EXTREMES OF A CLASS OF NON-HOMOGENEOUS GAUSSIAN RANDOM FIELDS

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Abstract: This contribution establishes exact tail asymptotics of $\sup_{(s,t)\in \boldsymbol{E}} X(s,t)$ for a large class of non-homogeneous Gaussian random fields X on a bounded convex set $\boldsymbol{E} \subset \mathbb{R}^2$, with variance function that attains its maximum on a segment on $\boldsymbol{E}$. These findings extend the classical results for homogeneous Gaussian random fields and Gaussian random fields with unique maximum point of the variance. Applications of our result include the derivation of the exact tail asymptotics of the Shepp statistics for stationary Gaussian processes, Brownian bridge and fractional Brownian motion as well as the exact tail asymptotic expansion for the maximum loss and span of stationary Gaussian processes.

Key Words: Extremes; non-homogeneous Gaussian random fields; Shepp statistics; fractional Brownian motion; maximum loss; span of Gaussian processes; Pickands constant; Piterbarg constant; generalized Pickands-Piterbarg constant.
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1. Introduction

Consider the fractional Brownian motion (fBm) incremental random field

$$X_\alpha(s,t) = B_\alpha(s+t) - B_\alpha(s), \quad (s,t) \in [0,\infty)^2,$$

where $\{B_\alpha(t), t \in \mathbb{R}\}$ is a standard fBm with Hurst index $\alpha/2 \in (0,1]$ which is a centered self-similar Gaussian process with stationary increments and covariance function

$$\mathrm{Cov}(B_\alpha(t), B_\alpha(s)) = \frac{1}{2}(|t|^\alpha + |s|^\alpha - \mid t-s \mid^\alpha), \quad s,t \in \mathbb{R}.$$

For the case $\alpha = 1$ both $X_\alpha(s,t)$ and its standardised version $X^*_\alpha(s,t) = X_\alpha(s,t)/t^{\alpha/2}$ appear naturally as limit models, see e.g., [8]. In the literature

$$Y_\alpha(t) = \sup_{s\in[0,S]} X_\alpha(s,t)$$

is referred to as the Shepp statistics of fBm, whereas $Y^*_\alpha(t) = \sup_{s\in[0,S]} X^*_\alpha(s,t)$ as the standardised Shepp statistics. Distributional results for Y^*_1 are derived in [28], see also [27] and Theorem 3.2 in [8]. Other important results for the Shepp statistics of Brownian motion and related quantities are presented in [11, 14, 29]. The first known result for the extremes of the Shepp statistics of Brownian motion goes back to [32], which is complemented in [17] for the case of fBm with $\alpha \in (0,1)$. In view of the aforementioned papers for any $\alpha \in (0,1]$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,1]^2} X_\alpha(s,t) > u\right) = C_\alpha u^{4/\alpha-2}\Psi(u)(1+o(1)), \quad u\to\infty \tag{1}$$

holds with C_α a positive constant and $\Psi(\cdot)$ the survival function of an $N(0,1)$ random variable. There is no result for the case $\alpha \in (1,2)$ in the literature; we shall cover this gap in Proposition 3.5.

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Results for the tail asymptotics of supremum of the standardised Shepp statistics can be derived using the findings of [7] and [20], see also [18, 19]. However, this is not the case for the tail asymptotics of the supremum of the Shepp statistics Y_α; no theoretical results in the literature can be applied for this case. This is due to the fact that on $[0,1]^2$ the variance of X_α attains its maximum at an infinite number of points, i.e., its maximal value is attained for any $s \in [0,1]$ and $t = 1$. In the asymptotic theory of Gaussian random fields, if the random field has a non-constant variance function, which attains its maximum at a unique (or finite) number of points, then under the so-called Piterbarg conditions, the exact tail asymptotics of supremum of Gaussian random fields with certain (E, α) structures for the variance and the correlation functions are derived by relying on the Double-Sum method, see e.g., the standard monograph [24].
The principle aim of this contribution is to extend Piterbarg's asymptotic theory for Gaussian random fields to the case where the maximum of the variance function on a bounded convex set $\boldsymbol{E}$ is attained on finite number of disjoint segments on $\boldsymbol{E}$. In particular, we assume that $\{X(s,t), (s,t) \in \boldsymbol{E}\}$, $\boldsymbol{E} = [0,S] \times [0,T], S, T > 0$, is a centered Gaussian random field with variance function $\sigma^2(s,t) = \mathrm{Var}(X(s,t))$ that satisfies the following assumption.

Assumption A1. There exists some positive function $\sigma(t)$ which attains its unique maximum on $[0,T]$ at T, and further

$$\sigma(s,t) = \sigma(t), \quad \forall (s,t) \in \boldsymbol{E}, \quad \sigma(t) = 1 - b(T-t)^\beta (1 + o(1)), \quad t \uparrow T \tag{2}$$

hold for some $\beta, b > 0$.
We shall impose the following assumption on the correlation function $r(s,t,s',t') = \mathbb{E}\left(\overline{X}(s,t)\overline{X}(s',t')\right)$ where $\overline{X}(s,t) = X(s,t)/\sigma(s,t)$:
Assumption A2. There exist constants $a_1 > 0, a_2 > 0, a_3 \neq 0$ and $\alpha_1, \alpha_2 \in (0,2]$ such that

$$r(s,t,s',t') = 1 - \left(|a_1(s-s')|^{\alpha_1} + |a_2(t-t') + a_3(s-s')|^{\alpha_2}\right)(1+o(1)) \tag{3}$$

holds uniformly with respect to $s, s' \in [0,S]$, as $|s-s'| \to 0, t, t' \uparrow T$, and further, there exists some constant $\delta_0 \in (0,T)$ such that

$$r(s,t,s',t') < 1 \tag{4}$$

holds for any $s, s' \in [0,S]$ satisfying $s \neq s'$, and $t, t' \in [\delta_0, T]$.

Note that in **A2** we assume that $a_3 \neq 0$, which includes a large class of correlation functions with (E, α) structure dealt with in [24]; the classical case $a_3 = 0$ is discussed in Remark 2.3.
Our main result, presented in Theorem 2.2 (and stated in higher generality in Remarks 2.4), derives the exact tail asymptotic behaviour of supremum of non-homogeneous Gaussian random fields X satisfying **A1-A2** and a Hölder condition formulated below in Assumption **A3**.
As an illustration to the derived theory, we analyze exact asymptotics of the tail distribution of extremes of Shepp statistics, the maximum loss and the span for a large class of Gaussian processes.

Organization of the paper: Our principal findings are presented in Section 2 followed by two sections dedicated to applications and examples. All the proofs are relegated to Section 5 and Appendix.

2. Main Results

In this section we are concerned with the asymptotics of

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right), \qquad u\to\infty$$

discussing first the case that $\boldsymbol{E} = [0,S]\times[0,T]$.

Pickands and Piterbarg Lemmas (cf. [24]) are fundamental in the analysis of the tail asymptotic behaviour of supremum of non-smooth centered Gaussian processes and Gaussian random fields. Restricting ourselves to the case that $\{X(t), t\ge 0\}$ is a centered stationary Gaussian process with a.s. continuous sample paths and correlation function $r(t)$, such that $r(t) = 1 - t^\alpha(1+o(1))$ as $t\to 0$, with $\alpha\in(0,2]$, and $r(t)<1$ for all $t>0$, in view of the seminal papers by J. Pickands III (see [21, 22]), for any $T\in(0,\infty)$

$$\mathbb{P}\left(\sup_{t\in[0,T]} X(t) > u\right) = \mathcal{H}_\alpha T u^{\frac{2}{\alpha}} \Psi(u)(1+o(1)), \qquad u\to\infty. \tag{5}$$

Here $\mathcal{H}_\alpha$ is the *Pickands constant* defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty}\frac{1}{T}\mathcal{H}_\alpha[0,T]\in(0,\infty), \text{ with } \mathcal{H}_\alpha[0,T] = \mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - t^\alpha\right)\right)\right).$$

The derivation of (5) is based on *Pickands Lemma* which states that

$$\mathbb{P}\left(\sup_{t\in[0,u^{-\frac{2}{\alpha}}T]} X(t) > u\right) = \mathcal{H}_\alpha[0,T]\Psi(u)(1+o(1)), \qquad u\to\infty. \tag{6}$$

In [23] V.I. Piterbarg rigorously proved Pickands theorem and further derived a crucial extension of (6) which we shall refer to as *Piterbarg Lemma*; it states that

$$\mathbb{P}\left(\sup_{t\in[0,u^{-\frac{2}{\alpha}}T]} \frac{X(t)}{1+bt^\alpha} > u\right) = \mathcal{P}_\alpha^b[0,T]\Psi(u)(1+o(1)), \qquad u\to\infty \tag{7}$$

holds for any $b>0$ with

$$\mathcal{P}_\alpha^b[0,T] = \mathbb{E}\left(\exp\left(\sup_{t\in[0,T]}\left(\sqrt{2}B_\alpha(t) - (1+b)t^\alpha\right)\right)\right)\in(0,\infty).$$

The positive constant (referred to as *Piterbarg constant*) given by

$$\mathcal{P}_\alpha^b = \lim_{T\to\infty}\mathcal{P}_\alpha^b[0,T]\in(0,\infty)$$

appears naturally when dealing with the extremes of non-stationary Gaussian processes or Gaussian random fields, see e.g., [24] and our main result below. It is known that $\mathcal{H}_1 = 1$, $\mathcal{H}_2 = 1/\sqrt{\pi}$, and

$$\mathcal{P}_1^b = 1+\frac{1}{b}, \qquad \mathcal{P}_2^b = \frac{1}{2}\left(1+\sqrt{1+\frac{1}{b}}\right), \quad b>0 \tag{8}$$

see e.g., [2, 10, 12, 16, 15, 13].

We note in passing that for stationary Gaussian processes [3] and [5] presented new elegant proofs of (5) without using Pickands Lemma. The following extension of Pickands and Piterbarg Lemmas plays an important role in our analysis. Hereafter we denote by $\tilde{B}_\alpha$ and B_α two independent fBm's defined on $\mathbb{R}$ with Hurst index $\alpha/2\in(0,1]$. Recall that $\Psi(\cdot)$ denotes the survival function of an $N(0,1)$ random variable; we write below $\Gamma(\cdot)$ for the Euler Gamma function.

Lemma 2.1. *Let $\{\eta(s,t),(s,t)\in[0,\infty)^2\}$ be a centered homogeneous Gaussian random field with covariance function*

$$r_\eta(s,t)=\exp\Big(-|a_1s|^{\alpha_1}-|a_2t-a_3s|^{\alpha_2}\Big),\quad (s,t)\in[0,\infty)^2,$$

where constants $\alpha_i\in(0,2], i=1,2,\ a_1>0, a_2>0, a_3\in\mathbb{R}$. Let further b,S,T be three positive constants. If $\beta\ge\alpha_2\ge\alpha_1$, then for any positive measurable function $g(u),u>0$ satisfying $\lim_{u\to\infty}g(u)/u=1$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,Su^{-\frac{2}{\alpha_1}}]\times[0,Tu^{-\frac{2}{\alpha_2}}]}\frac{\eta(s,t)}{1+bt^\beta}>g(u)\right)=\mathcal{H}_Y^b[S,T]\Psi(g(u))(1+o(1)),\qquad u\to\infty, \tag{9}$$

where

$$\mathcal{H}_Y^d[S,T]=\mathbb{E}\left(\exp\left(\sup_{(s,t)\in[0,S]\times[0,T]}\Big(\sqrt{2}Y(s,t)-\sigma_Y^2(s,t)-d(t)\Big)\right)\right)\in(0,\infty), \tag{10}$$

with $\sigma_Y^2(s,t)=Var(Y(s,t))$ and

$$Y(s,t)=\left\{\begin{array}{ll} Y_1(s,t):=\tilde{B}_{\alpha_1}(a_1s)+B_{\alpha_2}(a_2t-a_3s), & \alpha_1=\alpha_2,\\ Y_2(s,t):=\tilde{B}_{\alpha_1}(a_1s)+B_{\alpha_2}(a_2t), & \alpha_1<\alpha_2,\end{array}\right.\quad d(t)=\left\{\begin{array}{ll} 0, & \beta>\alpha_2,\\ bt^\beta, & \beta=\alpha_2,\end{array}\right.\quad (s,t)\in[0,\infty)^2. \tag{11}$$

Using the definition of Y_1 and Y_2 appearing in (11) we shall determine, for given a_i's, α_i's and b,β as above, the following constants (referred to as *generalized Pickands-Piterbarg constants*)

$$\mathcal{M}_{Y,\beta}^b=\lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{S}\mathcal{H}_Y^b[S,T]\in(0,\infty),$$

and

$$\widetilde{\mathcal{M}}_{Y,\beta}^b=\lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{S}\mathbb{E}\left(\exp\left(\sup_{(s,t)\in[0,S]\times[-T,T]}\Big(\sqrt{2}Y(s,t)-\sigma_Y^2(s,t)-bt^\beta\Big)\right)\right)\in(0,\infty).$$

Here $\mathcal{M}_{Y,\beta}^b$ and $\widetilde{\mathcal{M}}_{Y,\beta}^b$ are defined only for $\beta=\alpha_2$. Note that we suppress a_i's and α_i's in the definition of $\mathcal{M}_{Y,\beta}^b$ and $\widetilde{\mathcal{M}}_{Y,\beta}^b$ since they appear directly in the definition of Y.

Additional to **A1** and **A2** we shall impose the following Hölder condition, which in the literature is called *regularity*; see [24].

Assumption A3. There exist positive constants $\rho_1,\rho_2,\gamma,\mathcal{Q}$ such that

$$\mathbb{E}\left((X(s,t)-X(s',t'))^2\right)\quad\le\quad\mathcal{Q}\Big(|t-t'|^\gamma+|s-s'|^\gamma\Big)$$

holds for all $t,t'\in[\rho_1,T], s,s'\in[0,S]$ satisfying $|s-s'|<\rho_2$.

We present next our main result.

Theorem 2.2. *Let $\{X(s,t),(s,t)\in\boldsymbol{E}\},\boldsymbol{E}=[0,S]\times[0,T]$ be a centered Gaussian random field with a.s. continuous sample paths. Suppose that assumptions* **A1-A3** *are satisfied with the parameters mentioned therein. Then, as $u\to\infty$,*

i) if $\beta>\max(\alpha_1,\alpha_2)$

$$\mathbb{P}\left(\sup_{(s,t)\in\boldsymbol{E}}X(s,t)>u\right)\quad=\quad S\,\Gamma\left(1/\beta+1\right)\prod_{k=1}^2(a_k\mathcal{H}_{\alpha_k})b^{-\frac{1}{\beta}}u^{\frac{2}{\alpha_2}+\frac{2}{\alpha_1}-\frac{2}{\beta}}\Psi(u)(1+o(1)); \tag{12}$$

ii) if $\beta=\alpha_2=\alpha_1$

$$\mathbb{P}\left(\sup_{(s,t)\in\boldsymbol{E}}X(s,t)>u\right)\quad=\quad S\mathcal{M}_{Y_1,\alpha_1}^b u^{\frac{2}{\alpha_1}}\Psi(u)(1+o(1)); \tag{13}$$

iii) if $\beta = \alpha_2 > \alpha_1$

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = S a_1 a_2 \mathcal{P}_{\alpha_2}^{b a_2^{-\alpha_2}} \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u)(1+o(1)); \tag{14}$$

iv) if $\beta < \alpha_2 = \alpha_1$

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = S(a_1^{\alpha_1} + |a_3|^{\alpha_1})^{\frac{1}{\alpha_1}} \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u)(1+o(1)); \tag{15}$$

v) if $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = S a_1 \mathcal{H}_{\alpha_1} u^{\frac{2}{\alpha_1}} \Psi(u)(1+o(1)); \tag{16}$$

vi) if $\beta = \alpha_1 > \alpha_2$

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = S a_1 \mathcal{P}_{\alpha_1}^{b\left(\frac{|a_3|}{a_1 a_2}\right)^{\alpha_1}} \mathcal{H}_{\alpha_2} u^{\frac{2}{\alpha_2}} \Psi(u)(1+o(1)); \tag{17}$$

vii) if $\beta < \alpha_1$ and $\alpha_2 < \alpha_1$

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = S\,|a_3|\,\mathcal{H}_{\alpha_2} u^{\frac{2}{\alpha_2}} \Psi(u)(1+o(1)).$$

Remark 2.3. *If $a_3 = 0$, then there are only three scenarios to be considered. In particular if $\beta > \alpha_2$, then* (12) *holds. If $\beta = \alpha_2$, then* (14) *holds, whereas if $\beta < \alpha_2$, then* (16) *is valid.*

Remarks 2.4. *a) Let $\boldsymbol{E}$ be any bounded convex subset of $\mathbb{R}^2$. Assume that on $\boldsymbol{E}$ the maximum of the standard deviation $\sigma(s,t)$ is attained only on a segment $\boldsymbol{L}$ which is inside of $\boldsymbol{E}$, parallel to s-axis and of length ℓ. Then the claims of Theorem 2.2 are still vilad, by replacing S with ℓ in Cases i)-vii), $\Gamma(\cdot)$ with $2\Gamma(\cdot)$ in Case i), $\mathcal{M}^b_{Y_1,\alpha_1}$ with $\widetilde{\mathcal{M}}^b_{Y_1,\alpha_1}$ in Cases ii), $\mathcal{P}_{\alpha_2}^{b a_2^{-\alpha_2}}$ with $\widetilde{\mathcal{P}}_{\alpha_2}^{b a_2^{-\alpha_2}}$ in Case iii), and $\mathcal{P}_{\alpha_1}^{b(|a_3|/(a_1a_2))^{\alpha_1}}$ with $\widetilde{\mathcal{P}}_{\alpha_1}^{b(|a_3|/(a_1a_2))^{\alpha_1}}$ in Case vi), respectively. Here $\widetilde{\mathcal{P}}^b_\alpha$, with $b > 0$ and $\alpha \in (0,2]$ is the Piterbarg constant defined on the real line, i.e.,*

$$\widetilde{\mathcal{P}}^b_\alpha = \lim_{T\to\infty} \mathbb{E}\left(\exp\left(\sup_{t\in[-T,T]}\left(\sqrt{2}B_\alpha(t) - (1+b)t^\alpha\right)\right)\right) \in (0,\infty).$$

b) Assume that on $\boldsymbol{E}$ the maximum of the standard deviation $\sigma(s,t)$ is attained only on n segments $\{\boldsymbol{L}_i\}_{i=1}^n$ which are inside or on the boundary of $\boldsymbol{E}$, and parallel to s-axis. By the convexity of $\boldsymbol{E}$, we can always find n non-adjacent convex sets $\{\boldsymbol{E}_i\}_{i=1}^n$ such that $\boldsymbol{L}_i \subset \boldsymbol{E}_i \subset \boldsymbol{E}$, $i = 1, \cdots, n$. If further for any $i \neq j$

$$\sup_{(s,t)\in\boldsymbol{E}_i,(s',t')\in\boldsymbol{E}_j} r(s,t,s',t') < 1 \tag{18}$$

holds, then

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}} X(s,t) > u\right) = \sum_{i=1}^n \mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}_i} X(s,t) > u\right)(1+o(1)), \quad u\to\infty. \tag{19}$$

Additionally, suppose that on each $\{\boldsymbol{E}_i\}_{i=1}^n$ the assumptions **A1-A3** *are satisfied. Then an explicit expression for* (19) *can be established by applying the results in Theorem 2.2 and Remark a) above.*

c) Similar results can also be obtained when the segments $\{\boldsymbol{L}_i\}_{i=1}^n$, where the maximum of $\sigma(s,t)$ is attained, are non-parallel and disjoint. Specifically, we see from Remark b) that it is sufficient to consider the asymptotics of

$$\mathbb{P}\left(\sup_{(s,t)\in \boldsymbol{E}_i} X(s,t) > u\right), \quad u\to\infty, \quad i = 1,\cdots,n,$$

respectively. Let $(s,t)^\top$ be the transpose of (s,t). Then, for any $i=1,\cdots,n$, there is a non-degenerate lower triangular (rotation) matrix $A_i\in\mathbb{R}^{2\times 2}$ such that the maximum of the variance of $X((A_i(s,t)^\top)^\top)$ on $A_i^{-1}\boldsymbol{E}_i=\{(\tilde{s},\tilde{t}):(\tilde{s},\tilde{t})^\top=A_i^{-1}(s,t)^\top,(s,t)\in\boldsymbol{E}_i\}$ is attained on a line parallel to s-axis or t-axis. Consequently, similar results as in Theorem 2.2 can be obtained if certain assumptions as **A1-A3** *are satisfied by each $\{X((A_i(s,t)^\top)^\top),(s,t)\in A_i^{-1}\boldsymbol{E}_i\}$.*

We conclude this section with an example, which illustrates the existence of all the cases discussed in Theorem 2.2.

Example 2.5. *Consider a Gaussian random field defined as*

$$Z(s,t)=\frac{1}{\sqrt{2}}(Y(s+t)-X(s))(1-b(T-t)^{\beta}),\quad (s,t)\in[0,S]\times[0,T],$$

where b,β are two positive constants, and X,Y are two independent centered stationary Gaussian processes with covariance functions r_X,r_Y satisfying as $t\to 0$

$$r_X(t)=1-a_1t^{\alpha_1}(1+o(1)),\quad r_Y(t)=1-a_2t^{\alpha_2}(1+o(1))$$

for some constants $a_i>0,\alpha_i\in(0,2],i=1,2$. Further, assume that

$$r_X(s)<1,\ \forall\ s\in(0,S],\qquad r_Y(t)<1,\ \forall\ t\in(0,S+T].$$

It follows that the assumptions of Theorem 2.2 are satisfied by $\{Z(s,t),(s,t)\in[0,S]\times[0,T]\}$.

3. Extremes of Shepp Statistics

For a given centered Gaussian process $\{X(t),t\ge 0\}$ we shall define the incremental random field Z by

$$Z(s,t)=X(s+t)-X(s),\quad (s,t)\in[0,S]\times[0,T]. \tag{20}$$

The asymptotic analysis of the supremum of the Shepp statistics

$$Y(t)=\sup_{s\in[0,S]}Z(s,t),t\in[0,T]$$

boils down to the study of the tail asymptotics of the double-supremum $\sup_{(s,t)\in[0,S]\times[0,T]}Z(s,t)$. In this section we shall consider several important examples which can be analysed utilising the theory developed in Section 2.

3.1. **Stationary Gaussian processes.** Consider the Gaussian random field Z as in (20) where X is a centered stationary Gaussian process with covariance function r_X satisfying the following conditions:

S1: $r_X(t)$ attains its minimum on $[0,T]$ at the unique point $t=T$;

S2: there exist positive constants α_1,a_1,a_2 and $\alpha_2\in(0,2)$ such that

$$r_X(t)=r_X(T)+a_1(T-t)^{\alpha_1}(1+o(1)),\quad t\to T,\qquad r_X(t)=1-a_2t^{\alpha_2}(1+o(1)),\quad t\to 0;$$

S3: $r_X(s)<1$ for any $s\in(0,S+T]$.

Proposition 3.1. *Let $\{Z(s,t),(s,t)\in[0,S]\times[0,T]\}$ be an incremental random field given as in (20) with r_X satisfying* **S1-S3**. *Suppose that r_X is twice continuously differentiable on $[\mu,T]$ for some $\mu\in(0,T)$, $\left|r_X''(T)\right|\in(0,\infty)$, and let*

$b_i = a_i/\rho_T^2, i = 1, 2$ *with* $\rho_T = \sqrt{2(1 - r_X(T))}$. *Then, as* $u \to \infty$,

(*i*) *if* $\alpha_1 > \alpha_2$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) > u\right) = S\Gamma(1/\alpha_1+1)\mathcal{H}_{\alpha_2}^2 b_2^{\frac{2}{\alpha_2}} b_1^{-\frac{1}{\alpha_1}} \left(\frac{u}{\rho_T}\right)^{\frac{4}{\alpha_2}-\frac{2}{\alpha_1}} \Psi\left(\frac{u}{\rho_T}\right)(1+o(1));$$

(*ii*) *if* $\alpha_1 = \alpha_2$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) > u\right) = S\mathcal{M}_{Y,\alpha_1}^{b_1} \left(\frac{u}{\rho_T}\right)^{\frac{2}{\alpha_2}} \Psi\left(\frac{u}{\rho_T}\right)(1+o(1)),$$

where

$$Y(s,t) := \tilde{B}_{\alpha_2}\left(b_2^{\frac{1}{\alpha_2}} s\right) + B_{\alpha_2}\left(b_2^{\frac{1}{\alpha_2}} t - b_2^{\frac{1}{\alpha_2}} s\right),\ (s,t)\in[0,\infty)^2;$$

(*iii*) *if* $\alpha_1 < \alpha_2$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} Z(s,t) > u\right) = S(2b_2)^{\frac{1}{\alpha_2}}\mathcal{H}_{\alpha_2} \left(\frac{u}{\rho_T}\right)^{\frac{2}{\alpha_2}} \Psi\left(\frac{u}{\rho_T}\right)(1+o(1)).$$

We present two important examples that illustrate Proposition 3.1.

Example 3.2. *(Slepian process) Consider* X *to be the Slepian process, i.e.,*

$$X(t) = B_1(t+1) - B_1(t), \quad t\in[0,\infty),$$

with B_1 *the standard Brownian motion.It follows that the assumptions of Proposition 3.1 are satisfied, hence as* $u\to\infty$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,1]\times[0,\frac{1}{2}]} Z(s,t) > u\right) = \mathcal{M}_{Y,1}^1 u^2\Psi(u)(1+o(1))$$

holds with $Y(s,t) := \tilde{B}_1(s) + B_1(t-s),\ (s,t)\in(0,\infty)^2$.

Example 3.3. *(Ornstein-Uhlenbeck process) Consider a centered stationary Gaussian process* X *with covariance function* $r(t) = e^{-t}, t \ge 0$. *Then following Proposition 3.1*

$$\mathbb{P}\left(\sup_{(s,t)\in[0,1]^2} Z(s,t) > u\right) = \mathcal{M}_{Y,1}^{b_1} b_1 u^2\Psi(\sqrt{b_1}u)(1+o(1)), \quad u\to\infty,$$

with $b_1 = e^{-1}/(2(1-e^{-1}))$, $b_2 = 1/(2(1-e^{-1}))$ *and* $Y(s,t) := \tilde{B}_1(b_2 s) + B_1(b_2 t - b_2 s),\ (s,t)\in(0,\infty)^2$.

3.2. **Brownian bridge.** In this section we analyze

$$Z(s,t) = X(s+t) - X(s), \quad s, s+t\in[0,1], \tag{21}$$

where $X(s) := B_1(s) - sB_1(1), s\in[0,1]$ is a Brownian bridge (recall B_1 is a standard Brownian motion). Clearly, X is non-stationary and therefore we cannot apply Proposition 3.1 for this case.

Proposition 3.4. *If* $\{Z(s,t), (s,t)\in[0,1/2]^2\}$ *is given by* (21), *then*

$$\mathbb{P}\left(\sup_{(s,t)\in[0,\frac{1}{2}]^2} Z(s,t) > u\right) = 2^{\frac{5}{2}}\sqrt{\pi}u^3\Psi(2u)(1+o(1)), \quad u\to\infty. \tag{22}$$

3.3. **Fractional Brownian motion.** Consider the fBm incremental random field

$$Z(s,t) = B_\alpha(s+t) - B_\alpha(s), \quad (s,t) \in [0,S] \times [0,1], \tag{23}$$

where B_α is the fBm with Hurst index $\alpha/2 \in (0,1)$.

The following proposition extends the main result of [17] to the whole range of $\alpha \in (0,2)$.

Proposition 3.5. *Let $\{Z(s,t),(s,t) \in [0,S] \times [0,1]\}$ be given as in* (23). *We have, as $u \to \infty$,*

(i) if $\alpha \in (0,1)$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,1]} Z(s,t) > u\right) = S 2^{1-2/\alpha} \alpha^{-1} \mathcal{H}_\alpha^2 u^{\frac{4}{\alpha}-2} \Psi(u)(1+o(1)); \tag{24}$$

(ii) if $\alpha = 1$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,1]} Z(s,t) > u\right) = S \mathcal{M}_{Y,1}^{\frac{1}{2}} u^2 \Psi(u)(1+o(1)), \tag{25}$$

with

$$Y(s,t) := \tilde{B}_1\left(2^{-1}s\right) + B_1\left(2^{-1}(t-s)\right), \quad (s,t) \in [0,\infty)^2;$$

(iii) if $\alpha \in (1,2)$

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,1]} Z(s,t) > u\right) = S \mathcal{H}_\alpha u^{\frac{2}{\alpha}} \Psi(u)(1+o(1)). \tag{26}$$

4. Extremes of maximum loss and span of Gaussian processes

Let $\{\xi(t), t \in [0,1]\}$ be a Gaussian process with a.s. continuous sample paths. The maximum loss of the process ξ is given by

$$\chi_1(\xi) = \max_{0\le s\le t\le 1} (\xi(s) - \xi(t)),$$

and its span is defined as

$$\chi_2(\xi) = \max_{t\in[0,1]} \xi(t) - \min_{t\in[0,1]} \xi(t).$$

The notion of the maximum loss of certain Gaussian processes (e.g., Brownian motion and fBm, etc.) plays an important role in finance and insurance modelling, see e.g., [30], [31] and references therein.

In this section, as an application of Theorem 2.2 and Remarks 2.4, we derive exact tail asymptotics of the maximum loss for both stationary Gaussian process (in Proposition 4.1) and for Brownian bridge (in Proposition 4.2). The exact tail asymptotics of the span $\chi_2(\xi)$ when ξ is a centered stationary Gaussian process with covariance function that satisfies certain regular conditions is obtained in [26]. The same result should be retrieved, using first a time scaling and then resorting to Remarks 2.4. This observation is confirmed in Proposition 4.1 below.

Hereafter assume that $\{\xi(t), t \in [0,1]\}$ is a centered stationary Gaussian process with covariance function $r_\xi(s)$ satisfying the following conditions:

S1': $r_\xi(t)$ attains its minimum on $[0,1]$ at unique point $t_m \in (0,1)$;

S2': there exist positive constants a_1, a_2, α_1 and $\alpha_2 \in (0,2)$ such that

$$r_\xi(t) = r_\xi(t_m) + a_1 |t - t_m|^{\alpha_1} (1+o(1)), \quad t \to t_m$$

and

$$r_\xi(t) = 1 - a_2 t^{\alpha_2}(1+o(1)), \quad t \to 0;$$

S3': $r_\xi(t) < 1$ for any $t \in (0,1]$.

Proposition 4.1. *Let $\{\xi(t), t \in [0,1]\}$ be a centered stationary Gaussian process with covariance function $r_\xi(t)$ satisfying* **S1'-S3'**. *If $r_\xi(t)$ is twice continuously differentiable on interval $[t_m - \mu, t_m + \mu]$ for some positive small constant μ, then*

$$\begin{aligned} \mathbb{P}\left(\chi_2(\xi) > u\right) &= 2\mathbb{P}\left(\chi_1(\xi) > u\right) \\ &= 2^{2-\frac{4}{\alpha_2}+\frac{2}{\alpha_1}}(1-t_m)\mathcal{H}_{\alpha_2}^2 a_2^{\frac{2}{\alpha_2}}(1-r_\xi(t_m))^{2-\frac{4}{\alpha_2}+\frac{2}{\alpha_1}} u^{\frac{4}{\alpha_2}-\frac{2}{\alpha_1}} \Psi\left(\frac{u}{\sqrt{2(1-r_\xi(t_m))}}\right)(1+o(1)) \end{aligned} \tag{27}$$

as $u \to \infty$.

Proposition 4.2. *If $\{X(t), t \in [0,1]\}$ is the Brownian bridge given in* (21)*, then as $u \to \infty$*

$$\mathbb{P}\left(\chi_2(X) > u\right) = 2\mathbb{P}\left(\chi_1(X) > u\right) = 2^{\frac{9}{2}}\sqrt{\pi}u^3\Psi(2u)(1+o(1)). \tag{28}$$

Remarks 4.3. *a) The claim in* (27) *is consistent with Theorem 2.1 in* [26].
b) Let B_α be a standard fBm and consider its maximum loss $\chi_1(B_\alpha)$ and span $\chi_2(B_\alpha)$. The variance function of the random field $X_1(s,t) := B_\alpha(t) - B_\alpha(s)$ is given by

$$\sigma_{X_1}^2(s,t) = |t-s|^\alpha, \quad (s,t) \in [0,1]^2$$

and attains its maximum only at points $(0,1)$ and $(1,0)$. Therefore, Theorem 8.2 in [24] *yields that, as $u \to \infty$,*
(i) if $\alpha \in (0,1)$

$$\mathbb{P}\left(\chi_2(B_\alpha) > u\right) = 2\mathbb{P}\left(\chi_1(B_\alpha) > u\right) = 2^{3-2/\alpha}\alpha^{-2}\mathcal{H}_\alpha^2 u^{\frac{4}{\alpha}-4}\Psi(u)(1+o(1));$$

(ii) if $\alpha = 1$

$$\mathbb{P}\left(\chi_2(B_\alpha) > u\right) = 2\mathbb{P}\left(\chi_1(B_\alpha) > u\right) = 8\Psi(u)(1+o(1));$$

(iii) if $\alpha \in (1,2)$

$$\mathbb{P}\left(\chi_2(B_\alpha) > u\right) = 2\mathbb{P}\left(\chi_1(B_\alpha) > u\right) = 2\Psi(u)(1+o(1)).$$

5. Proofs

Proof of Lemma 2.1: The claim follows by a direct application of Lemma 6.1 given in Appendix. □

Proof of Theorem 2.2: As it will be seen at the end of the proof, by symmetry, Cases *vi*) and *vii*) follow from the claims of Cases *iii*) and *v*), respectively. Thus, we shall first focus on the proof of Cases *i*)-*v*). In view of assumption **A1** there exist some $\theta \in (0,1)$ and $\rho_0 \geq \rho_1$ (ρ_1 is as in **A3**) such that

$$\sup_{(s,t)\in[0,S]\times[0,\rho_0]} \sigma(s,t) < \theta.$$

For $\delta(u)=(\ln u/u)^{2/\beta}, u>0$ we may write

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[T-\delta(u),T]} X(s,t)>u\right) &\le \mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} X(s,t)>u\right)\\ &\le \mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[T-\delta(u),T]} X(s,t)>u\right)+\pi_1(u)+\pi_2(u),\end{aligned}$$

where

$$\pi_1(u):=\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,\rho_0]} X(s,t)>u\right),\quad \pi_2(u):=\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[\rho_0,T-\delta(u)]} X(s,t)>u\right).$$

We shall mainly focus on the analysis of

$$\pi(u):=\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[T-\delta(u),T]} X(s,t)>u\right),\qquad u\to\infty \tag{29}$$

and show that for $i=1,2$

$$\pi_i(u)=o(\pi(u)),\qquad u\to\infty, \tag{30}$$

which then implies

$$\mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} X(s,t)>u\right)=\pi(u)(1+o(1)),\qquad u\to\infty.$$

The asymptotics of (29) will be investigated for the Cases i)-v) separately by using a case-specific approach.

<u>Case i) $\beta>\max(\alpha_1,\alpha_2)$</u>: For space saving we consider only the case that $\alpha_1=\alpha_2=:\alpha$; the other cases can be shown with similar arguments. Following the idea of [25] choose first a constant $\alpha_0\in(\alpha,\beta)$, and denote

$$\triangle_{ij}=\triangle_i\times\triangle_j,\quad \triangle_{ij}^T=\triangle_i\times(T-\triangle_j),\ \text{with } \triangle_i=[iu^{-\frac{2}{\alpha_0}},(i+1)u^{-\frac{2}{\alpha_0}}],\ i=0,1,\cdots.$$

Set further

$$\tilde N_1(u)=\left\lfloor Su^{\frac{2}{\alpha_0}}\right\rfloor+1,\qquad \tilde N_2(u)=\left\lfloor(\ln u)^{\frac{2}{\beta}}u^{\frac{2}{\alpha_0}-\frac{2}{\beta}}\right\rfloor+1,$$

where $\lfloor\cdot\rfloor$ stands for the ceiling function. By Bonferroni's inequality we have that

$$\begin{aligned}\sum_{i=0}^{\tilde N_1(u)}\sum_{j=0}^{\tilde N_2(u)}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T} X(s,t)>u\right) &\ge \pi(u)\\ &\ge \sum_{i=0}^{\tilde N_1(u)-1}\sum_{j=0}^{\tilde N_2(u)-1}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T} X(s,t)>u\right)-\Sigma_1(u),\end{aligned}\tag{31}$$

with

$$\Sigma_1(u)=\sum\sum_{\substack{0\le i,i'\le\tilde N_1(u)-1,0\le j,j'\le\tilde N_2(u)-1\\(i,j)\ne(i',j')}}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T} X(s,t)>u,\ \sup_{(s,t)\in\triangle_{i'j'}^T} X(s,t)>u\right).$$

For any $\varepsilon\in(0,1)$ and all u large (set $b_{\pm\varepsilon}:=b(1\pm\varepsilon)$)

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T} X(s,t)>u\right)\le\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\frac{X(s,T-t)}{\sigma(s,T-t)}>u_{j-}\right),\quad u_{j-}=u(1+b_{-\varepsilon}(ju^{-\frac{2}{\alpha_0}})^\beta),$$

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T} X(s,t)>u\right)\ge\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\frac{X(s,T-t)}{\sigma(s,T-t)}>u_{j+}\right),\quad u_{j+}=u(1+b_{+\varepsilon}((j+1)u^{-\frac{2}{\alpha_0}})^\beta).$$

Let $\{\eta_{\pm\varepsilon}(s,t),(s,t)\in[0,\infty)^2\}$ with ε as above be centered stationary Gaussian random fields with covariance functions

$$r_{\eta_{\pm\varepsilon}}(s,t)=\exp\left(-(1\pm\varepsilon)^\alpha\Big(|a_1s|^\alpha+|a_2t+a_3s|^\alpha\Big)\right),\quad (s,t)\in[0,\infty)^2,$$

respectively. By Slepian's Lemma (see e.g., [6] or [4]) for all u large

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\frac{X(s,T-t)}{\sigma(s,T-t)}>u_{j-}\right)\quad\le\quad\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\eta_{+\varepsilon}(s,T-t)>u_{j-}\right).$$

In view of Theorem 7.2 in [24] as $u\to\infty$

$$\begin{aligned}
\pi(u)&\le\sum_{i=0}^{\tilde N_1(u)}\sum_{j=0}^{\tilde N_2(u)}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\eta_{+\varepsilon}(s,T-t)>u_{j-}\right)\\
&=(1+\varepsilon)^2a_1a_2\mathcal{H}_\alpha^2u^{-\frac{4}{\alpha_0}}\sum_{i=0}^{\tilde N_1(u)}\sum_{j=0}^{\tilde N_2(u)}u_{j-}^{\frac{4}{\alpha}}\Psi(u_{j-})(1+o(1))\\
&=(1+\varepsilon)^2a_1a_2\mathcal{H}_\alpha^2Su^{-\frac{2}{\alpha_0}+\frac{4}{\alpha}}\Psi(u)\sum_{j=0}^{\tilde N_2(u)}\exp\left(-b_{-\varepsilon}(ju^{\frac{2}{\beta}-\frac{2}{\alpha_0}})^\beta\right)(1+o(1))\\
&=(1+\varepsilon)^2a_1a_2\mathcal{H}_\alpha^2Su^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)\int_0^\infty\exp\left(-b_{-\varepsilon}x^\beta\right)dx(1+o(1)).
\end{aligned}\tag{32}$$

Similarly, we obtain

$$\begin{aligned}
\sum_{i=0}^{\tilde N_1(u)-1}\sum_{j=0}^{\tilde N_2(u)-1}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T}X(s,t)>u\right)&\ge\sum_{i=0}^{\tilde N_1(u)-1}\sum_{j=0}^{\tilde N_2(u)-1}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}}\eta_{-\varepsilon}(s,T-t)>u_{j+}\right)\\
&\ge(1-\varepsilon)^2a_1a_2\mathcal{H}_\alpha^2Su^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)\int_0^\infty\exp\left(-b_{+\varepsilon}x^\beta\right)dx(1+o(1)).
\end{aligned}\tag{33}$$

Next, we deal with the double sum part $\Sigma_1(u)$. Denote the distance of two non-empty sets $A,B\subset\mathbb{R}^n$ by

$$\rho(A,B)=\inf_{x\in A,y\in B}||x-y||,$$

with $||\cdot||$ the Euclidean distance. We see from (3) that there exists a positive constant ρ_3 such that

$$\begin{aligned}
\frac{3}{2}\Big(|a_1(s-s')|^\alpha+|a_2(t-t')+a_3(s-s')|^\alpha\Big)&\ge1-r(s,t,s',t')\\
&\ge\frac{1}{2}\Big(|a_1(s-s')|^\alpha+|a_2(t-t')+a_3(s-s')|^\alpha\Big)
\end{aligned}\tag{34}$$

for $|s-s'|\le2\rho_3,|T-t|\le2\rho_3$ and $|T-t'|\le2\rho_3$. It follows further from (4) that there exists some $\theta_0\in(0,1)$ such that

$$\sup_{\substack{0\le i,i'\le\tilde N_1(u)-1,0\le j,j'\le\tilde N_2(u)-1\\ \rho(\triangle_i,\triangle_{i'})>\rho_3}}\ \sup_{\substack{(s,t)\in\triangle_{ij}^T\\ (s',t')\in\triangle_{i'j'}^T}}r(s,t,s',t')<\theta_0.$$

Next, we divide the double sum part $\Sigma_1(u)$ as follows

$$\Sigma_1(u)\quad=\quad\Sigma_{1,1}(u)+\Sigma_{1,2}(u)+\Sigma_{1,3}(u),\quad u\ge0,$$

where $\Sigma_{1,1}(u)$ is the sum taken on $\rho(\triangle_i,\triangle_{i'})>\rho_3$, $\Sigma_{1,2}(u)$ is the sum taken on $\rho(\triangle_{ij}^T,\triangle_{i'j'}^T)=0$ and $\Sigma_{1,3}(u)$ is the sum taken on $u^{-2/\alpha_0}\le\rho(\triangle_{ij}^T,\triangle_{i'j'}^T)$ and $\rho(\triangle_i,\triangle_{i'})\le\rho_3$. We first give the estimation of $\Sigma_{1,1}(u)$. For $\xi(s,t,s',t'):=\overline{X}(s,t)+\overline{X}(s',t')$ we have

$$\mathbb{E}\left(\xi^2(s,t,s',t')\right)=4-2(1-r(s,t,s',t'))\tag{35}$$

implying

$$\sup_{\substack{0\le i,i'\le \tilde N_1(u)-1, 0\le j,j'\le \tilde N_2(u)-1\\ \rho(\triangle_i,\triangle_{i'})>\rho_3}} \sup_{\substack{(s,t)\in\triangle^T_{ij}\\ (s',t')\in\triangle^T_{i'j'}}} \mathbb{E}\left(\xi^2(s,t,s',t')\right) \le 4-2(1-\theta_0)<4.$$

Further we have

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}} X(s,t)>u, \sup_{(s,t)\in\triangle^T_{i'j'}} X(s,t)>u\right) &\le \mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}} \overline{X}(s,t)>u, \sup_{(s,t)\in\triangle^T_{i'j'}} \overline{X}(s,t)>u\right)\\ &\le \mathbb{P}\left(\sup_{\substack{(s,t)\in\triangle^T_{ij}\\ (s',t')\in\triangle^T_{i'j'}}} \xi(s,t,s',t')>2u\right).\end{aligned}$$

By Borell-TIS inequality (see [1] or [24]), for u sufficiently large

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}} X(s,t)>u, \sup_{(s,t)\in\triangle^T_{i'j'}} X(s,t)>u\right) \le \exp\left(-\frac{(u-a)^2}{2-(1-\theta_0)}\right),$$

where $a=\mathbb{E}\left(\sup_{(s,t),(s't')\in[0,S]\times[0,T]}\xi(s,t,s',t')\right)<\infty$. Thus

$$\limsup_{u\to\infty}\frac{\Sigma_{1,1}(u)}{u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)}=0. \tag{36}$$

The summand of $\Sigma_{1,2}(u)$ is equal to

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}} X(s,t)>u\right)+\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{i'j'}} X(s,t)>u\right)-\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}\cup\triangle^T_{i'j'}} X(s,t)>u\right).$$

Since $\rho(\triangle^T_{ij},\triangle^T_{i'j'})=0$, we have for $(s,t)\in\triangle^T_{ij}\cup\triangle^T_{i'j'}$ and sufficiently large u

$$u(1+b_{-\varepsilon}((j-1)_+u^{-\frac{2}{\alpha_0}})^\beta)=:\tilde u_{j-}\le\frac{u}{\sigma(s,t)}\le\tilde u_{j+}:=u(1+b_{+\varepsilon}((j+2)u^{-\frac{2}{\alpha_0}})^\beta)$$

Using again Theorem 7.2 in [24] for the last term we have

$$\mathbb{P}\left(\sup_{(s,t)\in\triangle^T_{ij}\cup\triangle^T_{i'j'}} X(s,t)>u\right)\ge 2(1-\varepsilon)^2a_1a_2\mathcal{H}^2_\alpha u^{-\frac{4}{\alpha_0}}\tilde u_{j+}^{\frac{4}{\alpha}}\Psi(\tilde u_{j+})(1+o(1))$$

as $u\to\infty$. Consequently, noting that for any $\triangle^T_{ij}$ there are at most 8 sets of the form $\triangle^T_{i'j'}$ in $[0,S]\times[T-\delta(u),T]$ adjecent with it, we conclude that

$$\Sigma_{1,2}(u) \le 8\sum_{i=0}^{\tilde N_1(u)}\sum_{j=0}^{\tilde N_2(u)}\left(2(1+\varepsilon)^2a_1a_2\mathcal{H}^2_\alpha u^{-\frac{4}{\alpha_0}}\tilde u_{j-}^{\frac{4}{\alpha}}\Psi(\tilde u_{j-})-2(1-\varepsilon)^2a_1a_2\mathcal{H}^2_\alpha u^{-\frac{4}{\alpha_0}}\tilde u_{j+}^{\frac{4}{\alpha}}\Psi(\tilde u_{j+})\right)(1+o(1))$$

and thus similar arguments as in (32) yield

$$\limsup_{\varepsilon\to0}\limsup_{u\to\infty}\frac{\Sigma_{1,2}(u)}{u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)}=0. \tag{37}$$

Finally, we estimate $\Sigma_{1,3}(u)$. Since $u^{-2/\alpha_0}\le\rho(\triangle^T_{ij},\triangle^T_{i'j'})$ and $\rho(\triangle_i,\triangle_{i'})\le\rho_3$, it follows in view of (34) that

$$\inf_{\substack{0\le i,i'\le \tilde N_1(u)-1, 0\le j,j'\le \tilde N_2(u)-1\\ \rho(\triangle_i,\triangle_{i'})\le\rho_3}} \inf_{\substack{(s,t)\in\triangle^T_{ij},(s',t')\in\triangle^T_{i'j'}\\ u^{-\frac{2}{\alpha_0}}\le\rho(\triangle^T_{ij},\triangle^T_{i'j'})}} \left(1-r(s,t,s',t')\right)\ge\frac12\nu u^{-\frac{2\alpha}{\alpha_0}}$$

for some positive constant ν, and thus

$$\sup_{\substack{0\le i,i'\le \tilde N_1(u)-1,0\le j,j'\le \tilde N_2(u)-1\\ \rho(\triangle_i,\triangle_{i'})\le\rho_3}}\ \sup_{\substack{(s,t)\in\triangle_{ij}^T,(s',t')\in\triangle_{i'j'}^T\\ u^{-\frac{2}{\alpha_0}}\le\rho(\triangle_{ij}^T,\triangle_{i'j'}^T)}} \mathbb{E}\left(\xi^2(s,t,s',t')\right)\le 4-\nu u^{-\frac{2\alpha}{\alpha_0}}.$$

Consequently, using Piterbarg inequality (cf. Theorem 8.1 in [24] or Theorem 8.1 in [25]) for the summand of $\Sigma_{1,3}(u)$ we obtain

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in\triangle_{ij}^T}X(s,t)>u,\sup_{(s,t)\in\triangle_{i'j'}^T}X(s,t)>u\right)&\le\mathbb{P}\left(\sup_{\substack{(s,t)\in\triangle_{ij}^T\\(s',t')\in\triangle_{ij}^T}}\xi(s,t,s',t')>2u\right)\\ &= o\left(\exp\left(-\frac{1}{16}\nu u^{-2\frac{\alpha_0-\alpha}{\alpha_0}}\right)\right)u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u),\end{aligned}$$

which implies that

$$\limsup_{u\to\infty}\frac{\Sigma_{1,3}(u)}{u^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)}\le\limsup_{u\to\infty}\sum\sum_{\substack{0\le i,i'\le\tilde N_1(u)-1,0\le j,j'\le\tilde N_2(u)-1\\(i,j)\neq(i',j')}} o\left(\exp\left(-\frac{1}{16}\nu u^{-2\frac{\alpha_0-\alpha}{\alpha_0}}\right)\right)=0. \tag{38}$$

Hence, in view of (31-33), (36-38) and by letting $\varepsilon\to0$ we conclude that

$$\pi(u)=a_1a_2\mathcal{H}_\alpha^2Su^{\frac{4}{\alpha}-\frac{2}{\beta}}\Psi(u)\int_0^\infty\exp\left(-bx^\beta\right)dx(1+o(1)),\qquad u\to\infty.$$

<u>Case $ii)$ $\beta=\alpha_1=\alpha_2$</u>: In order to simplify notation we set $\alpha:=\alpha_1=\alpha_2$. Let S_1,T_1 be two positive constants and define

$$\begin{aligned}&\widehat{\Delta}_i=[iS_1u^{-\frac{2}{\alpha}},(i+1)S_1u^{-\frac{2}{\alpha}}],\ i=0,\cdots,N_1(u),\quad \widetilde{\Delta}_i=[iT_1u^{-\frac{2}{\alpha}},(i+1)T_1u^{-\frac{2}{\alpha}}],\ i=0,\cdots,N_2(u),\\ &\overline{\triangle}_{ij}=\widehat{\Delta}_i\times\widetilde{\Delta}_j,\quad \overline{\triangle}_{ij}^T=\widehat{\Delta}_i\times(T-\widetilde{\Delta}_j),\end{aligned}$$

where

$$N_1(u)=\left\lfloor\frac{S}{S_1}u^{\frac{2}{\alpha}}\right\rfloor+1,\qquad N_2(u)=\left\lfloor\frac{(\ln u)^{\frac{2}{\beta}}}{T_1}\right\rfloor+1.$$

Again, Bonferroni's inequality implies

$$\begin{aligned}\Sigma_2(u)+\sum_{i=0}^{N_1(u)}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T}X(s,t)>u\right)&\ge\pi(u)\\ &\ge\sum_{i=0}^{N_1(u)-1}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T}X(s,t)>u\right)-\Sigma_3(u),\end{aligned}\tag{39}$$

where

$$\begin{aligned}\Sigma_2(u)&=\sum_{i=0}^{N_1(u)}\sum_{j=1}^{N_2(u)}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}^T}X(s,t)>u\right)\\ \Sigma_3(u)&=\sum\sum_{0\le i<i'\le N_1(u)-1}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T}X(s,t)>u,\sup_{(s,t)\in\overline{\triangle}_{i'0}^T}X(s,t)>u\right).\end{aligned}$$

Since our approach is of asymptotic nature, for any fixed $0 \le i \le N_1(u)$, the local structures of the variance and correlation of the Gaussian random field X on $\overline{\triangle}_{i0}^T$ are the only necessary properties influencing the asymptotics. Therefore,

$$\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T} X(s,t) > u\right) = \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}} \frac{\eta(s,t)}{1+bt^\beta} > u\right)(1+o(1))$$

as $u \to \infty$, where $\{\eta(s,t), (s,t) \in [0,S]\times[0,T]\}$ is the same as in Lemma 2.1. Hence Lemma 2.1 implies

$$\sum_{i=0}^{N_1(u)} \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T} X(s,t) > u\right) = \frac{S}{S_1} u^{\frac{2}{\alpha}} \mathcal{H}_{Y_1}^b[S_1,T_1]\Psi(u)(1+o(1)), \qquad u \to \infty. \tag{40}$$

Similarly

$$\sum_{i=0}^{N_1(u)-1} \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{i0}^T} X(s,t) > u\right) = \frac{S}{S_1} u^{\frac{2}{\alpha}} \mathcal{H}_{Y_1}^b[S_1,T_1]\Psi(u)(1+o(1)), \qquad u \to \infty. \tag{41}$$

Note that, for any $c, d \in \mathbb{R}$

$$|c+d|^p \le |c|^p + |d|^p, \qquad \text{if } \ p \in (0,1],$$
$$|c+d|^p \le 2^{p-1}(|c|^p + |d|^p), \qquad \text{if } \ p \in (1,\infty).$$

In view of Slepian's Lemma

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}^T} X(s,t) > u\right) &\le \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}} \eta(s,t) > u(1+b(jT_1u^{-\frac{2}{\alpha}})^\beta)\right)(1+o(1))\\ &\le \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}} \tilde{\eta}(s,t) > u(1+b(jT_1u^{-\frac{2}{\alpha}})^\beta)\right)(1+o(1))\end{aligned}$$

as $u \to \infty$, where $\{\tilde{\eta}(s,t), (s,t) \in [0,S]\times[0,T]\}$ is a centered homogeneous Gaussian random field with covariance function

$$r_{\tilde{\eta}}(s,t) = \exp\left(-|\tilde{a}_1 s|^\alpha - |\tilde{a}_2 t|^\alpha\right), \quad (s,t) \in [0,S]\times[0,T],$$

with $\tilde{a}_1 = (a_1^\alpha + 2\,|a_3|^\alpha)^{1/\alpha}$ and $\tilde{a}_2 = 2^{1/\alpha}a_2$. It follows further, using Lemma 2.1, that

$$\begin{aligned}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}^T} X(s,t) > u\right) &\le \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}_{ij}} \tilde{\eta}(s,t) > u(1+b(jT_1u^{-\frac{2}{\alpha}})^\beta)\right)(1+o(1))\\ &= \mathcal{H}_{\tilde{Y}_2}^0[S_1,T_1]\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2(1+2b(jT_1u^{-\frac{2}{\alpha}})^\beta)}{2}\right)(1+o(1))\\ &= \mathcal{H}_{\tilde{Y}_2}^0[S_1,T_1]\exp\left(-b(jT_1)^\beta\right)\Psi(u)(1+o(1))\end{aligned}$$

as $u \to \infty$, where $\mathcal{H}_{\tilde{Y}_2}^0[S_1,T_1]$ is defined in a similar way as $\mathcal{H}_{Y_2}^0[S_1,T_1]$ with $a_i, i = 1,2$ replaced by $\tilde{a}_i, i = 1,2$. Consequently

$$\Sigma_2(u) \le \sum_{j=1}^\infty \frac{S}{S_1} u^{\frac{2}{\alpha}} \mathcal{H}_{\tilde{Y}_2}^0[S_1,T_1]\exp\left(-b(jT_1)^\beta\right)\Psi(u)(1+o(1)). \tag{42}$$

From (4) there exists some $\theta_1 \in (0,1)$ such that

$$\sup_{\substack{1\le i<i'\le N_1(u)\\ \rho(\widehat{\Delta}_i,\widehat{\Delta}_{i'})>\rho_3}} \ \sup_{\substack{s\in\widehat{\Delta}_i, s'\in\widehat{\Delta}_{i'}\\ t,t'\in[0,T]}} r(s,t,s',t') < \theta_1,$$

where ρ_3 is the same as in (34). Below we shall re-write $\Sigma_3(u)$ as

$$\Sigma_3(u) = \Sigma_{3,1}(u) + \Sigma_{3,2}(u) + \Sigma_{3,3}(u), \quad u \ge 0,$$

where $\Sigma_{3,1}(u)$ is the sum taken on $\rho(\widehat{\Delta}_i, \widehat{\Delta}_{i'}) > \rho_3$, $\Sigma_{3,2}(u)$ is the sum taken on $i' = i+1$, and $\Sigma_{3,3}(u)$ is the sum taken on $i' > i+1$ and $\rho(\widehat{\Delta}_i, \widehat{\Delta}_{i'}) \le \rho_3$. First note that the estimation of $\Sigma_{3,1}(u)$ can be derived similarly to that of $\Sigma_{1,1}(u)$ in Case a) and thus for u sufficiently large

$$\Sigma_{3,1}(u) \le \frac{S^2}{S_1^2} u^{\frac{4}{\alpha}} \exp\left(-\frac{(u-a)^2}{2-(1-\theta_1)}\right), \tag{43}$$

where a is the same as in (36). Next, we consider $\Sigma_{3,3}(u)$. In view of (34) and (35) it follows that for $s \in \widehat{\Delta}_i, s' \in \widehat{\Delta}_{i'}, t, t' \in T - \widetilde{\Delta}_0$ and u large enough

$$2 \le \mathbb{E}\left(\xi^2(s,t,s',t')\right) \le 4 - |a_1(i'-i)S_1|^{\alpha} u^{-2}. \tag{44}$$

Further set $\overline{\xi}(s,t,s',t') = \xi(s,t,s',t')/\sqrt{\mathrm{Var}(\xi(s,t,s',t'))}$. Following similar argument as in the proof of Lemma 6.3 in [24], we obtain that

$$\mathbb{E}\left(\overline{\xi}(s,t,s',t') - \overline{\xi}(v,w,v',w')\right)^2 \le 4(\mathbb{E}\left(\overline{X}(s,t) - \overline{X}(v,w)\right)^2 + \mathbb{E}\left(\overline{X}(s',t') - \overline{X}(v',w')\right)^2).$$

Moreover, from (34) we see that, for u sufficiently large

$$\mathbb{E}\left(\overline{X}(s,t) - \overline{X}(v,w)\right)^2 \quad \le \quad 3\Big(|\tilde{a}_1(s-v)|^{\alpha} + |\tilde{a}_2(t-w)|^{\alpha}\Big)$$

implying thus

$$\mathbb{E}\left(\overline{\xi}(s,t,s',t') - \overline{\xi}(v,w,v',w')\right)^2 \le 2(1 - r_\zeta(s-v, t-w, s'-v', t'-w')), \tag{45}$$

where

$$r_\zeta(s,t,s',t') = \exp\left(-7\Big(|\tilde{a}_1 s|^{\alpha} + |\tilde{a}_2 t|^{\alpha} + |\tilde{a}_1 s'|^{\alpha} + |\tilde{a}_2 t'|^{\alpha}\Big)\right)$$

is the covariance function of the homogeneous Gaussian random field $\{\zeta(s,t,s',t'), (s,t,s',t') \in (0,\infty)^4\}$. Consequently, (44), (45) and Slepian's Lemma imply

$$\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t) > u, \sup_{(s,t)\in\overline{\triangle}^T_{i'0}} X(s,t) > u\right) \quad \le \quad \mathbb{P}\left(\sup_{\substack{(s,t)\in\overline{\triangle}^T_{i0}\\(s',t')\in\overline{\triangle}^T_{i'0}}} \zeta(s,t,s',t') > \frac{2u}{\sqrt{4 - |a_1(i'-i)S_1|^{\alpha} u^{-2}}}\right).$$

We obtain further from a similar lemma as Lemma 2.1 (cf. Lemma 6.1 in [24]) that

$$\mathbb{P}\left(\sup_{\substack{(s,t)\in\overline{\triangle}^T_{i0}\\(s',t')\in\overline{\triangle}^T_{i'0}}} \zeta(s,t,s',t') > \frac{2u}{\sqrt{4 - |a_1(i'-i)S_1|^{\alpha} u^{-2}}}\right) = (\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[S_1,T_1])^2 \frac{1}{\sqrt{2\pi}u} \times \exp\left(-\frac{4u^2}{2(4 - |a_1(i'-i)S_1|^{\alpha} u^{-2})}\right)(1+o(1)),$$

where $\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[S_1,T_1]$ is defined in a similar way as $\mathcal{H}^0_{Y_2}[S_1,T_1]$ with a_1, a_2 replaced by $7^{1/\alpha}\tilde{a}_1, 7^{1/\alpha}\tilde{a}_2$, respectively. Consequently, for all large u

$$\Sigma_{3,3}(u) \le \frac{S}{S_1} \sum_{j\ge 1} (\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[S_1,T_1])^2 \exp\left(-\frac{1}{8}|a_1 j S_1|^{\alpha}\right) u^{\frac{2}{\alpha}} \Psi(u)(1+o(1)). \tag{46}$$

Next, we consider $\Sigma_{3,2}(u)$. For any u positive

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t)>u, \sup_{(s,t)\in\overline{\Delta}^T_{(i+1)0}} X(s,t)>u\right)\\
&\le \mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t)>u, \sup_{(s,t)\in[(i+1)S_1u^{-\frac{2}{\alpha}},(i+1)S_1u^{-\frac{2}{\alpha}}+\sqrt{S_1}u^{-\frac{2}{\alpha}}]\times(T-\widetilde{\Delta}_0)} X(s,t)>u\right)\\
&+\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t)>u, \sup_{(s,t)\in[(i+1)S_1u^{-\frac{2}{\alpha}}+\sqrt{S_1}u^{-\frac{2}{\alpha}},(i+2)S_1u^{-\frac{2}{\alpha}}]\times(T-\widetilde{\Delta}_0)} X(s,t)>u\right)
\end{aligned}$$

and further

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t)>u, \sup_{(s,t)\in\overline{\Delta}^T_{(i+1)0}} X(s,t)>u\right)\\
&\le \mathcal{H}^0_{\tilde{Y}_2}[\sqrt{S_1},T_1]\Psi(u)+(\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[\sqrt{S_1},T_1])^2\exp\left(-\frac{1}{8}\left|a_1\sqrt{S_1}\right|^\alpha\right)\Psi(u)(1+o(1)).
\end{aligned}$$

Therefore, for all large u

$$\Sigma_{3,2}(u)\le \frac{S}{S_1}\left(\mathcal{H}^0_{\tilde{Y}_2}[\sqrt{S_1},T_1]+(\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[\sqrt{S_1},T_1])^2\exp\left(-\frac{1}{8}\left|a_1\sqrt{S_1}\right|^\alpha\right)\right)u^{\frac{2}{\alpha}}\Psi(u)(1+o(1)). \tag{47}$$

Consequently, from (39–43) and (46–47) we conclude that for any $S_i,T_i,i=1,2$

$$\begin{aligned}
&{S_1}^{-1}\mathcal{H}^b_{Y_1}[S_1,T_1]+\sum_{j=1}^\infty {S_1}^{-1}\mathcal{H}^0_{\tilde{Y}_2}[S_1,T_1]\exp\left(-b(jT_1)^\beta\right)\\
&\ge \limsup_{u\to\infty}\frac{\pi(u)}{Su^{\alpha/2}\Psi(u)}\ge\liminf_{u\to\infty}\frac{\pi(u)}{Su^{\alpha/2}\Psi(u)}\\
&\ge {S_2}^{-1}\mathcal{H}^b_{Y_1}[S_2,T_2]-{S_2}^{-1}(\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[S_2,T_2])^2\sum_{j\ge1}\exp\left(-\frac{1}{8}\left|a_1jS_2\right|^\alpha\right)\\
&\quad -{S_2}^{-1}\left(\mathcal{H}^0_{\tilde{Y}_2}[\sqrt{S_2},T_2]+(\tilde{\mathcal{H}}^0_{\tilde{Y}_2}[\sqrt{S_2},T_2])^2\exp\left(-\frac{1}{8}\left|a_1\sqrt{S_2}\right|^\alpha\right)\right).
\end{aligned}$$

Therefore, by similar arguments as in the proof of Theorem D.2 in [24] we conclude that

$$0<\mathcal{M}^b_{Y_1,\alpha_1}\le\limsup_{u\to\infty}\frac{\pi(u)}{Su^{\alpha/2}\Psi(u)}\le\liminf_{u\to\infty}\frac{\pi(u)}{Su^{\alpha/2}\Psi(u)}\le\mathcal{M}^b_{Y_1,\alpha_1}<\infty$$

establishing the claim.

Case *iii)* $\beta=\alpha_2>\alpha_1$: Note that $\mathcal{M}^b_{Y_2,\beta}$ can be given in terms of Piterbarg and Pickands constants as

$$\mathcal{M}^b_{Y_2,\beta}=\lim_{T\to\infty}\lim_{S\to\infty}\frac{1}{S}\mathcal{H}^b_{Y_2}[S,T]=a_1a_2\mathcal{P}^{ba_2^{-\alpha_2}}_{\alpha_2}\mathcal{H}_{\alpha_1}.$$

The proof for this case can be established using step-by-step the same arguments as in Case *ii*).

Case *iv)* $\beta<\alpha_2=\alpha_1$: In order to make use of the notation introduced in Case *ii*) we set $\alpha:=\alpha_1=\alpha_2$. First note that $\delta(u)<T_1u^{-2/\alpha}$, which implies

$$\begin{aligned}
\pi(u) &\le \mathbb{P}\left(\sup_{(s,t)\in[0,S]\times(T-\widetilde{\Delta}_0)} X(s,t)>u\right)\\
&\le \sum_{i=0}^{N_1(u)}\mathbb{P}\left(\sup_{(s,t)\in\overline{\triangle}^T_{i0}} X(s,t)>u\right)\\
&\le \frac{S}{S_1}u^{\frac{2}{\alpha}}\mathcal{H}^0_{Y_1}[S_1,T_1]\Psi(u)(1+o(1))
\end{aligned}$$

as $u \to \infty$. Further, by Assumptions **A1**–**A2** we have that $\mathbb{E}\left((X(s,T))^2\right) = 1, \forall s \in [0,S]$ and

$$r(s,T,s',T) = 1 - \left(a_1^{\alpha} + |a_3|^{\alpha}\right) |s-s'|^{\alpha} (1+o(1))$$

holds uniformly with respect to $s, s' \in [0,S]$, as $|s-s'| \to 0$. This means that $\{X(s,T), s \in [0,S]\}$ is a locally stationary Gaussian process. Therefore, in view of Theorem 7.1 in [24]

$$\begin{aligned} \pi(u) &\geq \mathbb{P}\left(\sup_{s\in[0,S]} X(s,T) > u\right) \\ &= S(a_1^{\alpha} + |a_3|^{\alpha})^{\frac{1}{\alpha}} \mathcal{H}_{\alpha} u^{\frac{2}{\alpha}} \Psi(u)(1+o(1)), \qquad u \to \infty. \end{aligned}$$

Letting $T_1 \to 0, S_1 \to \infty$, we conclude that

$$0 < \lim_{u\to\infty} \frac{\pi(u)}{S u^{\frac{2}{\alpha}} \Psi(u)} = (a_1^{\alpha} + |a_3|^{\alpha})^{\frac{1}{\alpha}} \mathcal{H}_{\alpha} < \infty.$$

Case $v)$ $\beta < \alpha_2$ and $\alpha_1 < \alpha_2$: The claim follows with identical arguments as in the proof of Case $iv)$.

In order to complete the proof of Cases $i)$-$v)$ we only need to show (30), for which it is sufficient to give the following upper bounds for $\pi_1(u)$ and $\pi_2(u)$. By Borell-TIS inequality, for u large enough

$$\pi_1(u) \leq \exp\left(-\frac{\left(u - \mathbb{E}\left(\sup_{(s,t)\in[0,S]\times[0,\rho_0]} X(s,t)\right)\right)^2}{2\theta^2}\right). \tag{48}$$

Further, by Assumption **A3** applying Piterbarg inequality we obtain, as $u \to \infty$

$$\begin{aligned} \pi_2(u) &\leq \mathcal{Q} u^{\frac{4}{\gamma}-1} \exp\left(-\frac{u^2}{2\sigma^2(T-\delta(u))}\right) \\ &= \mathcal{Q} u^{\frac{4}{\gamma}-1} \exp\left(-\frac{u^2}{2}\right) \exp\left(-b(\ln u)^2\right)(1+o(1)), \end{aligned} \tag{49}$$

where $\mathcal{Q}$ is some positive constant not depending on u. Therefore, the proof of Cases $i)$-$v)$ is complete.

Next, we consider Cases $vi)$-$vii)$. We introduce a time scaling of the Gaussian random field $\{X(s,t), (s,t) \in \boldsymbol{E}\}$ by matrix $B = \begin{pmatrix} a_3 & a_2 \\ 0 & a_2 \end{pmatrix}$, i.e., let $Z(s,t) := X((s-t)/a_3, t/a_2)$. By this time scaling, we have

$$\mathbb{P}\left(\sup_{(s,t)\in\boldsymbol{E}} X(s,t) > u\right) = \mathbb{P}\left(\sup_{(s,t)\in\boldsymbol{K}} Z(s,t) > u\right), \tag{50}$$

where $\boldsymbol{K}$ is a region on $\mathbb{R}^2$ with vertices at points $(0,0)$, (a_2T, a_2T), $(a_3S, 0)$ and $(a_3S + a_2T, a_2T)$. The Gaussian random field $\{Z(s,t), (s,t) \in \boldsymbol{K}\}$ has the following properties:

P1) The standard deviation function $\sigma_Z(s,t)$ of $\{Z(s,t), (s,t) \in \boldsymbol{K}\}$ satisfies

$$\sigma_Z(s,t) = 1 - \frac{b}{a_2^{\beta}} (a_2T - t)^{\beta} (1+o(1)), \quad t \uparrow a_2T.$$

P2) The correlation function $r_Z(s,t,s',t')$ of $\{Z(s,t), (s,t) \in \boldsymbol{K}\}$ satisfies

$$r_Z(s,t,s',t') = 1 - \left(|s-s'|^{\alpha_2} + \left|\frac{a_1}{a_3}(t-t') - \frac{a_1}{a_3}(s-s')\right|^{\alpha_1}\right)(1+o(1))$$

for any $(s,t), (s',t') \in \boldsymbol{K}$ such that $|s-s'| \to 0$ and $t, t' \uparrow a_2T$, and further there exists some $\delta_0 \in (0,T)$ such that

$$r(s,t,s',t') < 1$$

holds for any $(s,t),(s',t') \in \boldsymbol{K}_0$ satisfying $s \neq s'$. Here $\boldsymbol{K}_0$ is a region on $\mathbb{R}^2$ with vertices at points $(a_2\delta_0, a_2\delta_0)$, (a_2T, a_2T), $(a_3S + a_2\delta_0, a_2\delta_0)$ and $(a_3S + a_2T, a_2T)$.

P3) There exist positive constants $\mathcal{Q}, \gamma, \rho_1$ and ρ_2 such that

$$\mathbb{E}\left((Z(s,t) - Z(s',t'))^2\right) \le \mathcal{Q}(|s-s'|^\gamma + |t-t'|^\gamma)$$

holds for any $(s,t),(s',t') \in \boldsymbol{K}$ satisfying $a_2T - t < \rho_1$, $a_2T - t' < \rho_1$ and $|s-s'| < \rho_2$.

Note that in the above proof the most important structural property of the set $\boldsymbol{E}$ is that the segment $\boldsymbol{L} = \{(s,t) \in \boldsymbol{E} : t = T\}$ is on the boundary of $\boldsymbol{E}$, which is also the case for $\{Z(s,t), (s,t) \in \boldsymbol{K}\}$. Therefore, in view of the above properties of $\{Z(s,t), (s,t) \in \boldsymbol{K}\}$, the claims of the Cases *vi*) and *vii*) follow by an application of the claims of Cases *iii*) and *v*). The proof is complete. □

Proof of Proposition 3.1: The variance function of Z is given by

$$\sigma_Z^2(s,t) = 2(1 - r_X(t))$$

and attains its maximum on $[0,S] \times \{T\}$. Therefore, it is sufficient to consider the asymptotics of

$$\Pi(u) := \mathbb{P}\left(\sup_{(s,t)\in[0,S]\times[0,T]} Z^*(s,t) > \tilde{u}\right), \qquad u \to \infty,$$

with

$$\tilde{u} := \frac{u}{\rho_T}, \quad \text{and} \quad Z^*(s,t) := \frac{Z(s,t)}{\rho_T},$$

where $\rho_T = \sqrt{2(1 - r_X(T))} > 0$. The asymptotics of $\Pi(u)$ follows from Theorem 2.2 by checking the assumptions **A1-A3**. The standard deviation function of Z^* satisfies

$$\sigma_{Z^*}(s,t) = \frac{\sqrt{2(1 - r_X(t))}}{\rho_T} = 1 - \frac{a_1}{2(1 - r_X(T))}(T-t)^{\alpha_1}(1 + o(1)), \quad t \to T,$$

whereas for its correlation function we have

$$R_{Z^*}(s,t,s',t') = \frac{r_X(|s+t-s'-t'|) - r_X(|s-s'-t'|) - r_X(|s+t-s'|) + r_X(|s-s'|)}{2\sqrt{(1-r_X(t))(1-r_X(t'))}}. \tag{51}$$

Since $r_X(t)$ is twice continuously differentiable in $[\mu, T]$ and $\left|r_X^{''}(T)\right| \in (0,\infty)$ for some constant $\mathcal{Q}_1$ we have

$$|r_X(t') - r_X(|s-s'-t'|) + r_X(t) - r_X(|s+t-s'|)| \le \mathcal{Q}_1(|t-t'+s-s'|^2 + |s-s'|^2)(1+o(1))$$

as $t, t' \to T$, $|s-s'| \to 0$. Consequently, $\alpha_2 \in (0,2)$ implies

$$R_{Z^*}(s,t,s',t') = 1 - \frac{a_2}{\rho_T^2}\left(|t-t'+s-s'|^{\alpha_2} + |s-s'|^{\alpha_2}\right)(1+o(1)) \tag{52}$$

as $t, t' \to T$, $|s-s'| \to 0$. Next, for any fixed $\varepsilon_0 > 0$, we have from **S3** that there exists some θ_0 such that

$$r_X(|s-s'|) \le \theta_0 < 1$$

for any $s, s' \in [0,S]$ satisfying $|s-s'| > \varepsilon_0$. Further, from **S2** we obtain that there exists some positive constant δ_0 such that

$$2\sqrt{(1-r_X(t))(1-r_X(t'))} \ge \rho_T^2 - \frac{1-\theta_0}{2} > 0$$

for any $t,t'\in[\delta_0,T]$. Hence

$$R_{Z^*}(s,t,s',t') \le \frac{1+\theta_0-2r_X(T)}{\rho_T^2-\frac{1-\theta_0}{2}}<1 \tag{53}$$

for any $t,t'\in[\delta_0,T]$, $s,s'\in[0,S]$ satisfying $|s-s'|>\varepsilon_0$ and thus both **A1** and **A2** are satisfied. It follows that

$$\mathbb{E}\left(Z^*(s,t)-Z^*(s',t')\right)^2 \le 2\mathbb{E}\left(\overline{Z}(s,t)-\overline{Z}(s',t')\right)^2+\frac{2}{\rho_T^2}(\sigma_Z(s,t)-\sigma_Z(s',t'))^2.$$

Therefore, the differentiability of $r_X(t)$, assumption **S2** and (52) imply that there exist some positive constants $\rho_1,\rho_2,\mathcal{Q}_3,\mathcal{Q}_4$ such that

$$\begin{aligned}\mathbb{E}\left(Z^*(s,t)-Z^*(s',t')\right)^2 &\le \mathcal{Q}_3(|t-t'+s-s'|^{\alpha_2}+|s-s'|^{\alpha_2}+|t-t'|^{2\min(\alpha_1,1)})\\ &\le \mathcal{Q}_4(|t-t'|^{\min(2\alpha_1,\alpha_2)}+|s-s'|^{\min(2\alpha_1,\alpha_2)})\end{aligned}$$

for all $s,s'\in[0,S],t,t'\in[\rho_1,T]$ satisfying $|s-s'|<\rho_2$, hence the proof is complete. □

Proofs of Proposition 3.4 and Proposition 3.5: Note first that the standard deviation of the incremental random field Z of the Brownian bridge satisfies

$$\sigma_Z(s,t)=(t(1-t))^{\frac12}=\frac12-\left(t-\frac12\right)^2(1+o(1)),\quad t\to\frac12. \tag{54}$$

Furthermore, for its correlation function we have

$$r_Z(s,s',t,t') = 1-2(|t-t'+s-s'|+|s-s'|)(1+o(1)) \tag{55}$$

as $t,t'\to 1/2,\ |s-s'|\to 0$.

For the fBm incremental random field Z we have for its standard deviation

$$\sigma_Z(s,t)=t^{\frac{\alpha}{2}}=1-\frac{\alpha}{2}(1-t)(1+o(1)),\quad t\to 1.$$

As shown in [25] the correlation function r_Z of Z satisfies

$$r_Z(s,s',t,t') = 1-\frac12(|t-t'+s-s'|^{\alpha}+|s-s'|^{\alpha})(1+o(1))$$

as $t,t'\to 1,\ |s-s'|\to 0$. Hence for both cases **A1-A3** are fulfilled, and thus the claims follow by a direct application of Theorem 2.2. □

Proofs of Proposition 4.1 and Proposition 4.2: By a linear time change using the matrix $A\in\mathbb{R}^{2\times2}$ given by

$$A=\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}$$

we have for any $u>0$

$$\mathbb{P}\left(\chi_2(\xi)>u\right)=\mathbb{P}\left(\sup_{(s,t)\in A[0,1]^2}(\xi(t+s)-\xi(s))>u\right).$$

Here the set $A[0,1]^2=\{(\tilde{s},\tilde{t}):(\tilde{s},\tilde{t})^\top=A(s,t)^\top,(s,t)\in[0,1]^2\}$ is bounded and convex. The variance function of the random field $\{\xi(t+s)-\xi(s),(s,t)\in A[0,1]^2\}$ is $2(1-r_\xi(|t|))$ which attains its unique maximum on the set $A[0,1]^2$ on two lines $\boldsymbol{L}_1=\{(s,t)\in A[0,1]^2:t=t_m\}$ and $\boldsymbol{L}_2=\{(s,t)\in A[0,1]^2:t=-t_m\}$. Note that the differentiability of $r_\xi(t)$ implies $\alpha_1\ge 2>\alpha_2$. Therefore, the claim in (27) follows from Remarks 2.4 b); the conditions therein can be established directly as in the proof of Proposition 3.1 except (18) for $i=1, j=2$, which can also be confirmed by a similar argument as in (53). Further, since

$$\mathbb{P}(\chi_2(X) > u) = \mathbb{P}\left(\sup_{(s,t)\in A[0,1]^2} (X(t+s) - X(s)) > u\right)$$

in view of (54) and (55) we conclude that the claim in (28) follows immediately from Remarks 2.4 b), and thus the proof is complete. □

6. Appendix

Let $\mathbf{D}$ be a compact set in $\mathbb{R}^2$ such that $(0,0)\in\mathbf{D}$, and let $\{\xi_u(s,t), (s,t)\in\mathbf{D}\}$, $u>0$ be a family of centered Gaussian random fields with a.s. continuous sample paths. The next lemma is proved based on the classical approach rooted in the ideas of [21, 22], see also [9], Lemma 1; in particular, it implies the claim of Lemma 2.1.

Lemma 6.1. *Let $d(\cdot)$ be a nonnegative continuous function on $[0,\infty)$ and let $g(u), u>0$ be a positive function satisfying $\lim_{u\to\infty} g(u)/u = 1$. Assume that the variance function $\sigma^2_{\xi_u}$ of ξ_u satisfies the following conditions*

$$\sigma_{\xi_u}(0,0) = 1 \text{ for all large } u, \quad \lim_{u\to\infty} \sup_{(s,t)\in\mathbf{D}} \left|u^2(1-\sigma_{\xi_u}(s,t)) - d(t)\right| = 0,$$

and there exist some positive constants G, ν, u_0 such that, for all $u > u_0$

$$u^2\, Var(\xi_u(s,t) - \xi_u(s',t')) \le G(|s-s'|^\nu + |t-t'|^\nu)$$

holds uniformly with respect to $(s,t),(s',t')\in\mathbf{D}$. If further there exists a centered Gaussian random field $\{Y(s,t),(s,t)\in(0,\infty)^2\}$ with a.s. continuous sample paths and $Y(0,0)=0$ such that

$$\lim_{u\to\infty} u^2\, Var(\xi_u(s,t) - \xi_u(s',t')) = 2\, Var(Y(s,t) - Y(s',t')), \quad \forall (s,t),(s',t')\in\mathbf{D},$$

then

$$\mathbb{P}\left(\sup_{(s,t)\in\mathbf{D}} \xi_u(s,t) > g(u)\right) = \mathcal{H}^d_Y[\mathbf{D}]\Psi(g(u))(1+o(1)), \quad u\to\infty, \tag{56}$$

where

$$\mathcal{H}^d_Y[\mathbf{D}] = \mathbb{E}\left(\exp\left(\sup_{(s,t)\in\mathbf{D}}\left(\sqrt{2}Y(s,t) - \sigma^2_Y(s,t) - d(t)\right)\right)\right).$$

Proof of Lemma 6.1: For large u we have

$$\begin{aligned}&\mathbb{P}\left(\sup_{(s,t)\in\mathbf{D}} \xi_u(s,t) > g(u)\right)\\ &= \frac{1}{\sqrt{2\pi}g(u)}\exp\left(-\frac{(g(u))^2}{2}\right)\int_{-\infty}^{\infty} e^{w-\frac{w^2}{2(g(u))^2}}\mathbb{P}\left(\sup_{(s,t)\in\mathbf{D}}\xi_u(s,t) > g(u)\Big|\xi_u(0,0) = g(u) - \frac{w}{g(u)}\right)dw.\end{aligned}\tag{57}$$

Let

$$R_{\xi_u}(s,t,s',t') = \mathbb{E}\left(\xi_u(s,t)\xi_u(s',t')\right), \quad (s,t),(s',t')\in\mathbf{D}$$

be the covariance function of ξ_u. The conditional random field

$$\left\{\xi_u(s,t)\Big|\xi_u(0,0) = g(u) - \frac{w}{g(u)}, (s,t)\in\mathbf{D}\right\}$$

has the same distribution as

$$\left\{\xi_u(s,t) - R_{\xi_u}(s,t,0,0)\xi_u(0,0) + R_{\xi_u}(s,t,0,0)\left(g(u) - \frac{w}{g(u)}\right), (s,t) \in \mathbf{D}\right\}.$$

Thus, the integrand in (57) can be rewritten as

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,t)\in\mathbf{D}}\left(\xi_u(s,t) - R_{\xi_u}(s,t,0,0)\xi_u(0,0) + R_{\xi_u}(s,t,0,0)\left(g(u) - \frac{w}{g(u)}\right)\right) > g(u)\right)\\
&= \mathbb{P}\left(\sup_{(s,t)\in\mathbf{D}}\left(\chi_u(s,t) - (g(u))^2(1 - R_{\xi_u}(s,t,0,0)) + w(1 - R_{\xi_u}(s,t,0,0))\right) > w\right),
\end{aligned}$$

where

$$\chi_u(s,t) = g(u)(\xi_u(s,t) - R_{\xi_u}(s,t,0,0)\xi_u(0,0)).$$

Next, the following convergence

$$(g(u))^2(1 - R_{\xi_u}(s,t,0,0)) - w(1 - R_{\xi_u}(s,t,0,0)) \to \sigma_Y^2(s,t) + d(t), \ \ u \to \infty$$

holds, for any $w \in \mathbb{R}$, uniformly with respect to $(s,t) \in \mathbf{D}$. Moreover,

$$\begin{aligned}
\mathbb{E}\left(\left(\chi_u(s,t) - \chi_u(s',t')\right)^2\right) &= (g(u))^2\left(\mathbb{E}\left(\left(\xi_u(s,t) - \xi_u(s',t')\right)^2\right) - (R_{\xi_u}(s,t,0,0) - R_{\xi_u}(s',t',0,0))^2\right)\\
&\to 2\mathrm{Var}(Y(s,t) - Y(s',t')), \ \ u \to \infty
\end{aligned}$$

holds for any $(s,t),(s',t') \in \mathbf{D}$. Hence the claim follows by using the same arguments as in the proof of Lemma 6.1 in [24] or those in the proof of Lemma 1 in [9]. $\square$

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