\infty p(u)^{-\alpha_0} e^{-bu^\gamma} du = K. \tag{34}$$

Combining (21), (29), and (34) gives the desired upper bound.

The lower bound is obtained along the same lines: since S_2 and S_3 are asymptotically negligible, we shall evaluate only S_1 starting again from (19). As in (23), we have

$$\begin{aligned} e_{n,j}^2 &= B_{j-1} (v_j/d_n)^{\alpha(t_{j-1})} \frac{v_j}{d_n} (1+\mathrm{o}(1)) \\ &\ge B_{j-1} (n d_n p(u_{j-1}))^{-\alpha(t_{j-1})} \ \frac{v_j}{d_n} (1+\mathrm{o}(1)) \\ &= B_{j-1} n^{-\alpha_0} n^{-(\alpha(t_{j-1})-\alpha_0)} p(u_{j-1})^{-\alpha(t_{j-1})} d_n^{-\alpha_0-1} d_n^{\alpha_0-\alpha(t_{j-1})} v_j (1+\mathrm{o}(1)) \\ &= B_{j-1} l_n^{-1} n^{-(\alpha(t_{j-1})-\alpha_0)} p(u_{j-1})^{-\alpha(t_{j-1})} d_n^{\alpha_0-\alpha(t_{j-1})} v_j (1+\mathrm{o}(1)). \end{aligned} \tag{35}$$

Recall that for $j \le J_1$ and the constants B_{j-1} are uniformly close to B. Moreover, by using (30), we have for large n,

$$d_n^{\alpha_0-\alpha(t_{j-1})} \ge d_n^{-(b+\varepsilon)t_{j-1}^\gamma} \ge d_n^{-(b+\varepsilon)(U/d_n)^\gamma} = 1+\mathrm{o}(1).$$

Hence, (35) yields

$$\begin{aligned} e_{n,j}^2 &\ge (B-\varepsilon) l_n^{-1} n^{-(b+\varepsilon)t_{j-1}^\gamma} \ p(u_{j-1})^{-\alpha_0} p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} \ v_j \\ &= (B-\varepsilon) l_n^{-1} \ n^{-(b+\varepsilon)(u_{j-1}/d_n)^\gamma} \ p(u_{j-1})^{-\alpha_0} p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} \ v_j, \end{aligned} \tag{36}$$

where as before

$$n^{-(b+\varepsilon)(u_{j-1}/d_n)^\gamma} = n^{-(b+\varepsilon)u_{j-1}^\gamma/\log n} = \exp\{-(b+\varepsilon)u_{j-1}^\gamma\}.$$

Since $p(\cdot)$ is non-increasing and $\{v_j\}$ is non-decreasing, we also have an integral bound

$$
\begin{aligned}
&\exp\{-(b+\varepsilon)u_{j-1}^{\gamma}\}p(u_{j-1})^{-\alpha_0}v_j \\
&= \exp\{(b+\varepsilon)[u_{j-2}^{\gamma}-u_{j-1}^{\gamma}]\}p(u_{j-1})^{-\alpha_0}\exp\{-(b+\varepsilon)u_{j-2}^{\gamma}\}v_j \\
&\geq \exp\{(b+\varepsilon)[u_{j-2}^{\gamma}-u_{j-1}^{\gamma}]\}\inf_{u_{j-2}\leq u\leq u_{j-1}}\left(p(u)^{-\alpha_0}e^{-(b+\varepsilon)u^{\gamma}}\right)v_{j-1} \\
&\geq \exp\{(b+\varepsilon)[u_{j+1}^{\gamma}-u_{j-1}^{\gamma}]\}\int_{u_{j-2}}^{u_{j-1}}p(u)^{-\alpha_0}e^{-(b+\varepsilon)u^{\gamma}}du. \qquad (37)
\end{aligned}
$$

We have already seen that the exponential factor in (37) is negligible.

Finally, for $u_j \leq U$, the fact that $d_n \to \infty$ implies (cf. (33))

$$
p(u_{j-1})^{-(\alpha(t_{j-1})-\alpha_0)} \geq \min\{1, p(0)^{-\max_{0\leq t\leq U/d_n}(\alpha(t)-\alpha_0)}\} = 1+\mathrm{o}(1). \qquad (38)
$$

By plugging (37) and (38) into (36), and summing up the resulting bounds over $j \leq J_1$, we obtain

$$
S_1 \geq (B-2\varepsilon)l_n^{-1}\int_0^U p(u)^{-\alpha_0}e^{-(b+\varepsilon)u^{\gamma}}du \qquad \text{as } n\to\infty.
$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we arrive at

$$
\liminf_{n\to\infty} l_n S_1 \geq B\int_0^U p(u)^{-\alpha_0}e^{-bu^{\gamma}}du.
$$

Finally,

$$
\liminf_{n\to\infty} l_n e_n^2 \geq \sup_{U>0}\liminf_{n\to\infty} l_n S_1 \geq B\int_0^{\infty} p(u)^{-\alpha_0}e^{-bu^{\gamma}}du = K. \qquad (39)
$$

This is the desired lower bound. □

Proof of Theorem 2:
Applying the notation of Theorem 1, we have for an interval approximation error

$$
e_{n,j}^2 = B_{j-1}\, w_j^{\alpha(t_{j-1})+1}\,(1+r_{n,j}), \quad w_j = t_j - t_{j-1}, j = 1,\ldots,n,
$$

where $\max_j r_{n,j} = \mathrm{o}(1)$ as $n\to\infty$. Now for a small enough $\rho > 0$, similarly to Theorem 1, we get

$$
\int_{\rho}^{1} e_n(t)^2 dt \leq C/n^{\alpha_1},\ C>0, \qquad \alpha_1 := \inf_{t\in[\rho,1]}\alpha(t) > \alpha_0,
$$

that is only $e_{n,j}$ such that $[t_{j-1},t_j] \subset [0,\rho]$ are relevant for the asymptotics, say, $e_{n,j}, j = 1,\ldots,J = J(\rho,n)$. Let us denote the approximation rate $L_n := n^{\alpha_0}(\log n)^{1/\gamma}$. Next, for $S_1 := \sum_{j=1}^{J} e_{n,j}$ and for a small enough ρ, we have by continuity of the design density $p(\cdot)$ and by the mean value theorem

$$
\begin{aligned}
e_{n,j}^2 &= B_{j-1}\,(np(\eta_j))^{-\alpha(t_{j-1})} w_j\,(1+\mathrm{o}(1)) \\
&\le \frac{B}{p(0)^{\alpha_0}}(1+\varepsilon)\,n^{-\alpha_0}\int_{t_{j-2}}^{t_{j-1}} e^{-(b-\varepsilon)t^{\gamma}\log n}dt\,(1+\mathrm{o}(1)) \\
&= L_n^{-1}\,\frac{B}{p_0^{\alpha_0}}(1+\varepsilon)\int_{u_{j-2}}^{u_{j-1}} e^{-(b-\varepsilon)u^{\gamma}}du\,(1+\mathrm{o}(1)),
\end{aligned}
$$

where $p_0 := p(0)$. By summing up, we obtain

$$
\limsup_{n\to\infty} L_n S_1 \le \frac{B}{p_0^{\alpha_0}}(1+\varepsilon)\int_0^{\infty} e^{-(b-\varepsilon)u^{\gamma}}du = \frac{B}{p_0^{\alpha_0}}(1+\varepsilon)\frac{\Gamma(1/\gamma+1)}{(b-\varepsilon)^{1/\gamma}}.
$$

Hence,

$$
\limsup_{n\to\infty} L_n e_n^2 = \limsup_{n\to\infty} L_n S_1 \le \frac{B}{p_0^{\alpha_0}}\,(1+\varepsilon)\,\frac{\Gamma(1/\gamma+1)}{(b-\varepsilon)^{1/\gamma}}.
$$

Since ε can be chosen arbitrary small, we get

$$
\limsup_{n\to\infty} L_n e_n^2 \le \frac{B}{p_0^{\alpha_0}}\,\frac{\Gamma(1/\gamma+1)}{b^{1/\gamma}} = K_1.
$$

The lower bound follows from similar arguments. This completes the proof. □

Acknowledgements We are grateful to both anonimous referees for useful advice concerning this work and eventual future development of the subject.

Research of E. Hashorva was partially supported by the Swiss National Science Foundation grant 200021-140633/1 and RARE -318984 (an FP7 Marie Curie IRSES Fellowship). Research of M. Lifshits was supported by RFBR grant 13-01-00172 and by SPbSU grant 6.38.672.2013. Research of O. Seleznjev was supported by the Swedish Research Council grant 2009-4489.

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