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EXTENDED GAUSSIAN THRESHOLD DEPENDENT RISK MODELS

Bai Long

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

EXTENDED GAUSSIAN THRESHOLD DEPENDENT RISK MODELS

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Docteur ès Sciences Actuarielles

par

Long BAI

Directeur de thèse
Prof. Enkelejd Hashorva

Jury

Prof. Olivier Cadot, Président
Prof. François Dufresne, expert interne
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Prof. Krzysztof Dębicki, expert externe
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La thèse est intitulée :

EXTENDED GAUSSIAN THRESHOLD DEPENDENT RISK MODELS

Lausanne, le 4 juin 2018

Le doyen

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Lausanne, June 2018 *Long Bai*

Contents

Chapter 1

Introduction and Notation

One of the primary topics in finance and insurance is the investigation of risk models, via stochastic analysis and quantitative estimation of the ruin related indications, such as ruin probability, ruin time and some other important risk measures, which provides crucial information for actuaries and decision makers.

In Chapter 2 we are concerned with the asymptotic behaviour, as $u \to \infty$, of $\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > u\right\}$, where $X_u(t), t \in [0,T], u > 0$ is a family of centered Gaussian processes with continuous trajectories. A key application of our findings concerns $\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t)) > u\right\}$, as $u \to \infty$, for X a centered Gaussian process and g some measurable trend function. Further applications include the approximation of both the ruin time and the ruin probability of the Brownian motion risk model with constant force of interest. This part also give us the main idea to solve the problems of Gaussian related models with trend.

Next in Chapter 3 and Chapter 4, we consider the Parisian ruin of Brownian motion risk models which is a development of the Brownian motion risk model in Chapter 2. Let $B(t), t \in \mathbb{R}$ be a standard Brownian motion. Define a risk process

$$R_u^\delta(t) = e^{\delta t}\left(u + c\int_0^t e^{-\delta s}ds - \sigma\int_0^t e^{-\delta s}dB(s)\right), t \geq 0, \tag{1.1}$$

where $u \geq 0$ is the initial reserve, $\delta \geq 0$ is the force of interest, $c > 0$ is the rate of premium and $\sigma > 0$ is a volatility factor. For $S \in (0,\infty)$ in Chapter 3 and $S = \infty$ in Chapter 4, we obtain an approximation of the Parisian ruin probability

$$\mathcal{K}_S^\delta(u, T_u) := \mathbb{P}\left\{\inf_{t\in[0,S]}\sup_{s\in[t,t+T_u]} R_u^\delta(s) < 0\right\},$$

as $u \to \infty$ where T_u is a bounded function. Further, we show that the Parisian ruin time of this risk process can be approximated by an exponential random variable. Our results are new even for the classical ruin probability and ruin time which correspond to $T_u \equiv 0$ in the Parisian setting. When $S = \infty$, it turns out that the Parisian ruin probability decays exponentially as u tends to infinity and is a decreasing function of the force of interest for u large. Moreover, we obtain the approximations of Parisian ruin time.

With motivation from [49], in Chapter 5 we derive the exact tail asymptotics of $\alpha(t)$-locally stationary Gaussian processes with non-constant variance functions. We show that some certain variance functions lead to qualitatively new results.

Based on our analysis of one-dimensional related Gaussian risk model, in Chapter 6 we focus on the vector-valued scenario. Let $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t)), t \in \mathcal{T} \subset \mathbb{R}$ be a centered vector-valued Gaussian process with independent components and continuous trajectories, and $\boldsymbol{h}(t) = (h_1(t), \ldots, h_n(t)), t \in \mathcal{T}$ be a vector-valued continuous function. We investigate the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}\min_{1\leq i\leq n}(X_i(t)+h_i(t)) > u\right\}$$

as $u \to \infty$. As an illustration to the derived results we analyze two important classes of $\boldsymbol{X}(t)$: with locally-stationary structure and with varying variances of the coordinates, and calculate exact asymptotics of simultaneous ruin probability and ruin time in a fractional Brownian risk model.

Another problem related to vector-valued Gaussian processes, the L^P norm of Gaussian processes with trend, is investigated in Chapter 7. For $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$ and $g(t)$ a continuous function, the asymptotics of tail distribution of $\|\boldsymbol{X}(t)\|_p$ have been investigated in numerous literatures. In this chapter we are concerned with the exact tail asymptotics of $\|\boldsymbol{X}(t)\|_p^c$, $c > 0$, with trend $g(t)$ over $[0, T]$. Both scenarios that $\boldsymbol{X}(t)$ is locally stationary and non-stationary are considered. Important examples include $\sum_{i=1}^n |X_i(t)| + g(t)$ and chi-square processes with trend, i.e., $\sum_{i=1}^n X_i^2(t) + g(t)$. These results are of interest in applications in engineering, insurance and statistics, etc.

Further, extending our ideas to the scenario of two dimensional Gaussian fields with trend, we consider the drawdown and drawup of fractional Brownian motion with trend in Chapter 8, which corresponds to the logarithm of geometric fractional Brownian motion representing the stock price in financial market. We derive the asymptotics of tail probabilities of the maximum drawdown and maximum drawup as the threshold goes to infinity, respectively. It turns out that the extremes of drawdown leads to new scenarios of asymptotics depending on Hurst index of fractional Brownian motion.

In the former results, we notice that the Pickands and Piterbarg constants play a pivotal role. Numerous papers are focus on the Pickands related constants, but the analysis about Piterbarg constants, especially the quantitative analysis are rare. Hence in Chapter 9, we investigate generalised Piterbarg constants

$$\mathcal{P}_{\alpha,\delta}^h = \lim_{T\to\infty} \mathbb{E}\left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha-h(t)} \right\}$$

determined in terms of a fractional Brownian motion B_α with Hurst index $\alpha/2 \in (0, 1]$, the non-negative constant δ and a continuous function h. We show that these constants, similarly to generalised Pickands constants, appear naturally in the tail asymptotic behaviour of supremum of Gaussian processes. Further, we derive several bounds for $\mathcal{P}_{\alpha,\delta}^h$ and in special cases explicit formulas are obtained.

Through this thesis, the notation always has the following definition, unless we redefined them. First is Pickands-type constant defined by

$$\mathcal{H}_\alpha = \lim_{T\to\infty} \frac{1}{T}\mathcal{H}_\alpha[0, T], \quad \text{with } \mathcal{H}_\alpha[S, T] = \mathbb{E}\left\{ \sup_{t\in[0,T]} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha} \right\}, \quad T > S, \tag{1.2}$$

where $S, T \in \mathbb{R}$ are constants with $S < T$ and B_α is an fBm. Further, define for $f \in C_0^*([S, T])$ and a positive constant a

$$\mathcal{P}_{\alpha,a}^f[S, T] = \mathbb{E}\left\{ \sup_{t\in[S,T]} e^{\sqrt{2a}B_\alpha(t)-a|t|^\alpha-f(t)} \right\}, \tag{1.3}$$

and set

$$\mathcal{P}_{\alpha,a}^f[0, \infty) = \lim_{T\to\infty} \mathcal{P}_{\alpha,a}^f[0, T], \quad \mathcal{P}_{\alpha,a}^f(-\infty, \infty) = \lim_{S\to-\infty, T\to\infty} \mathcal{P}_{\alpha,a}^f[S, T].$$

The finiteness of $\mathcal{P}_{\alpha,a}^f[0, \infty)$ and $\mathcal{P}_{\alpha,a}^f(-\infty, \infty)$ is guaranteed under weak assumptions on f, which will be shown in the proof of Theorem 2.2.1, see [134, 76, 77, 43, 114, 13, 116, 118, 47, 63, 37, 65, 40, 121, 57, 64, 44, 79, 34] for various properties of $\mathcal{H}_\alpha$ and $\mathcal{P}_{\alpha,a}^f[0, \infty)$.

In our notation, $\sim$ means asymptotic equivalence when the argument tends to 0 (or ∞). Below $\Phi(\cdot)$ and $\Psi(\cdot)$ stand for the distribution function and survival function of an $N(0, 1)$ random variable, respectively. Note that $\Psi(u) \sim \frac{1}{\sqrt{2\pi}u} e^{-\frac{u^2}{2}}, u \to \infty$. Denote by $\Gamma(\cdot)$ the gamma function and $\mathbb{I}_{\{\cdot\}}$ the indicator function.

Chapter 2

Extremes of Threshold-Dependent Gaussian Processes[1]

2.1 Introduction

Let $X(t), t \geq 0$ be a centered Gaussian process with continuous trajectories. An important problem in applied and theoretical probability is the determination of the asymptotic behavior of

$$p(u) = \mathbb{P}\left\{ \sup_{t \in [0,T]} (X(t) + g(t)) > u \right\}, \quad u \to \infty \tag{2.1}$$

for some $T > 0$ and $g(t), t \in [0, T]$ a bounded measurable function. For instance, if $g(t) = -ct$, then in the context of risk theory $p(u)$ has interpretation as the ruin probability over the finite-time horizon $[0, T]$. Dually, in the context of queueing theory, $p(u)$ is related to the buffer overload problem; see e.g., [53, 47, 63, 84, 40].

For the special case that $g(t) = 0, t \in [0, T]$ the exact asymptotics of (2.1) is well-known for both locally stationary and general non-stationary Gaussian processes, see e.g., [115, 118, 132, 18, 119, 9, 80, 49, 54, 121, 24, 23, 5]. Commonly, for X a centered non-stationary Gaussian process it is assumed that the standard deviation function σ is such that $t_0 = \arg\max_{t \in [0,T]} \sigma(t)$ is unique and $\sigma(t_0) = 1$. Additionally, if the correlation function r and the standard deviation function σ satisfy (hereafter $\sim$ means asymptotic equivalence)

$$1 - r(s, t) \sim a|t - s|^{\alpha}, \qquad 1 - \sigma(t_0 + t) \sim b|t|^{\beta}, \qquad s, t \to t_0 \tag{2.2}$$

for some a, b, β positive and $\alpha \in (0, 2]$, then we have (see [119][Theorem D.3])

$$p(u) \sim C_0 u^{(\frac{2}{\alpha} - \frac{2}{\beta})_+} \mathbb{P}\left\{X(t_0) > u\right\}, \quad u \to \infty, \tag{2.3}$$

where $(x)_+ = \max(0, x)$ and

$$C_0 = \begin{cases} a^{1/\alpha} b^{-1/\beta} \Gamma(1/\beta + 1) \mathcal{H}_{\alpha}, & \text{if } \alpha < \beta, \\ \mathcal{P}_{\alpha,a}^{b|t|^{\alpha}}, & \text{if } \alpha = \beta, \\ 1, & \text{if } \alpha > \beta. \end{cases}$$

The more general case with non-zero g has also been considered in the literature for both finite- and infinite-time horizon; see e.g., [123, 33, 125, 53, 82, 91]. However, most of the aforementioned contributions related to finite-time horizon treat only restrictive trend functions g. For instance, in [123][Theorem 3] a Hölder-type condition for g is assumed, which excludes important cases of g that appear in applications. The restrictions are often so severe that simple cases such as the Brownian bridge with drift considered in Example 2.3.3 below cannot be covered.

A key difficulty when dealing with $p(u)$ is that $X + g$ is not a centered Gaussian process. It is however possible to

[1] This chapter is based on L. Bai, K. Dębicki, E. Hashorva, and L. Ji (2018): Extremes of Threshold-Dependent Gaussian Processes, published in the *Science China Mathematics*, to appear.

get rid of the trend function g since for any bounded function g and all large u (2.1) can be re-written as

$$p_T(u) = \mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > u\right\}, \quad X_u(t) = \frac{X(t)}{1-g(t)/u}, \quad t \in [0,T]. \tag{2.4}$$

The advantage of the above rearrangement is that, for each large u, the process $X_u(t)$, $t \in [0,T]$ is centered. However, $X_u(t)$ depends on the threshold u, which makes the analysis more complicated than in the classical centered case (2.2).

Our principal result is Theorem 2.2.2 which derives the asymptotics of $p_T(u)$ for quite general families of centered Gaussian processes X_u under tractable assumptions on the variance and correlation functions of X_u. To this end, using tailored double sum method, in Theorem 2.2.1 we first derive the asymptotics of

$$p_\Delta(u) = \mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t) > u\right\}, \quad u \to \infty$$

for some short compact intervals $\Delta(u)\subseteq[0,T]$, $u > 0$, for which $p_T(u) \sim p_\Delta(u)$, as $u \to \infty$.

The idea of transformation of the original problem into the crossing probability of some threshold-dependent Gaussian process and then application of the double sum technique was used also in several contributions that deal with analogs of (2.1) for infinite time horizon, i.e. for $T = \infty$; see e.g., [47, 63, 89–91]. However, the transformation used there needs different time-scaling than proposed in this contribution, i.e. is of the form $\tilde{X}_u(t) = X(ut)/(1+g(ut)/u)$. Then the asymptotics of $p_\infty(u)$, as $u \to \infty$, is usually concentrated around $t_u := \arg\max_{t\in[0,\infty)} \mathrm{var}\left(\tilde{X}_u(t)\right)$, with the local structure of variance

$$\frac{\mathrm{var}\,(X_u(t))}{\mathrm{var}\,(X_u(t_u))} = 1 - h(u)\sigma_\eta(t-t_u)(1+o(1)), \tag{2.5}$$

as $t \to t_u$, where $\sigma_\eta^2 = \mathrm{var}(\eta(t))$ and η is some Gaussian process with stationary increments. The factorization present on the right hand side of (2.5) simplifies next steps of the analysis, which is usually based on the double sum technique. In this paper we focus on finite-time case $T < \infty$, which requires transformation like in (2.4), where the local structure of the variance function of X_u has more complicated form than (2.5); see assumption **A2** in Section 2.2. It is worth mentioning that a slightly different transformation than (2.4) has also been adopted in, e.g., [53, 82] when dealing with finite-time case; however, in those contributions lower and upper bounds are derived to reduce the difficulty of the problem, for which some Hölder-type condition on g has to be imposed.

Theorem 2.2.2 extends partial results analyzed in literature, as e.g. in [53], from the class of Gaussian processes with stationary increments with specific drift to more general family of Gaussian processes with general drift functions. More specifically, applications of our main results include new results for a class of locally stationary Gaussian processes with general trend (Proposition 2.3.1) and that of Proposition 2.3.3 for the class of non-stationary Gaussian processes with trend, as well as those of their corollaries. For instance, a direct application of Proposition 2.3.3 yields the asymptotics of (2.1) for a non-stationary X with standard deviation function σ and correlation function r satisfying (2.2) with $t_0 = argmax_{t\in[0,T]}\sigma(t)$. If further the trend function g is continuous in a neighborhood of t_0, $g(t_0) = \max_{t\in[0,T]} g(t)$ and

$$g(t) \sim g(t_0) - c|t-t_0|^\gamma, \quad t \to t_0 \tag{2.6}$$

for some positive constants c, γ, then (2.3) holds with C_0 specified in Theorem 2.3.5 and β, u being substituted by $\min(\beta, 2\gamma)$ and $u - g(t_0)$ respectively. As an application of the derived results, in Section 2.3.3 we find asymptotics of *ruin probability* in a Gaussian risk model with constant force of interest.

Complementary, we investigate asymptotic properties of the first passage time (ruin time) of $X(t) + g(t)$ to u on the finite-time interval $[0,T]$, given the process has ever exceeded u during $[0,T]$. Here all the derived results are new. In particular, for

$$\tau_u = \inf\{t \geq 0 : X(t) > u - g(t)\}, \tag{2.7}$$

with $\inf\{\emptyset\} = \infty$, we are interested in the approximate distribution of $\tau_u|\tau_u \le T$, as $u \to \infty$. Normal and exponential approximations of various Gaussian models have been discussed in [91, 81, 41, 42, 45]. In this paper, we derive general results for the approximations of the conditional passage time in Propositions 2.3.2, 2.3.6. The asymptotics of $p_\Delta(u)$ for $\Delta(u)$ displayed in Theorem 2.2.1 plays a key role in the derivation of these results.

Organisation of the rest of the paper: In Section 2, the tail asymptotics of the supremum of a family of centered Gaussian processes indexed by u are given. Several applications and examples are displayed in Section 3. Finally, we present all the proofs in Section 4 and Section 5.

2.2 Main Results

Let $X_u(t), t \in \mathbb{R}, u > 0$ be a family of threshold-dependent centered Gaussian processes with continuous trajectories, variance functions σ_u^2 and correlation functions r_u. Our main results concern the asymptotics of slight generalization of $p_\Delta(u)$ and $p_T(u)$ for families of centered Gaussian processes X_u satisfying some regularity conditions for variance and covariance respectively.

Let $C_0^*(E)$ be the set of continuous real-valued functions defined on the interval E such that $f(0) = 0$ and for some $\epsilon_2 > \epsilon_1 > 0$

$$\lim_{|t|\to\infty, t\in E} f(t)/|t|^{\epsilon_1} = \infty, \qquad \lim_{|t|\to\infty, t\in E} f(t)/|t|^{\epsilon_2} = 0, \tag{2.8}$$

if $\sup\{x : x \in E\} = \infty$ or $\inf\{x : x \in E\} = -\infty$.

In the following $\mathcal{R}_\alpha$ denotes the set of regularly varying functions at 0 with index $\alpha \in \mathbb{R}$, see [69, 129, 136] for details.

We shall impose the following assumptions where $\Delta(u)$ is a compact interval:

A1: For any large u, there exists a point $t_u \in \mathbb{R}$ such that $\sigma_u(t_u) = 1$.

A2: There exists some $\lambda > 0$ such that

$$\lim_{u\to\infty} \sup_{t\in\Delta(u)} \left| \frac{\left(\frac{1}{\sigma_u(t_u+t)} - 1\right) u^2 - f(u^\lambda t)}{f(u^\lambda t) + 1} \right| = 0 \tag{2.9}$$

holds for some non-negative continuous function f with $f(0) = 0$.

A3: There exists $\rho \in \mathcal{R}_{\alpha/2}, \alpha \in (0, 2]$ such that

$$\lim_{u\to\infty} \sup_{\substack{s,t\in\Delta(u)\\ t\neq s}} \left| \frac{1 - r_u(t_u + s, t_u + t)}{\rho^2(|t - s|)} - 1 \right| = 0.$$

In the rest of the paper we tacitly assume that

$$\eta := \lim_{s\to 0} \frac{\rho^2(s)}{s^{2/\lambda}} \in [0, \infty],$$

with λ given in **A2**.

Remarks 2.2.1. i) If f satisfies $f(0) = 0$ and $f(t) > 0, t \neq 0$, then

$$\lim_{u\to\infty} \sup_{t\in\Delta(u), t\neq 0} \left| \frac{\frac{1}{\sigma_u(t_u+t)} - 1}{u^{-2} f(u^\lambda t)} - 1 \right| = 0$$

for some $\lambda > 0$ implies that (2.9) is valid.

ii) Condition **A2** is crucial for getting precise tail asymptotics of $\sup_{t\in\Delta(u)} X_u(t_u + t)$ given in Theorem 2.2.1. More precisely, together with **A3** it guarantees that the conditional process, which plays a key role in main steps of the proof of Theorem 2.2.1, weakly converges to $\sqrt{2a}B_\alpha(t) - a|t|^\alpha - f(t)$ for some appropriately chosen $a > 0$, shaping the form of the asymptotic constant in the derived asymptotics; see (1.3). Assumption **A3** extends (2.2) allowing local behavior of the correlation to behave according to the class of regularly varying functions.

Using that $\sigma_u(t_u) = 1$, assumption **A2** covers the case $\sigma_u(t_u + t) = 1 - cu^{-\gamma} t^\beta (1 + o(1))$ for suitably chosen γ, β and power function f. For example, if $t_u = 0, \sigma_u(t) = 1 - t^2$ and $\Delta(u) = [0, u^{-1}]$, then (2.9) holds with $f(t) = t^2$ and $\lambda = 1$.

For the regularly varying function $\rho(\cdot)$, we denote by $\overleftarrow{\rho}(\cdot)$ its asymptotic inverse (which is asymptotically unique). Further, we set $0 \cdot \infty = 0$ and $u^{-\infty} = 0$ if $u > 0$.
In the next theorem we shall consider two functions $x_1(u), x_2(u), u \in \mathbb{R}$ such that $x_1(\frac{1}{t}) \in \mathcal{R}_{\mu_1}$, $x_2(\frac{1}{t}) \in \mathcal{R}_{\mu_2}$ with $\mu_1, \mu_2 \geq \lambda$, and

$$\lim_{u\to\infty} u^{\lambda} x_i(u) = x_i \in [-\infty, \infty], i = 1, 2, \quad \text{with} \quad x_1 < x_2. \tag{2.10}$$

Theorem 2.2.1. *Let $X_u(t), t \in \mathbb{R}$ be a family of centered Gaussian processes with variance functions σ_u^2 and correlation functions r_u. If **A1-A3** are satisfied with $\Delta(u) = [x_1(u), x_2(u)]$, and $f \in C_0^*([x_1, x_2])$, then for M_u satisfying $M_u \sim u, u \to \infty$, we have*

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t_u + t) > M_u\right\} \sim C\left(u^{\lambda}\overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}} \Psi(M_u), \quad u \to \infty, \tag{2.11}$$

where

$$C = \begin{cases} \mathcal{H}_\alpha \int_{x_1}^{x_2} e^{-f(t)} dt, & \text{if } \eta = \infty, \\ \mathcal{P}_{\alpha,\eta}^f [x_1, x_2], & \text{if } \eta \in (0, \infty), \\ \sup_{t\in[x_1,x_2]} e^{-f(t)}, & \text{if } \eta = 0, \end{cases} \tag{2.12}$$

and $\mathcal{P}_{\alpha,\eta}^f(-\infty, \infty) \in (0, \infty)$.

Remark. Let $\alpha \in (0, 2], a > 0$ be given. If $f \in C_0^*([x_1, x_2])$ for $x_1, x_2, y \in \mathbb{R}, x_1 < x_2$, as shown in Appendix, we have, with $f_y(t) := f(y + t), t \in \mathbb{R}$

$$\mathcal{P}_{\alpha,a}^f [x_1, x_2] = \mathcal{P}_{\alpha,a}^{f_y}[x_1 - y, x_2 - y], \quad \mathcal{P}_{\alpha,a}^f [x_1, \infty) = \mathcal{P}_{\alpha,a}^{f_y}[x_1 - y, \infty). \tag{2.13}$$

In particular, if $f(t) = ct, c > 0$, then for any $x \in \mathbb{R}$

$$\mathcal{P}_{\alpha,a}^{ct}[x, \infty) = \mathcal{P}_{\alpha,a}^{cx+ct}[0, \infty) = e^{-cx}\mathcal{P}_{\alpha,a}^{ct}[0, \infty).$$

Next, for any fixed $T \in (0, \infty)$, in order to analyse $p_T(u)$ we shall suppose that:
A1': For all large u, $\sigma_u(t)$ attains its maximum over $[0, T]$ at a unique point t_u such that

$$\sigma_u(t_u) = 1 \quad \text{and} \quad \lim_{u\to\infty} t_u = t_0 \in [0, T].$$

A4: For all u large enough

$$\inf_{t\in[0,T]\setminus(t_u+\Delta(u))} \frac{1}{\sigma_u(t)} \geq 1 + \frac{p(\ln u)^q}{u^2} \tag{2.14}$$

holds for some constants $p > 0, q > 1$.

A5: For some positive constants $G, \varsigma > 0$

$$\mathbb{E}\left\{(\overline{X}_u(t) - \overline{X}_u(s))^2\right\} \leq G|t - s|^{\varsigma}$$

holds for all $s, t \in \{x \in [0, T] : \sigma(x) \neq 0\}$ and $\overline{X}_u(t) = \frac{X_u(t)}{\sigma_u(t)}$.

Below we define for λ given in **A2** and ν, d positve

$$\Delta(u)=\begin{cases}[0,\delta_u] & \text{if } \ t_u\equiv 0,\\ [-t_u,\delta_u], & \text{if } \ t_u\sim du^{-\nu} \text{ and } \nu\ge\lambda,\\ [-\delta_u,\delta_u], & \text{if } \ t_u\sim du^{-\nu} \text{ or } T-t_u\sim du^{-\nu} \text{ when } \nu<\lambda, \text{ or } t_0\in(0,T),\\ [-\delta_u,T-t_u], & \text{if } \ T-t_u\sim du^{-\nu} \text{ and } \nu\ge\lambda,\\ [-\delta_u,0] & \text{if } \ t_u=T,\end{cases} \tag{2.15}$$

where $\delta_u=\left(\frac{(\ln u)^q}{u}\right)^\lambda$ with q given in **A4**.

Theorem 2.2.2. *Let $X_u(t), t\in[0,T]$ be a family of centered Gaussian processes with variance functions σ_u^2 and correlation functions r_u. Assume that **A1'**,**A2**-**A5** are satisfied with $\Delta(u)=[c_1(u),c_2(u)]$ given in (2.15) and*

$$\lim_{u\to\infty} c_i(u)u^\lambda = x_i\in[-\infty,\infty], i=1,2, \quad x_1<x_2.$$

If $f\in C_0^*([x_1,x_2])$, then for M_u such that $\lim_{u\to\infty} M_u/u=1$ we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t)>M_u\right\}\sim C\left(u^\lambda\overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}}\Psi(M_u), \quad u\to\infty, \tag{2.16}$$

where C is the same as in (2.12) if $\eta\in(0,\infty]$ and $C=1$ if $\eta=0$.

Remark. In the case that $\Delta(u)$ does not depend on the time horizon T and $t_0<\infty$, the asymptotic result in (2.16) in some cases allows for replacement of T by ∞. In this case, Theorem 2.2.2 can be applied directly for the asymptotics of the tail probability of maximum over infinite-time horizon of Gaussian processes with trend, under appropriate conditions on variance of $X(t)$ or/and trend function $g(t)$ as $t\to\infty$.

2.3 Applications

2.3.1 Locally stationary Gaussian processes with trend

In this section we consider the asymptotics of (2.1) for $X(t), t\in[0,T]$ a centered locally stationary Gaussian process with unit variance and correlation function r satisfying

$$\lim_{\varepsilon\to0}\sup_{t\in[0,T],|h|<\varepsilon}\left|\frac{1-r(t,t+h)}{|h|^\alpha}-a(t)\right|=0 \tag{2.17}$$

with $\alpha\in(0,2]$, $a(\cdot)$ a positive continuous function on $[0,T]$ and further

$$r(s,t)<1, \ \forall s,t\in[0,T] \text{ and } s\neq t. \tag{2.18}$$

We refer to e.g., [16, 18, 87, 119, 22] for results on locally stationary Gaussian processes. Extensions of this class to $\alpha(t)$-locally stationary processes are discussed in [49, 83, 10].
Regarding the continuous trend function g, we define $g_m=\max_{t\in[0,T]} g(t)$ and set

$$H:=\{s\in[0,T]: g(s)=g_m\}.$$

Set below, for any $t_0\in[0,T]$

$$Q_{t_0}=1+\mathbb{I}_{\{t_0\in(0,T)\}}, \quad w_{t_0}=\begin{cases}-\infty, & \text{if } t_0\in(0,T),\\ 0, & \text{if } t_0=0 \text{ or } t_0=T.\end{cases} \tag{2.19}$$

Proposition 2.3.1. *Suppose that (2.17) and (2.18) hold for a centered locally stationary Gaussian process $X(t), t\in[0,T]$ and let $g:[0,T]\to\mathbb{R}$ be a continuous function.*

i) If $H=\{t_0\}$ and (2.6) holds, then as $u\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}\sim C_{t_0}u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi\left(u-g_m\right), \tag{2.20}$$

where (set $a=a(t_0)$)

$$C_{t_0}=\begin{cases} Q_{t_0}a^{1/\alpha}c^{-1/\gamma}\Gamma(1/\gamma+1)\mathcal{H}_\alpha, & \text{if } \alpha<2\gamma,\\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0},\infty), & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma.\end{cases}$$

ii) If $H=[A,B]\subset[0,T]$ with $0\le A<B\le T$, then as $u\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}\sim\mathcal{H}_\alpha\int_A^B(a(t))^{1/\alpha}dt u^{\frac{2}{\alpha}}\Psi\left(u-g_m\right).$$

Remarks 2.3.1. i) If $H=\{t_1,\ldots,t_n\}$, then as mentioned in [119], the tail distribution of the corresponding supremum is easily obtained assuming that for each t_i the assumptions of Theorem 2.3.1 statement i) hold, implying that

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}\sim\Big(\sum_{j=1}^n C_{t_j}\Big)u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi\left(u-g_m\right),\quad u\to\infty.$$

ii) The novelty of Theorem 2.3.1 statement i) is that for the trend function g only a polynomial local behavior around t_0 is assumed. In the literature so far only the case that (2.6) holds with $\gamma=2$ has been considered (see [125]).

iii) By the proof of Proposition 2.3.1 statement i), if $g(t)$ is a measurable function which is continuous in a neighborhood of t_0 and smaller than $g_m-\varepsilon$ for some $\varepsilon>0$ in the rest part over $[0,T]$, then the results still hold.

We present below the approximation of the conditional passage time $\tau_u|\tau_u\le T$ with τ_u defined in (2.7).

Proposition 2.3.2. *Suppose that (2.17) and (2.18) hold for a centered locally stationary Gaussian process $X(t),t\in[0,T]$. Let $g:[0,T]\to\mathbb{R}$ be a continuous function, $H=\{t_0\}$ and (2.6) holds.*
i) If $t_0\in[0,T)$, then for any $x\in(w_{t_0},\infty)$

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u-t_0)\le x\middle|\tau_u\le T\right\}\sim\begin{cases}\frac{\gamma c^{1/\gamma}\int_{w_{t_0}}^x e^{-c|t|^\gamma}dt}{Q_{t_0}\Gamma(1/\gamma)}, & \text{if } \alpha<2\gamma,\\ \frac{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0},x]}{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0},\infty)}, & \text{if } \alpha=2\gamma,\\ \sup_{t\in[w_{t_0},x]}e^{-c|t|^\gamma}, & \text{if } \alpha>2\gamma.\end{cases}$$

ii) If $t_0=T$, then for any $x\in(-\infty,0)$

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u-t_0)\le x\middle|\tau_u\le T\right\}\sim\begin{cases}\frac{\gamma c^{1/\gamma}\int_{-x}^\infty e^{-c|t|^\gamma}dt}{\Gamma(1/\gamma)}, & \text{if } \alpha<2\gamma,\\ \frac{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[-x,\infty)}{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[0,\infty)}, & \text{if } \alpha=2\gamma,\\ e^{-c|x|^\gamma}, & \text{if } \alpha>2\gamma.\end{cases}$$

Example 2.3.1. Let $X(t),t\in[0,T]$ be a centered stationary Gaussian process with unit variance and correlation function r that satisfies $r(t)=1-a|t|^\alpha(1+o(1)),\ t\to0$ for some $a>0$, $\alpha\in(0,2]$, and $r(t)<1$, for all $t\in(0,T]$. Let τ_u be defined as in (2.7) with $g(t)=-ct,c>0$. Then we have

$$\mathbb{P}\left\{\max_{t\in[0,T]}(X(t)-ct)>u\right\}\sim u^{(\frac{2}{\alpha}-1)_+}\Psi(u)\begin{cases}c^{-1}a^{1/\alpha}\mathcal{H}_\alpha, & \alpha\in(0,2),\\ \mathcal{P}_{\alpha,a}^{ct}[0,\infty), & \alpha=2,\end{cases}$$

and for any x positive

$$\mathbb{P}\left\{u\tau_u \le x \Big| \tau_u \le T\right\} \sim \begin{cases} 1-e^{-cx}, & \alpha\in(0,2),\\ \frac{\mathcal{P}^{ct}_{\alpha,a}[0,x]}{\mathcal{P}^{ct}_{\alpha,a}[0,\infty)}, & \alpha=2.\end{cases}$$

Example 2.3.2. Let $X(t), t>0$ be a standardized fBm, i.e., $X(t)=B_\alpha(t)/t^{\alpha/2}$ with B_α an fBm. Let c,T be positive constants. Then for any $n\in\mathbb{N}$, we have

$$\mathbb{P}\left\{\max_{t\in[T,(n+1)T]}\left(X(t)+c\sin\left(\frac{2\pi t}{T}\right)\right)>u\right\}\sim\left(\sum_{j=1}^{n}a_j^{\frac{1}{\alpha}}\right)\mathcal{H}_\alpha\frac{T}{\sqrt{2c\pi}}u^{\frac{2}{\alpha}-\frac{1}{2}}\Psi(u-c),$$

where $a_j=\frac{1}{2}\left(\frac{(4j+1)T}{4}\right)^{-\alpha}, j=1,\ldots,n$.

2.3.2 Non-stationary Gaussian processes with trend

In this section we consider the asymptotics of (2.1) for $X(t), t\in[0,T]$ a centered Gaussian process with non-constant variance function σ^2. Define below whenever $\sigma(t)\neq 0$

$$\overline{X}(t):=\frac{X(t)}{\sigma(t)},\quad t\in[0,T],$$

and set for a continuous function g

$$m_u(t):=\frac{\sigma(t)}{1-g(t)/u},\quad t\in[0,T],\quad u>0. \tag{2.21}$$

Proposition 2.3.3. *Let X and g be as above. Assume that $t_u=argmax_{t\in[0,T]}m_u(t)$ is unique with $\lim_{u\to\infty}t_u=t_0$ and $\sigma(t_0)=1$. Further, we suppose that **A2-A5** are satisfied with $\sigma_u(t)=\frac{m_u(t)}{m_u(t_u)}$, $r_u(s,t)=r(s,t)$, $\overline{X}_u(t)=\overline{X}(t)$ and $\Delta(u)=[c_1(u),c_2(u)]$ given in (2.15). If in **A2** $f\in C_0^*([x_1,x_2])$ and*

$$\lim_{u\to\infty}c_i(u)u^\lambda=x_i\in[-\infty,\infty], i=1,2,\quad x_1<x_2,$$

then we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}\sim C\left(u^\lambda\overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}}\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right),\quad u\to\infty, \tag{2.22}$$

where C is the same as in (2.12) when $\eta\in(0,\infty]$ and $C=1$ when $\eta=0$.

Remarks 2.3.2. i) Theorem 2.3.3 extends [123][Theorem 3] and the results of [53] where (2.1) was analyzed for special X with stationary increments and special trend function g.

ii) The assumption that $\sigma(t_0)=1$ is not essential in the proof. In fact, for the general case where $\sigma(t_0)\neq 1$ we have that (2.22) holds with

$$C=\begin{cases}\sigma_0^{-\frac{2}{\alpha}}\mathcal{H}_\alpha\int_{x_1}^{x_2}e^{-\sigma_0^{-2}f(t)}dt, & \text{if }\ \eta=\infty,\\ \mathcal{P}_{\alpha,\sigma_0^{-2}\eta}^{\sigma_0^{-2}f}[x_1,x_2], & \text{if }\ \eta\in(0,\infty),\\ 1, & \text{if }\ \eta=0,\end{cases}\qquad \sigma_0=\sigma(t_0).$$

Proposition 2.3.4. *Under the notation and assumptions of Theorem 2.3.3 without assuming* **A3,A5**, *if X is differentiable in the mean square sense such that*

$$r(s,t)<1, s\neq t,\quad \mathbb{E}\left\{X'^2(t_0)\right\}>\sigma'^2(t_0),$$

and $\mathbb{E}\left\{X'^2(t)\right\} - \sigma'^2(t)$ *is continuous in a neighborhood of* t_0*, then* (2.22) *holds with*

$$\alpha = 2, \quad \rho^2(t) = \frac{1}{2}\left(\mathbb{E}\left\{X'^2(t_0)\right\} - \sigma'^2(t_0)\right)t^2.$$

The next result is an extension of a classical theorem concerning the extremes of non-stationary Gaussian processes discussed in the Introduction, see [119][Theorem D.3].

Proposition 2.3.5. *Let* $X(t), t \in [0,T]$ *be a centered Gaussian process with correlation function* r *and variance function* σ^2 *such that* $t_0 = argmax_{t\in[0,T]}\sigma(t)$ *is unique with* $\sigma(t_0) = \sigma > 0$*. Suppose that* g *is a bounded measurable function being continuous in a neighborhood of* t_0 *such that* (2.6) *holds. If further* (2.2) *is satisfied, then*

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t)) > u\right\} \sim C_0 u^{(\frac{2}{\alpha}-\frac{2}{\beta^*})_+}\Psi\left(\frac{u-g(t_0)}{\sigma}\right), \tag{2.23}$$

where $\beta^* = \min(\beta, 2\gamma)$*,*

$$C_0 = \begin{cases} \sigma^{-2/\alpha}a^{1/\alpha}\mathcal{H}_\alpha\int_{w_{t_0}}^\infty e^{-f(t)}dt, & \text{if } \alpha < \beta^*, \\ \mathcal{P}^f_{\alpha,\sigma^{-2}a}[w_{t_0},\infty), & \text{if } \alpha = \beta^*, \\ 1, & \text{if } \alpha > \beta^*, \end{cases}$$

with $f(t) = \frac{b}{\sigma^3}|t|^\beta \mathbb{I}_{\{\beta=\beta^*\}} + \frac{c}{\sigma^2}|t|^\gamma \mathbb{I}_{\{2\gamma=\beta^*\}}$ *and* w_{t_0} *defined in* (2.19)*.*

Proposition 2.3.6. *i) Under the conditions and notation of Theorem 2.3.3, for any* $x \in [x_1, x_2]$ *we have*

$$\lim_{u\to\infty}\mathbb{P}\left\{u^\lambda(\tau_u - t_u) \le x \middle| \tau_u \le T\right\} = \begin{cases} \frac{\int_{x_1}^x e^{-f(t)}dt}{\int_{x_1}^{x_2} e^{-f(t)}dt}, & \text{if } \eta = \infty, \\ \frac{\mathcal{P}^f_{\alpha,\eta}[x_1,x]}{\mathcal{P}^f_{\alpha,\eta}[x_1,x_2]}, & \text{if } \eta \in (0,\infty), \\ \sup_{t\in[x_1,x]} e^{-f(t)}, & \text{if } \eta = 0. \end{cases} \tag{2.24}$$

ii) Under the conditions and notation of Theorem 2.3.5, if $t_0 \in [0,T)$*, then for* $x \in (w_{t_0}, \infty)$

$$\lim_{u\to\infty}\mathbb{P}\left\{u^{2/\beta^*}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} = \begin{cases} \frac{\int_{w_{t_0}}^x e^{-f(t)}dt}{\int_{w_{t_0}}^\infty e^{-f(t)}dt}, & \text{if } \alpha < \beta^*, \\ \frac{\mathcal{P}^f_{\alpha,a}[w_{t_0},x]}{\mathcal{P}^f_{\alpha,a}[w_{t_0},\infty)}, & \text{if } \alpha = \beta^*, \\ \sup_{t\in[w_{t_0},x]} e^{-f(t)}, & \text{if } \alpha > \beta^*, \end{cases}$$

and if $t_0 = T$*, then for* $x \in (-\infty, 0)$

$$\lim_{u\to\infty}\mathbb{P}\left\{u^{2/\beta^*}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} = \begin{cases} \frac{\int_{-x}^\infty e^{-f(t)}dt}{\int_0^\infty e^{-f(t)}dt}, & \text{if } \alpha < \beta^*, \\ \frac{\mathcal{P}^f_{\alpha,a}[-x,\infty)}{\mathcal{P}^f_{\alpha,a}[0,\infty)}, & \text{if } \alpha = \beta^*, \\ e^{-f(x)}, & \text{if } \alpha > \beta^*. \end{cases}$$

Example 2.3.3. Let $X(t) = B(t) - tB(1), t \in [0,1]$, where $B(t)$ is a standard Brownian motion and suppose that τ_u is defined by (2.7) with $g(t) = -ct$. Then

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct) > u\right\} \sim e^{-2(u^2+cu)}, \tag{2.25}$$

$$\mathbb{P}\left\{u\left(\tau_u - \frac{u}{c+2u}\right) \le x \middle| \tau_u \le 1\right\} \sim \Phi(4x), \quad x \in (-\infty,\infty).$$

We note that according to [20][Lemma 2.7], the result in (2.25) is actually exact, i.e. for any $u > 0$,

$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}=e^{-2(u^2+cu)}$.
Now, let $T=1/2$. It appears that the asymptotics in this case is different, i.e.,

$$\mathbb{P}\left\{\sup_{t\in[0,1/2]}(X(t)-ct)>u\right\}\sim\Phi(c)e^{-2(u^2+cu)}, \tag{2.26}$$

and

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\le x\Big|\tau_u\le\frac{1}{2}\right\}\sim\frac{\Phi(4x)}{\Phi(c)},\quad x\in(-\infty,c/4].$$

Similarly, we have

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\left(X(t)+\frac{c}{2}-c\left|t-\frac{1}{2}\right|\right)>u\right\}\sim 2\Psi(c)e^{-2(u^2-cu)} \tag{2.27}$$

and

$$\mathbb{P}\left\{u\left(\tau_u-\frac{1}{2}\right)\le x\Big|\tau_u\le 1\right\}\sim\frac{\int_{-\infty}^{4x}e^{-\frac{(|t|+c)^2}{2}}dt}{2\sqrt{2\pi}\Psi(c)},\quad x\in(-\infty,\infty).$$

We conclude this section with an application of Theorem 2.3.3 to the calculation of the ruin probability of a Brownian motion risk model with constant force of interest over infinite-time horizon.

2.3.3 Ruin probability in Gaussian risk model

Consider risk reserve process $U(t)$, with interest rate δ modeled by

$$U(t)=ue^{\delta t}+c\int_0^t e^{\delta(t-v)}dv-\sigma\int_0^t e^{\delta(t-v)}dB(v),\quad t\ge 0,$$

where c,δ,σ are some positive constants and B is a standard Brownian motion. The corresponding ruin probability over infinite-time horizon is defined as

$$p(u)=\mathbb{P}\left\{\inf_{t\in[0,\infty)}U(t)<0\right\}.$$

For this model we also define the ruin time $\tau_u=\inf\{t\ge 0: U(t)<0\}$. Set below

$$h(t)=\frac{\delta}{\sigma^2}\left(\sqrt{t+r^2}-r\right)^2,\quad t\in[0,\infty),\quad r=\frac{c}{\delta}.$$

We present next approximations of the ruin probability and the conditional ruin time $\tau_u|\tau_u<\infty$ as $u\to\infty$.

Theorem 2.3.1. *As $u\to\infty$*

$$p(u)\sim\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,\infty\right)\Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2+4cu}\right) \tag{2.28}$$

and for $x\in(-r^2,\infty)$

$$\mathbb{P}\left\{u^2\left(e^{-2\delta\tau_u}-\left(\frac{c}{\delta u+c}\right)^2\right)\le x\big|\tau_u<\infty\right\}\sim\frac{\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,x\right]}{\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,\infty\right)}.$$

Remark. According to [78] (see also [68]) for any c,δ positive we have

$$\mathbb{P}\left\{\inf_{t\in[0,\infty)}U(t)<0\right\}=\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right)\Big/\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right). \tag{2.29}$$

By (2.28) and (2.13)

$$\begin{aligned}\mathbb{P}\left\{\inf_{t\in[0,\infty]} U(t)<0\right\} &\sim \mathbb{E}\left\{\sup_{t\in[-r^2,\infty)} \exp\left(\sqrt{\frac{2\delta}{\sigma^2}}B(t)-\frac{\delta}{\sigma^2}\left(\sqrt{t+r^2}-r\right)^2-\frac{\delta}{\sigma^2}|t|\right)\right\}\Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2+4cu}\right)\\ &\sim \mathbb{E}\left\{\sup_{t\in[-\frac{c^2}{\sigma^2\delta},\infty)} \exp\left(\sqrt{2}B(t)-\left(t+\frac{c^2}{\sigma^2\delta}\right)+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t+\frac{c^2}{\sigma^2\delta}}-|t|\right)\right\}\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right)\\ &= \mathbb{E}\left\{\sup_{t\in[0,\infty)} \exp\left(\sqrt{2}B\left(t\right)-2t+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t}\right)\right\}\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right),\end{aligned}$$

which combined with (2.29) implies that for any c,δ,σ positive

$$\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right)\mathbb{E}\left\{\sup_{t\in[0,\infty)}\exp\left(\sqrt{2}B\left(t\right)-2t+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t}\right)\right\}=1. \tag{2.30}$$

2.4 Proofs

In the proofs presented in this section $\mathbb{C}_i, i\in\mathbb{N}$ are some positive constants which may be different from line to line. We first give two preliminary lemmas, which play an important role in the proof of Theorem 2.2.1.

Lemma 2.4.1. *If ρ be a regularly varying function at 0 with index $\alpha/2\in(0,1]$, then there exists a centered stationary Gaussian process $\xi(t), t\in\mathbb{R}$ with unit variance, continuous sample paths and correlation function r satisfying*

$$1-r(t)\sim a\rho^2(|t|),\quad t\to 0,\ a>0. \tag{2.31}$$

Moreover, if f is a continuous function, and K_u is a family of countable index sets, then for

$$Z_u(t):=\frac{\xi(\overleftarrow{\rho}(u^{-1})t)}{1+u^{-2}f(\overleftarrow{\rho}(u^{-1})u^{\lambda}t)},\quad t\in[S_1,S_2],$$

where $\lambda>0$ and $-\infty<S_1<S_2<\infty$, we we have

$$\lim_{u\to\infty}\sup_{k\in K_u}\left|\frac{1}{\Psi(M_k(u))}\mathbb{P}\left\{\sup_{t\in[S_1,S_2]}Z_u(t)>M_k(u)\right\}-\mathcal{R}_\eta^f[S_1,S_2]\right|=0, \tag{2.32}$$

provided that $M_k(u), k\in K_u$ is such that

$$\lim_{u\to\infty}\sup_{k\in K_u}\left|\frac{M_k(u)}{u}-1\right|=0, \tag{2.33}$$

where $\eta:=\lim_{t\downarrow 0}\frac{\rho^2(t)}{t^{2/\lambda}}\in(0,\infty]$ and $h(t)=f(\eta^{-1/\alpha}t)$ for $\eta\in(0,\infty)$, $h(t)=f(0)$ for $\eta=\infty$ and

$$\mathcal{R}_\eta^f[S_1,S_2]:=\mathbb{E}\left\{\sup_{t\in[S_1,S_2]}e^{\sqrt{2a}B_\alpha(t)-a|t|^\alpha-f(\eta^{-1/\alpha}t)}\right\}=\begin{cases}\mathcal{H}_\alpha[a^{1/\alpha}S_1,a^{1/\alpha}S_2] & f(\cdot)\equiv 0,\\ \mathcal{P}_{\alpha,a}^h[S_1,S_2] & \textit{otherwise.}\end{cases}$$

Proof of Lemma 2.4.1 The existence of ξ is guaranteed by the Assertion in [89][p.265] and follows from [73, 74]. Next, set $\eta^{-1/\alpha}=0$ if $\eta=\infty$ and set further

$$q_u:=\overleftarrow{\rho}(u^{-1}). \tag{2.34}$$

The proof follows by checking the conditions of [60][Theorem 2.1] where the results still holds if we omit the requirements $f(0)=0$ and $0\in[S_1,S_2]$. By (2.33)

$$\lim_{u\to\infty}\inf_{k\in K_u}M_k(u)=\infty.$$

By continuity of f we have

$$\lim_{u\to\infty} \sup_{k\in K_u, t\in[S_1,S_2]} \left| M_k^2(u)u^{-2}f(q_u u^{\lambda} t) - f(\eta^{-1/\alpha}t) \right| = 0. \tag{2.35}$$

Moreover, (2.31) implies

$$\mathrm{var}(\xi(q_u t) - \xi(q_u t')) = 2 - 2r\left(|q_u(t-t')|\right) \sim 2a\rho^2\left(|q_u(t-t')|\right), \ u\to\infty,$$

holds for $t,t'\in[S_1,S_2]$. Thus by (2.33)

$$\lim_{u\to\infty} \sup_{k\in K_u} \sup_{t\neq t'\in[S_1,S_2]} \left| \frac{M_k^2(u)}{u^2} \frac{\mathrm{var}(\xi(q_u t) - \xi(q_u t'))}{2a\rho^2\left(|q_u(t-t')|\right)} - 1 \right| = 0. \tag{2.36}$$

Since $\rho^2 \in \mathcal{R}_\alpha$ which satisfies the uniform convergence theorem (UCT) for regularly varying function, see, e.g., [19], i.e.,

$$\lim_{u\to\infty} \sup_{t,t'\in[S_1,S_2]} \left| u^2\rho^2\left(|q_u(t-t')|\right) - |t-t'|^\alpha \right| = 0, \tag{2.37}$$

and further by the Potter's bound for ρ^2, see [19] we have

$$\limsup_{u\to\infty} \sup_{\substack{t,t'\in[S_1,S_2]\\ t\neq t'}} \frac{u^2\rho^2\left(|q_u(t-t')|\right)}{|t-t'|^{\alpha-\varepsilon_1}} \le \mathbb{C}_1 \max\left(|S_1-S_2|^{\alpha-\varepsilon_1}, |S_1-S_2|^{\alpha+\varepsilon_1}\right) < \infty, \tag{2.38}$$

where $\varepsilon_1 \in (0, \min(1,\alpha))$. We know that for $\alpha\in(0,2]$

$$||t|^\alpha - |t'|^\alpha| \le \mathbb{C}_2 |t-t'|^{\alpha\wedge 1}, \ t,t'\in[S_1,S_2]. \tag{2.39}$$

By (2.31) for any small $\epsilon>0$, when u large enough

$$r(q_u t) \le 1 - \rho^2(q_u|t|)(1-\epsilon), \quad r(q_u t) \ge 1 - \rho^2(q_u|t|)(1+\epsilon) \tag{2.40}$$

hold for $t\in[S_1,S_2]$, then by (2.33) for u large enough

$$\begin{aligned}
&\sup_{k\in K_u} \sup_{|t-t'|<\varepsilon, t,t'\in[S_1,S_2]} M_k^2(u)\mathbb{E}\left\{[\xi(q_u t) - \xi(q_u t')]\xi(0)\right\} \\
&\le \mathbb{C}_3 u^2 \sup_{|t-t'|<\varepsilon, t,t'\in[S_1,S_2]} |r(q_u t) - r(q_u t')| \\
&\le \mathbb{C}_3 \sup_{|t-t'|<\varepsilon, t,t'\in[S_1,S_2]} \left(|u^2\rho^2(q_u|t|) - u^2\rho^2(q_u|t'|)| + \epsilon|u^2\rho^2(q_u|t|)| + \epsilon|u^2\rho^2(q_u|t'|)| \right) \\
&\le \mathbb{C}_3 \sup_{|t-t'|<\varepsilon, t,t'\in[S_1,S_2]} \Big(|u^2\rho^2\left(|q_u(t)|\right) - |t|^\alpha| + |u^2\rho^2\left(|q_u(t')|\right) - |t'|^\alpha| + ||t|^\alpha - |t'|^\alpha| \\
&\quad + \mathbb{C}_4\epsilon\left(|t|^{\alpha-\varepsilon_1} + |t'|^{\alpha-\varepsilon_1}\right)\Big) && (2.41)\\
&\le \mathbb{C}_5 \varepsilon^{\alpha\wedge 1} + \mathbb{C}_6\epsilon, \ u\to\infty && (2.42)\\
&\to 0, \varepsilon\to 0, \epsilon\to 0,
\end{aligned}$$

where in (2.41) we use (2.38) and (2.42) follows from (2.37) and (2.39).
Hence the proof follows from [60][Theorem 2.1]. □

Lemma 2.4.2. *Let $Z_u(s,t), (s,t)\in\mathbb{R}^2$ be a centered stationary Gaussian field with unit variance and correlation function $r_{Z_u}(\cdot,\cdot)$ satisfying*

$$1 - r_{Z_u}(s,t) = \exp\left(-au^{-2}\left(\left|s\right|^{\alpha/2} + \left|t\right|^{\alpha/2}\right)\right), \quad (s,t)\in\mathbb{R}^2, \tag{2.43}$$

with $a>0$. If K_u is some countable index sets, then for $M_k(u), k\in K_u$ satisfying (2.33) and for any $S_1,S_2,T_1,T_2\ge 0$

such that $\max(S_1,S_2)>0, \max(T_1,T_2)>0$, *we have*

$$\lim_{u\to\infty}\sup_{k\in K_u}\left|\frac{1}{\Psi(M_k(u))}\mathbb{P}\left\{\sup_{(s,t)\in D} Z_u(s,t)>M_k(u)\right\}-\mathcal{F}(S_1,S_2,T_1,T_2)\right|=0,$$

where $D=[-S_1,S_2]\times[-T_1,T_2]$ *and*

$$\mathcal{F}(S_1,S_2,T_1,T_2)=\mathcal{H}_{\alpha/2}[-a^{2/\alpha}S_1,a^{2/\alpha}S_2]\mathcal{H}_{\alpha/2}[-a^{2/\alpha}T_1,a^{2/\alpha}T_2].$$

Proof of Lemma 2.4.2 The proof follows by checking the conditions of [45][Lemma 5.3]. Since by (2.43)

$$\begin{aligned}\mathrm{var}(Z_u(s,t)-Z_u(s',t')) &= 2-2r_{Z_u}\left((s-s'),(t-t')\right)\\ &\sim au^{-2}\left(|s-s'|^{\alpha/2}+|t-t'|^{\alpha/2}\right),\end{aligned}$$

we obtain

$$\lim_{u\to\infty}\sup_{k\in K_u}\sup_{(s,t)\neq(s',t')\in D}\left|M_k^2(u)\frac{\mathrm{var}(Z_u(s,t)-Z_u(s',t'))}{2a(|s-s'|^{\alpha/2}+|t-t'|^{\alpha/2})}-1\right|=0. \tag{2.44}$$

Further, since for $\alpha/2\in(0,1]$

$$||t|^{\alpha/2}-|t'|^{\alpha/2}|\le\mathbb{C}_1|t-t'|^{\alpha/2},\quad\text{and}\quad ||s|^{\alpha/2}-|s'|^{\alpha/2}|\le\mathbb{C}_2|s-s'|^{\alpha/2}$$

hold for $t,t'\in[-T_1,T_2], s,s'\in[-S_1,S_2]$, we have by (2.43)

$$\begin{aligned}&\sup_{k\in K_u}\sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\(s,t),(s',t')\in D}} M_k^2(u)\mathbb{E}\left\{[Z_u(s,t)-Z_u(s',t')]Z_u(0,0)\right\}\\ &\le\mathbb{C}_3u^2\sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\(s,t),(s',t')\in D}}|r_{Z_u}(s,t)-r_{Z_u}(s',t')|\\ &\le\mathbb{C}_4a\sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\(s,t),(s',t')\in D}}||s|^{\alpha/2}+|t|^{\alpha/2}-|s'|^{\alpha/2}-|t'|^{\alpha/2}|\\ &\le\mathbb{C}_4a\sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\(s,t),(s',t')\in D}}\left(||s|^{\alpha/2}-|s'|^{\alpha/2}|+||t|^{\alpha/2}-|t'|^{\alpha/2}|\right)\\ &\le\mathbb{C}_5\varepsilon^{\alpha/2}\to0,\ u\to\infty,\varepsilon\to0.\end{aligned}$$

Hence the claim follows from [45][Lemma 5.3]. □

Proof of Theorem 2.2.1 We have from **A3** (recall the definition of q_u in (2.34))

$$\lim_{t\to0}\frac{\rho^2(t)}{t^{2/\lambda}}=\eta\in[0,\infty],\quad\lim_{u\to\infty}u^\lambda q_u=\eta^{-\lambda/2}.$$

Without loss of generality, we consider only the case $t_u=0$ for u large enough. By **A2** for $t\in\Delta(u)$, for sufficiently large u,

$$\frac{1}{\mathcal{F}_{u,+\varepsilon}(t)}\le\sigma_u(t)\le\frac{1}{\mathcal{F}_{u,-\varepsilon}(t)},\quad\mathcal{F}_{u,\pm\varepsilon}(t)=1+u^{-2}\left[(1\pm\varepsilon)f(u^\lambda t)\pm\varepsilon\right] \tag{2.45}$$

for small constant $\varepsilon\in(0,1)$. Since further

$$\pi(u):=\mathbb{P}\left\{\sup_{t\in\Delta(u)}X_u(t)>M_u\right\}=\mathbb{P}\left\{\sup_{t\in\Delta(u)}\overline{X_u}(t)\sigma_u(t)>M_u\right\}, \tag{2.46}$$

we have

$$\pi(u) \le \mathbb{P}\left\{ \sup_{t\in\Delta(u)} \frac{\overline{X_u}(t)}{\mathcal{F}_{u,-\varepsilon}(t)} > M_u \right\}, \quad \pi(u) \ge \mathbb{P}\left\{ \sup_{t\in\Delta(u)} \frac{\overline{X_u}(t)}{\mathcal{F}_{u,+\varepsilon}(t)} > M_u \right\}.$$

Set for some positive constant S

$$I_k(u) = [kq_uS, (k+1)q_uS], \quad k \in \mathbb{Z}.$$

Further, define

$$\mathcal{G}_{u,+\varepsilon}(k) = M_u \sup_{s\in I_k(u)} \mathcal{F}_{u,+\varepsilon}(s), \;\; N_1(u) = \left\lfloor \frac{x_1(u)}{Sq_u} \right\rfloor - \mathbb{I}_{\{x_1\le 0\}},$$
$$\mathcal{G}_{u,-\varepsilon}(k) = M_u \inf_{s\in I_k(u)} \mathcal{F}_{u,-\varepsilon}(s), \;\; N_2(u) = \left\lfloor \frac{x_2(u)}{Sq_u} \right\rfloor + \mathbb{I}_{\{x_2\le 0\}}.$$

In view of [89], we can find centered stationary Gaussian processes $Y_{\pm\varepsilon}(t), t \in \mathbb{R}$ with continuous trajectories, unit variance and correlation function satisfying

$$r_{\pm\varepsilon}(t) = 1 - (1\pm\varepsilon)\rho^2(|t|)(1+o(1)), \quad t \to 0.$$

Case 1) $\eta = \infty$:
For any positive u

$$\sum_{k=N_1(u)+1}^{N_2(u)-1} \mathbb{P}\left\{ \sup_{t\in I_k(u)} X_u(t) > M_u \right\} - \sum_{i=1}^{2} \Lambda_i(u) \le \pi(u) \le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_k(u)} X_u(t) > M_u \right\}, \tag{2.47}$$

where

$$\Lambda_1(u) = \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_k(u)} X_u(t) > M_u, \sup_{t\in I_{k+1}(u)} X_u(t) > M_u \right\},$$

and

$$\Lambda_2(u) = \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{P}\left\{ \sup_{t\in I_k(u)} X_u(t) > M_u, \sup_{t\in I_l(u)} X_u(t) > M_u \right\}.$$

Set below

$$\Theta(u) = \frac{\mathcal{H}_\alpha}{u^\lambda q_u} \int_{x_1}^{x_2} e^{-f(t)} dt \Psi(M_u).$$

which is well-defined since $\int_{x_1}^{x_2} e^{-f(t)}dt < \infty$ follows by the assumption $f \in C_0^*([x_1, x_2])$. By Slepian inequality (see e.g., [1]), (2.46) and Lemma 2.4.1

$$\begin{aligned}
\sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_k(u)} X_u(t) > M_u \right\} &\le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_k(u)} \overline{X_u}(t) > \mathcal{G}_{u,-\varepsilon}(k) \right\} \\
&\le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_k(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k) \right\} \\
&= \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{ \sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k) \right\} \\
&\sim \sum_{k=N_1(u)}^{N_2(u)} \mathcal{H}_\alpha[0, (1+\varepsilon)^{1/\alpha} S] \Psi(\mathcal{G}_{u,-\varepsilon}(k)) \\
&\sim \mathcal{H}_\alpha[0, (1+\varepsilon)^{1/\alpha} S] \Psi(M_u) \sum_{k=N_1(u)}^{N_2(u)} e^{-M_u^2 u^{-2} \inf_{s\in I_k(u)} \left[(1-\varepsilon)f(u^\lambda s) - \varepsilon\right]} \\
&\sim \frac{\mathcal{H}_\alpha[0, (1+\varepsilon)^{1/\alpha} S]}{S u^\lambda q_u} \int_{x_1}^{x_2} e^{-(1-\varepsilon)f(t)+\varepsilon} dt \Psi(M_u) \\
&\sim \Theta(u), \quad u \to \infty, S \to \infty, \varepsilon \to 0.
\end{aligned} \tag{2.48}$$

Similarly, we derive that

$$\sum_{k=N_1(u)+1}^{N_2(u)-1} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t)>u\right\} \geq (1+o(1))\Theta(u), u\to\infty,\ S\to\infty,\ \varepsilon\to 0.$$

Moreover,

$$\begin{aligned}
\Lambda_1(u) &\leq \sum_{k=N_1(u)}^{N_2(u)} \left(\mathbb{P}\left\{\sup_{t\in I_k(u)} Y_{+\varepsilon}(t)>\widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\}+\mathbb{P}\left\{\sup_{t\in I_{k+1}(u)} Y_{+\varepsilon}(t)>\widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\}\right.\\
&\quad \left.-\mathbb{P}\left\{\sup_{t\in I_k(u)\cup I_{k+1}(u)} Y_{-\varepsilon}(t)>\overline{\mathcal{G}}_{u,+\varepsilon}(k)\right\}\right)\\
&\leq \sum_{k=N_1(u)}^{N_2(u)} \left(2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]-\mathcal{H}_\alpha[0,2(1-\varepsilon)^{1/\alpha}S]\right)\Psi(\widehat{\mathcal{G}}_{u,-\varepsilon}(k))\\
&\leq \left(2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]-\mathcal{H}_\alpha[0,2(1-\varepsilon)^{1/\alpha}S]\right)\sum_{k=N_1(u)}^{N_2(u)} \Psi(\widehat{\mathcal{G}}_{u,-\varepsilon}(k))\\
&= o(\Theta(u)),\ u\to\infty, S\to\infty, \varepsilon\to 0,
\end{aligned} \tag{2.49}$$

where

$$\widehat{\mathcal{G}}_{u,-\varepsilon}(k)=\min(\mathcal{G}_{u,-\varepsilon}(k),\mathcal{G}_{u,-\varepsilon}(k+1)),\quad \overline{\mathcal{G}}_{u,+\varepsilon}(k)=\max(\mathcal{G}_{u,+\varepsilon}(k),\mathcal{G}_{u,+\varepsilon}(k+1)).$$

By **A3** for any $(s,t)\in I_k(u)\times I_l(u)$ with $N_1(u)\leq k,l\leq N_2(u), l\geq k+2$ we have

$$2\leq \mathrm{var}\left(\overline{X_u}(s)+\overline{X_u}(t)\right)=4-2(1-r_u(s,t))\leq 4-\rho^2(|t-s|)\leq 4-\mathbb{C}_1u^{-2}|(l-k-1)S|^{\alpha/2}$$

and for $(s,t),(s',t')\in I_k(u)\times I_l(u)$ with $N_1(u)\leq k,l\leq N_2(u)$

$$\begin{aligned}
&1-Cov\left(\frac{\overline{X_u}(s)+\overline{X_u}(t)}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}},\frac{\overline{X_u}(s')+\overline{X_u}(t')}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right)\\
&=\frac{1}{2}\mathbb{E}\left\{\left(\frac{\overline{X_u}(s)+\overline{X_u}(t)}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}}-\frac{\overline{X_u}(s')+\overline{X_u}(t')}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right)^2\right\}\\
&=\frac{1}{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')+\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}\\
&\quad +Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)\left(\frac{1}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}}-\frac{1}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right)^2\\
&\leq 2\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')\right)^2\right\}+2\mathbb{E}\left\{\left(\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}+\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')+\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}\\
&\leq 8(1-r_u(s,s')+1-r_u(t,t'))\\
&=16u^{-2}\left(\left|\frac{s-s'}{q_u}\right|^{\alpha/2}+\left|\frac{t-t'}{q_u}\right|^{\alpha/2}\right).
\end{aligned}$$

In view of our assumptions, we can find centered homogeneous Gaussian random fields $Z_u(s,t)$ with correlation

$$r_{Z_u}(s,t)=\exp\left(-32u^{-2}\left(\left|\frac{s}{q_u}\right|^{\alpha/2}+\left|\frac{t}{q_u}\right|^{\alpha/2}\right)\right).$$

Slepian inequality, Lemma 2.4.2 and (2.48) imply

$$\Lambda_2(u) \leq \sum_{N_1(u)\leq k,l\leq N_2(u),l\geq k+2} \mathbb{P}\left\{\sup_{s\in I_k(u)} X_u(s)>M_u,\ \sup_{t\in I_l(u)} X_u(t)>M_u\right\}$$

$$
\begin{aligned}
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2} \mathbb{P}\left\{\sup_{(s,t)\in I_k(u)\times I_l(u)} (\overline{X_u}(s)+\overline{X_u}(t)) > 2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2} \mathbb{P}\left\{\sup_{(s,t)\in I_0(u)\times I_0(u)} Z_u(s,t) > \frac{2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)}{\sqrt{4-\mathbb{C}_1u^{-2}|(l-k-1)S|^{\alpha/2}}}\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2} \left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2 \Psi\left(\frac{2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)}{\sqrt{4-\mathbb{C}_1u^{-2}|(l-k-1)S|^{\alpha/2}}}\right)\\
&\le 2\sum_{k=N_1(u)}^{N_2(u)} \sum_{l=1}^{N_2(u)-N_1(u)} \left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2 \Psi\left(\frac{2\mathcal{G}_{u,-\varepsilon}(k)}{\sqrt{4-\mathbb{C}_1u^{-2}(lS)^{\alpha/2}}}\right)\\
&\le 2\sum_{k=N_1(u)}^{N_2(u)} \left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2 \Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\sum_{l=1}^{\infty} e^{-\mathbb{C}_2(lS)^{\alpha/2}}\\
&\le 2\mathcal{H}_{\alpha/2}32^{2/\alpha}Se^{-\mathbb{C}_3S^{\alpha/2}} \sum_{k=N_1(u)}^{N_2(u)} \mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\\
&= o(\Theta(u)), \quad u\to\infty, S\to\infty, \varepsilon\to 0,
\end{aligned}
\tag{2.50}
$$

where $\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)=\min(\mathcal{G}_{u,-\varepsilon}(k),\mathcal{G}_{u,-\varepsilon}(l))$. Combing (2.47)-(2.49) with (2.50), we obtain

$$\pi(u)\sim\Theta(u), \quad u\to\infty.$$

Case 2) $\eta\in(0,\infty)$: This implies $\lambda=2/\alpha$.
Set for any small constant $\theta\in(0,1)$ and any constant $S_1>0$

$$S_1^*=\begin{cases} -S_1, & \text{if } x_1=-\infty;\\ (x_1+\theta)\eta^{1/\alpha}, & \text{if } x_1\in(-\infty,\infty),\end{cases} \quad S_2^*=\begin{cases} (x_2-\theta)\eta^{1/\alpha}, & \text{if } x_2\in(-\infty,\infty);\\ S_1, & \text{if } x_2=\infty,\end{cases} \tag{2.51}$$

$$S_1^{**}=\begin{cases} -S, & \text{if } x_1=-\infty;\\ (x_1-\theta)\eta^{1/\alpha}, & \text{if } x_1\in(-\infty,\infty),\end{cases} \quad S_2^{**}=\begin{cases} (x_2+\theta)\eta^{1/\alpha}, & \text{if } x_2\in(-\infty,\infty);\\ S, & \text{if } x_2=\infty.\end{cases} \tag{2.52}$$

With $K^*=[q_uS_1^*,q_uS_2^*]$ and $K^{**}=[q_uS_1^{**},q_uS_2^{**}]$ we have for any $S_1>0$ and u large enough

$$\pi(u)\ge\mathbb{P}\left\{\sup_{t\in K^*} X_u(t)>M_u\right\}, \tag{2.53}$$

$$\pi(u)\le\mathbb{P}\left\{\sup_{t\in K^{**}} X_u(t)>M_u\right\}+\sum_{\substack{k=N_1(u)\\ k\neq 0,-1}}^{N_2(u)}\mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t)>M_u\right\}. \tag{2.54}$$

Using Slepian inequality and Lemma 2.4.1, we have that

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\in K^*} X_u(t)>M_u\right\} &\ge \mathbb{P}\left\{\sup_{t\in K^*} \frac{Y_{-\varepsilon}(t)}{\mathcal{F}_{u,+\varepsilon}(t)}>M_u\right\}\\
&\sim \mathcal{P}_{\alpha,1}^{h_{+\varepsilon}}[S_1^*,S_2^*]\Psi(M_u), \ u\to\infty,
\end{aligned}
$$

where $h_{\pm\varepsilon}(t)=(1\pm\varepsilon)f(\eta^{-1/\alpha}t)\pm\varepsilon$, and similarly

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{t\in K^{**}} X_u(t)>M_u\right\} &\le \mathbb{P}\left\{\sup_{t\in K^{**}} \frac{Y_{+\varepsilon}(t)}{\mathcal{F}_{u,-\varepsilon}(t)}>M_u\right\}\\
&\sim \mathcal{P}_{\alpha,1}^{h_{-\varepsilon}}[S_1^{**},S_2^{**}]\Psi(M_u), \ u\to\infty.
\end{aligned}
\tag{2.55}
$$

Moreover, in light of (2.8), the Slepian inequality and Lemma 2.4.1

$$\begin{aligned}\sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\} &\le \sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \frac{Y_{+\varepsilon}(t)}{\mathcal{F}_{u,-\varepsilon}(t)} > M_u\right\}\\ &\le \sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\}\\ &\sim \sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\\ &\sim \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(M_u)\sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} e^{-\inf_{s\in[k,k+1]}\left((1-\varepsilon)f(s\eta^{-1/\alpha}S)-\varepsilon\right)}\\ &\sim \mathbb{C}_4\mathcal{H}_\alpha\Psi(M_u)Se^{-\mathbb{C}_5(\eta^{-1/\alpha}S)^{\epsilon_1/2}}e^{\varepsilon}\\ &= o\left(\Psi(M_u)\right),\ u\to\infty, S\to\infty, \varepsilon\to 0. \end{aligned} \tag{2.56}$$

Letting $\varepsilon\to 0,\ S_1\to\infty,\ S\to\infty$, and $\theta\to 0$ we obtain

$$\pi(u)\sim \mathcal{P}^f_{\alpha,\eta}[x_1,x_2]\Psi(M_u),\quad u\to\infty.$$

Next, if we set $x_1(u)=-\left(\frac{\ln u}{u}\right)^\lambda, x_2(u)=\left(\frac{\ln u}{u}\right)^\lambda$, then

$$x_1=-\infty,\quad x_2=\infty,\quad S_1^*=-S_1,\quad S_2^*=S_1,\quad S_1^{**}=-S,\quad S_2^{**}=S.$$

Inserting (2.55), (2.56) into (2.54) and letting $\varepsilon\to 0$ leads to

$$\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)}\le \mathcal{P}^f_{\alpha,\eta}[-S,S]+\mathbb{C}_4\mathcal{H}_\alpha Se^{-\mathbb{C}_5(\eta^{-1/\alpha}S)^{\epsilon_1/2}}<\infty.$$

By (2.53), we have

$$\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)}\ge \mathcal{P}^f_{\alpha,\eta}[-S_1,S_1]>0.$$

Letting $S_1\to\infty, S\to\infty$ we obtain

$$\mathcal{P}^f_{\alpha,\eta}(-\infty,\infty)\in(0,\infty),\quad \pi(u)\sim\mathcal{P}^f_{\alpha,\eta}(-\infty,\infty)\Psi(M_u),\quad u\to\infty.$$

Case 3) $\eta=0$: Note that

$$\pi(u)\le \mathbb{P}\left\{\sup_{t\in((I_{-1}(u)\cup I_0(u))\cap\Delta(u))}\overline{X_u}(t)\sigma_u(t)>M_u\right\}+\sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)}\mathbb{P}\left\{\sup_{t\in I_k(u)}\overline{X_u}(t)\sigma_u(t)>M_u\right\}=: J_1(u)+J_2(u).$$

By (2.45)

$$\frac{1}{\mathcal{F}_{u,+\varepsilon}(t)}\le\sigma_u(t)\le\frac{1}{\mathcal{F}_{u,-\varepsilon}(t)}\le\frac{1}{1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^\lambda s)-\varepsilon]} \tag{2.57}$$

holds for all $t\in\Delta(u)$. Hence Lemma 2.4.1 implies

$$J_1(u)\ \le\ \mathbb{P}\left\{\sup_{t\in[-q_uS,q_uS]}\overline{X_u}(t)>M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^\lambda s)-\varepsilon]\right)\right\}$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\sup_{t\in[-q_uS,q_uS]} Y_{+\varepsilon}(t) > M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^\lambda s)-\varepsilon]\right)\right\}\\
&\sim \mathcal{H}_\alpha[0,2(1+\varepsilon)^{1/\alpha}S]\Psi\left(M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^\lambda s)-\varepsilon]\right)\right)\\
&\sim \mathcal{H}_\alpha[0,2(1+\varepsilon)^{1/\alpha}S]\Psi\left(M_u\right)e^{-(1-\varepsilon)\omega^*+\varepsilon}\\
&\sim \Psi\left(M_u\right)e^{-\omega^*},\ u\to\infty,\ S\to 0,\ \varepsilon\to 0,
\end{aligned}
$$

where $\omega^* = \inf_{t\in[x_1,x_2]} f(t)$. Furthermore, by Lemma 2.4.1, for any $x>0$

$$
\begin{aligned}
J_2(u) &\le \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \sim \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\\
&\le 2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(M_u)\sum_{k=1}^{\infty} e^{-(1-2\varepsilon)(kxS)^{\epsilon_1/2}+2\varepsilon}\\
&\le \mathbb{C}_6\mathcal{H}_\alpha\Psi(M_u)Se^{-\mathbb{C}_7(xS)^{\epsilon_1/2}} = o\left(\Psi(M_u)\right),\ u\to\infty, x\to\infty,\ S\to 0, \qquad (2.58)
\end{aligned}
$$

hence

$$
\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)} \le e^{-\omega^*}, \quad u\to\infty.
$$

Next, since $f\in C_0^*([x_1,x_2])$ there exists $y(u)\in\Delta(u)$ satisfying

$$
\lim_{u\to\infty} y(u)u^\lambda = y \in \{z\in[x_1,x_2]: f(z)=\omega^*\}.
$$

Consequently, in view of (2.57)

$$
\begin{aligned}
\pi(u) &\ge \mathbb{P}\left\{X_u(y(u)) > M_u\right\}\\
&\ge \mathbb{P}\left\{\overline{X}_u(y(u)) > M_u(1+[(1+\varepsilon)f(u^\lambda y(u))+\varepsilon]u^{-2})\right\}\\
&= \Psi\left(M_u(1+(1+\varepsilon)[f(u^\lambda y(u))+\varepsilon]u^{-2})\right)\\
&\sim \Psi\left(M_u\right)e^{-f(y)},\ u\to\infty,\ \varepsilon\to 0,
\end{aligned}
$$

which implies that

$$
\pi(u)\sim\Psi\left(M_u\right)e^{-\omega^*}, \quad u\to\infty
$$

establishing the proof. □

Proof of Theorem 2.2.2 Clearly, for any $u>0$

$$
\pi(u)\le\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > M_u\right\}\le \pi(u)+\pi_1(u),
$$

where with $D(u) := [0,T]\setminus(t_u+\Delta(u))$,

$$
\pi(u) := \mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t_u+t) > M_u\right\}, \quad \pi_1(u) := \mathbb{P}\left\{\sup_{t\in D(u)} X_u(t) > M_u\right\}.
$$

Next, we derive an upper bound for $\pi_1(u)$ which will finally imply that

$$
\pi_1(u) = o(\pi(u)), \quad u\to\infty. \qquad (2.59)
$$

Thus by **A4**, **A5** and Piterbarg inequality (see e.g., [119][Theorem 8.1], [122][Theorem 3] and [45][Lemma 5.1])

$$
\pi_1(u) = \mathbb{P}\left\{\sup_{t\in D(u)} \overline{X}_u(t)\sigma_u(t) > M_u\right\}
$$

$$\begin{aligned} &\leq \mathbb{P}\left\{\sup_{t\in D(u)} \overline{X}_u(t) > M_u + \mathbb{C}_1\frac{p(\ln u)^q}{u}\right\} \\ &\leq \mathbb{C}_2 T M_u^{2/\varsigma}\Psi\left(M_u + \mathbb{C}_1\frac{p(\ln u)^q}{u}\right) \\ &= o\left(\Psi\left(M_u\right)\right), \quad u\to\infty. \end{aligned} \tag{2.60}$$

Since **A1'** implies **A1**, by Theorem 2.2.1 and **A2**, **A3**, we have

$$\pi(u) \sim \Psi\left(M_u\right)\begin{cases} \frac{\mathcal{H}_\alpha}{u^\lambda q_u}\int_{x_1}^{x_2} e^{-f(t)}dt, & \text{if } \eta=\infty, \\ \mathcal{P}_{\alpha,\eta}^f[x_1,x_2], & \text{if } \eta\in(0,\infty), \quad u\to\infty, \\ 1, & \text{if } \eta=0, \end{cases} \tag{2.61}$$

where the result of case $\eta = 0$ comes from the fact that $f(t) \geq 0$ for $t \in [x_1, x_2]$, $f(0) = 0$ and $0 \in [x_1, x_2]$. Consequently, it follows from (2.60) and (2.61) that (2.59) holds, and thus the proof is complete. □

Proof of Theorem 2.3.1 Without loss of generality we assume that $g_m = g(t_0) = 0$.

i) We present first the proof for $t_0 \in (0, T)$. Let $\Delta(u) = [-\delta(u), \delta(u)]$, where $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^{1/\gamma}$ with some large $q > 1$. By (2.6) for u large enough and some small $\varepsilon \in (0, 1)$

$$1+\frac{(1-\varepsilon)c|t|^\gamma}{u} \leq \frac{1}{\sigma_u(t+t_0)} := \frac{u-g(t+t_0)}{u} = 1-\frac{g(t+t_0)}{u} \leq 1+\frac{(1+\varepsilon)c|t|^\gamma}{u} \tag{2.62}$$

holds for all $t \in [-\theta, \theta], \theta > 0$. It follows that

$$\Pi(u) \leq \mathbb{P}\left\{\sup_{t\in[0,T]} (X(t)+g(t)) > u\right\} \leq \Pi(u)+\Pi_1(u),$$

with

$$\Pi_1(u) := \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta]} (X(t)+g(t)) > u\right\},$$

and

$$\Pi(u) := \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]} (X(t)+g(t)) > u\right\} = \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]} X(t)\frac{u}{u-g(t)} > u\right\}.$$

By (2.62), we may further write

$$\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq 0}\left|\frac{\frac{1}{\sigma_u(t_0+t)}-1}{cu^{-1}|t|^\gamma}-1\right| = \lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq 0}\left|\frac{\frac{1}{\sigma_u(t_0+t)}-1}{cu^{-2}|u^{1/\gamma}t|^\gamma}-1\right| = 0,$$

and

$$\inf_{t\in[-\theta,\theta]\setminus\Delta(u)}\frac{1}{\sigma_u(t+t_0)} \geq 1+\frac{(1-\varepsilon)c(\ln u)^q}{u^2}.$$

In addition, from (2.17) we have that

$$\lim_{u\to\infty}\sup_{\substack{s,t\in\Delta(u)\\ t\neq s}}\left|\frac{1-r(t_0+t,t_0+s)}{a|t-s|^\alpha}-1\right| = 0,$$

and

$$\sup_{s,t\in[t_0-\theta,t_0+\theta]}\mathbb{E}\left\{X(t)-X(s))^2\right\} \leq \sup_{s,t\in[t_0-\theta,t_0+\theta]}(2-2r(s,t)) \leq \mathbb{C}_1|t-s|^\alpha$$

hold when θ is small enough. Therefore, by Theorem 2.2.2

$$\Pi(u) \sim u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi(u)\begin{cases}\mathcal{H}_\alpha a^{\frac{1}{\alpha}}\int_{w_{t_0}}^{\infty} e^{-c|t|^\gamma}dt, & \text{if } \alpha<2\gamma,\\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0},\infty), & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma.\end{cases}$$

Moreover, since $g_\theta := \sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]} g(t)<0$ we have

$$\Pi_1(u)\le \mathbb{P}\left\{\sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]} X(t)>u-g_\theta\right\}\sim \mathcal{H}_\alpha\int_0^T\frac{1}{a(t)}dt\; u^{\frac{2}{\alpha}}\Psi(u-g_\theta)=o(\Pi(u)),\;\; u\to\infty,$$

hence the claims follow.
For $t_0=0$ and $t_0=T$, we just need to replace $\Delta(u)$ by $\Delta(u)=[0,\delta(u)]$ and $\Delta(u)=[-\delta(u),0]$, respectively.
ii) Applying [119][Theorem 7.1] we obtain

$$\mathbb{P}\left\{\sup_{t\in[A,B]}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in[A,B]}X(t)>u\right\}\sim\int_A^B (a(t))^{1/\alpha}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u).$$

Set $\Delta_\varepsilon=[A-\varepsilon,B+\varepsilon]\cap[0,T]$ for some $\varepsilon>0$, then we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\} &\ge \mathbb{P}\left\{\sup_{t\in[A,B]}(X(t)+g(t))>u\right\},\\ \mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\} &\le \mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}(X(t)+g(t))>u\right\}+\mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}(X(t)+g(t))>u\right\}.\end{aligned}$$

Since g is a continuous function and $g_\varepsilon:=\sup_{t\in[0,T]\setminus\Delta_\varepsilon} g(t)<0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}(X(t)+g(t))>u\right\} &\le \mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}X(t)>u-g_\varepsilon\right\}\\ &\le \mathbb{C}_2u^{2/\alpha}\Psi(u-g_\varepsilon)=o\left(u^{2/\alpha}\Psi(u)\right),\quad u\to\infty,\varepsilon\to0.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}(X(t)+g(t))>u\right\} &\le \mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}X(t)>u\right\}\sim\int_{A-\varepsilon}^{B+\varepsilon}(a(t))^{\frac{1}{\alpha}}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u)\\ &\sim \int_A^B (a(t))^{\frac{1}{\alpha}}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u),\quad u\to\infty,\varepsilon\to0.\end{aligned}$$

Hence the claims follow. □

Proof of Theorem 2.3.2 We give the proof only for $t_0=0$. In this case, $x\in(0,\infty)$. By definition

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u-t_0)\le x\middle|\tau_u\le T\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,u^{-1/\gamma}x]}(X(t)+g(t))>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}}.$$

Set $\Delta(u)=[0,u^{-1/\gamma}x]$. For all large u

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in\Delta(u)}X(t)\frac{u}{u-g(t)}>u\right\}.$$

Denote $X_u(t)=X(t)\frac{u}{u-g(t)}$ and $\sigma_u(t)=\frac{u}{u-g(t)}$. As in the proof of Theorem 2.3.1 i), by Theorem 2.2.1 we obtain

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(t)+g(t))>u\right\}\sim u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi(u)\begin{cases}a^{\frac{1}{\alpha}}\mathcal{H}_\alpha\int_0^x e^{-c|t|^\gamma}dt, & \text{if } \alpha<2\gamma,\\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[0,x], & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma.\end{cases}$$

Consequently, by Theorem 2.3.1 statement i), the results follow. □

PROOF OF THEOREM 2.3.3 Clearly, for any $u>0$

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,T]}\overline{X}(t)\frac{m_u(t)}{m_u(t_u)}>\frac{u-g(t_u)}{\sigma(t_u)}\right\},$$

and **A1'** is satisfied. By the continuity of $\sigma(t)$, $\lim_{u\to\infty}t_u=t_0$ and $\sigma(t_0)=1$, we have that for u large enough

$$\sigma(t_u)>0,\ \text{and}\ \frac{u-g(t_u)}{\sigma(t_u)}\sim u,\ u\to\infty.$$

Set next

$$X_u(t)=\overline{X}(t)\frac{m_u(t)}{m_u(t_u)},\quad t\in[0,T],$$

which has standard deviation function $\sigma_u(t)=\frac{m_u(t_u+t)}{m_u(t_u)}$ and correlation function $r_u(s,t)=r(s,t)$ satisfying assumptions **A2**–**A4**. Further, $\overline{X}_u(t)=\overline{X}(t)$ implies **A5**. Hence the claims follow from Theorem 2.2.2. □

PROOF OF THEOREM 2.3.4 For all large u

$$1-r(t_u+t,t_u+s)=\frac{\mathbb{E}\left\{[X(t_u+t)-X(t_u+s)]^2\right\}-[\sigma(t_u+t)-\sigma(t_u+s)]^2}{2\sigma(t_u+t)\sigma(t_u+s)}. \tag{2.63}$$

Using that

$$\begin{aligned}\mathbb{E}\left\{[X(t_u+t)-X(t_u+s)]^2\right\}&=\mathbb{E}\left\{X'^2(t_u+s)\right\}(t-s)^2+o((t-s)^2),\\ [\sigma(t_u+t)-\sigma(t_u+s)]^2&=\sigma'^2(t_u+t)(t-s)^2+o((t-s)^2),\end{aligned}$$

we have, as $u\to\infty$

$$1-r(t_u+t,t_u+s)=\frac{\mathbb{E}\left\{X'^2(t_u+t)\right\}-\sigma'^2(t_u+t)}{2\sigma(t_u+t)\sigma(t_u+s)}(t-s)^2+o((t-s)^2).$$

Since $D(s,t):=\frac{\mathbb{E}\left\{X'^2(t)\right\}-\sigma'^2(t)}{2\sigma(s)\sigma(t)}$ is continuous at (t_0,t_0), then setting $D=D(t_0,t_0)$ we obtain

$$\lim_{u\to\infty}\sup_{\substack{t\in\Delta(u),s\in\Delta(u)\\ t\neq s}}\left|\frac{1-r(t_u+t,t_u+s)}{D|t-s|^2}-1\right|=0,$$

which implies that **A3** is satisfied. Next we suppose that $\sigma(t)>\frac{1}{2}$ for any $t\in[0,T]$, since if we set $E_1=\{t\in[0,T]:\sigma(t)\le\frac{1}{2}\}$, by Borell-TIS inequality

$$\mathbb{P}\left\{\sup_{t\in E_1}(X(t)+g(t))>u\right\}\le\exp\left(-2\left(u-\sup_{t\in[0,T]}g(t)-\mathbb{C}_1\right)^2\right)=o\left(\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right)\right)$$

as $u\to\infty$, where $\mathbb{C}_1=\mathbb{E}\left\{\sup_{t\in[0,T]}X(t)\right\}<0$. Further by (2.63)

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(s))^2\right\}\le 2-2r(t,s)\le 4\left(\sup_{\theta\in[0,T]}\mathbb{E}\left\{X'^2(\theta)\right\}(t-s)^2-\inf_{\theta\in[0,T]}\sigma'^2(\theta)(t-s)^2\right),$$

then **A5** is satisfied. Consequently, the conditions of Theorem 2.3.3 are satisfied and hence the claim follows. □

PROOF OF THEOREM 2.3.5 Without loss of generality we assume that $g(t)$ satisfies (2.6) with $g(t_0)=0$.
First we present the proof for $t_0\in(0,T)$. Clearly, m_u attains its maximum at the unique point t_0. Further, we have

$$\frac{m_u(t_0)}{m_u(t_0+t)}-1=\frac{1}{\sigma(t_0+t)}(1-\sigma(t_0+t))-\frac{g(t_0+t)}{u\sigma(t_0+t)}.$$

Consequently, by (2.2) and (2.6)

$$\frac{m_u(t_0)}{m_u(t_0+t)} = 1 + \left(b|t|^\beta + \frac{c}{u}|t|^\gamma\right)(1+o(1)), \quad t \to 0 \tag{2.64}$$

holds for all large u. Further, set $\Delta(u) = [-\delta(u), \delta(u)]$, where $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^{2/\beta^*}$ for some constant $q > 1$ with $\beta^* = \min(\beta, 2\gamma)$, and let $f(t) = b|t|^\beta \mathbb{I}_{\{\beta=\beta^*\}} + c|t|^\gamma \mathbb{I}_{\{2\gamma=\beta^*\}}$. We have

$$\lim_{u\to\infty} \sup_{t\in\Delta(u), t\neq 0} \left| \frac{\left(\frac{m_u(t_0)}{m_u(t_0+t)} - 1\right) u^2 - f(u^{2/\beta^*} t)}{f(u^{2/\beta^*} t) + \mathbb{I}_{\{\beta\neq 2\gamma\}}} \right| = 0. \tag{2.65}$$

By (2.2)

$$\mathbb{E}\left\{(\overline{X}(t) - \overline{X}(s))^2\right\} = \mathbb{E}\left\{(\overline{X}(t))^2\right\} + \mathbb{E}\left\{(\overline{X}(s))^2\right\} - 2\mathbb{E}\left\{\overline{X}(t)\overline{X}(s)\right\} = 2 - 2r(s,t) \le \mathbb{C}_1 |t-s|^\alpha \tag{2.66}$$

holds for $s, t \in [t_0 - \theta, t_0 + \theta]$, with $\theta > 0$ sufficiently small. By (2.64), for any $\varepsilon > 0$

$$\frac{m_u(t_0)}{m_u(t_0+t)} \ge 1 + \mathbb{C}_2(1-\varepsilon)\frac{(\ln u)^q}{u} \tag{2.67}$$

holds for all $t \in [-\theta, \theta] \setminus \Delta(u)$. Further

$$\Pi(u) := \mathbb{P}\left\{\sup_{t\in[t_0-\theta, t_0+\theta]} (X(t) + g(t)) > u\right\} \le \mathbb{P}\left\{\sup_{t\in[0,T]} (X(t)+g(t)) > u\right\} \le \Pi(u) + \Pi_1(u),$$

with

$$\Pi_1(u) := \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta, t_0+\theta])} (X(t)+g(t)) > u\right\}.$$

By(2.2), (2.65)-(2.67) which imply **A2–A5** and Theorem 2.3.3, we have

$$\Pi(u) \sim u^{(\frac{2}{\alpha} - \frac{2}{\beta^*})_+} \Psi(u) \begin{cases} \mathcal{H}_\alpha a^{1/\alpha} \int_{w_{t_0}}^\infty e^{-f(t)} dt, & \text{if } \alpha < \beta^*, \\ \mathcal{P}_{\alpha,a}^f [w_{t_0}, \infty), & \text{if } \alpha = \beta^*, \\ 1, & \text{if } \alpha > \beta^*. \end{cases} \tag{2.68}$$

In order to complete the proof it suffices to show that

$$\Pi_1(u) = o(\Pi(u)).$$

Since $\sigma_\theta := \max_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])} \sigma(t) < 1$, by the Borell-TIS inequality we have

$$\Pi_1(u) \le \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])} X(t) > u\right\} \le \exp\left(-\frac{(u-\mathbb{C}_3)^2}{2\sigma_\theta^2}\right) = o(\Pi(u)),$$

where $\mathbb{C}_3 = \mathbb{E}\left\{\sup_{t\in[0,T]} X(t)\right\} < \infty$.
For the cases $t_0 = 0$ and $t_0 = T$, we just need to replace $\Delta(u)$ by $[0, \delta(u)]$ and $[-\delta(u), 0]$, respectively. Hence the proof is complete. □

PROOF OF THEOREM 2.3.6 i) We shall present the proof only for the case $t_0 \in (0,T)$. In this case, $[x_1, x_2] = \mathbb{R}$. By definition, for any $x \in \mathbb{R}$

$$\mathbb{P}\left\{u^\lambda(\tau_u - t_u) \le x \middle| \tau_u \le T\right\} = \frac{\mathbb{P}\left\{\sup_{t\in[0, t_u + u^{-\lambda}x]} (X(t)+g(t)) > u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]} (X(t)+g(t)) > u\right\}}.$$

For $u>0$ define

$$X_u(t)=\overline{X}(t_u+t)\frac{m_u(t_u+t)}{m_u(t_u)},\quad \sigma_u(t)=\frac{m_u(t_u+t)}{m_u(t_u)}.$$

As in the proof of Theorem 2.3.3, we obtain

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}X_u(t)>\frac{u-g(t_u)}{\sigma(t_u)}\right\},$$

and **A1'**, **A2**–**A5** are satisfied with $\Delta(u)=[-\delta_u,u^{-\lambda}x]$. Clearly, for any $u>0$

$$\pi(u)\le\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}X_u(t)>\frac{u-g(t_u)}{\sigma(t_u)}\right\}\le\pi(u)+\pi_1(u),$$

where

$$\pi(u)=\mathbb{P}\left\{\sup_{t\in[t_u-\delta(u),t_u+u^{-\lambda}x]}X_u(t)>\frac{u-g(t_u)}{\sigma(t_u)}\right\},\quad \pi_1(u)=\mathbb{P}\left\{\sup_{t\in[0,t_u-\delta(u)]}X_u(t)>\frac{u-g(t_u)}{\sigma(t_u)}\right\}.$$

Applying Theorem 2.2.1 we have

$$\pi(u)\sim\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right)\begin{cases}\frac{\mathcal{H}_\alpha}{u^\lambda q_u}\int_{-\infty}^x e^{-f(t)}dt, & \text{if }\ \eta=\infty,\\ \mathcal{P}_{\alpha,\eta}^f(-\infty,x], & \text{if }\ \eta\in(0,\infty),\\ \sup_{t\in(-\infty,x]}e^{-f(t)}, & \text{if }\ \eta=0.\end{cases}\tag{2.69}$$

In view of (2.60)

$$\pi_1(u)=o\left(\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right)\right),\quad u\to\infty,$$

hence

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}(X(t)+g(t))>u\right\}\sim\pi(u),\quad u\to\infty$$

and thus the claim follows by (2.69) and Theorem 2.3.3.
ii) We give the proof of $t_0=T$. In this case $x\in(-\infty,0)$ implying

$$\mathbb{P}\left\{u^{2/\beta^*}(\tau_u-T)\le x\middle|\tau_u\le T\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t))>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}}.$$

Set $\delta_u=\left(\frac{(\ln u)^q}{u}\right)^{2/\beta^*}$ for some $q>1$ and let

$$\Delta(u)=[-\delta_u,u^{-2/\beta^*}x],\quad \sigma_u(t)=\frac{m_u(t)}{m_u(T)},$$

with

$$m_u(t)=\frac{\sigma(t)}{1-g(t)/u},\qquad X_u(t)=\overline{X}(t)\frac{m_u(t)}{m_u(T)}.$$

For all large u, we have

$$\pi(u)\le\mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t))>u\right\}\le\pi(u)+\mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]}(X(t)+g(t))>u\right\},$$

where

$$\pi(u):=\mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(T+t)+g(T+t))>u\right\}=\mathbb{P}\left\{\sup_{t\in\Delta(u)}X_u(T+t)>u\right\}.$$

As in the proof of Theorem 2.3.5 it follows that the Assumptions **A2–A5** hold with $\Delta(u) = [-\delta_u, u^{-2/\beta^*}x]$. Hence an application of Theorem 2.2.1 yields

$$\pi(u) \sim u^{(\frac{2}{\alpha}-\frac{2}{\beta^*})_+}\Psi(u) \begin{cases} a^{1/\alpha}\mathcal{H}_\alpha \int_{-x}^{\infty} e^{-f(t)}dt, & \text{if } \alpha < \beta^*, \\ \mathcal{P}^f_{\alpha,a}[-x,\infty), & \text{if } \alpha = \beta^*, \\ e^{-f(x)}, & \text{if } \alpha > \beta^*. \end{cases} \tag{2.70}$$

In view of (2.60)

$$\mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]}(X(t)+g(t)) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]} X_u(t) > u\right\} = o\left(\Psi(u)\right), \quad u\to\infty$$

implying

$$\mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t)) > u\right\} \sim \pi(u), \quad u\to\infty.$$

Consequently, the proof follows by (2.70) and Theorem 2.3.5. □

Proof of Theorem 2.3.1 Set next $A(t) = \int_0^t e^{-\delta v}dB(v)$ and define

$$\widetilde{U}(t) = u + c\int_0^t e^{-\delta v}dv - \sigma A(t), \quad t\geq 0.$$

Since

$$\sup_{t\in[0,\infty)} \mathbb{E}\left\{[A(t)]^2\right\} = \frac{1}{2\delta}$$

implying $\sup_{t\in[0,\infty)}\mathbb{E}\left\{|A(t)|\right\} < \infty$, then by the martingale convergence theorem in [112] we have that $\widetilde{U}(\infty) := \lim_{t\to\infty}\widetilde{U}(t)$ exists and is finite almost surely. Clearly, for any $u>0$

$$\begin{aligned} p(u) &= \mathbb{P}\left\{\inf_{t\in[0,\infty)}\widetilde{U}(t) < 0\right\} \\ &= \mathbb{P}\left\{\sup_{t\in[0,\infty]}\left(\sigma A(t) - c\int_0^t e^{-\delta v}dv\right) > u\right\} \\ &= \mathbb{P}\left\{\sup_{t\in[0,1]}\left(\sigma A(-\frac{1}{2\delta}\ln t) - \frac{c}{\delta}(1-t^{\frac{1}{2}})\right) > u\right\}. \end{aligned}$$

The proof will follow by applying Theorem 2.3.3, hence we check next the assumptions therein for this specific model. Below, we set $Z(t) = \sigma A(-\frac{1}{2\delta}\ln t)$ with variance function given by

$$V_Z^2(t) = Var\left(\sigma\int_0^{-\frac{1}{2\delta}\ln t} e^{-\delta v}dB(v)\right) = \frac{\sigma^2}{2\delta}(1-t), \quad t\in[0,1].$$

We show next that for u sufficiently large, the function

$$M_u(t) := \frac{uV_Z(t)}{G_u(t)} = \frac{\frac{\sigma}{\sqrt{2\delta}}\sqrt{1-t}}{1+\frac{c}{\delta u}(1-t^{1/2})}, \quad 0\leq t\leq 1,$$

with $G_u(t) := u + \frac{c}{\delta}(1-t^{\frac{1}{2}})$ attains its maximum at the unique point $t_u = \left(\frac{c}{\delta u+c}\right)^2$. In fact, we have

$$\begin{aligned} [M_u(t)]_t := \frac{dM_u(t)}{dt} &= \frac{dV_Z(t)}{dt}\cdot\frac{u}{G_u(t)} - \frac{V_Z(t)}{G_u^2(t)}\left(-\frac{cu}{2\delta}t^{-\frac{1}{2}}\right) = \frac{u}{2G_u^2(t)V_z(t)}\left[\frac{dV_Z^2(t)}{dt}G_u(t) + V_Z^2(t)\frac{ct^{-\frac{1}{2}}}{\delta}\right] \\ &= \frac{u\sigma^2 t^{-1/2}}{4\delta G_u^2(t)V_Z(t)}\left[\frac{c}{\delta} - \left(u+\frac{c}{\delta}\right)t^{\frac{1}{2}}\right]. \end{aligned} \tag{2.71}$$

Letting $[M_u(t)]_t = 0$, we get $t_u = \left(\frac{c}{\delta u + c}\right)^2$. By (2.71), $[M_u(t)]_t > 0$ for $t \in (0, t_u)$ and $[M_u(t)]_t < 0$ for $t \in (t_u, 1]$, so t_u is the unique maximum point of $M_u(t)$ over $[0,1]$. Further

$$M_u := M_u(t_u) = \frac{\sigma u}{\sqrt{2\delta u^2 + 4cu}} = \frac{\sigma}{\sqrt{2\delta}}(1 + o(1)), \ u \to \infty.$$

We set $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^2$ for some $q > 1$, and $\Delta(u) = [-t_u, \delta(u)]$. Next we check the assumption **A2**. It follows that

$$\frac{M_u}{M_u(t_u+t)} - 1 = \frac{[G_u(t_u+t)V_Z(t_u)]^2 - [G_u(t_u)V_Z(t_u+t)]^2}{V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u) + V_Z(t_u+t)G_u(t_u)]}.$$

We further write

$$\begin{aligned}
&[G_u(t_u+t)V_Z(t_u)]^2 - [G_u(t_u)V_Z(t_u+t)]^2 \\
&\quad = \left[\left(u + \frac{c}{\delta}\right) - \frac{c}{\delta}\sqrt{t_u+t}\right]^2 \frac{\sigma^2}{2\delta}(1-t_u) - \left[\left(u + \frac{c}{\delta}\right) - \frac{c}{\delta}\sqrt{t_u}\right]^2 \frac{\sigma^2}{2\delta}(1-t_u-t) \\
&\quad = \left(u + \frac{c}{\delta}\right)^2 \frac{\sigma^2}{2\delta} t - 2\left(u + \frac{c}{\delta}\right)\frac{c\sigma^2}{2\delta^2}(\sqrt{t_u+t} - \sqrt{t_u})(1-t_u) - \frac{c^2\sigma^2}{2\delta^3}t \\
&\quad = \left(u + \frac{c}{\delta}\right)^2 \frac{\sigma^2}{2\delta} t(1-t_u) - 2\left(u + \frac{c}{\delta}\right)^2 \frac{\sigma^2}{2\delta}(1-t_u)\sqrt{t_u}(\sqrt{t_u+t} - \sqrt{t_u}) \\
&\quad = \frac{\sigma^2}{2\delta}\left[\left(u + \frac{c}{\delta}\right)^2 - \left(\frac{c}{\delta}\right)^2\right](\sqrt{t+t_u} - \sqrt{t_u})^2 \\
&\quad = \frac{\sigma^2}{2\delta}\left(u^2 + \frac{2c}{\delta}u\right)(\sqrt{t+t_u} - \sqrt{t_u})^2.
\end{aligned}$$

Since for any $t \in \Delta(u)$

$$\sqrt{\frac{\sigma^2}{2\delta}(1 - t_u - \delta(u))} \le V_Z(t_u+t) \le \sqrt{\frac{\sigma^2}{2\delta}}, \ u + \frac{c}{\delta} - \frac{c}{\delta}\sqrt{t_u + \delta(u)} \le G_u(t_u+t) \le u + \frac{c}{\delta},$$

we have for all large u

$$V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u) + V_Z(t_u+t)G_u(t_u)] \le \frac{\sigma^2}{\delta}\left(u + \frac{c}{\delta}\right)^2$$

and

$$\begin{aligned}
V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u) + V_Z(t_u+t)G_u(t_u)] &\ge \frac{\sigma^2}{\delta}(1 - t_u - \delta(u))\left(u + \frac{c}{\delta} - \frac{c}{\delta}\sqrt{t_u + \delta(u)}\right)^2 \\
&\ge \frac{\sigma^2}{\delta}\left[\left(u + \frac{c}{\delta}\right)^2 - u\right].
\end{aligned}$$

Thus as $u \to \infty$

$$\inf_{t \in \Delta(u), t \ne 0} \frac{M_u/M_u(t_u+t) - 1}{\frac{1}{2}\left(\sqrt{u^2 t + \frac{c^2}{\delta^2}} - \frac{c}{\delta}\right)^2 u^{-2}} - 1 \ge \frac{\frac{1}{2}\frac{u^2 + \frac{2c}{\delta}u}{(u + \frac{c}{\delta})^2}(\sqrt{t+t_u} - \sqrt{t_u})^2}{\frac{1}{2}\left(\sqrt{t + \frac{c^2}{(\delta u)^2}} - \frac{c}{\delta u}\right)^2} - 1 \ge \frac{u^2 + \frac{2c}{\delta}u}{(u + \frac{c}{\delta})^2} - 1 \to 0, \tag{2.72}$$

where we used the fact that for $t \in \Delta(u)$

$$(\sqrt{t+t_u} - \sqrt{t_u})^2 \ge \left(\sqrt{t + \frac{c^2}{(\delta u)^2}} - \frac{c}{\delta u}\right)^2.$$

Furthermore, since

$$0 \le \frac{\sqrt{t+t_u} - \sqrt{t_u}}{\sqrt{t + \frac{c^2}{(\delta u)^2}} - \frac{c}{\delta u}} - 1 = \frac{\sqrt{t + \frac{c^2}{(\delta u)^2}} + \frac{c}{\delta u}}{\sqrt{t+t_u} + \sqrt{t_u}} - 1 \le \frac{\sqrt{t + \frac{c^2}{(\delta u)^2}} - \sqrt{t+t_u}}{\sqrt{t+t_u} + \sqrt{t_u}}$$

$$= \frac{\frac{c^2}{(\delta u)^2} - t_u}{(\sqrt{t+t_u}+\sqrt{t_u})(\sqrt{t+\frac{c^2}{(\delta u)^2}}+\sqrt{t+t_u})} \le \frac{\sqrt{\frac{c^2}{(\delta u)^2}-t_u}}{\sqrt{t_u}} = \sqrt{\left(1+\frac{c}{\delta u}\right)^2-1},$$

we have as $u \to \infty$

$$\begin{aligned}\sup_{t\in\Delta(u),t\neq 0} \frac{M_u/M_u(t_u+t)-1}{\frac{1}{2}\left(\sqrt{u^2t+\frac{c^2}{\delta^2}}-\frac{c}{\delta}\right)^2 u^{-2}} - 1 &\le \frac{\frac{1}{2}\frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2-u}(\sqrt{t+t_u}-\sqrt{t_u})^2}{\frac{1}{2}\left(\sqrt{t+\frac{c^2}{(\delta u)^2}}-\frac{c}{\delta u}\right)^2} - 1 \\ &\le \frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2-u}\left(1+\sqrt{\left(1+\frac{c}{\delta u}\right)^2-1}\right)^2 - 1 \to 0. \end{aligned} \tag{2.73}$$

Consequently, (2.72) and (2.73) imply

$$\lim_{u\to\infty} \sup_{t\in\Delta(u),t\neq 0} \left| \frac{M_u/M_u(t_u+t)-1}{\frac{1}{2}\left(\sqrt{u^2t+\frac{c^2}{\delta^2}}-\frac{c}{\delta}\right)^2 u^{-2}} - 1 \right| = 0. \tag{2.74}$$

Since for $0 \le t' \le t < 1$, the correlation function of $Z(t)$ equals

$$r(t,t') = \frac{\mathbb{E}\left\{(\sigma\int_0^{-\frac{1}{2\delta}\ln t} e^{-\delta v}dB(v))(\sigma\int_0^{-\frac{1}{2\delta}\ln t'} e^{-\delta v}dB(v))\right\}}{\sqrt{\frac{\sigma^2}{2\delta}(1-t)}\sqrt{\frac{\sigma^2}{2\delta}(1-t')}} = \frac{\sqrt{1-t}}{\sqrt{1-t'}} = 1 - \frac{t-t'}{\sqrt{1-t'}(\sqrt{1-t'}+\sqrt{1-t})},$$

we have

$$\begin{aligned}\sup_{t,t'\in\Delta(u),t'\neq t}\left|\frac{1-r(t_u+t,t_u+t')}{\frac{1}{2}|t-t'|}-1\right| &= \sup_{t,t'\in\Delta(u),t'\neq t}\left|\frac{2}{\sqrt{1-t-t_u}(\sqrt{1-t'-t_u}+\sqrt{1-t-t_u})}-1\right| \\ &\le \frac{1}{1-(\frac{c}{c+\delta u})^2-(\frac{(\ln u)^q}{u})^2} - 1 \to 0, \quad u\to\infty. \end{aligned} \tag{2.75}$$

Further, for some small $\theta \in (0,1)$, we obtain (set below $\overline{Z}(t) = \frac{Z(t)}{V_Z(t)}$)

$$\mathbb{E}\left(\overline{Z}(t)-\overline{Z}(t')\right)^2 = 2-2r(t,t') \le \mathbb{C}_1|t-t'| \tag{2.76}$$

for $t,t' \in [0,\theta]$. For all large u

$$\Pi(u) := \mathbb{P}\left\{\sup_{t\in[0,\theta]}\left(Z(t)-\frac{c}{\delta}(1-t^{\frac{1}{2}})\right) > u\right\} \le p(u) \le \Pi(u)+\widetilde{\Pi}(u),$$

where

$$\widetilde{\Pi}(u) := \mathbb{P}\left\{\sup_{t\in[\theta,1]}\left(Z(t)-\frac{c}{\delta}(1-t^{\frac{1}{2}})\right) > u\right\} \le \mathbb{P}\left\{\sup_{t\in[\theta,1]} Z(t) > u\right\}.$$

Moreover, for all large u

$$\begin{aligned}\frac{1}{M_u(t)} - \frac{1}{M_u} &\ge \frac{[G_u(t)V_Z(t_u)]^2-[G_u(t_u)V_Z(t)]^2}{2uV_Z^3(t_u)G_u(t_u)} = \frac{\frac{\sigma^2}{2\delta}(u^2+\frac{2c}{\delta}u)(\sqrt{t}-\sqrt{t_u})^2}{2u[\frac{\sigma^2}{2\delta}(1-t_u)]^{3/2}[u+\frac{c}{\delta}(1-\sqrt{t_u})]} \\ &\ge \mathbb{C}_2(\sqrt{t}-\sqrt{t_u})^2 \ge \frac{\mathbb{C}_2\delta^2(u)}{\left(\sqrt{\delta(u)+t_u}+\sqrt{t_u}\right)^2} \ge \mathbb{C}_3\frac{(\ln u)^{2q}}{u^2}\end{aligned}$$

holds for any $t \in [t_u+\delta(u),\theta]$, therefore

$$\inf_{t\in[t_u+\delta(u),\theta]} \frac{M_u}{M_u(t)} \ge 1+\mathbb{C}_3\frac{(\ln u)^q}{u^2}.$$

The above inequality combined with (2.74), (2.75), (2.76) and Theorem 2.3.3 yields

$$\Pi(u) \sim \mathcal{P}^h_{1,\delta/\sigma^2}\left[-\frac{c^2}{\delta^2}, \infty\right) \Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2 + 4cu}\right), \ u \to \infty.$$

Finally, since

$$\sup_{t\in[\theta,1]} V_Z^2(t) \leq \frac{\sigma^2}{2\delta}(1-\theta), \ \ \text{and} \ \ \mathbb{E}\left\{\sup_{t\in[\theta,1]} Z(t)\right\} \leq \mathbb{C}_4 < \infty,$$

by Borell-TIS inequality

$$\widetilde{\Pi}(u) \leq \mathbb{P}\left\{\sup_{t\in[\theta,1]} Z(t) > u\right\} \leq \exp\left(-\frac{\delta(u-\mathbb{C}_4)^2}{\sigma^2(1-\theta)}\right) = o(\Pi(u)), \quad u \to \infty,$$

which establishes the proof. Next, we consider that

$$\begin{aligned}
\mathbb{P}\left\{u^2\left(e^{-2\delta\tau_u} - \left(\frac{c}{\delta u + c}\right)^2\right) \leq x \Big| \tau_u < \infty\right\} &= \frac{\mathbb{P}\left\{\inf_{t\in[-\frac{1}{2\delta}\ln(t_u+u^{-2}x),\infty)} \widetilde{U}(t) < 0\right\}}{\mathbb{P}\left\{\inf_{t\in[0,\infty)} \widetilde{U}(t) < 0\right\}} \\
&= \frac{\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-2}x]}\left(\sigma A(-\frac{1}{2\delta}\ln t) - \frac{c}{\delta}(1-t^{\frac{1}{2}})\right) > u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,1]}\left(\sigma A(-\frac{1}{2\delta}\ln t) - \frac{c}{\delta}(1-t^{\frac{1}{2}})\right) > u\right\}} \\
&= \mathbb{P}\left\{u^2\left(\tau_u^* - t_u\right) \leq x \big| \tau_u^* < 1\right\},
\end{aligned}$$

where

$$\tau_u^* = \{t \in [0,1] : \sigma A(-\frac{1}{2\delta}\ln t) - \frac{c}{\delta}(1-t^{\frac{1}{2}}) > u\}.$$

The proof follows by Theorem 2.3.6 i). □

2.5 Some Technical Results

Proof of (2.13)**:** Let $\xi(t), t \in \mathbb{R}$ be a centered stationary Gaussian process with continuous sample paths, unit variance and correlation function r satisfying

$$1 - r(t) \sim a|t|^\alpha, \ t \to 0, \ \ a > 0, \ \alpha \in (0,2]. \tag{2.77}$$

In view of by Theorem 2.2.1, for $-\infty < x_1 < x_2 < \infty$ and $f \in C_0^*([x_1,x_2])$ we have

$$\mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}x_1,u^{-2/\alpha}x_2]} \frac{\xi(t)}{1+u^{-2}f(u^{2/\alpha}t)} > u\right\} \sim \Psi(u)\mathcal{P}^f_{\alpha,a}[x_1,x_2], \quad u \to \infty$$

and for any $y \in \mathbb{R}$

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}x_1,u^{-2/\alpha}x_2]} \frac{\xi(t)}{1+u^{-2}f(u^{2/\alpha}t)} > u\right\} \\
&= \mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}(x_1-y),u^{-2/\alpha}(x_2-y)]} \frac{\xi(t+yu^{-2/\alpha})(1+u^{-2}f(y))}{1+u^{-2}f(y+u^{2/\alpha}t)} > u(1+u^{-2}f(y))\right\} \\
&\sim \Psi(u(1+u^{-2}f(y)))\mathcal{P}^{f_y(t)-f(y)}_{\alpha,a}[x_1-y,x_2-y] \\
&\sim \Psi(u)\mathcal{P}^{f_y(t)}_{\alpha,a}[x_1-y,x_2-y].
\end{aligned}$$

Let

$$Z_u(t) = \frac{\xi(t+yu^{-2/\alpha})(1+u^{-2}f(y))}{1+u^{-2}f(y+u^{2/\alpha}t)}, \quad t \in [u^{-2/\alpha}(x_1-y), u^{-2/\alpha}(x_2-y)]$$

and denote its variance function by $\sigma^2_{Z_u}(t)$. Then

$$\left(\frac{1}{\sigma_{Z_u}(t)}-1\right)u^2=\left(\frac{1+u^{-2}f(y+u^{2/\alpha}t)}{1+u^{-2}f(y)}-1\right)u^2=\frac{f(y+u^{2/\alpha}t)-f(y)}{1+u^{-2}f(y)},$$

i.e.,

$$\lim_{u\to\infty}\sup_{t\in[u^{-2/\alpha}(x_1-y),u^{-2/\alpha}(x_2-y)]}\left|\frac{\left(\frac{1}{\sigma_{Z_u}(t)}-1\right)u^2}{f(y+u^{2/\alpha}t)-f(y)}-1\right|=0.$$

Consequently, we have

$$\mathcal{P}^f_{\alpha,a}[x_1,x_2]=\mathcal{P}^{f_y}_{\alpha,a}[x_1-y,x_2-y].$$

Further, letting $x_2\to\infty$ yields $\mathcal{P}^f_{\alpha,a}[x_1,\infty)=\mathcal{P}^{f_y}_{\alpha,a}[x_1-y,\infty)$. This completes the proof. □

Proof of Example 2.3.1: We have $t_0=0,\gamma=1,g_m=0$. In view of Theorem 2.3.1 statement i)

$$\mathbb{P}\left\{\max_{t\in[0,T]}(X(t)-ct)>u\right\}\sim\Psi(u)\begin{cases}c^{-1}a^{1/\alpha}u^{2/\alpha-1}\mathcal{H}_\alpha, & \alpha\in(0,2),\\ \mathcal{P}^{ct}_{\alpha,a}[0,\infty), & \alpha=2.\end{cases}$$

Since for all large u

$$\mathbb{P}\left\{u\tau_u\le x\middle|\tau_u\le T\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,u^{-1}x]}(X(t)-g(t))>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)-g(t))>u\right\}},$$

then using Theorem 2.3.2, we obtain for $x\in(0,\infty)$

$$\lim_{u\to\infty}\mathbb{P}\left\{u\tau_u\le x\middle|\tau_u\le T\right\}=\begin{cases}\frac{\int_0^x e^{-ct}dt}{\int_0^\infty e^{-ct}dt}, & \alpha\in(0,2),\\ \frac{\mathcal{P}^{ct}_{\alpha,a}[0,x]}{\mathcal{P}^{ct}_{\alpha,a}[0,\infty)}, & \alpha=2.\end{cases}$$

Proof of Example 2.3.2: We have that $X(t)=\frac{B_\alpha(t)}{\sqrt{Var(B_\alpha(t))}}$ is locally stationary with correlation function

$$r_X(t,t+h)=\frac{|t|^\alpha+|t+h|^\alpha-|h|^\alpha}{2|t(t+h)|^{\alpha/2}}=1-\frac{1}{2t^\alpha}|h|^\alpha+o(|h|^\alpha),\quad h\to 0$$

for any $t>0$. Since $g(t)=c\sin\left(\frac{2\pi t}{T}\right),t\in[T,(n+1)T]$ attains its maximum at $t_j=\frac{(4j+1)T}{4},j\le n$ and

$$g(t)=c-2c\left(\frac{\pi}{T}\right)^2|t-t_j|^2(1+o(1)),\ t\to t_j,\ j\le n$$

the claim follows by applying Remarks 2.3.1 statement i). □

Proof of Example 2.3.3: First note that the variance function of $X(t)$ is given by $\sigma^2(t)=t(1-t)$ and correlation function is given by $r(t,s)=\frac{\sqrt{s(1-t)}}{\sqrt{t(1-s)}},0\le s<t\le 1$.

Case 1) The proof of (2.25): Clearly, $m_u(t):=\frac{\sqrt{t(1-t)}}{1+ct/u}$ attains its maximum over $[0,1]$ at the unique point $t_u=\frac{u}{c+2u}\in(0,1)$ which converges to $t_0=\frac{1}{2}$ as $u\to\infty$, and $m_u^*:=m_u(t_u)=\frac{1}{2\sqrt{1+c/u}}$. Furthermore, we have

$$\begin{aligned}\frac{m_u^*}{m_u(t)}-1&=\frac{u+ct}{\sqrt{t(1-t)}}\frac{\sqrt{t_u(1-t_u)}}{u+ct_u}-1=\frac{(u+ct)\sqrt{t_u(1-t_u)}-(u+ct_u)\sqrt{t(1-t)}}{\sqrt{t(1-t)}(u+ct_u)}\\&=\frac{(u+ct)^2t_u(1-t_u)-(u+ct_u)^2t(1-t)}{\sqrt{t(1-t)}(u+ct_u)[(u+ct)\sqrt{t_u(1-t_u)}+(u+ct_u)\sqrt{t(1-t)}]}.\end{aligned}\tag{2.78}$$

Setting $\Delta(u)=\left[-\frac{(\ln u)^q}{u},\frac{(\ln u)^q}{u}\right]$, and $(t_u+\Delta(u))\subset[0,\frac{1}{2}]$ for all large u, we have

$$(u+ct)^2t_u(1-t_u)-(u+ct_u)^2t(1-t)\;=\;u^2[(t_u-t_u^2)-(t-t^2)]+2cutt_u(t-t_u)+c^2tt_u(t-t_u)$$

$$= \quad (t-t_u)^2 u(u+c) \tag{2.79}$$

and

$$\frac{u^4}{2\left(u+\frac{c}{2}\right)^2} - u^{-1/2} \le 2(u+ct)^2[t(1-t)] \le \frac{1}{2}\left(u+\frac{c}{2}\right)^2$$

for all $t \in (t_u + \Delta(u))$. Then

$$\lim_{u\to\infty} \sup_{t\in\Delta(u), t\neq 0} \left| \frac{m_u^*/m_u(t_u+t)-1}{2t^2} - 1 \right| = \lim_{u\to\infty} \sup_{t\in\Delta(u), t\neq 0} \left| \frac{m_u^*/m_u(t_u+t)-1}{2(ut)^2 u^{-2}} - 1 \right| = 0. \tag{2.80}$$

Furthermore, since

$$r(t,s) = \frac{\sqrt{s(1-t)}}{\sqrt{t(1-s)}} = 1 + \frac{\sqrt{s(1-t)} - \sqrt{t(1-s)}}{\sqrt{t(1-s)}} = 1 - \frac{t-s}{\sqrt{t(1-s)}(\sqrt{s(1-t)}+\sqrt{t(1-s)})},$$

and

$$\frac{1}{2} - \frac{1}{u} \le \sqrt{t(1-s)}(\sqrt{s(1-t)}+\sqrt{t(1-s)}) \le \frac{1}{2} + \frac{1}{u}$$

for all $s < t,\ s,t \in (t_u + \Delta(u))$, we have

$$\lim_{u\to\infty} \sup_{\substack{t,s\in\Delta(u)\\ t\neq s}} \left| \frac{1-r(t_u+t, t_u+s)}{2|t-s|} - 1 \right| = 0.$$

Next for some small $\theta \in (0, \frac{1}{2})$, we have

$$\mathbb{E}\left\{(\overline{X}(t) - \overline{X}(s))^2\right\} = 2(1-r(t,s)) \le \frac{|t-s|}{(\frac{1}{2}-\theta)^2}$$

holds for all $s,t \in [\frac{1}{2}-\theta, \frac{1}{2}+\theta]$. Moreover, by (2.78), (2.79) and

$$2(u+ct)^2[t(1-t)] \le 2\left[u + c\left(\frac{1}{2}+\theta\right)\right]^2 \left(\frac{1}{2}+\theta\right)^2$$

for all $t \in [\frac{1}{2}-\theta, \frac{1}{2}+\theta]$, we have that for any $t \in [\frac{1}{2}-\theta, \frac{1}{2}+\theta] \setminus (t_u + \Delta(u))$

$$\frac{m_u^*}{m_u(t)} - 1 \ge \frac{(\ln u)^{2q}}{2[u+c(\frac{1}{2}+\theta)]^2(\frac{1}{2}+\theta)^2},$$

and further

$$\frac{m_u^*}{m_u(t)} \ge 1 + \mathbb{C}_1 \frac{(\ln u)^q}{u^2}, \quad t \in [\frac{1}{2}-\theta, \frac{1}{2}+\theta] \setminus (t_u + \Delta(u)). \tag{2.81}$$

Consequently, by Theorem 2.3.3

$$\mathbb{P}\left\{\sup_{t\in[t_0-\theta, t_0+\theta]} (X(t)-ct) > u\right\} \sim 8\mathcal{H}_1 u \int_{-\infty}^{\infty} e^{-8t^2} dt \Psi\left(2\sqrt{cu+u^2}\right) \sim e^{-2(u^2+cu)}.$$

In addition, since $\sigma_\theta := \max_{t\in[0,1]/[t_0-\theta,t_0+\theta]} \sigma(t) < \sigma(t_0) = \frac{1}{2}$, by Borell-TIS inequality

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]} (X(t)-ct) > u\right\} &\le \mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]} X(t) > u\right\} \le \exp\left(-\frac{\left(u - \mathbb{E}\left\{\sup_{t\in[0,1]} X(t)\right\}\right)^2}{2\sigma_\theta^2}\right) \\ &= o(e^{-2(u^2+cu)}).\end{aligned} \tag{2.82}$$

Thus, by the fact that

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\geq \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}$$

and

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\leq \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}+\mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\},$$

we conclude that

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\sim e^{-2(u^2+cu)}.$$

For any $u>0$

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\leq x\middle|\tau_u\leq 1\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-1}x]}(X(t)-ct)>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}}$$

and by Theorem 2.2.1

$$\mathbb{P}\left\{\sup_{t\in\left[t_u-\frac{(\ln u)^q}{u},t_u+u^{-1}x\right]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{x}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right).$$

The above combined with (2.81) and (2.82) implies that as $u\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-1}x]}(X(t)-ct)>u\right\}\sim\mathbb{P}\left\{\sup_{t\in\left[t_u-\frac{(\ln u)^q}{u},t_u+u^{-1}x\right]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{x}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right).$$

Consequently,

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\leq x\middle|\tau_u\leq 1\right\}\sim\frac{\int_{-\infty}^{x}e^{-8t^2}dt}{\int_{-\infty}^{\infty}e^{-8t^2}dt}=\Phi(4x),\quad x\in(-\infty,\infty).$$

Case 2) The proof of (2.26): We have $t_u=\frac{u}{c+2u}\in(0,\frac{1}{2})$, which converges to $t_0=\frac{1}{2}$ as $u\to\infty$. Since

$$\frac{1}{2}-t_u\sim\frac{c}{4u},\quad u\to\infty,$$

by Theorem 2.3.3

$$\mathbb{P}\left\{\sup_{t\in[0,1/2]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{c/4}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right)\sim\Phi(c)e^{-2(u^2+cu)}.$$

As for the proof of Case 1) we obtain further

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\leq x\middle|\tau_u\leq\frac{1}{2}\right\}\sim\frac{\int_{-\infty}^{x}e^{-8t^2}dt}{\int_{-\infty}^{c/4}e^{-8t^2}dt}\sim=\Phi(4x)\Phi(c),\quad x\in(-\infty,c/4].$$

Case 3) The proof of (2.27): The variance function $\sigma^2(t)$ is maximal for $t\in[0,1]$ at the unique point $t_0=\frac{1}{2}$, which is also the unique maximum point of $\frac{c}{2}-c\left|t-\frac{1}{2}\right|, t\in[0,1]$. Furthermore,

$$\sigma(t)=\sqrt{t(1-t)}\sim\frac{1}{2}-\left(t-\frac{1}{2}\right)^2,\quad t\to\frac{1}{2}$$

and

$$r(t,s)\sim 1-2|t-s|,\quad s,t\to\frac{1}{2}.$$

By Theorem 2.3.5 as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\left(X(t)+\frac{c}{2}-c\left|t-\frac{1}{2}\right|\right)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{\infty}e^{-\left(8t^2+4c|t|\right)}dt\Psi\left(2u-c\right)\sim 2\Psi(c)e^{-2(u^2-cu)}$$

and in view of Theorem 2.3.6 ii)

$$\mathbb{P}\left\{u\left(\tau_u-\frac{1}{2}\right)\leq x\middle|\tau_u\leq 1\right\}\sim\frac{\int_{-\infty}^{x}e^{-\left(8|t|^2+4c|t|\right)}dt}{\int_{-\infty}^{\infty}e^{-(8|t|^2+4c|t|)}dt},\ u\to\infty.$$

□

Chapter 3

Parisian Ruin of Brownian Motion Risk Model over a Finite-Time Horizon[1]

3.1 Introduction

In a theoretical insurance model the surplus process $R_u(t)$ can be defined by

$$R_u(t) = u + ct - X(t), \quad t \geq 0,$$

see [69], where $u \geq 0$ is the initial reserve, $c > 0$ is the rate of premium and $X(t), t \geq 0$ denotes the aggregate claims process. More specifically, we assume that the aggregate claims process is a Brownian motion, i.e., $X(t) = \sigma B(t)$, $\sigma > 0$. Due to the nature of the financial market, we shall consider a more general surplus process including interest rate, see [130], called a risk reserve process with constant force of interest, i.e., $R_u^\delta(t),\ t \geq 0$, in (1.1). See [130, 41, 86] for more studies on risk models with force of interest.
During the time horizon $[0, S], S \in (0, \infty]$, the classical ruin probability is defined as below

$$\psi_S^\delta(u) := \mathbb{P}\left\{ \inf_{t\in[0,S]} R_u^\delta(t) < 0 \right\}, \tag{3.1}$$

see [69, 89, 91, 63]. In [68, 78] the exact formula of $\psi_\infty^\delta(u)$ for $\delta > 0$ is shown to be

$$\psi_\infty^\delta(u) = \frac{\Psi\left(\sqrt{\frac{2\delta}{\sigma^2}}u + \sqrt{\frac{2c^2}{\sigma^2\delta}}\right)}{\Psi\left(\sqrt{\frac{2c^2}{\sigma^2\delta}}\right)}, \quad u > 0.$$

For $\delta = 0$, the exact value of $\psi_\infty^0(u)$ is well-known (cf. [62]) with

$$\psi_\infty^0(u) = e^{-\frac{2cu}{\sigma^2}}, \quad u > 0.$$

In the literature, there are no results for the classical ruin probability in the case of finite time horizon, i.e., $S \in (0, \infty)$. For $S \in (0, \infty)$, with motivation from the recent contributions [42, 43] we shall investigate in this paper the Parisian ruin probability over the time period $[0, S]$ defined as

$$\mathcal{K}_S^\delta(u, T_u) := \mathbb{P}\left\{ \inf_{t\in[0,S]} \sup_{s\in[t,t+T_u]} R_u^\delta(s) < 0 \right\}, \tag{3.2}$$

[1] This chapter is based on L. Bai, and L. Luo (2017): Parisian ruin of the Brownian motion risk model with constant force of interest, published in the *Statistics & Probability Letters*, Volume 120, 34-44.

where $T_u \ge 0$ models the pre-specified time. Our assumption on T_u is that

$$\lim_{u\to\infty} T_u u^2 = T \in [0,\infty)$$

and thus $\psi_S^\delta(u)$ is a special case of $\mathcal{K}_S^\delta(u,T_u)$ with $T_u \equiv 0$.
Another quantity of interest is the conditional distribution of the ruin time for the surplus process $R_u^\delta(t)$. The classical ruin time, e.g., [41, 81, 91], is defined as

$$\tau(u) = \inf\{t>0 : R_u^\delta(t) < 0\}. \tag{3.3}$$

Here as in [42] we define the Parisian ruin time of the risk process $R_u^\delta(t)$ by

$$\eta(u) = \inf\{t \ge T_u : t-\kappa_{t,u} \ge T_u, R_u^\delta(t) < 0\}, \quad \text{with } \kappa_{t,u} = \sup\{s\in[0,t] : R_u^\delta(s) \ge 0\}, \tag{3.4}$$

and $\tau(u)$ is a special case of $\eta(u)$ with $T_u \equiv 0$.
Brief organization of the rest of the paper: In Section 2 we first present our main results on the asymptotics of $\mathcal{K}_S^\delta(u,T_u)$ as $u\to\infty$ and then we display the approximation of the Parisian ruin time. All the proofs are relegated to Section 3.

3.2 Main results

Before giving the main results, we shall introduce a generalized Piterbarg constant as

$$\widetilde{\mathcal{P}}(T) = \lim_{\lambda\to\infty} \widetilde{\mathcal{P}}(\lambda,T), \quad T \ge 0, \tag{3.5}$$

where for $\lambda, T \ge 0$

$$\widetilde{\mathcal{P}}(\lambda,T) = \mathbb{E}\left\{\sup_{t\in[0,\lambda]} \inf_{s\in[0,T]} e^{\sqrt{2}B(t-s)-|t-s|-(t-s)}\right\}.$$

Note further that the classical Piterbarg constant $\mathcal{P}_{1,1}^t[0,\infty)$ equals $\widetilde{\mathcal{P}}(0)$ and $\mathcal{P}_{1,1}^t[0,\infty)=2$, see [38, 13, 82].

Theorem 3.2.1. *For $\delta>0, S>0$ and $\lim_{u\to\infty} T_u u^2 = T \in [0,\infty)$, we have*

$$\mathcal{K}_S^\delta(u,T_u) \sim \widetilde{\mathcal{P}}(aT)\Psi\left(\frac{\sqrt{2\delta}(u+\frac{c}{\delta}(1-e^{-\delta S}))}{\sigma\sqrt{1-e^{-2\delta S}}}\right), \ u\to\infty, \tag{3.6}$$

where $a := \frac{2\delta^2 e^{-2\delta S}}{\sigma^2(1-e^{-2\delta S})^2}$.

Remarks 3.2.1. a) When $T_u \equiv 0$, $\mathcal{K}_S^\delta(u,T_u)$ reduces to the classical ruin probability $\psi_S^\delta(u)$, and by Theorem 3.2.1 with $T=0$

$$\mathcal{K}_S^\delta(u,0) = \psi_S^\delta(u) \sim 2\Psi\left(\frac{\sqrt{2\delta}(u+\frac{c}{\delta}(1-e^{-\delta S}))}{\sigma\sqrt{1-e^{-2\delta S}}}\right), \ u\to\infty.$$

b) If $\delta=0$

$$\begin{aligned}\mathcal{K}_S^0(u,T_u) &= \mathbb{P}\left\{\inf_{t\in[0,S]} \sup_{s\in[t,t+T_u]} (u+cs-\sigma B(s)) < 0\right\}\\ &\sim \widetilde{\mathcal{P}}(bT)\Psi\left(\frac{u+cS}{\sigma\sqrt{S}}\right), \ u\to\infty,\end{aligned} \tag{3.7}$$

where $b := \frac{1}{2\sigma^2 S^2}$ and we used the result of Corollary 3.4 (ii) in [43].

Further, if $\delta = 0$ and $T_u \equiv 0$, by (3.7) with $T = 0$, we get the asymptotic result of the classical ruin probability

$$\psi_S^0(u) \sim 2\Psi\left(\frac{u+cS}{\sigma\sqrt{S}}\right), \ u \to \infty. \tag{3.8}$$

In fact, [62] gave the exact result of $\psi_S^0(u),\ u > 0$, i.e.,

$$\begin{aligned}\psi_S^0(u) &= \Psi\left(\frac{u+cS}{\sigma\sqrt{S}}\right) + e^{-\frac{2cu}{\sigma^2}}\Phi\left(\frac{cS-u}{\sigma\sqrt{S}}\right)\\ &\sim 2\Psi\left(\frac{u+cS}{\sigma\sqrt{S}}\right), \ u \to \infty,\end{aligned}$$

which follows from

$$\lim_{u\to\infty}\frac{e^{-\frac{2cu}{\sigma^2}}\Phi(\frac{cS-u}{\sigma\sqrt{S}})}{\Psi(\frac{u+cS}{\sigma\sqrt{S}})} = \lim_{u\to\infty}\frac{-\frac{2c}{\sigma^2}e^{-\frac{2cu}{\sigma^2}}\Phi(\frac{cS-u}{\sigma\sqrt{S}}) - \frac{1}{\sigma\sqrt{2\pi S}}e^{-(\frac{u+cS}{\sigma\sqrt{S}})^2/2}}{-\frac{1}{\sigma\sqrt{2\pi S}}e^{-(\frac{u+cS}{\sigma\sqrt{S}})^2/2}} = 1.$$

Our next result discusses the approximation of the conditional ruin time.

Theorem 3.2.2. *Let $\eta(u)$ satisfy (3.4), under the assumptions of Theorem 3.2.1, we have for any $x > 0$ and $\delta \geq 0$,*

$$\mathbb{P}\left\{u^2(S+T_u-\eta(u)) > x \middle| \eta(u) \leq S+T_u\right\} \sim \begin{cases}\exp(-ax), & \text{if } \delta > 0,\\ \exp(-bx), & \text{if } \delta = 0,\end{cases} \quad u \to \infty,$$

where $a := \frac{2\delta^2 e^{-2\delta S}}{\sigma^2(1-e^{-2\delta S})^2}$ and $b := \frac{1}{2\sigma^2 S^2}$.

Remark. If $T_u \equiv 0$, then $\eta(u) = \tau(u)$ and by Theorem 3.2.2, we obtain as $u \to \infty$

$$\mathbb{P}\left\{u^2(S-\tau(u)) > x \mid \tau(u) \leq S\right\} \sim \begin{cases}\exp(-ax), & \text{if } \delta > 0,\\ \exp(-bx), & \text{if } \delta = 0.\end{cases}$$

Hereafter we assume that $\mathbb{C}_i, i \in \mathbb{N}$ are some positive constants.

3.3 Proofs

Proof of Theorem 3.2.1 For $S > 0$ and u large enough

$$\begin{aligned}\mathcal{K}_S^\delta(u, T_u) &= \mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\left(\sigma\int_0^s e^{-\delta z}dB(z) - c\int_0^s e^{-\delta z}dz\right) > u\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\overline{X}(s)\frac{f_u(S)}{f_u(s)} > f_u(S)\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}X_u(s) > f_u(S)\right\},\end{aligned}$$

with

$$X(s) = \sigma\int_0^s e^{-\delta z}dB(z),\ \overline{X}(s) = \frac{X(s)}{\sigma_X(s)},\ f_u(s) = \frac{u+\frac{c}{\delta}(1-e^{-\delta s})}{\sigma_X(s)} \text{ and } X_u(s) = \overline{X}(s)\frac{f_u(S)}{f_u(s)},$$

where $\sigma_X^2(s)$ is the variance of $X(s)$ with $\sigma_X^2(s) = \frac{\sigma^2}{2\delta}(1-e^{-2\delta s})$.
Set $\rho(u) = \left(\frac{\ln u}{u}\right)^2$ and for any $\lambda > 0$, Bonferroni inequality yields

$$\Pi_0(u) := \mathbb{P}\left\{\sup_{t\in[S-\lambda u^{-2},S]}\inf_{s\in[t,t+T_u]}X_u(s) > f_u(S)\right\} \leq \mathcal{K}_S^\delta(u,T_u) \leq \Pi_0(u)+\Pi_1(u)+\Pi_2(u), \tag{3.9}$$

where

$$\Pi_1(u)=\mathbb{P}\left\{\sup_{t\in[0,S-\rho(u)]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\},\quad \Pi_2(u)=\mathbb{P}\left\{\sup_{t\in[S-\rho(u),S-\lambda u^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\}.$$

First we give some upper bounds of $\Pi_1(u)$ and $\Pi_2(u)$ which finally show that

$$\Pi_1(u)+\Pi_2(u)=o\left(\Pi_0(u)\right),\quad u\to\infty. \tag{3.10}$$

For all u large

$$\begin{aligned}\mathbb{E}\left\{(X_u(t_1)-X_u(t_2))^2\right\}&=\mathbb{E}\left\{\left(X(t_1)\frac{f_u(S)}{u+\frac{c}{\delta}\left(1-e^{-\delta t_1}\right)}-X(t_2)\frac{f_u(S)}{u+\frac{c}{\delta}\left(1-e^{-\delta t_2}\right)}\right)^2\right\}\\&\le \mathbb{C}_1\mathbb{E}\left\{\left(\int_{t_1}^{t_2}e^{-\delta z}dB(z)\right)^2\right\}+\mathbb{C}_2\left(\frac{u+\frac{c}{\delta}(1-e^{-\delta S})}{u+\frac{c}{\delta}(1-e^{-\delta t_1})}-\frac{u+\frac{c}{\delta}(1-e^{-\delta S})}{u+\frac{c}{\delta}(1-e^{-\delta t_2})}\right)^2\\&\le \mathbb{C}_3|t_1-t_2|,\quad t_1<t_2,\ t_1,t_2\in(0,S].\end{aligned}$$

Moreover,

$$\sup_{t\in[0,S-\rho(u)]}\mathrm{var}\left(X_u(t)\right)=\sup_{t\in[0,S-\rho(u)]}\left(\frac{f_u(S)}{f_u(t)}\right)^2=\frac{f_u^2(S)}{f_u^2(S-\rho(u))},$$

where we use the fact that $f_u(t)$ is a decreasing function for $t\in[0,S]$ when u large enough. Therefore, by Theorem 8.1 in [119], we obtain

$$\Pi_1(u)\le\mathbb{P}\left\{\sup_{t\in[0,S-\rho(u)]}X_u(t)>f_u(S)\right\}\le\mathbb{C}_4u^2\Psi\left(f_u(S-\rho(u))\right), \tag{3.11}$$

and direct calculation yields that

$$\begin{aligned}u^2\Psi\left(f_u(S-\rho(u))\right)&\le\frac{u^2}{\sqrt{2\pi}f_u(S)}e^{-\frac{f_u^2(S)}{2}\left(\frac{f_u^2(S-\rho(u))}{f_u^2(S)}-1\right)}e^{-\frac{f_u^2(S)}{2}}\\&\sim u^2e^{-a(\ln u)^2}\Psi(f_u(S))=o\left(\Psi(f_u(S))\right),\quad u\to\infty,\end{aligned}$$

where $a=\frac{2\delta^2e^{-2\delta S}}{\sigma^2(1-e^{-2\delta S})^2}$ and we use the fact that

$$1-\frac{f_u(S)}{f_u(S-t)}\sim\frac{\delta e^{-2\delta S}}{1-e^{-2\delta S}}t,\quad t\to0. \tag{3.12}$$

Set

$$\triangle_k=\left[k\lambda u^{-2},(k+1)\lambda u^{-2}\right],\ k\in\mathbb{N},\quad\text{and}\quad N(u)=\left\lfloor\lambda^{-1}\rho(u)u^2\right\rfloor,$$

where $\lfloor\cdot\rfloor$ stands for the ceiling function, then

$$\begin{aligned}\Pi_2(u)&\le\mathbb{P}\left\{\sup_{t\in[S-\rho(u),S-\lambda u^{-2}]}X_u(t)>f_u(S)\right\}\\&=\mathbb{P}\left\{\sup_{t\in[\lambda u^{-2},\rho(u)]}X_u(S-t)>f_u(S)\right\}\\&\le\sum_{k=1}^{N(u)}\mathbb{P}\left\{\sup_{t\in\Delta_k}X_u(S-t)>f_u(S)\right\}\\&\le\sum_{k=1}^{N(u)}\mathbb{P}\left\{\sup_{t\in\Delta_0}\overline{X}(S-t)>f_u(S-k\lambda u^{-2})\right\}\end{aligned}$$

$$\leq \sum_{k=1}^{N(u)} \mathbb{P}\left\{ \sup_{t\in[0,\lambda]} \overline{X}(S-u^{-2}t) > f_u(S-k\lambda u^{-2}) \right\}. \tag{3.13}$$

Clearly,

$$\inf_{1\leq k\leq N(u)} f_u(S-k\lambda u^{-2}) \to \infty, u\to\infty, \tag{3.14}$$

and for $t_1 < t_2,\ t_1, t_2 \in [0, S]$,

$$r_X(t_1,t_2) := \mathbb{E}\left\{\overline{X}(t_1)\overline{X}(t_2)\right\} = \sqrt{\frac{1-e^{-2\delta t_1}}{1-e^{-2\delta t_2}}}.$$

Further,

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{1\leq k\leq N(u)} \sup_{\substack{t_1\neq t_2,\\ t_1,t_2\in[0,\lambda]}} \left| f_u^2(S-k\lambda u^{-2}) \frac{\mathrm{var}\left(\overline{X}(S-u^{-2}t_1)-\overline{X}(S-u^{-2}t_2)\right)}{2a|t_1-t_2|} - 1 \right| \\
&= \lim_{u\to\infty} \sup_{1\leq k\leq N(u)} \sup_{\substack{t_1\neq t_2,\\ t_1,t_2\in[0,\lambda]}} \left| f_u^2(S-k\lambda u^{-2}) \frac{2-2r_X(S-u^{-2}t_1, S-u^{-2}t_2)}{2a|t_1-t_2|} - 1 \right| \\
&= 0,
\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}
&\sup_{1\leq k\leq N(u)} \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} f_u^2(S-k\lambda u^{-2}) \mathbb{E}\left\{ \left(\overline{X}(S-u^{-2}t_1)-\overline{X}(S-u^{-2}t_2)\right)\overline{X}(S) \right\} \\
&\leq \mathbb{C}_5 u^2 \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} \left| r_X(S-u^{-2}t_1, S) - r_X(S-u^{-2}t_2, S) \right| \\
&\leq \mathbb{C}_6 u^2 \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} \left| \sqrt{1-e^{-2\delta(S-u^{-2}t_1)}} - \sqrt{1-e^{-2\delta(S-u^{-2}t_2)}} \right| \\
&\leq \mathbb{C}_7 \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} |t_1-t_2| \to 0, \quad u\to\infty,\ \varepsilon\to 0.
\end{aligned} \tag{3.16}$$

According to (3.14), (3.15), (3.16) and Lemma 5.3 of [45], (3.13) is followed by

$$\Pi_2(u) \leq \mathbb{C}_8\lambda \sum_{k=1}^{N(u)} \Psi\left(f_u(S-k\lambda u^{-2})\right) \leq \mathbb{C}_9 \Psi(f_u(S))\lambda \sum_{k=1}^{\infty} e^{-\mathbb{C}_{10}k\lambda} = o\left(\Psi(f_u(S))\right),\ u\to\infty,\ \lambda\to\infty, \tag{3.17}$$

where the last inequality follows from (3.12).

Next we give the asymptotic behavior of $\Pi_0(u)$ as $u\to\infty$ based on an appropriate application of the Appendix in [43]. For any $\varepsilon_1 > 0$ and u large enough

$$\begin{aligned}
\Pi_0(u) &= \mathbb{P}\left\{ \sup_{t\in[S-\lambda u^{-2}, S]} \inf_{s\in[t,t+T_u]} X_u(s) > f_u(S) \right\} \\
&\leq \mathbb{P}\left\{ \sup_{t\in[S-\lambda u^{-2}, S]} \inf_{s\in[t,t+(1-\varepsilon_1)Tu^{-2}]} X_u(s) > f_u(S) \right\} \\
&= \mathbb{P}\left\{ \sup_{t\in[0,\lambda]} \inf_{s\in[0,(1-\varepsilon_1)T]} X_u(S+u^{-2}s-u^{-2}t) > f_u(S) \right\} \\
&= \mathbb{P}\left\{ \sup_{t\in[0,\lambda]} \inf_{s\in[0,(1-\varepsilon_1)T]} Y_u(t,s) > f_u(S) \right\} \\
&=: \Pi_0^+(u)
\end{aligned}$$

and

$$\Pi_0(u) \geq \mathbb{P}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[0,(1+\varepsilon_1)T]} Y_u(t,s) > f_u(S)\right\} =: \Pi_0^-(u),$$

where $Y_u(t,s) := X_u(S+u^{-2}s-u^{-2}t)$, for $(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$.
Since

$$\sigma_{Y_u}(t,s) := \sqrt{\mathrm{var}\left(Y_u(t,s)\right)} = \sqrt{\mathrm{var}(X_u(S+u^{-2}s-u^{-2}t))} = \frac{f_u(S)}{f_u(S+u^{-2}s-u^{-2}t)}$$

and (3.12), there exists $d(t,s)=\frac{\delta e^{-2\delta S}}{1-e^{-2\delta S}}(t-s)$ such that

$$\lim_{u\to\infty}\sup_{(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]}\left|u^2(1-\sigma_{Y_u}(t,s))-d(t,s)\right|=0. \tag{3.18}$$

Moreover, for $(t_1,s_1),(t_2,s_2)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$ and $s_1-t_1>s_2-t_2$,

$$\begin{aligned}
&\mathrm{var}(Y_u(t_1,s_1)-Y_u(t_2,s_2))\\
&\quad= f_u^2(S)\mathbb{E}\left\{\frac{X(S+u^{-2}s_1-u^{-2}t_1)}{u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)})}-\frac{X(S+u^{-2}s_2-u^{-2}t_2)}{u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)})}\right\}^2\\
&\quad= f_u^2(S)(J_1(u)+J_2(u)+J_3(u)),
\end{aligned}$$

where

$$\begin{aligned}
J_1(u) &= \mathbb{E}\left\{\frac{X(S+u^{-2}s_1-u^{-2}t_1)-X(S+u^{-2}s_2-u^{-2}t_2)}{u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)})}\right\}^2,\\
J_2(u) &= 2\frac{\frac{c}{\delta}(e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)}-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)})}{(u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)}))(u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)}))}\\
&\quad\times\mathbb{E}\left\{\left(\frac{X(S+u^{-2}s_1-u^{-2}t_1)-X(S+u^{-2}s_2-u^{-2}t_2)}{u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)})}\right)X(S+u^{-2}s_2-u^{-2}t_2)\right\}=0,\\
J_3(u) &= \left(\frac{\frac{c}{\delta}(e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)}-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)})}{(u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)}))(u+\frac{c}{\delta}(1-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)}))}\right)^2\mathbb{E}\left\{X(S+u^{-2}s_2-u^{-2}t_2)\right\}^2.
\end{aligned}$$

Since

$$\begin{aligned}
\lim_{u\to\infty}u^2f_u^2(S)J_1(u) &= \lim_{u\to\infty}f_u^2(S)\mathbb{E}\left\{X(S+u^{-2}s_1-u^{-2}t_1)-X(S+u^{-2}s_2-u^{-2}t_2)\right\}^2\\
&= \lim_{u\to\infty}\frac{u^2}{\frac{\sigma^2}{2\delta}(1-e^{-2\delta S})}\frac{\sigma^2}{2\delta}(e^{-2\delta(S+u^{-2}s_2-u^{-2}t_2)}-e^{-2\delta(S+u^{-2}s_1-u^{-2}t_1)})\\
&= \frac{2\delta e^{-2\delta S}}{1-e^{-2\delta S}}((s_1-s_2)-(t_1-t_2))\\
&= \frac{2\delta e^{-2\delta S}}{1-e^{-2\delta S}}\mathrm{var}\left(B(s_1-t_1)-B(s_2-t_2)\right),\\
\lim_{u\to\infty}u^2f_u^2(S)J_3(u) &\leq \lim_{u\to\infty}\mathbb{C}_{11}(e^{-\delta(S+u^{-2}s_1-u^{-2}t_1)}-e^{-\delta(S+u^{-2}s_2-u^{-2}t_2)})\mathbb{E}\left\{X(S+u^{-2}s_2-u^{-2}t_2)\right\}^2=0,
\end{aligned}$$

thus

$$\lim_{u\to\infty}u^2\mathrm{var}(Y_u(t_1,s_1)-Y_u(t_2,s_2)) = \frac{2\delta e^{-2\delta S}}{1-e^{-2\delta S}}\mathrm{var}\left(B(s_1-t_1)-B(s_2-t_2)\right). \tag{3.19}$$

Further, there exist some constant $G,u_0>0$, such that for any $u>u_0$

$$u^2\mathrm{var}(Y_u(t_1,s_1)-Y_u(t_2,s_2)) \leq G(|t_1-t_2|+|s_1-s_2|) \tag{3.20}$$

holds uniformly with respect to $(t_1,s_1),(t_2,s_2)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$. By (3.18), (3.19), (3.20), Lemma 5.1 in [43] and $\lim_{u\to\infty} f_u(S)/u = 1/\sigma_X(S)$, we obtain

$$\Pi_0^-(u) \sim \widetilde{\mathcal{P}}(a\lambda, a(1+\varepsilon_1)T)\Psi(f_u(S)),\ \ u\to\infty. \tag{3.21}$$

Similarly

$$\Pi_0^+(u) \sim \widetilde{\mathcal{P}}(a\lambda, a(1-\varepsilon_1)T)\Psi(f_u(S)),\ \ u\to\infty.$$

Letting $\varepsilon_1\to 0$ and $\lambda\to\infty$, we have

$$\Pi_0(u) \sim \widetilde{\mathcal{P}}(aT)\Psi(f_u(S)), u\to\infty.$$

The above combined with (3.11) and (3.17) drives (3.10), therefore by (3.9) the proof is complete. □

PROOF OF THEOREM 3.2.2 **Case 1** $\delta>0$: According to the definition of conditional probability, for any $x,u>0$

$$\begin{aligned}
&\mathbb{P}\left\{u^2(S+T_u-\eta(u))>x \mid \eta(u)\le S+T_u\right\}\\
&\quad=\frac{\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\left(\sigma\int_0^s e^{-\delta z}dB(z)-c\int_0^s e^{-\delta z}dz\right)>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}\left(\sigma\int_0^s e^{-\delta z}dB(z)-c\int_0^s e^{-\delta z}dz\right)>u\right\}}.
\end{aligned}\tag{3.22}$$

Using the same notation of $X(s)$, $\overline{X}(s)$, $f_u(s)$, $X_u(s)$, $\sigma_X(s)$ as in the proof of Theorem 2.1, we have for u large enough

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\left(\sigma\int_0^s e^{-\delta z}dB(z)-c\int_0^s e^{-\delta z}dz\right)>u\right\}\\
&\quad=\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\overline{X}(s)\frac{f_u(S)}{f_u(s)}>f_u(S)\right\}\\
&\quad=\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\},
\end{aligned}$$

Set $\rho(u)=\left(\frac{\ln u}{u}\right)^2$. For any $\lambda>0$, Bonferroni inequality yields

$$\Pi_0^*(u)\le\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\}\le\Pi_0^*(u)+\Pi_1^*(u)+\Pi_2^*(u),\tag{3.23}$$

where

$$\begin{aligned}
\Pi_0^*(u)&=\mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2},S-xu^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\},\\
\Pi_1^*(u)&=\mathbb{P}\left\{\sup_{t\in[0,S-\rho(u)]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\},\\
\Pi_2^*(u)&=\mathbb{P}\left\{\sup_{t\in[S-\rho(u),S-xu^{-2}-\lambda u^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\}.
\end{aligned}$$

By (3.11) and (3.17) in the proof of Theorem 3.2.1, we know

$$\Pi_1^*(u)=o\left(\Psi(f_u(S))\right),\quad u\to\infty,\tag{3.24}$$

and

$$\Pi_2^*(u)\le\mathbb{P}\left\{\sup_{t\in[S-\rho(u),S-\lambda u^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\}=o\left(\Psi(f_u(S))\right),\ u\to\infty,\ \lambda\to\infty.\tag{3.25}$$

Next we give the asymptotic behavior of $\Pi_0^*(u)$ as $u\to\infty$. For any $\varepsilon_1>0$ and u large enough

$$\begin{aligned}\Pi_0^*(u) &= \mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2},S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\overline{X}(s)\frac{f_u(S)}{f_u(s)}>f_u(S)\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2},S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\overline{X}(s)\frac{f_u(S-xu^{-2})}{f_u(s)}>f_u(S-xu^{-2})\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2},S-xu^{-2}]}\inf_{s\in[t,t+(1-\varepsilon_1)Tu^{-2}]}\overline{X}(s)\frac{f_u(S-xu^{-2})}{f_u(s)}>f_u(S-xu^{-2})\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[0,(1-\varepsilon_1)T]}\overline{X}(S+u^{-2}s-u^{-2}t-u^{-2}x)\frac{f_u(S-xu^{-2})}{f_u(S+u^{-2}s-u^{-2}t-u^{-2}x)}>f_u(S-xu^{-2})\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[0,(1-\varepsilon_1)T]}Y_u^*(t,s)>f_u(S-xu^{-2})\right\}\\ &=: \Pi_0^{*+}(u),\end{aligned}$$

and

$$\Pi_0^*(u)\ge \mathbb{P}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[0,(1+\varepsilon_1)T]}Y_u^*(t,s)>f_u(S-xu^{-2})\right\}=:\Pi_0^{*-}(u),$$

where $Y_u^*(t,s):=\overline{X}(S+u^{-2}s-u^{-2}t-u^{-2}x)\frac{f_u(S-xu^{-2})}{f_u(S+u^{-2}s-u^{-2}t-u^{-2}x)}$, $(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$ and $\sigma^2_{Y_u^*}(t,s):=\mathrm{var}(Y_u^*(t,s))=\left(\frac{f_u(S-xu^{-2})}{f_u(S+u^{-2}s-u^{-2}t-u^{-2}x)}\right)^2$.
Using the similar argumentation as (3.18) in the proof of Theorem 3.2.1, we have

$$\lim_{u\to\infty}\sup_{(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]}|u^2(1-\sigma_{Y_u^*}(t,s))-d(t,s)|=0,$$

with $d(t,s)=\frac{\delta e^{-2\delta S}}{1-e^{-2\delta S}}(t-s)$. Moreover, (3.19), (3.20) still hold for $Y_u^*(t,s)$ and $(t_1,s_1),(t_2,s_2)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$. By Lemma 5.1 in [43] and $\lim_{u\to\infty}f_u(S)/u=1/\sigma_X(S)$, we obtain

$$\Pi_0^{*-}(u)\sim\widetilde{\mathcal{P}}(a\lambda,a(1+\varepsilon_1)T)\Psi(f_u(S-xu^{-2}))\sim e^{-ax}\widetilde{\mathcal{P}}(a\lambda,a(1+\varepsilon_1)T)\Psi(f_u(S)),\ u\to\infty.$$

Similarly,

$$\Pi_0^{*+}(u)\sim e^{-ax}\widetilde{\mathcal{P}}(a\lambda,a(1-\varepsilon_1)T)\Psi(f_u(S)),\ u\to\infty.$$

Letting $\varepsilon_1\to0$ and $\lambda\to\infty$, we have

$$\Pi_0^*(u)\sim e^{-ax}\widetilde{\mathcal{P}}(aT)\Psi(f_u(S)),u\to\infty.$$

The above combined with (3.23), (3.24) and (3.25) derives that

$$\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}X_u(s)>f_u(S)\right\}\sim e^{-ax}\widetilde{\mathcal{P}}(aT)\Psi(f_u(S)),\ u\to\infty.$$

Thus, the claim follows by using the results of Theorem 3.2.1 and (3.22).
Case 2 $\delta=0$:

$$\mathbb{P}\left\{u^2(S+T_u-\eta(u))>x\middle|\eta(u)\le S+T_u\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}(\sigma B(s)-cs)>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,S]}\inf_{s\in[t,t+T_u]}(\sigma B(s)-cs)>u\right\}}.$$

For u large enough

$$\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}(\sigma B(s)-cs)>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s)>f_u(S)\right\},$$

with

$$X(s)=\sigma B(s),\ \overline{X}(s)=\frac{B(s)}{\sqrt{s}},\ f_u(s)=\frac{u+cs}{\sigma\sqrt{s}}\ \text{and}\ \ \widetilde{X}_u(s)=\overline{X}(s)\frac{f_u(S)}{f_u(s)}.$$

Set $\rho(u)=\left(\frac{\ln u}{u}\right)^2$. For any $\lambda>0$, Bonferroni inequality yields

$$\widetilde{\Pi}_0(u)\le\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s)>f_u(S)\right\}\le\widetilde{\Pi}_0(u)+\widetilde{\Pi}_1(u)+\widetilde{\Pi}_2(u),\tag{3.26}$$

where

$$\begin{aligned}\widetilde{\Pi}_0(u)&=\mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2},S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s)>f_u(S)\right\},\\ \widetilde{\Pi}_1(u)&=\mathbb{P}\left\{\sup_{t\in[0,S-\rho(u)]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s)>f_u(S)\right\},\\ \widetilde{\Pi}_2(u)&=\mathbb{P}\left\{\sup_{t\in[S-\rho(u),S-xu^{-2}-\lambda u^{-2}]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s)>f_u(S)\right\}.\end{aligned}$$

Notice that for u large enough

$$\begin{aligned}\mathbb{E}\left\{(\widetilde{X}_u(t_1)-\widetilde{X}_u(t_2))^2\right\}&=\frac{1}{S}\mathbb{E}\left\{\left(\frac{u+cS}{u+ct_1}B(t_1)-\frac{u+cS}{u+ct_2}B(t_2)\right)^2\right\}\\ &\le\mathbb{C}_{12}\mathbb{E}\left\{(B(t_1)-B(t_2))^2\right\}+\mathbb{C}_{13}\left(\frac{u+cS}{u+ct_1}-\frac{u+cS}{u+ct_2}\right)^2\\ &\le\mathbb{C}_{14}|t_1-t_2|,\quad t_1<t_2,\ t_1,t_2\in(0,S],\end{aligned}$$

and

$$\sup_{t\in[0,S-\rho(u)]}\operatorname{var}\left(\widetilde{X}_u(t)\right)=\sup_{t\in[0,S-\rho(u)]}\left(\frac{f_u(S)}{f_u(t)}\right)^2=\frac{f_u^2(S)}{f_u^2(S-\rho(u))},$$

where we use the fact that $f_u(t)$ is a decreasing function for $t\in[0,S]$ when u large enough. Moreover,

$$1-\frac{f_u(S)}{f_u(S-t)}\sim\frac{1}{2S}t,\quad t\to 0,$$

$$\inf_{1\le k\le N(u)}f_u(S-k\lambda u^{-2})\to\infty,u\to\infty,$$

and for $t_1<t_2,\ t_1,t_2\in[0,S]$,

$$r_{\widetilde{X}}(t_1,t_2):=\mathbb{E}\left\{\overline{X}(t_1)\overline{X}(t_2)\right\}=\sqrt{\frac{t_1}{t_2}}.$$

Then

$$\lim_{u\to\infty}\sup_{1\le k\le N(u)}\sup_{\substack{t_1\ne t_2,\\ t_1,t_2\in[0,\lambda]}}\left|f_u^2(S-k\lambda u^{-2})\frac{\operatorname{var}\left(\overline{X}(S-u^{-2}t_1)-\overline{X}(S-u^{-2}t_2)\right)}{2b|t_1-t_2|}-1\right|$$

$$= \lim_{u\to\infty} \sup_{1\le k\le N(u)} \sup_{\substack{t_1\neq t_2,\\ t_1,t_2\in[0,\lambda]}} \left| f_u^2(S-k\lambda u^{-2}) \frac{2-2r_{\widetilde{X}}(S-u^{-2}t_1, S-u^{-2}t_2)}{2b|t_1-t_2|} - 1 \right| = 0, \tag{3.27}$$

where $b = \frac{1}{2\sigma^2 S^2}$, and

$$\begin{aligned}
&\sup_{1\le k\le N(u)} \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} f_u^2(S-k\lambda u^{-2})\mathbb{E}\left\{\left(\overline{X}(S-u^{-2}t_1)-\overline{X}(S-u^{-2}t_2)\right)\overline{X}(S)\right\}\\
&\quad \le \mathbb{C}_{15}u^2 \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} \left|r_{\widetilde{X}}(S-u^{-2}t_1,S) - r_{\widetilde{X}}(S-u^{-2}t_2,S)\right|\\
&\quad \le \mathbb{C}_{16}u^2 \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} \left|\sqrt{S-u^{-2}t_1} - \sqrt{S-u^{-2}t_2}\right|\\
&\quad \le \mathbb{C}_{17} \sup_{\substack{|t_1-t_2|<\varepsilon\\ t_1,t_2\in[0,\lambda]}} |t_1-t_2| \to 0, \quad u\to\infty,\ \varepsilon\to 0.
\end{aligned} \tag{3.28}$$

By Theorem 8.1 in [119] and Lemma 5.3 in [45], using the similar argumentation as in the proof of Theorem 3.2.1, we derive

$$\widetilde{\Pi}_1(u) + \widetilde{\Pi}_2(u) = o\left(\Psi(f_u(S))\right), \ u\to\infty,\ \lambda\to\infty. \tag{3.29}$$

Next we give the asymptotic behavior of $\widetilde{\Pi}_0(u)$ as $u\to\infty$. For any $\varepsilon_1>0$ and u large enough

$$\begin{aligned}
\widetilde{\Pi}_0(u) &= \mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2}, S-xu^{-2}]} \inf_{s\in[t,t+T_u]} \overline{X}(s)\frac{f_u(S)}{f_u(s)} > f_u(S)\right\}\\
&= \mathbb{P}\left\{\sup_{t\in[S-xu^{-2}-\lambda u^{-2}, S-xu^{-2}]} \inf_{s\in[t,t+T_u]} \overline{X}(s)\frac{f_u(S-xu^{-2})}{f_u(s)} > f_u(S-xu^{-2})\right\}\\
&\le \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \inf_{s\in[0,(1-\varepsilon_1)T]} \widetilde{Y}_u(t,s) > f_u(S-xu^{-2})\right\}\\
&=: \widetilde{\Pi}_0^+(u)
\end{aligned}$$

and

$$\widetilde{\Pi}_0(u) \ge \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \inf_{s\in[0,(1+\varepsilon_1)T]} \widetilde{Y}_u(t,s) > f_u(S-xu^{-2})\right\} =: \widetilde{\Pi}_0^-(u),$$

where $\widetilde{Y}_u(t,s) := \overline{X}(S+u^{-2}s-u^{-2}t-u^{-2}x)\frac{f_u(S-xu^{-2})}{f_u(S+u^{-2}s-u^{-2}t-u^{-2}x)}$, for $(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$.
Using the similar argumentation as (3.18), (3.19) and (3.20) in the proof of Theorem 3.2.1, we obtain that

$$\lim_{u\to\infty} \sup_{(t,s)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]} |u^2(1-\sigma_{\widetilde{Y}_u}(t,s)) - \widetilde{d}(t,s)| = 0,$$

with $\widetilde{d}(t,s) = \frac{1}{2S}(t-s)$ and $\sigma_{\widetilde{Y}_u}(t,s) := \sqrt{\mathrm{var}(\widetilde{Y}_u(t,s))}$,

$$\lim_{u\to\infty} u^2\mathrm{var}(\widetilde{Y}_u(t_1,s_1) - \widetilde{Y}_u(t_2,s_2)) = \frac{1}{S}\mathrm{var}\left(B(s_1-t_1) - B(s_2-t_2)\right),$$

and for some constant G and all u large enough

$$u^2\mathrm{var}(\widetilde{Y}_u(t_1,s_1) - \widetilde{Y}_u(t_2,s_2)) \le G(|t_1-t_2| + |s_1-s_2|)$$

uniformly for $(t_1,s_1),(t_2,s_2)\in[0,\lambda]\times[0,(1+\varepsilon_1)T]$.
By Lemma 5.1 in [43] and $\lim_{u\to\infty} f_u(S)/u = \frac{1}{\sigma\sqrt{S}}$, we obtain

$$\widetilde{\Pi}_0^-(u) \sim \widetilde{\mathcal{P}}(b\lambda, b(1+\varepsilon_1)T)\Psi(f_u(S-xu^{-2})) \sim e^{-bx}\widetilde{\mathcal{P}}(b\lambda, b(1+\varepsilon_1)T)\Psi(f_u(S)),\ u\to\infty.$$

Similarly,

$$\widetilde{\Pi}_0^+(u) \sim e^{-bx}\widetilde{\mathcal{P}}(b\lambda, b(1-\varepsilon_1)T)\Psi(f_u(S)),\ u \to \infty.$$

Letting $\varepsilon_1 \to 0$ and $\lambda \to \infty$, we have

$$\widetilde{\Pi}_0(u) \sim e^{-bx}\widetilde{\mathcal{P}}(bT)\Psi(f_u(S)),\ u \to \infty.$$

The above combined with (3.26) and (3.29) leads to

$$\mathbb{P}\left\{\sup_{t\in[0,S-xu^{-2}]}\inf_{s\in[t,t+T_u]}\widetilde{X}_u(s) > f_u(S_x(u))\right\} \sim e^{-bx}\widetilde{\mathcal{P}}(bT)\Psi(f_u(S)),\ u \to \infty.$$

Using the above asymptotic equality and b) of Remarks 3.2.1, we obtain the results.

□

Chapter 4

Parisian Ruin of Brownian Motion Risk Model over an Infinite-Time Horizon[1]

4.1 Introduction

In the risk theory, the surplus process of an insurance company can be modeled by

$$R_u(t) = u + ct - X(t), \quad t \geq 0,$$

see [69], where $u \geq 0$ is the initial reserve, ct models the total premium received up to time t, and $X(t), t \geq 0$ denotes the aggregate claims process. One of the most important characteristics in risk theory is the ruin probability defined by

$$\mathbb{P}\left\{ \inf_{t\in[0,S]} R_u(t) < 0 \right\}.$$

Some contributions, i.e., [28, 31, 42, 43], extend this classical ruin probability to the so-called Parisian ruin probability which allows the surplus process to spend a pre-specified time under level zero before ruin is recognized. The name for this problem is borrowed from the Parisian option. Depending on the type of such an option, the prices are activated or canceled if the underlying asset stays above or below the barrier long enough in a row (see [26, 4] and [32]). We believe that the Parisian ruin probability could be a better measure of risk in many situations, giving insurance companies the chance to achieve solvency. Moreover, originated from Chapter 11 of the U.S. bankruptcy code, Parisian ruin is also considered as a theoretical description of the liquidation risk, see [103, 110]. Figure 4.1 in Appendix depicts both the classical ruin and Parisian ruin scenarios.

As in [42, 43], the Parisian ruin of $R_u(t)$ is defined by

$$\mathcal{K}_S(u,T) = \mathbb{P}\left\{ \inf_{t\in[0,S]} \sup_{s\in[t,t+T]} R_u(s) < 0 \right\}, \quad S \in (0,\infty], \tag{4.1}$$

where $T \in [0,\infty)$ models the pre-specified time. Calculation of the probability of Parisian ruin $\mathcal{K}_S(u,T)$ is more complex than the calculation of the classical ruin $\mathbb{P}\left\{\inf_{t\in[0,S]} R_u(s) < 0\right\}$. When $S = \infty$ and X is modelled by a specific class of Lévy processes, exact formulas for $\mathcal{K}_\infty(u,T)$ with $T \in (0,\infty)$ are derived in [28, 31]. See [27, 29, 30, 102] for some recent developments. But if X are not Lévy processes, such as Gaussian processes, exact formulas usually are very difficult to obtain. Some contributions such as [42, 43, 11] then focus on the asymptotic results.

For $X(t), t \geq 0$ a Gaussian process, the asymptotics of $\mathcal{K}_S(u,T)$ over finite-time horizon, i.e. $S \in (0,\infty)$, are investigated in [43]. Further, [42] showed the tail asymptotic results of $R_u(t)$ over infinite-time horizon, i.e. $S = \infty$ in (4.1), where $X(t)$ is a self-similar Gaussian process. In this paper considering the nature of the financial market, we introduce the force of interest δ into the model $R_u(t)$ as $R_u^\delta(t)$ in (1.1) when $X(t) = B(t)$. [11] gave an approximation

[1] This chapter is based on L. Bai (2018): Asymptotics of Parisian ruin of Brownian motion risk model over an infinite-time horizon, published in the *Scandinavian Actuarial Journal*, to appear.

of the Parisian ruin probability

$$\mathcal{K}_S^\delta(u,T) := \mathbb{P}\left\{ \inf_{t\in[0,S]} \sup_{s\in[t,t+T]} R_u^\delta(s) < 0 \right\}, \quad S \in (0,\infty),$$

as $u \to \infty$. See [130, 41, 86, 78] for more studies on risk models with force of interest. In the literature, no results are available for the approximation of Parisian ruin probability over infinite time horizon for $\delta > 0$. In this contribution we shall investigate the asymptotics of the Parisian ruin probability

$$\mathcal{K}^\delta(u,T) := \mathbb{P}\left\{ \inf_{t\geq 0} \sup_{s\in[t,t+T]} R_u^\delta(s) < 0 \right\},$$

as $u \to \infty$. The findings of this paper are mainly of theoretical relevance. Nonetheless, since we are able to derive the results for the Brownian motion setup, which is a benchmark model in actuarial practice and comes as the limiting model in approximations, our results have some importance also for actuarial practice and risk management.
When $T = 0$, according to [78] (see also [68]) we have

$$\mathcal{K}^\delta(u,0) = \Psi\left(\frac{\sqrt{2\delta}}{\sigma}\left(u+\frac{c}{\delta}\right)\right) \Big/ \Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right). \tag{4.2}$$

When $\delta = 0$ and $T \in [0,\infty)$, [42] showed that

$$\mathcal{K}^0(u,T) = \mathbb{P}\left\{ \inf_{t\geq 0} \sup_{s\in[t,t+T]} (u + cs - \sigma B(s)) < 0 \right\} \sim \mathcal{F}\left(\frac{2c^2T}{\sigma^2}\right) \exp\left(-\frac{2cu}{\sigma^2}\right), \ u \to \infty,$$

where

$$\mathcal{F}(T) = \lim_{\lambda\to\infty} \frac{1}{\lambda}\mathbb{E}\left\{ \sup_{t\in[0,\lambda]} \inf_{s\in[0,T]} e^{\sqrt{2}B(t+s)-(t+s)} \right\}.$$

Hereafter we make the convention that $\sup\{\emptyset\} = 0$ and $\inf\{\emptyset\} = \infty$.
Complementary, we investigate the conditional distribution of the ruin time for the surplus process $R_u^\delta(t)$. The classical ruin time, e.g., [41, 81, 91], is defined as

$$\tau(u) = \inf\{t > 0 : R_u^\delta(t) < 0\}. \tag{4.3}$$

Here as in [103, 42, 11] we define the Parisian ruin time of the risk process $R_u^\delta(t)$ by

$$\eta(u) = \inf\{t \geq T : t - \kappa_{t,u} \geq T, R_u^\delta(t) < 0\}, \quad \text{with } \kappa_{t,u} = \sup\{s \in [0,t] : R_u^\delta(s) \geq 0\}, \tag{4.4}$$

and $\eta(u) = \tau(u)$ when $T = 0$.
Brief outline of the rest of the paper: In Section 2 we present our main results on the asymptotics of $\mathcal{K}^\delta(u,T)$ and the approximation of the Parisian ruin time. All the proofs are relegated to Section 3.

4.2 Main results

Now we turn to our principal problem deriving below the asymptotic behaviour of $\mathcal{K}^\delta(u,T)$ as $u \to \infty$. For $\delta > 0$, setting

$$\widetilde{R}_u^\delta(s) = u + c\int_0^s e^{-\delta v}dv - \sigma\int_0^s e^{-\delta v}dB(v), \quad s \geq 0.$$

Since for $t \in (0,\infty)$

$$\mathbb{E}\left\{\left[\sigma\int_0^t e^{-\delta v}dB(v)\right]^2\right\} = \frac{\sigma^2}{2\delta}\left(1 - e^{-2\delta t}\right),$$

then

$$\sup_{t\in[0,\infty)} \mathbb{E}\left\{\left[\sigma\int_0^t e^{-\delta v}dB(v)\right]^2\right\}<\infty$$

implies that

$$\sup_{t\in[0,\infty)} \mathbb{E}\left\{\left|\sigma\int_0^t e^{-\delta v}dB(v)\right|\right\}<\infty,$$

by the martingale convergence theorem, see [112], $\widetilde{R}_u^\delta(\infty) := \lim_{t\to\infty}\widetilde{R}_u^\delta(t)$ exists and is finite almost surely. Thus for any $u>0$

$$\begin{aligned}\mathcal{K}^\delta(u,T) &= \mathbb{P}\left\{\inf_{t\in[0,\infty)}\sup_{s\in[t,t+T]}\widetilde{R}_u^\delta(s)<0\right\} = \mathbb{P}\left\{\inf_{t\in[0,\infty]}\sup_{s\in[t,t+T]}\widetilde{R}_u^\delta(s)<0\right\}\\ &= \mathbb{P}\left\{\sup_{t\in[0,\infty]}\inf_{s\in[t,t+T]}\left(\sigma\int_0^s e^{-\delta v}dB(v)-c\int_0^s e^{-\delta v}dv\right)>u\right\} =: \psi(u).\end{aligned}$$

Thus in the analysis of our main results, we consider $\psi(u)$.

Theorem 4.2.1. *For $\delta>0$ and $T\in[0,\infty)$, we have for any $\lambda>\frac{4c^2}{\delta^2}$*

$$\widetilde{\mathcal{P}}_a^f[0,\lambda] \le \lim_{u\to\infty}\mathcal{K}^\delta(u,T)\Big/\Psi\left(\frac{\sqrt{2}(\delta u+c)}{\sigma\sqrt{\delta}}\right) \le \left(\widetilde{\mathcal{P}}_a^f[0,\lambda]+\frac{2\lambda e^{\frac{c^2}{\sigma^2\delta}}}{e^{\frac{\delta\lambda}{4\sigma^2}}-1}\right), \tag{4.5}$$

and further letting $\lambda\to\infty$

$$\lim_{u\to\infty}\mathcal{K}^\delta(u,T)\Big/\Psi\left(\frac{\sqrt{2}(\delta u+c)}{\sigma\sqrt{\delta}}\right) = \widetilde{\mathcal{P}}_a^f[0,\infty),$$

where

$$\widetilde{\mathcal{P}}_a^f[0,\infty) := \lim_{\lambda\to\infty}\widetilde{\mathcal{P}}_a^f[0,\lambda] := \lim_{\lambda\to\infty}\mathbb{E}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[a,1]}\exp\left(\sqrt{2}B(st)-st-f(st)\right)\right\},$$

with $a=e^{-2\delta T}$ and $f(t)=t-\frac{2c}{\delta}\sqrt{t}$.

Remarks 4.2.1. i) By [13], when $\lambda\ge 0,\ a\in[0,1]$ and $f(t)$ is a continuous function satisfying $\lim_{t\to\infty}\frac{f(t)}{t^\epsilon}=\infty$ for some $\epsilon>0$, we have

$$\widetilde{\mathcal{P}}_a^f[0,\lambda]\in(0,\infty),\quad \widetilde{\mathcal{P}}_a^f[0,\infty)\in(0,\infty).$$

Note further that $\widetilde{\mathcal{P}}_0^f[0,\lambda]=e^{-f(0)}$ and

$$\widetilde{\mathcal{P}}_1^f[0,\lambda]=\mathbb{E}\left\{\sup_{t\in[0,\lambda]}\exp\left(\sqrt{2}B(t)-|t|-f(t)\right)\right\},$$

see e.g. [38, 13, 82] for more details of $\widetilde{\mathcal{P}}_a^f$.
ii) In Theorem 4.2.1, if $T=0,\ a=1$,we get the asymptotic result of the classical ruin probability, i.e., as $u\to\infty$

$$\begin{aligned}\mathcal{K}^\delta(u,0)=\mathbb{P}\left\{\inf_{t\ge 0}R_u^\delta(s)<0\right\} &\sim \mathbb{E}\left\{\sup_{t\in[0,\infty)}\exp\left(\sqrt{2}B(t)-t-f(t)\right)\right\}\Psi\left(\frac{\sqrt{2}(\delta u+c)}{\sigma\sqrt{\delta}}\right)\\ &= \widetilde{\mathcal{P}}_1^f[0,\infty)\Psi\left(\frac{\sqrt{2}(\delta u+c)}{\sigma\sqrt{\delta}}\right),\end{aligned} \tag{4.6}$$

which corresponds to the results in [12].

iii) Since $\mathcal{K}^\delta(u,T) \le \mathcal{K}^\delta(u,0)$, by (4.2) and Theorem 4.2.1 we have

$$\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right)\widetilde{\mathcal{P}}_a^f[0,\infty) \le 1.$$

Further, $T=0$ yields that

$$\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right)\widetilde{\mathcal{P}}_1^f[0,\infty) = 1.$$

iv) Since $\widetilde{\mathcal{P}}_a^f[0,\infty)$ is an expectation of the supremum of a process over an infinite time interval, it is difficult to simulate the exact result and (4.5) give the bounds over a finite time interval to simplify the simulation.

By (4.5), we get the upper and lower bounds of $\mathcal{K}^\delta(u,T)$. Table 4.1 is the simulated bounds of $\mathcal{K}^\delta(u,T)$. We notice that the bounds decrease when T increases. Moreover, $\widetilde{\mathcal{P}}_a^f[0,\lambda]$ is a decreasing function of T, and we get the same relation. In fact, when T is bigger, it means that we allow the surplus of a company to stay longer time under level zero before the ruin happens, thus the ruin probability should be decreasing. Further, we notice that the bounds decrease when δ increases. Since $\widetilde{\mathcal{P}}_a^f[0,\infty)$ is a decreasing function of $\delta>0$ and $\Psi\left(\frac{\sqrt{2}(\delta u+c)}{\sigma\sqrt{\delta}}\right)$ is decreasing when δ increases and $u \ge \frac{c}{\sqrt{2\delta}}$, the asymptotic of $\mathcal{K}^\delta(u,T)$ is also a decreasing function of δ. The effect of δ is not an intuitionistic result from the original risk model.

Table 4.1: The simulated bounds of $\mathcal{K}^\delta(u,T)$

u	c	σ	δ	T	λ	upper bound	lower bound
5	0.1	1	0.05	5	600	0.3760	0.3869
5	0.1	1	0.05	6	600	0.3657	0.3766
5	0.1	1	0.07	5	600	0.0489	0.0492
5	0.1	1	0.07	6	600	0.0392	0.0395
5	0.1	1	0.1	5	1000	0.0078	0.0078
5	0.1	1	0.1	6	1000	0.0073	0.0073
4	0.1	1	0.1	5	1000	0.0286	0.0286
4	0.1	1	0.1	6	1000	0.0258	0.0258

Next recall the Parisian ruin time $\eta(u)$ as in (4.4), and using the results in Theorem 4.2.1, we obtain the asymptotic conditional distribution of $\eta(u)$ as follows.

Theorem 4.2.2. *Let $\eta(u)$ satisfy (4.4). Under the assumptions and notation of Theorem 4.2.1, we have for $\delta>0$ and $x\in(0,\infty)$*

$$\lim_{u\to\infty}\mathbb{P}\left\{\eta(u) \le -\frac{1}{2\delta}\ln\left(t_u + xu^{-2}\right)\ \middle|\eta(u)<\infty\right\} = 1-\frac{\widetilde{\mathcal{P}}_a^f[0,\frac{\delta x}{\sigma^2}]}{\widetilde{\mathcal{P}}_a^f[0,\infty)}, \tag{4.7}$$

where $t_u = \left(\frac{c}{\delta u+c}\right)^2$.

Remarks 4.2.2. i) When $\delta=0$, [42] showed that for $x\in\mathbb{R}$

$$\lim_{u\to\infty}\mathbb{P}\left\{u^{-\frac{1}{2}}\left(\eta(u)-\frac{u}{c}\right)\le x\ \middle|\eta(u)<\infty\right\} = \Phi(cx),$$

where $\Phi(\cdot)$ denotes the distribution function of an $\mathcal{N}(0,1)$ random variable.
ii) When $T=0$, $\eta(u)=\tau(u)$, by (4.7), we have

$$\lim_{u\to\infty}\mathbb{P}\left\{u^2\left(e^{-2\delta\tau(u)}-\left(\frac{c}{\delta u+c}\right)^2\right)\le x\ \middle|\eta(u)<\infty\right\} = \frac{\widetilde{\mathcal{P}}_1^f[0,\frac{\delta x}{\sigma^2}]}{\widetilde{\mathcal{P}}_1^f[0,\infty)},$$

which corresponds to the result in [12].

4.3 Proofs

Hereafter we assume that $\mathbb{C}_i, i \in \mathbb{N}$ are positive constants.

PROOF OF THEOREM 4.2.1 Using a change of variable $s = -\frac{1}{2\delta}\ln s^*, s^* \in [t^* e^{-2\delta T}, t^*],\ t^* \in [0,1]$, we have

$$\begin{aligned}\psi(u) &= \mathbb{P}\left\{\sup_{t^*\in[0,1]}\inf_{s^*\in[t^*e^{-2\delta T},t^*]}\left(\sigma\int_0^{-\frac{1}{2\delta}\ln s^*} e^{-\delta v}dB(v) - c\int_0^{-\frac{1}{2\delta}\ln s^*} e^{-\delta v}dv\right) > u\right\}\\ &= \mathbb{P}\left\{\sup_{t^*\in[0,1]}\inf_{s^*\in[t^*e^{-2\delta T},t^*]} W(s^*) > u\right\},\end{aligned}$$

where

$$W(s) = \sigma\int_0^{-\frac{1}{2\delta}\ln s} e^{-\delta v}dB(v) - \frac{c}{\delta}(1 - s^{\frac{1}{2}}).$$

For simplicity, we still use s, t instead of s^*, t^*.

Below, we set $Z(s) = \sigma\int_0^{-\frac{1}{2\delta}\ln s} e^{-\delta v}dB(v)$ with variance function given by

$$V_Z^2(s) := \mathrm{var}\{Z(s)\} = \mathbb{E}\left\{(Z(s))^2\right\} = \mathbb{E}\left\{\sigma^2\int_0^{-\frac{1}{2\delta}\ln s} e^{-2\delta v}dv\right\} = \frac{\sigma^2}{2\delta}(1-s), \quad s\in[0,1].$$

We show next that for u sufficiently large

$$M_u(t) := \frac{uV_Z(t)}{G_u(t)} = \frac{\frac{\sigma}{\sqrt{2\delta}}\sqrt{1-t}}{1+\frac{c}{\delta u}(1-t^{1/2})}, \quad 0\le t\le 1,$$

with $G_u(t) := u + \frac{c}{\delta}(1-t^{\frac{1}{2}})$ attains its maximum at the unique point

$$t_u = \left(\frac{c}{\delta u + c}\right)^2.$$

In fact, we have for $t \in (0,1)$

$$\begin{aligned}[M_u(t)]_t := \frac{dM_u(t)}{dt} &= \frac{dV_Z(t)}{dt}\cdot\frac{u}{G_u(t)} - \frac{V_Z(t)}{G_u^2(t)}\left(-\frac{cu}{2\delta}t^{-\frac{1}{2}}\right)\\ &= \frac{u}{2G_u^2(t)V_z(t)}\left[\frac{dV_Z^2(t)}{dt}G_u(t) + V_Z^2(t)\frac{ct^{-\frac{1}{2}}}{\delta}\right]\\ &= \frac{u\sigma^2 t^{-1/2}}{4\delta G_u^2(t)V_Z(t)}\left[\frac{c}{\delta} - \left(u+\frac{c}{\delta}\right)t^{\frac{1}{2}}\right].\end{aligned} \tag{4.8}$$

Letting $[M_u(t)]_t = 0$, we get $t_u = \left(\frac{c}{\delta u + c}\right)^2$.

By (4.8), $[M_u(t)]_t > 0$ for $t \in (0, t_u)$ and $[M_u(t)]_t < 0$ for $t \in (t_u, 1)$, then t_u is the unique maximum point of $M_u(t)$ over $[0,1]$ and $t_u \to 0, u \to \infty$. Further

$$M_u := M_u(0) = \frac{\sigma\sqrt{\delta}u}{\sqrt{2}(\delta u + c)} = \frac{\sigma}{\sqrt{2\delta}}(1+o(1)),\ u\to\infty.$$

Set $\omega(u) = \left(\frac{\ln u}{u}\right)^2$, $\Delta(u) = [0, \omega(u)]$ and for a constant $\lambda > \frac{4c^2}{\delta^2}$

$$I_u(k) = \left[k\lambda u^{-2}, (k+1)\lambda u^{-2}\right],\ k\in\mathbb{N},\quad N(u) = \left\lfloor \lambda^{-1}(\ln u)^2\right\rfloor.$$

We have for u large enough

$$\Pi_0(u) \le \psi(u) \le \Pi_0(u) + \Pi_1(u) + \Pi_2(u) + \Pi_3(u), \tag{4.9}$$

where for $\theta \in (0,1)$

$$\Pi_0(u) = \mathbb{P}\left\{\sup_{t\in[0,\lambda u^{-2}]} \inf_{s\in[te^{-2\delta T},t]} W(s) > u\right\}, \quad \Pi_1(u) = \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_u(k)} \inf_{s\in[te^{-2\delta T},t]} W(s) > u\right\},$$
$$\Pi_2(u) = \mathbb{P}\left\{\sup_{t\in[\omega(u),\theta]} \inf_{s\in[te^{-2\delta T},t]} W(s) > u\right\}, \quad \Pi_3(u) = \mathbb{P}\left\{\sup_{t\in[\theta,1]} \inf_{s\in[te^{-2\delta T},t]} W(s) > u\right\}.$$

First we show the asymptotic of $\Pi_0(u)$. For u large enough

$$\Pi_0(u) = \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \inf_{s\in[a,1]} \overline{Z}(stu^{-2})\frac{M_u(stu^{-2})}{M_u} > \frac{u}{M_u}\right\}$$

where $\overline{Z}(t) = \frac{Z(t)}{V_Z(t)}$ and $a = e^{-2\delta T}$.
By Appendix, we have

$$\lim_{u\to\infty} \sup_{\substack{t\in[0,(\ln u)^2]\\ s\in[0,1]}} \left|\left(1 - \frac{M_u(stu^{-2})}{M_u}\right)u^2 - \frac{1}{2}f(st)\right| = 0. \tag{4.10}$$

where $f(t) = t - \frac{2c}{\delta}\sqrt{t}$. For $0 \le t_1 \le t_2 < 1$, the correlation function of $Z(t)$ equals

$$\begin{aligned} r(t_1,t_2) &= \frac{\mathbb{E}\left\{(\sigma\int_0^{-\frac{1}{2\delta}\ln t_1} e^{-\delta v}dB(v))(\sigma\int_0^{-\frac{1}{2\delta}\ln t_2} e^{-\delta v}dB(v))\right\}}{\sqrt{\frac{\sigma^2}{2\delta}(1-t_1)}\sqrt{\frac{\sigma^2}{2\delta}(1-t_2)}} \\ &= \frac{\sqrt{1-t_2}}{\sqrt{1-t_1}} = 1 - \frac{t_2-t_1}{\sqrt{1-t_1}(\sqrt{1-t_1}+\sqrt{1-t_2})}, \end{aligned} \tag{4.11}$$

which implies that

$$\begin{aligned} \sup_{t_1,t_2\in\Delta(u),t_1\neq t_2}\left|\frac{1-r(t_1,t_2)}{\frac{1}{2}|t_1-t_2|} - 1\right| &= \sup_{t_1,t_2\in\Delta(u),t_1\neq t_2}\left|\frac{2}{\sqrt{1-t_1}(\sqrt{1-t_1}+\sqrt{1-t_2})} - 1\right| \\ &\le \frac{1}{1-(\frac{c}{c+\delta u})^2 - (\frac{\ln u}{u})^2} - 1 \\ &\to 0, \quad u\to\infty. \end{aligned} \tag{4.12}$$

By Appendix, for $t_1,t_2 \in [0,\lambda]$ and $s_1,s_2 \in [0,1]$

$$\begin{aligned} \lim_{u\to\infty} u^2\mathrm{var}\left(\overline{Z}(s_1t_1u^{-2})\frac{M_u(s_1t_1u^{-2})}{M_u} - \overline{Z}(s_2t_2u^{-2})\frac{M_u(s_2t_2u^{-2})}{M_u}\right) &= |s_1t_1 - s_2t_2| \\ &= 2\mathrm{var}\left(\frac{1}{\sqrt{2}}B(s_1t_1) - \frac{1}{\sqrt{2}}B(s_2t_2)\right) \end{aligned} \tag{4.13}$$

For some small $\theta\in(0,1)$, by (4.11) we obtain that for $t_1,t_2\in[0,\theta]$

$$\mathbb{E}\left(\overline{Z}(t_1) - \overline{Z}(t_2)\right)^2 = 2 - 2r(t_1,t_2) \le \mathbb{C}_1|t_1-t_2| \tag{4.14}$$

holds. By (4.10), (4.12),(4.13), (4.14) and Lemma 5.1 in [43], as $u\to\infty$,

$$\Pi_0(u) \sim \mathbb{E}\left\{\sup_{t\in[0,\lambda]} \inf_{s\in[a,1]} \exp\left(\zeta(st)\right)\right\}\Psi\left(\frac{u}{M_u}\right), \tag{4.15}$$

where

$$\zeta(t) = \frac{\sqrt{2\delta}}{\sigma}B(t) - \frac{\delta}{\sigma^2}t - \frac{\delta}{\sigma^2}f(t).$$

Next we show that as $u \to \infty$,

$$\Pi_1(u) = o\left(\Pi_0(u)\right), \quad \Pi_2(u) = o\left(\Pi_0(u)\right), \quad \text{and} \quad \Pi_3(u) = o\left(\Pi_0(u)\right).$$

Let $Y(t), t \in \mathbb{R}$ be a stationary Gaussian process with continuous trajectories, unit variance and correlation function satisfying for a constant $\varepsilon_3 \in (0, \frac{1}{2})$

$$r_Y(t) = \exp\left(-\frac{(1+\varepsilon_3)}{2}|t|\right).$$

By (4.10) and Slepian inequality in [119], we have

$$\begin{aligned}
\Pi_1(u) &\le \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_u(k)} W(t) > u\right\}\\
&\le \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_u(k)} \overline{Z}(t) > \mathcal{A}_u(k)\right\}\\
&\le \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_u(k)} Y(t) > \mathcal{A}_u(k)\right\}\\
&= \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in [0,\lambda]} Y(u^{-2}t) > \mathcal{A}_u(k)\right\}
\end{aligned}$$

where $\mathcal{A}_u(k) := \frac{u}{M_u}\left(1 + \frac{1-\varepsilon_2}{2u^2}\left(k\lambda - \frac{2c}{\delta}\sqrt{k\lambda}\right)\right)$ and $\varepsilon_2 \in (0,1)$ is a small constant. We observe that

$$\inf_{1\le k\le N(u)} \mathcal{A}_u(k) \ge \frac{u}{M_u} \to \infty, \ u \to \infty. \tag{4.16}$$

Further,

$$\begin{aligned}
&\lim_{u\to\infty} \sup_{1\le k\le N(u)} \sup_{\substack{t_1\neq t_2,\\ t_1,t_2\in[0,\lambda]}} \left|\mathcal{A}_u^2(k)\frac{\mathrm{var}\left(Y(u^{-2}t_1) - Y(u^{-2}t_2)\right)}{\frac{2\delta(1+\varepsilon_3)}{\sigma^2}|t_1-t_2|} - 1\right|\\
&= \lim_{u\to\infty} \sup_{1\le k\le N(u)} \sup_{\substack{t_1\neq t_2,\\ t_1,t_2\in[0,\lambda]}} \left|\mathcal{A}_u^2(k)\frac{2-2r_Y(u^{-2}t_1 - u^{-2}t_2)}{\frac{2\delta(1+\varepsilon_3)}{\sigma^2}|t_1-t_2|} - 1\right|\\
&= 0,
\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}
&\sup_{1\le k\le N(u)} \sup_{\substack{|t_1-t_2|<\epsilon\\ t_1,t_2\in[0,\lambda]}} \mathcal{A}_u^2(k)\mathbb{E}\left\{\left(Y(u^{-2}t_1) - Y(u^{-2}t_2)\right)Y(0)\right\}\\
&\le \mathbb{C}_2 u^2 \sup_{\substack{|t_1-t_2|<\epsilon\\ t_1,t_2\in[0,\lambda]}} \left|r_Y(u^{-2}t_1) - r_Y(u^{-2}t_2)\right|\\
&\le \mathbb{C}_3 u^2 \sup_{\substack{|t_1-t_2|<\epsilon\\ t_1,t_2\in[0,\lambda]}} \left|\frac{1+\varepsilon_3}{2}u^{-2}(t_1-t_2)\right|\\
&\le \mathbb{C}_4 \sup_{\substack{|t_1-t_2|<\epsilon\\ t_1,t_2\in[0,\lambda]}} |t_1-t_2| \to 0, \quad u\to\infty, \ \epsilon\to 0.
\end{aligned} \tag{4.18}$$

According to (4.16), (4.17), (4.18) and Lemma 5.3 of [45], we have as $u \to \infty, \varepsilon_2 \to 0$,

$$\begin{aligned}
\Pi_1(u) &\sim \lambda \sum_{k=1}^{N(u)} \Psi\left(\mathcal{A}_u(k)\right)\\
&\le \lambda \sum_{k=1}^{N(u)} \frac{1}{\sqrt{2\pi}\mathcal{A}_u(k)} e^{-\frac{\mathcal{A}_u^2(k)}{2}}
\end{aligned}$$

$$\begin{aligned}
&\le \lambda \sum_{k=1}^{N(u)} \frac{M_u}{\sqrt{2\pi}u} \exp\left(-\frac{u^2}{2M_u^2}\left(1+\frac{1-\varepsilon_2}{u^2}\left(k\lambda - \frac{2c}{\delta}\sqrt{k\lambda}\right) - \frac{2\varepsilon_2}{u^2}\right)\right) \\
&\sim \lambda\Psi\left(\frac{u}{M_u}\right) e^{\frac{\varepsilon_2}{M_u^2}} \sum_{k=1}^{N(u)} \exp\left(-\frac{1-\varepsilon_2}{2M_u^2}\left(k\lambda - \frac{2c}{\delta}\sqrt{k\lambda}\right)\right) \\
&\le 2\Psi\left(\frac{u}{M_u}\right)\lambda \sum_{k=1}^{\infty} e^{-\frac{\delta k}{4\sigma^2}\lambda + \frac{c^2}{\sigma^2\delta}} \\
&= \frac{2\lambda e^{\frac{c^2}{\sigma^2\delta}}}{e^{\frac{\delta\lambda}{4\sigma^2}}-1}\Psi\left(\frac{u}{M_u}\right). \qquad (4.19)
\end{aligned}$$

Moreover, for all u large

$$\begin{aligned}
\frac{1}{M_u(t)} - \frac{1}{M_u} &\ge \frac{[G_u(t)V_Z(0)]^2 - [G_u(0)V_Z(t)]^2}{2uV_Z^3(0)G_u(0)} \\
&= \frac{\frac{\sigma^2}{2\delta}\left\{\left[\left(u+\frac{c}{\delta}\right)^2 + \left(\frac{c}{\delta}\right)^2\right]t - 2\left(u+\frac{c}{\delta}\right)\frac{c}{\delta}\sqrt{t}\right\}}{2u(\frac{\sigma^2}{2\delta})^{3/2}(u+\frac{c}{\delta})} \\
&\ge \mathbb{C}_8 t \\
&\ge \mathbb{C}_8 \frac{(\ln u)^2}{u^2}
\end{aligned}$$

holds for any $t \in [\omega(u), \theta]$, therefore

$$\sup_{t\in[\omega(u),\theta]} M_u(t) \le \left(\frac{1}{M_u} + \mathbb{C}_8\frac{(\ln u)^2}{u^2}\right)^{-1}.$$

Thus the above inequality combined with (4.14) and Theorem 8.1 in [119] derives that

$$\begin{aligned}
\Pi_2(u) &\le \mathbb{P}\left\{\sup_{t\in[\omega(u),\theta]} \overline{Z}(t)M_u(t) > u\right\} \\
&\le \mathbb{P}\left\{\sup_{t\in[0,\theta]} \overline{Z}(t) > u\left(\frac{1}{M_u} + \mathbb{C}_8\frac{(\ln u)^2}{u^2}\right)\right\} \\
&\le \mathbb{C}_9 u^2 \Psi\left(u\left(\frac{1}{M_u} + \mathbb{C}_8\frac{(\ln u)^2}{u^2}\right)\right) \\
&= o\left(\Psi\left(\frac{u}{M_u}\right)\right), \quad u\to\infty. \qquad (4.20)
\end{aligned}$$

Finally, since

$$\sup_{t\in[\theta,1]} V_Z^2(t) \le \frac{\sigma^2}{2\delta}(1-\theta), \quad \text{and} \quad \mathbb{E}\left\{\sup_{t\in[\theta,1]} Z(t)\right\} \le \mathbb{C}_{10} < \infty,$$

by Borell inequality in [1]

$$\Pi_3(u) \le \mathbb{P}\left\{\sup_{t\in[\theta,1]} Z(t) > u\right\} \le \exp\left(-\frac{\delta(u-\mathbb{C}_{10})^2}{\sigma^2(1-\theta)}\right) = o\left(\Psi\left(\frac{u}{M_u}\right)\right), \quad u\to\infty, \qquad (4.21)$$

which combined with (4.9), (4.15), (4.19) and (4.20) shows that when u large enough for any $\lambda > \frac{4c^2}{\delta^2}$

$$\begin{aligned}
\psi(u) &\ge \mathbb{E}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[a,1]} \exp\left(\zeta(st)\right)\right\}\Psi\left(\frac{u}{M_u}\right), \\
\psi(u) &\le \left(\mathbb{E}\left\{\sup_{t\in[0,\lambda]}\inf_{s\in[a,1]} \exp\left(\zeta(st)\right)\right\} + \frac{2\lambda e^{\frac{c^2}{\sigma^2\delta}}}{e^{\frac{\delta\lambda}{4\sigma^2}}-1}\right)\Psi\left(\frac{u}{M_u}\right).
\end{aligned}$$

Consequently, letting $\lambda \to \infty$, we have

$$\begin{aligned}\psi(u) &\sim \mathbb{E}\left\{ \sup_{t\in[0,\infty)} \inf_{s\in[a,1]} \exp\left(\zeta(st)\right)\right\} \Psi\left(\frac{\sqrt{2}(\delta u + c)}{\sigma\sqrt{\delta}}\right) \\ &= \mathbb{E}\left\{ \sup_{t\in[0,\infty)} \inf_{s\in[a,1]} \exp\left(\sqrt{2}B(st) - st - f(st)\right)\right\} \Psi\left(\frac{\sqrt{2}(\delta u + c)}{\sigma\sqrt{\delta}}\right), \quad u\to\infty.\end{aligned}$$

□

Proof of Theorem 4.2.2 We use the same notation as in the proof of Theorem 4.2.1. For $x \in (0,\infty)$ and $u > 0$

$$\begin{aligned}&\mathbb{P}\left\{ u^2\left(e^{-2\delta\eta(u)} - \left(\frac{c}{\delta u + c}\right)^2\right) \le x \middle| \eta(u) < \infty\right\} \\ &= \frac{\mathbb{P}\left\{\inf_{t\in[-\frac{1}{2\delta}\ln(u^{-2}x),\infty)} \sup_{s\in[t,t+T]} \widetilde{R}_u^\delta(s) < 0\right\}}{\mathbb{P}\left\{\inf_{t\in[0,\infty)} \sup_{s\in[t,t+T]} \widetilde{R}_u^\delta(s) < 0\right\}} \\ &= \frac{\mathbb{P}\left\{\sup_{t^*\in[0,u^{-2}x]} \inf_{s^*\in[t^*e^{-2\delta T},t^*]} W(s^*) > u\right\}}{\mathbb{P}\left\{\sup_{t^*\in[0,1]} \inf_{s^*\in[t^*e^{-2\delta T},t^*]} W(s^*) > u\right\}} \\ &=: \frac{\psi_x(u)}{\psi(u)}.\end{aligned}$$

For $\psi_x(u)$, using the similar argumentation about $\Pi_0(u)$ as in the proof of Theorem 4.2.1 with $\lambda = x$, we obtain

$$\begin{aligned}\psi_x(u) &\sim \mathbb{E}\left\{ \sup_{t\in[0,\infty)} \inf_{s\in[a,1]} \exp\left(\zeta(st)\right)\right\} \Psi\left(\frac{\sqrt{2}(\delta u + c)}{\sigma\sqrt{\delta}}\right) \\ &= \mathbb{E}\left\{ \sup_{t\in[0,\frac{\delta x}{\sigma^2}]} \inf_{s\in[a,1]} \exp\left(\sqrt{2}B(st) - st - f(st)\right)\right\} \Psi\left(\frac{\sqrt{2}(\delta u + c)}{\sigma\sqrt{\delta}}\right), \quad u\to\infty.\end{aligned}$$

Then we have

$$\lim_{u\to\infty} \mathbb{P}\left\{ u^2\left(e^{-2\delta\eta_u} - \left(\frac{c}{\delta u + c}\right)^2\right) \le x \middle| \eta_u < \infty\right\} = \lim_{u\to\infty} \frac{\psi_x(u)}{\psi(u)} = \frac{\widetilde{\mathcal{P}}_a^f[0,\frac{\delta x}{\sigma^2}]}{\widetilde{\mathcal{P}}_a^f[0,\infty)}.$$

Thus the final result follows.

□

4.4 Some Technical Results

This section is dedicated to the proof of (4.10) and (4.13).
The proof of (4.10):
We have

$$1 - \frac{M_u(t)}{M_u} = \frac{[G_u(t)V_Z(0)]^2 - [G_u(0)V_Z(t)]^2}{G_u(t)V_Z(0)[V_Z(t)G_u(0) + G_u(t)V_Z(0)]}$$

and

$$\begin{aligned}[G_u(t)V_Z(0)]^2 - [G_u(0)V_Z(t)]^2 &= \left[\left(u + \frac{c}{\delta}\right) - \frac{c}{\delta}\sqrt{t}\right]^2 \frac{\sigma^2}{2\delta} - \left(u + \frac{c}{\delta}\right)^2 \frac{\sigma^2}{2\delta}(1-t) \\ &= \left(u + \frac{c}{\delta}\right)^2 \frac{\sigma^2}{2\delta}t - 2\left(u + \frac{c}{\delta}\right)\frac{c\sigma^2}{2\delta^2}\sqrt{t} + \frac{c^2\sigma^2}{2\delta^3}t \\ &= \frac{\sigma^2}{2\delta}\left\{\left[\left(u + \frac{c}{\delta}\right)^2 + \left(\frac{c}{\delta}\right)^2\right]t - 2\left(u + \frac{c}{\delta}\right)\frac{c}{\delta}\sqrt{t}\right\}.\end{aligned}$$

Since for any $t \in \Delta(u)$

$$\sqrt{\frac{\sigma^2}{2\delta}(1-\delta(u))} \le V_Z(t) \le \sqrt{\frac{\sigma^2}{2\delta}}, \quad u + \frac{c}{\delta} - \frac{c}{\delta}\sqrt{\delta(u)} \le G_u(t) \le u + \frac{c}{\delta},$$

then for all large u

$$V_Z(0)G_u(t)[G_u(t)V_Z(0) + V_Z(t)G_u(0)] \le \frac{\sigma^2}{\delta}\left(u + \frac{c}{\delta}\right)^2$$

and

$$\begin{aligned} V_Z(0)G_u(t)[G_u(t)V_Z(0) + V_Z(t)G_u(0)] &\ge \frac{\sigma^2}{\delta}(1-\delta(u))\left(u + \frac{c}{\delta} - \frac{c}{\delta}\sqrt{\delta(u)}\right)^2 \\ &\ge \frac{\sigma^2}{\delta}\left[\left(u + \frac{c}{\delta}\right)^2 - u\right]. \end{aligned}$$

Setting $f(t) = t - \frac{2c}{\delta}\sqrt{t}$, we have for $t \in [0, (\ln u)^2],\ s \in [0,1]$

$$\begin{aligned} &\left(1 - \frac{M_u(stu^{-2})}{M_u}\right)u^2 - \frac{1}{2}f(st) \\ &\le \frac{\frac{1}{2}\left\{\left[\left(u+\frac{c}{\delta}\right)^2 + \left(\frac{c}{\delta}\right)^2\right]st - \frac{2c}{\delta}\left(u+\frac{c}{\delta}\right)u\sqrt{st}\right\}}{\left(u+\frac{c}{\delta}\right)^2 - u} - \frac{1}{2}\left(st - \frac{2c}{\delta}\sqrt{st}\right) \\ &= \frac{1}{2}\frac{\left(\left(\frac{c}{\delta}\right)^2 + u\right)st + \frac{2c}{\delta}\sqrt{st}\left(\left(u+\frac{c}{\delta}\right)^2 - u - (u+\frac{c}{\delta})u\right)}{\left(u+\frac{c}{\delta}\right)^2 - u} \\ &\le \frac{1}{2}\frac{\left(\left(\frac{c}{\delta}\right)^2 + u\right)(\ln u)^2 + \frac{2c\ln u}{\delta}\left(|\frac{c}{\delta} - 1|u + \frac{c^2}{\delta^2}\right)}{\left(u+\frac{c}{\delta}\right)^2 - u} \to 0, \quad u \to \infty. \end{aligned}$$

and

$$\begin{aligned} &\left(1 - \frac{M_u(stu^{-2})}{M_u}\right)u^2 - \frac{1}{2}f(st) \\ &\ge \frac{\frac{1}{2}\left\{\left[\left(u+\frac{c}{\delta}\right)^2 + \left(\frac{c}{\delta}\right)^2\right]st - \frac{2c}{\delta}\left(u+\frac{c}{\delta}\right)u\sqrt{st}\right\}}{\left(u+\frac{c}{\delta}\right)^2} - \frac{1}{2}\left(st - \frac{2c}{\delta}\sqrt{st}\right) \\ &= \frac{1}{2}\frac{\left(\frac{c}{\delta}\right)^2 st + \frac{2c}{\delta}\sqrt{st}\left(\left(u+\frac{c}{\delta}\right)^2 - (u+\frac{c}{\delta})u\right)}{\left(u+\frac{c}{\delta}\right)^2} \\ &= \frac{1}{2}\frac{\left(\frac{c}{\delta}\right)^2 st + \frac{2c\sqrt{st}}{\delta}\left(\frac{c}{\delta}u + \frac{c^2}{\delta^2}\right)}{\left(u+\frac{c}{\delta}\right)^2} \ge 0. \end{aligned}$$

Then (4.10) follows.

The proof of (4.13):

For $t_1, t_2 \in [0, \lambda]$ and $s_1, s_2 \in [0,1]$

$$\begin{aligned} &u^2 \mathrm{var}\left(\overline{Z}(s_1t_1u^{-2})\frac{M_u(s_1t_1u^{-2})}{M_u} - \overline{Z}(s_2t_2u^{-2})\frac{M_u(s_2t_2u^{-2})}{M_u}\right) \\ &= \frac{u^2}{M_u^2}\mathbb{E}\left\{\frac{Z(s_1t_1u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})} - \frac{Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})}\right\}^2 \\ &= \frac{u^2}{M_u^2}\mathbb{E}\left\{\frac{Z(s_1t_1u^{-2}) - Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})} + \frac{Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})} - \frac{Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})}\right\}^2 \\ &= \frac{u^2}{M_u^2}\left(J_1(u) + J_2(u) + J_3(u)\right), \end{aligned}$$

where

$$J_1(u) = \mathbb{E}\left\{\left(\frac{Z(s_1t_1u^{-2}) - Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})}\right)^2\right\},$$

$$J_2(u) = 2\left(\frac{1}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})} - \frac{1}{1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})}\right)\mathbb{E}\left\{\frac{(Z(s_1t_1u^{-2}) - Z(s_2t_2u^{-2}))Z(s_2t_2u^{-2})}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})}\right\}$$
$$= 0,$$

$$J_3(u) = \left(\frac{1}{1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})} - \frac{1}{1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})}\right)^2\mathbb{E}\left\{\left(Z(s_2t_2u^{-2})\right)^2\right\}.$$

Since for $t_1, t_2 \in [0, \lambda]$ and $s_1, s_2 \in (0, 1]$

$$\begin{aligned}
\lim_{u\to\infty}\frac{u^2}{M_u^2}J_1(u) &= \lim_{u\to\infty}\frac{u^2}{M_u^2(1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}}))^2}\mathbb{E}\left\{\left(Z(s_1t_1u^{-2}) - Z(s_2t_2u^{-2})\right)^2\right\} \\
&= \lim_{u\to\infty}\frac{u^2}{M_u^2(1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}}))^2}\left(\sigma^2\int_{-\frac{1}{2\delta}\ln(s_2t_2u^{-2})}^{-\frac{1}{2\delta}\ln(s_1t_1u^{-2})} e^{-2\delta v}dv\right) \\
&= \lim_{u\to\infty}\frac{\sigma^2|s_2t_2 - s_1t_1|}{2\delta M_u^2(1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}}))^2} \\
&= |s_1t_1 - s_2t_2|, \\
\lim_{u\to\infty}\frac{u^2}{M_u^2}J_3(u) &= \lim_{u\to\infty}\frac{\sigma^2(1-s_2t_2u^{-2})u^2}{2\delta M_u^2}\left(\frac{\frac{c}{\delta u}\left(\sqrt{s_1t_1u^{-2}} - \sqrt{s_2t_2u^{-2}}\right)}{\left(1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})\right)\left(1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})\right)}\right)^2 \\
&= \lim_{u\to\infty}\frac{\sigma^2(1-s_2t_2u^{-2})}{2\delta M_u^2u^2}\left(\frac{\frac{c}{\delta}\left(\sqrt{s_1t_1} - \sqrt{s_2t_2}\right)}{\left(1+\frac{c}{\delta u}(1-\sqrt{s_1t_1u^{-2}})\right)\left(1+\frac{c}{\delta u}(1-\sqrt{s_2t_2u^{-2}})\right)}\right)^2 \\
&= 0.
\end{aligned}$$

Thus we have (4.13).

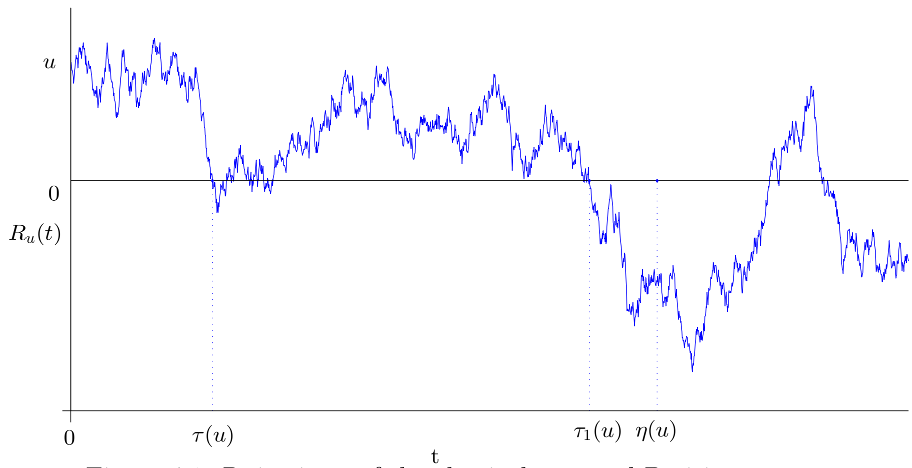


Figure 4.1: Ruin times of the classical case and Parisian case

Figure 4.1 shows the classical ruin time $\tau(u)$ and Parisian ruin time $\eta(u)$ of a surplus process $R_u(t)$ where $\eta(u) - \tau_1(u) = T$ is the pre-specified time under level zero.

Chapter 5

Extremes of $\alpha(t)$-Locally Stationary Gaussian Processes with Non-Constant Variances[1]

5.1 Introduction and Main Result

For $X(t),\ t \in [0,T],\ T>0$ a centered stationary Gaussian process with unit variance and continuous sample paths Pickands derived in [116] that

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \sim T\mathcal{H}_\alpha a^{1/\alpha} u^{2/\alpha} \mathbb{P}\left\{X(0) > u\right\}, \qquad u \to \infty, \tag{5.1}$$

provided that the correlation function r satisfies (2.77) and

$$r(t) < 1,\ \forall\, t \neq 0, \tag{5.2}$$

with $\alpha \in (0,2]$.

The deep contribution [18] introduced the class of locally stationary Gaussian processes with index α, i.e., a centered Gaussian process $X(t), t \in [0,T]$ with a constant variance function, say equal to 1, and correlation function satisfying (2.17).

Clearly, the class of locally stationary Gaussian processes includes the stationary ones. It allows for some minor fluctuations of dependence at t and at the same time keeps stationary structure at the local scale. See [18, 21, 87] for studies on the locally stationary Gaussian processes with index α.

In [49] the tail asymptotics of the supremum of $\alpha(t)$-locally stationary Gaussian processes are investigated. Such processes and random fields are of interest in various applications, see [49] and the recent contributions [8, 83, 85]. Following the definition in [49], a centered Gaussian process $X(t), t \in [0,T]$ with continuous sample paths and unit variance is $\alpha(t)$-locally stationary if the correlation function $r(\cdot,\cdot)$ satisfies the following conditions:

(i) $\alpha(t) \in C([0,T])$ and $\alpha(t) \in (0,2]$ for all $t \in [0,T]$;

(ii) $a(t) \in C([0,T])$ and $0 < \inf\{a(t) : t \in [0,T]\} \leq \sup\{a(t) : t \in [0,T]\} < \infty$;

(iii) uniformly for $t \in [0,T]$

$$1 - r(t,t+h) = a(t)|h|^{\alpha(t)} + o(|h|^{\alpha(t)}),\ h \to 0,$$

where $f(t) \in C(\mathcal{T})$ means that $f(t)$ is continuous on $\mathcal{T} \subset \mathbb{R}$.

In this paper, we shall consider the case that the variance function $\sigma^2(t) = Var(X(t))$ is not constant, assuming

[1] This chapter is based on L. Bai (2017): Extremes of $\alpha(t)$-Locally Stationary Gaussian Processes with Non-Constant Variances, published in the *Journal of Mathematical Analysis and Applications,* Volume 446, 248-263.

instead that:
(iv) $\sigma(t)$ attains its maximum equal to 1 over $[0,T]$ at the unique point $t_0 \in [0,T]$ and for some constants $c, \gamma > 0$,

$$\frac{1}{\sigma(t)} = 1 + ce^{-|t-t_0|^{-\gamma}}(1+o(1)), \quad t \to t_0.$$

A crucial assumption in our result is that similar to the variance function, the function $\alpha(t)$ has a certain behaviour around the extreme point t_0. Specifically, as in [49] we shall assume:
(v) there exist $\beta, \delta, \ b > 0$ such that

$$\alpha(t+t_0) = \alpha(t_0) + b|t|^\beta + o(|t|^{\beta+\delta}), \quad t \to 0.$$

Remark. We remark that t_0 does not need to be the unique point such that $\alpha(t)$ is minimal on $[0,T]$, which is different from [49]. For instance, $[0,T] = [0,2\pi]$, $t_0 = 0$ and $\alpha(t) = 1 + \frac{1}{2}\sin(t)$, then 0 is not the minimum point of $\alpha(t)$ over $[0,2\pi]$ which means assumptions about $\alpha(t)$ in [49] are not satisfied but assumption (v) here is satisfied with

$$\alpha(t) = 1 + \frac{1}{2}|t| + o(|t|^{\frac{3}{2}}), \ t \to 0.$$

Below we set $\alpha := \alpha(t_0)$, $a := a(t_0)$ and define $0^a = \infty$ for $a < 0$. Our main result is stated in the next theorem.

Theorem 5.1.1. *If a centered Gaussian process $X(t), t \in [0,T]$ with continuous sample paths is such that the assumptions (i)-(v) are valid, then we have as $u \to \infty$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \sim \widehat{I} a^{1/\alpha} \mathcal{H}_\alpha u^{2/\alpha} (\ln u)^{-\frac{1}{\gamma\wedge\beta}} \Psi(u) \begin{cases} 2^{-1/\gamma}, & \text{if } \gamma < \beta, \\ \int_0^{2^{-1/\gamma}} e^{\frac{-2bx^\beta}{\alpha^2}} dx, & \text{if } \gamma = \beta, \\ \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}} dx, & \text{if } \gamma > \beta, \end{cases}$$

where $\gamma \wedge \beta = \min(\gamma, \beta)$ and

$$\widehat{I} = \begin{cases} 1, & \text{if } t_0 = 0 \text{ or } t_0 = T, \\ 2, & \text{if } t_0 \in (0,T). \end{cases}$$

Remark. i) If $\alpha(t) \equiv \alpha$ for all t in a small neighborhood of t_0, the asymptotic of $\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\}$ is the same as in the case of $\gamma < \beta$ in Theorem 5.1.1.
ii) The result of case $\gamma > \beta$ in Theorem 5.1.1 is the same as the $\alpha(t)$-locally stationary scenario in [49], which means that $\sigma(t)$ varies so slow in a small neighborhood of t_0 that $X(t)$ can be considered as $\alpha(t)$-locally stationary in this small neighborhood.

The following example is a straightforward application of Theorem 5.1.1.

Example 5.1.1. Here we consider a multifractional Brownian motion $B_{H(t)}(t), \ t \geq 0$, i.e., a centered Gaussian process with covariance function

$$\mathbb{E}\left\{B_{H(t)}(t) B_{H(s)}(s)\right\} = \frac{1}{2} D(H(s)+H(t)) \left[|s|^{H(s)+H(t)} + |t|^{H(s)+H(t)} - |t-s|^{H(s)+H(t)}\right],$$

where $D(x) = \frac{2\pi}{\Gamma(x+1)\sin\left(\frac{\pi x}{2}\right)}$ and $H(t)$ is a Hölder function of exponent λ such that $0 < H(t) < \min(1,\lambda)$ for $t \in [0,\infty)$. For constants T_1, T_2 with $0 < T_1 < T_2$, define

$$\overline{B}_{H(t)}(t) := \frac{B_{H(t)}(t)}{\sqrt{\operatorname{var}(B_{H(t)}(t))}}, \ t \in [T_1, T_2],$$

and

$$\sigma(t) := 1 - e^{-|t-t_0|^{-\gamma}}, \ t \in [T_1, T_2],$$

with some $t_0 \in (T_1, T_2)$ and $\gamma > 0$.

By [49], $\overline{B}_{H(t)}(t),\ t\in[T_1,T_2]$, is a $2H(t)$-locally stationary Gussian process with correlation function

$$r(t,t+h)=1-\frac{1}{2}t^{-2H(t)}|h|^{2H(t)}+o(|h|^{2H(t)}),\quad h\to 0.$$

Further, we assume that there exist $\beta,\delta,b>0$ such that $H(t+t_0)=H(t_0)+bt^\beta+o(t^{\beta+\delta})$, as $t\to 0$. Then

$$\mathbb{P}\left\{\sup_{t\in[T_1,T_2]}\sigma(t)\overline{B}_{H(t)}(t)>u\right\}\sim 2^{1-1/2H}\frac{\mathcal{H}_{2H}}{t_0}u^{1/H}(\ln u)^{-\frac{1}{\gamma\wedge\beta}}\Psi(u)\begin{cases}2^{-1/\gamma}, & \text{if } \gamma<\beta,\\ \int_0^{2^{-1/\gamma}}e^{\frac{-bx^\beta}{H^2}}dx, & \text{if } \gamma=\beta,\\ \int_0^\infty e^{\frac{-bx^\beta}{H^2}}dx, & \text{if } \gamma>\beta,\end{cases}\quad u\to 0.$$

with $H:=H(t_0)$.

5.2 Proofs

In the rest of the paper, we focus on the case when $t_0=0$. The complementary scenario when $t_0\in(0,T]$ follows by analogous argumentation.

Lemma 5.2.1. *Under the assumptions of Theorem 5.1.1 we have*

$$\mathbb{P}\left\{\sup_{t\in[0,T]}X(t)>u\right\}\sim\mathbb{P}\left\{\sup_{t\in[0,\delta_1(u)]}X(t)>u\right\},\ u\to\infty. \tag{5.3}$$

Moreover, there exists a constant $C>0$ such that for all sufficiently large u

$$\mathbb{P}\left\{\sup_{t\in[\delta_2(u),T]}X(t)>u\right\}\le CTu^{2/\alpha}(\ln u)^{-4/3\beta}\Psi\left(u\right), \tag{5.4}$$

where for some constant $q>1$

$$\delta_1(u)=\left(\frac{1}{2\ln u-q\ln\ln u}\right)^{1/\gamma}\quad \textit{and}\quad \delta_2(u)=\left(\frac{\alpha^2(\ln(\ln u))}{\beta(\ln u)}\right)^{1/\beta}. \tag{5.5}$$

By (5.4), in the proof of Theorem 5.1.1, we derive that, as $u\to\infty$,

$$\mathbb{P}\left\{\sup_{t\in[\delta_2(u),T]}X(t)>u\right\}=o\left(\mathbb{P}\left\{\sup_{t\in[0,\delta_2(u)]}X(t)>u\right\}\right). \tag{5.6}$$

Since $\delta_1(u)\to 0,\delta_2(u)\to 0$ as $u\to\infty$ and $a(t)$ is continuous, without loss of generality, we may assume that $a(t)\equiv a(0)=a$ for $t\in([0,\delta_1(u)]\cup[0,\delta_2(u)])$. Moreover, by assumption (iv), we know that $\sigma(t)>0$ for $t\in([0,\delta_1(u)]\cup[0,\delta_2(u)])$. Below we use notation $\overline{X}(t)=\frac{X(t)}{\sigma(t)}$ for all t such that $\sigma(t)$ is positive.

Proof of Theorem 5.1.1 First we derive the asymptotic of

$$\pi(u):=\mathbb{P}\left\{\sup_{t\in\Delta(u)}X(t)>u\right\},$$

as $u\to\infty$, where $\Delta(u)=[0,\delta(u)]$ and

$$\delta(u)=\begin{cases}\delta_1(u), & \text{if } \gamma\le\beta,\\ \delta_2(u), & \text{if } \gamma>\beta,\end{cases}$$

with $\delta_1(u)$ and $\delta_2(u)$ in (5.5), which combined with Lemma 5.2.1 finally shows that

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \sim \pi(u). \tag{5.7}$$

In the following $\mathbb{Q}_i,\ i \in \mathbb{N}$, are some positive constants. For some $S > 0$, let $Y_{\nu,u}(t), t \in [0, S]$ be a family of centered stationary Gaussian processes with

$$Cov\left(Y_{\nu,u}(s), Y_{\nu,u}(t)\right) = \exp\left(-(1-\nu)au^{-2}|s-t|^{\alpha+2b\delta^{\beta}(u)}\right),$$

for $\nu \in (0,1), u > 0$ such that $\alpha + 2b\delta^{\beta}(u) \le 2$ and $s,t \in [0,S]$. Further, let $Z_{\nu,u}(t), t \in [0,S]$ be another family of centered stationary Gaussian processes with

$$Cov\left(Z_{\nu,u}(s), Z_{\nu,u}(t)\right) = \exp\left(-(1+\nu)au^{-2}|s-t|^{\alpha}\right),$$

for $\nu \in (0,1), u > 0$ and $s,t \in [0,S]$. Due to assumptions (i) and (v), α is strictly smaller than 2, which guarantees that covariance function of $Y_{\nu,u}(t), t \in [0,S]$ and $Z_{\nu,u}(t), t \in [0,S]$ are positive-definite. Hence the introduced families of Gaussian processes exist.
By assumption (iv), for any small $\varepsilon \in (0,1)$

$$1 + (1-\varepsilon)ce^{-|t|^{-\gamma}} \le \frac{1}{\sigma(t)} \le 1 + (1+\varepsilon)ce^{-|t|^{-\gamma}}, \tag{5.8}$$

holds for $t \in [0, \delta(u)]$.
Case 1: $\gamma < \beta$. Set for any $\epsilon \in (0,1)$ and all u large

$$N(0) = N(u,0) := \left\lfloor \frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor, \quad N_\epsilon(u) = \left\lfloor (1-\epsilon)\frac{\delta_1(u)u^{2/\alpha}}{S} \right\rfloor = \left\lfloor \frac{(1-\epsilon)u^{2/\alpha}}{(2\ln u - q\ln\ln u)^{1/\gamma}S} \right\rfloor,$$
$$B_j(u) = B_{j,0}(u) = \left[j\frac{S}{u^{2/\alpha}}, (j+1)\frac{S}{u^{2/\alpha}}\right], \ j \in \mathbb{N}, \ \mathcal{G}_u^{\pm\varepsilon} = u\left(1 + (1\pm\varepsilon)ce^{-((1-\epsilon)\delta_1(u))^{-\gamma}}\right).$$

We notice the fact that

$$\Psi(\mathcal{G}_u^{\pm\varepsilon}) \sim \Psi(u), \ u \to \infty,$$

and

$$I_1(u) \le \pi(u) \le I_1(u) + I_2(u), \tag{5.9}$$

where

$$I_1(u) = \mathbb{P}\left\{\sup_{t\in[0,(1-\epsilon)\delta_1(u)]} X(t) > u\right\}, \quad I_2(u) = \mathbb{P}\left\{\sup_{t\in[(1-\epsilon)\delta_1(u),\delta_1(u)]} X(t) > u\right\}.$$

Then by Bonferroni's inequality, (5.8), Lemma 5.3.1 with $k = 0$ and Lemma 5.3.2

$$\begin{aligned}
I_1(u) &\le \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{\sup_{t\in B_j(u)} X(t) > u\right\} \\
&\le \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{\sup_{t\in B_j(u)} \overline{X}(t) > \mathcal{G}_u^{-\varepsilon}\right\} \\
&\le \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{\sup_{t\in[jS,(j+1)S]} \overline{X}(tu^{-2/\alpha}) > \mathcal{G}_u^{-\varepsilon}\right\} \\
&\le \sum_{j=0}^{N_\epsilon(u)} \mathbb{P}\left\{\sup_{t\in[0,S]} Z_{u,\nu}(t) > \mathcal{G}_u^{-\varepsilon}\right\}
\end{aligned}$$

$$\begin{aligned}
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right] \Psi\left(\mathcal{G}_u^{-\varepsilon}\right) \\
&\sim \sum_{j=0}^{N_\epsilon(u)} \mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right] \Psi(u) \\
&\sim (1-\epsilon)u^{2/\alpha}\delta_1(u)\frac{\mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]}{S}\Psi(u) \\
&\sim (1-\epsilon)((1+\nu)a)^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}\delta_1(u)\Psi(u),\ u\to\infty,\ S\to\infty. \qquad (5.10)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\sum_{j=0}^{N_\epsilon(u)-1}\mathbb{P}\left\{\sup_{t\in B_j(u)} X(t)>u\right\} &\geq \sum_{j=0}^{N_\epsilon(u)-1}\mathbb{P}\left\{\sup_{t\in[0,S]} Y_{u,\nu}(t)>\mathcal{G}_u^{+\varepsilon}\right\} \\
&\sim (1-\epsilon)((1-\nu)a)^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}\delta_1(u)\Psi(u),\ u\to\infty,\ S\to\infty. \qquad (5.11)
\end{aligned}$$

Since

$$I_1(u)\geq \sum_{j=0}^{N_\epsilon(u)-1}\mathbb{P}\left\{\sup_{t\in B_j(u)} X(t)>u\right\} - \sum_{0\leq j<k\leq N_\epsilon(u)}\mathbb{P}\left\{\sup_{t\in B_j(u)} X(t)>u,\ \sup_{t\in B_k(u)} X(t)>u\right\}, \qquad (5.12)$$

and by [49][Lemma 4.5]

$$\begin{aligned}
\sum_{0\leq j<k\leq N_\epsilon(u)}\mathbb{P}\left\{\sup_{t\in B_j(u)} X(t)>u,\ \sup_{t\in B_k(u)} X(t)>u\right\} &\leq \sum_{0\leq j<k\leq N_\epsilon(u)}\mathbb{P}\left\{\sup_{t\in B_j(u)} \overline{X}(t)>u,\ \sup_{t\in B_k(u)} \overline{X}(t)>u\right\} \\
&= o\left(u^{2/\alpha}\delta_1(u)\Psi(u)\right),\ u\to\infty,\ S\to\infty,\ \epsilon\to 0. \qquad (5.13)
\end{aligned}$$

Thus inserting (5.11) and (5.13) into (5.12), we have

$$\lim_{u\to\infty}\frac{I_1(u)(2\ln u - q\ln\ln u)^{1/\gamma}}{u^{2/\alpha}\Psi(u)}\geq (1-\epsilon)((1-\nu)a)^{1/\alpha}\mathcal{H}_\alpha,$$

which combined with (5.10) gives that

$$I_1(u)\sim\frac{a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}}{(2\ln u - q\ln\ln u)^{1/\gamma}}\Psi(u), u\to\infty,\ \nu\to 0,\ \epsilon\to 0. \qquad (5.14)$$

By (iii) and (v), we have for all u large

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(s))^2\right\} = 2-2r(s,t)\leq \mathbb{Q}_1|s-t|^\alpha,$$

uniformly holds for $s,t\in[(1-\epsilon)\delta_1(u),\delta_1(u)]$. By Piterbarg inequality for u large enough, see e.g., [119][Theorem 8.1] or an extension in [45][Lemma 5.1]

$$I_2(u)\leq\mathbb{P}\left\{\sup_{t\in[(1-\epsilon)\delta_1(u),\delta_1(u)]}\overline{X}(t)>u\right\}\leq\mathbb{Q}_2\epsilon\delta_1(u)u^{2/\alpha}\Psi(u), \qquad (5.15)$$

which implies

$$\lim_{\epsilon\to 0}\lim_{u\to\infty}\frac{I_2(u)(2\ln u - q\ln\ln u)^{1/\gamma}}{u^{2/\alpha}\Psi(u)}=0.$$

Combining this equation with (5.9) and (5.14), we get

$$\pi(u)\sim\frac{a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}}{(2\ln u - q\ln\ln u)^{1/\gamma}}\Psi(u)\sim a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}(2\ln u)^{-1/\gamma}\Psi(u),\ u\to\infty.$$

Case 2: $\gamma = \beta$. Set

$$d_k = d_k(u) := \left(\frac{k}{\ln(u)(\ln\ln(u))^{1/\beta}}\right)^{1/\beta}, \quad A_k = A_k(u) := [d_k, d_{k+1}].$$

Further let $M_\epsilon(u) = \max(k \in \mathbb{N} : d_k \le (1-\epsilon)\delta_1(u))$ for some $\epsilon \in (0,1)$, then $M_\epsilon(u) \to \infty,\ u \to \infty$. Clearly

$$\bigcup_{k=0}^{M_\epsilon(u)-1} A_k \subset [0,(1-\epsilon)\delta_1(u)] \subset \bigcup_{k=0}^{M_\epsilon(u)} A_k.$$

We divide each interval A_k into subintervals of length $S/u^{2/\alpha(d_k)}$, i.e.,

$$B_{j,k} = B_{j,k}(u) := \left[d_k + j\frac{S}{u^{2/\alpha(d_k)}}, d_k + (j+1)\frac{S}{u^{2/\alpha(d_k)}}\right]$$

for $j = 0, 1, \ldots, N(k)$, where $N(k) = N(k,u) := \left\lfloor \frac{d_{k+1}-d_k}{S} u^{2/\alpha(d_k)} \right\rfloor$. Notice that

$$\bigcup_{k=0}^{N(k)-1} B_{j,k} \subset A_k \subset \bigcup_{k=0}^{N(k)} B_{j,k}.$$

We have

$$I_1(u) \le \pi(u) \le I_1(u) + I_2(u), \tag{5.16}$$

where

$$I_1(u) \quad = \quad \mathbb{P}\left\{\sup_{t\in[0,(1-\epsilon)\delta_1(u)]} X(t) > u\right\}, \quad I_2(u) = \mathbb{P}\left\{\sup_{t\in[(1-\epsilon)\delta_1(u),\delta_1(u)]} X(t) > u\right\}.$$

Then by Bonferroni's inequality

$$\begin{aligned} I_1(u) \quad &\ge \quad \sum_{k=0}^{M_\epsilon(u)-1}\sum_{j=0}^{N(k)-1} \mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u\right\} - \sum_{\substack{(j,k),(j',k')\in\mathcal{L}\\(j,k)\prec(j',k')}} \mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u, \sup_{t\in B_{j',k'}} X(t) > u\right\} \\ &=: \quad J_1(u) - J_2(u), \end{aligned} \tag{5.17}$$

where $\mathcal{L} = \{(j,k) : 0 \le k \le M_\epsilon(u) - 1, 0 \le j \le N(k) - 1\}$ and

$$(j,k) \prec (j',k') \text{ iff } (k < k') \vee (k = k' \wedge j < j'),$$

and by (5.8), Lemma 5.3.1 and Lemma 5.3.2

$$\begin{aligned} I_1(u) \quad &\le \quad \sum_{k=0}^{M_\epsilon(u)}\sum_{j=0}^{N(k)} \mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u\right\} \\ &\le \quad \sum_{k=0}^{M_\epsilon(u)}\sum_{j=0}^{N(k)} \mathbb{P}\left\{\sup_{t\in B_{j,k}} \overline{X}(t) > \mathcal{G}_u^{-\varepsilon}\right\} \\ &\le \quad \sum_{k=0}^{M_\epsilon(u)}\sum_{j=0}^{N(k)} \mathbb{P}\left\{\sup_{t\in[0,S]} Z_{\nu,u}(t) > \mathcal{G}_u^{-\varepsilon}\right\} \\ &\sim \quad \sum_{k=0}^{M_\epsilon(u)}\sum_{j=0}^{N(k)} \mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]\Psi\left(\mathcal{G}_u^{-\varepsilon}\right) \\ &\sim \quad \sum_{k=0}^{M_\epsilon(u)} \frac{d_{k+1}-d_k}{S} u^{2/\alpha(d_k)} \mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]\Psi(u) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta}(d_{k+1}-d_k) e^{(\ln u)\left(\frac{2(\alpha-\alpha(d_k))}{\alpha\alpha(d_k)}\right)} \\
&\leq \frac{\mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta}(d_{k+1}-d_k) e^{\frac{-2(1-\varepsilon_1)(\ln u)\left(bd_k^\beta - d_k^{\beta+\delta}\right)}{\alpha^2}} \\
&\leq \frac{\mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]}{S} \frac{u^{2/\alpha}}{(\ln u)^{1/\beta}} \Psi(u) \\
&\quad \times \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta}(d_{k+1}-d_k) e^{\frac{-2(1-\varepsilon_1)b\left((\ln u)^{1/\beta} d_k\right)^\beta}{\alpha^2}} e^{\frac{2(1-\varepsilon_1)(\ln u) d^{\beta+\delta}_{M_\epsilon(u)+1}}{\alpha^2}},
\end{aligned}
$$

as $u \to \infty$, where $\varepsilon_1 \in (0,1)$ is a small constant.
Moreover, using that $d_{M_\epsilon(u)} \leq (1-\epsilon)\delta_1(u)$ and $\lim_{u\to\infty} (\ln u)\delta_1(u)^{\beta+\delta} = 0$, we observe that

$$
\lim_{u\to\infty} e^{\frac{2(1-\varepsilon_1)(\ln u) d^{\beta+\delta}_{M_\epsilon(u)+1}}{\alpha^2}} = 1.
$$

Finally, since

$$
\lim_{u\to\infty} \sup_{k=0,\ldots,M_\epsilon(u)} (\ln u)^{1/\beta}(d_{k+1}-d_k) = 0
$$

and

$$
\lim_{u\to\infty} (\ln u)^{1/\beta} d_{M_\epsilon(u)+1} = (1-\epsilon)\left(\frac{1}{2}\right)^{1/\beta},
$$

we obtain

$$
\lim_{u\to\infty} \sum_{k=0}^{M_\epsilon(u)} (\ln u)^{1/\beta}(d_{k+1}-d_k) e^{\frac{-2(1-\varepsilon_1)b\left((\ln u)^{1/\beta} d_k\right)^\beta}{\alpha^2}} = \int_0^{(1-\epsilon)\left(\frac{1}{2}\right)^{1/\beta}} e^{\frac{-2(1-\varepsilon_1)bx^\beta}{\alpha^2}} dx.
$$

Thus

$$
\lim_{u\to\infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} \leq \frac{\mathcal{H}_\alpha\left[0, S((1+\nu)a)^{1/\alpha}\right]}{S} \int_0^{(1-\epsilon)\left(\frac{1}{2}\right)^{1/\beta}} e^{\frac{-2(1-\varepsilon_1)bx^\beta}{\alpha^2}} dx, \tag{5.18}
$$

and letting $S \to \infty, \varepsilon_1, \nu \to 0$, and $\epsilon \to 0$, we get the upper bound. Similarly, we derive that

$$
\lim_{\epsilon\to 0} \lim_{S\to\infty} \lim_{u\to\infty} \frac{J_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} \geq a^{1/\alpha}\mathcal{H}_\alpha \int_0^{\left(\frac{1}{2}\right)^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}} dx. \tag{5.19}
$$

By [49] [Lemma 4.5]

$$
\begin{aligned}
J_2(u) &= \sum_{\substack{(j,k),(j',k')\in\mathcal{L} \\ (j,k)\prec(j',k')}} \mathbb{P}\left\{ \sup_{t\in B_{j,k}} X(t) > u, \sup_{t\in B_{j',k'}} X(t) > u \right\} \\
&\leq \sum_{\substack{(j,k),(j',k')\in\mathcal{L} \\ (j,k)\prec(j',k')}} \mathbb{P}\left\{ \sup_{t\in B_{j,k}} \overline{X}(t) > u, \sup_{t\in B_{j',k'}} \overline{X}(t) > u \right\} \\
&= o\left(u^{2/\alpha}(\ln u)^{-1/\beta}\Psi(u)\right), \ u\to\infty, \ S\to\infty, \epsilon\to 0.
\end{aligned} \tag{5.20}
$$

Thus inserting (5.19) and (5.20) into (5.17), we get

$$
\lim_{\epsilon\to 0} \lim_{S\to\infty} \lim_{u\to\infty} \frac{I_1(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} \geq a^{1/\alpha}\mathcal{H}_\alpha \int_0^{\left(\frac{1}{2}\right)^{1/\beta}} e^{\frac{-2(1-\varepsilon_1)bx^\beta}{\alpha^2}} dx. \tag{5.21}
$$

By (5.15)

$$
\lim_{\epsilon\to 0} \lim_{u\to\infty} \frac{I_2(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} = 0. \tag{5.22}
$$

Hence according to (5.16), (5.18), (5.21), and (5.22), we have

$$\pi(u) \sim a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}(\ln u)^{-1/\beta}\Psi(u)\int_0^{\left(\frac{1}{2}\right)^{1/\beta}} e^{\frac{-2bx^\beta}{\alpha^2}}dx,\ u\to\infty.$$

Case 3: $\gamma > \beta$. We consider $\pi(u) = \mathbb{P}\left\{\sup_{t\in[0,\delta_2(u)]} X(t) > u\right\}$ with

$$\delta_2(u) = \left(\frac{\alpha^2(\ln(\ln u))}{\beta(\ln u)}\right)^{1/\beta}.$$

Set for some $\varepsilon > 0$

$$\mathcal{F}_u^{\pm\varepsilon} = u\left(1+(1\pm\varepsilon)ce^{-(\delta_2(u))^{-\gamma}}\right),\ \mathcal{K} = \{t\in[0,T] : \sigma(t)\neq 0\},$$

and we observe that

$$\Psi\left(\mathcal{F}_u^{\pm\varepsilon}\right)\sim\Psi(u),\ u\to\infty.$$

By [49][Theorem 2.1]

$$\begin{aligned}\pi(u) &\leq \mathbb{P}\left\{\sup_{t\in[0,\delta_2(u)]}\overline{X}(t) > u\right\}\\ &\leq \mathbb{P}\left\{\sup_{t\in\mathcal{K}}\overline{X}(t) > u\right\}\\ &\sim a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}(\ln u)^{-\frac{1}{\beta}}\int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}}dx\Psi(u),\ u\to\infty.\end{aligned}\tag{5.23}$$

Let $d_k, A_k, B_{j,k}, N(k)$ be the same as in **Case 2** and $M(u) = \max(k\in\mathbb{N} : d_k\leq\delta_2(u))$. Clearly

$$\bigcup_{k=0}^{M(u)-1} A_k\subset[0,\delta_2(u)]\subset\bigcup_{k=0}^{M(u)} A_k,\quad \bigcup_{k=0}^{N(k)-1} B_{j,k}\subset A_k\subset\bigcup_{k=0}^{N(k)} B_{j,k},$$

and by Bonferroni's inequality

$$\begin{aligned}\pi(u) &\geq \sum_{k=0}^{M(u)-1}\sum_{j=0}^{N(k)-1}\mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u\right\} - \sum_{\substack{(j,k),(j',k')\in\mathcal{L}'\\(j,k)\prec(j',k')}}\mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u,\ \sup_{t\in B_{j',k'}} X(t) > u\right\}\\ &=: J_1'(u) - J_2'(u),\end{aligned}\tag{5.24}$$

where $\mathcal{L}' = \{(j,k) : 0\leq k\leq M(u)-1, 0\leq j\leq N(k)-1\}$.
By (5.8), Lemma 5.3.1, Lemma 5.3.2 and similar argumentation as (5.19) with $\mathcal{G}_u^{\pm\varepsilon}$ replaced by $\mathcal{F}_u^{\pm\varepsilon}$ and the fact that $(\ln u)^{1/\beta}d_{M(u)+1}\to\infty$, $u\to\infty$, we get

$$\lim_{S\to\infty}\lim_{u\to\infty}\frac{J_1'(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)}\geq a^{1/\alpha}\mathcal{H}_\alpha\int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}}dx.\tag{5.25}$$

By[49][Lemma 4.5]

$$\begin{aligned}J_2'(u) &= \sum_{\substack{(j,k),(j',k')\in\mathcal{L}'\\(j,k)\prec(j',k')}}\mathbb{P}\left\{\sup_{t\in B_{j,k}} X(t) > u,\ \sup_{t\in B_{j',k'}} X(t) > u\right\}\\ &\leq \sum_{\substack{(j,k),(j',k')\in\mathcal{L}'\\(j,k)\prec(j',k')}}\mathbb{P}\left\{\sup_{t\in B_{j,k}}\overline{X}(t) > u,\ \sup_{t\in B_{j',k'}}\overline{X}(t) > u\right\}\\ &= o\left(u^{2/\alpha}(\ln u)^{-1/\beta}\Psi(u)\right),\ u\to\infty.\end{aligned}\tag{5.26}$$

Hence inserting (5.25) and (5.26) into (5.24), we have

$$\lim_{u\to\infty} \frac{\pi(u)(\ln u)^{1/\beta}}{u^{2/\alpha}\Psi(u)} \geq a^{1/\alpha}\mathcal{H}_\alpha \int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}}dx,$$

which combined with (5.23) gives that

$$\pi(u) \sim a^{1/\alpha}\mathcal{H}_\alpha u^{2/\alpha}(\ln u)^{-1/\beta}\Psi(u)\int_0^\infty e^{\frac{-2bx^\beta}{\alpha^2}}dx,\ u\to\infty.$$

Consequently, according to Lemma 5.2.1 and

$$\pi(u) \leq \mathbb{P}\left\{\sup_{t\in[0,T]} X(t) > u\right\} \leq \pi(u) + \mathbb{P}\left\{\sup_{t\in[\delta(u),T]} X(t) > u\right\},$$

(5.7) is proved and all claims follow. □

5.3 Some technical results

In this section we present the proofs of the lemmas used in the proof of Theorem 5.1.1.
Proof of Lemma 5.2.1 Below $\mathbb{Q}_k,\ k=0,1,2\ldots$, are some positive constants.
Step 1: First we prove (5.3). By the continuity of $\sigma(t)$ in [0,T], for any small enough constant $0<\theta<1$

$$\sup_{t\in[\theta,T]} \sigma(t) =: \rho(\theta) < \sigma(t_0) = \sigma(0) = 1.$$

Then by Borell inequality in [1]

$$\mathbb{P}\left\{\sup_{t\in[\theta,T]} X(t) > u\right\} \leq \exp\left(-\frac{(u-\mathbb{Q}_0)^2}{2\rho^2(\theta)}\right) = o\left(\Psi\left(u\right)\right),$$

as $u\to\infty$, where $\mathbb{Q}_0 = \mathbb{E}\left\{\sup_{t\in[0,T]} X(t)\right\} < \infty$.
By assumption (iv), for any small $\varepsilon\in(0,1)$, when θ small enough

$$1+(1-\varepsilon)ce^{-|t|^{-\gamma}} \leq \frac{1}{\sigma(t)} \leq 1+(1+\varepsilon)ce^{-|t|^{-\gamma}},$$

holds for $t\in[0,\theta]$. Then

$$\frac{1}{\sigma(t)} \geq 1+(1-\varepsilon)ce^{-|t|^{-\gamma}} \geq 1+(1-\varepsilon)cu^{-2}(\ln u)^q$$

uniformly holds for $t\in[\delta_1(u),\theta]$.
Moreover by assumption (i) and (iii), when θ small enough

$$\begin{aligned}\mathbb{E}\left\{(X(t)-X(s))^2\right\} &= \mathbb{E}\left\{X^2(t)\right\} + \mathbb{E}\left\{X^2(s)\right\} - 2\mathbb{E}\left\{X(t)X(s)\right\}\\ &\leq 2-2(1-2a(t)|t-s|^{\alpha(t)})\\ &\leq \mathbb{Q}_1|t-s|^{\varsigma}\end{aligned}$$

holds uniformly for $s,t\in[0,\theta]$, where $\mathbb{Q}_1 = \sup_{t\in[0,\theta]} 4a(t)$ and $\varsigma = \inf_{t\in[0,\theta]}\alpha(t) > 0$.
Then by Piterbarg inequality

$$\mathbb{P}\left\{\sup_{t\in[\delta_1(u),\theta]} X(t) > u\right\} \leq \mathbb{Q}_2\theta u^{2/\varsigma}\Psi(u[1+(1-\varepsilon)cu^{-2}(\ln u)^q]) = o\left(\Psi\left(u\right)\right),\quad u\to\infty.$$

Further, since

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t)>u\right\} \le \mathbb{P}\left\{\sup_{t\in[0,\delta_1(u)]} X(t)>u\right\} + \mathbb{P}\left\{\sup_{t\in[\delta_1(u),\theta]} X(t)>u\right\} + \mathbb{P}\left\{\sup_{t\in[\theta,T]} X(t)>u\right\},$$

and

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t)>u\right\} \ge \mathbb{P}\left\{\sup_{t\in[0,\delta_1(u)]} X(t)>u\right\} \ge \mathbb{P}\left\{X(0)>u\right\} = \Psi\left(u\right),$$

we get

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X(t)>u\right\} \sim \mathbb{P}\left\{\sup_{t\in[0,\delta_1(u)]} X(t)>u\right\}, \quad u\to\infty.$$

Step 2: Next we prove (5.4). When $\gamma\le\beta$, since $\delta_1(u)=o(\delta_2(u))$, as $u\to\infty$ and by **Step 1**

$$\mathbb{P}\left\{\sup_{t\in[\delta_1(u),T]} X(t)>u\right\} = o\left(\Psi\left(u\right)\right), \quad u\to\infty.$$

Then for u large enough, (5.4) is obvious.
When $\gamma>\beta$, for u large enough, we have $\delta_2(u)<\delta_1(u)$ and

$$\mathbb{P}\left\{\sup_{t\in[\delta_2(u),T]} X(t)>u\right\} \le \mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} X(t)>u\right\} + \mathbb{P}\left\{\sup_{t\in[\delta_1(u),T]} X(t)>u\right\}.$$

By **Step 1**, we know for all u large

$$\mathbb{P}\left\{\sup_{t\in[\delta_1(u),T]} X(t)>u\right\} \le \Psi(u),$$

and then we just need to deal with $\mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} X(t)>u\right\}$.
Since $\delta_1(u)\to 0,\ u\to\infty$, then by assumption (v)

$$\alpha(t)>\alpha+\frac{3}{4}b(\delta_2(u))^{\beta}$$

holds for all $t\in[\delta_2(u),\delta_1(u)]$ when u large enough.
Let $\eta_u=u^{-2/\left(\alpha+\frac{3}{4}b(\delta_2(u))^{\beta}\right)}$. For sufficiently large u and $s,t\in[\delta_2(u),\delta_1(u)]$, there exists a constant $\mathbb{Q}_3>0$ such that

$$1-r(s,t)\le 1-e^{-\mathbb{Q}_3|s-t|^{\alpha+\frac{3}{4}b(\delta_2(u))^{\beta}}}.$$

Let $Y_u(t), t\ge 0$ be a family of centered stationary Gaussian processes with correlation functions

$$r_Y(s,t)=e^{-\mathbb{Q}_3|s-t|^{\alpha+\frac{3}{4}b(\delta_2(u))^{\beta}}}.$$

Then from Slepian's inequality we get for any constant $S>0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} X(t)>u\right\} &\le \mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} \frac{X(t)}{\sigma(t)}>u\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} Y_u(t)>u\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in[0,S]} Y_u(t)>u\right\}\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{i=0}^{\lfloor S\eta_u^{-1}\rfloor+1} \mathbb{P}\left\{\sup_{t\in[i\eta_u,(i+1)\eta_u]} Y_u(t)>u\right\}\\
&\leq (\lfloor S\eta_u^{-1}\rfloor+1)\mathbb{P}\left\{\sup_{t\in[0,\eta_u]} Y_u(t)>u\right\},
\end{aligned}
$$

for sufficiently large u. Notice that for each $s,t\in[0,1]$

$$1-r_Y(\eta_u t,\eta_u s)=\mathbb{Q}_3u^{-2}|s-t|^{\alpha+\frac{3}{4}b(\delta_2(u))^\beta}(1+o(1))=\mathbb{Q}_3u^{-2}|s-t|^{\alpha}(1+o(1)),\ u\to\infty.$$

Hence, from [119][Lemma D.1]

$$\mathbb{P}\left\{\sup_{t\in[0,\eta_u]} Y_u(t)>u\right\}\sim\mathcal{H}_\alpha[1]\Psi(u),$$

as $u\to\infty$. Combining this with the fact that

$$
\begin{aligned}
\eta_u^{-1} &= u^{2/\left(\alpha+\frac{3}{4}\delta_2(u)\right)}=u^{2/\alpha}u^{2/\left(\alpha+\frac{3}{4}\delta_2(u)\right)-2/\alpha}=u^{2/\alpha}u^{-\frac{3}{2}(\delta_2(u))^\beta/\left(\alpha\left(\alpha+\frac{3}{4}(\delta_2(u))^\beta\right)\right)}\\
&= u^{2/\alpha}u^{-\frac{3}{2}\frac{\alpha^2(\ln(\ln u))}{\beta(\ln u)}/\left(\alpha\left(\alpha+\frac{3}{4}(\delta_2(u))^\beta\right)\right)}\leq u^{2/\alpha}u^{-\frac{4}{3}\frac{\ln(\ln u)}{\beta(\ln u)}}=u^{2/\alpha}(\ln u)^{-4/(3\beta)},
\end{aligned}
$$

we get for some constant $\mathbb{Q}_4$ and all u large enough

$$\mathbb{P}\left\{\sup_{t\in[\delta_2(u),\delta_1(u)]} X(t)>u\right\}\leq\mathbb{Q}_4Su^{2/\alpha}(\ln u)^{-4/3\beta}\Psi(u).$$

Then the result follows. □

Lemma 5.3.1. *Under the notation in the proof of Theorem 5.1.1, for* $(j,k)\in\mathfrak{U}=\{(j,k):0\leq k\leq M^*(u),0\leq j\leq N(k)\}$ *and* $\lim_{u\to\infty}\frac{f(u)}{u}=1$*, there exists* u_0 *such that for each* $u\geq u_0$
1) $\mathbb{P}\left\{\sup_{t\in B_{j,k}}\overline{X}(t)>f(u)\right\}\geq\mathbb{P}\left\{\sup_{t\in[0,S]}Y_{\nu,u}(t)>f(u)\right\}$*;*
2) $\mathbb{P}\left\{\sup_{t\in B_{j,k}}\overline{X}(t)>f(u)\right\}\leq\mathbb{P}\left\{\sup_{t\in[0,S]}Z_{\nu,u}(t)>f(u)\right\}$*,*
where

$$M^*(u)=\begin{cases}0, & \text{if } \gamma<\beta,\\ M_\epsilon(u), & \text{if } \gamma=\beta,\\ M(u), & \text{if } \gamma>\beta.\end{cases}$$

Proof of Lemma 5.3.1 Since the proofs of scenarios $\gamma<\beta$, $\gamma=\beta$, and $\gamma>\beta$ are similar, we only present the proof of $\gamma=\beta$. Set $X_{j,k,u}(t)=\overline{X}\left(d_k+\frac{jS+t}{u^{2/\alpha(d_k)}}\right)$, then $\sup_{t\in B_{j,k}}\overline{X}(t)\overset{d}{=}\sup_{t\in[0,S]}X_{j,k,u}(t)$. It is enough to analyze the supremum of $X_{j,k,u}(t)$.
1) For sufficiently large u and $s,t\in[0,T]$

$$
\begin{aligned}
1-Cov\left(X_{j,k,u}(s),X_{j,k,u}(t)\right) &= 1-Cov\left(\overline{X}\left(d_k+\frac{jS+s}{u^{2/\alpha(d_k)}}\right),\overline{X}\left(d_k+\frac{jS+t}{}\right)\right)\\
&\geq (1-\nu/2)^{1/3}a\left|u^{-2/\alpha(d_k)}(s-t)\right|^{\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)}\\
&= (1-\nu/2)^{1/3}au^{-2\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)/\alpha(d_k)}\left|(s-t)\right|^{\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)}\\
&= (1-\nu/2)^{1/3}a\times I_1\times I_2. \qquad (5.27)
\end{aligned}
$$

We deal with I_1 and I_2 separately. For sufficiently large u, uniformly with respect to k,

$$
\begin{aligned}
I_1 &= u^{-2\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)/\alpha(d_k)}\\
&= u^{-2}u^{2\left(\alpha(d_k)-\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)\right)/\alpha(d_k)}\\
&= u^{-2}e^{2(\ln u)\left(\alpha(d_k)-\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)\right)/\alpha(d_k)}
\end{aligned}
$$

$$\geq \quad u^{-2}(1-\nu/2)^{1/3}, \tag{5.28}$$

where the last inequality follows from the fact that

$$\begin{aligned}(\ln u)\left|\alpha(d_k)-\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)\right| &\leq (\ln u)\left(\left|b(d_k)^\beta - b\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)^\beta\right| + 2\delta_1^{\beta+\delta}(u)\right)\\ &\leq (\ln u)\left(\frac{b}{(\ln u)(\ln\ln u)^{1/\beta}} + 2\delta_1^{\beta+\delta}(u)\right)\\ &\leq \frac{b}{(\ln\ln u)^{1/\beta}} + 2(\ln u)\left(\frac{1}{2\ln u - q\ln\ln u}\right)^{\frac{\beta+\delta}{\gamma}} \to 0,\ u\to\infty.\end{aligned}$$

For I_2, we need to prove that

$$I_2 \geq (1-\nu/2)^{1/3}|s-t|^{\alpha+2b\delta_1^\beta(u)}. \tag{5.29}$$

Assumption (v) implies that

$$\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right) < \alpha + 2b\delta_1^\beta(u) \tag{5.30}$$

for each $(j,k)\in\mathfrak{U}$. Thus if $|s-t|<1$, then (5.29) holds immediately. If $1\leq|s-t|\leq S$, then by (5.30)

$$\begin{aligned}I_2 &= |(s-t)|^{\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)}\\ &\geq T^{\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)-\alpha-2b\delta_1^\beta(u)}|s-t|^{\alpha+2b\delta_1^\beta(u)}\\ &\geq T^{-2b\delta_1^\beta(u)}|s-t|^{\alpha+2b\delta_1^\beta(u)}\\ &\geq (1-\nu/2)^{1/3}|s-t|^{\alpha+2b\delta_1^\beta(u)}\end{aligned}$$

for sufficiently large u. The above combined with (5.27), (5.28) and (5.29) gives that for sufficiently large u, uniformly with respect to $(j,k)\in\mathfrak{U}$,

$$1-Cov\left(X_{j,k,u}(s),X_{j,k,u}(t)\right) \geq (1-\nu/2)au^{-2}|s-t|^{\alpha+2b\delta_1^\beta(u)} \geq 1-Cov\left(Y_{\nu,u}(s),Y_{\nu,u}(t)\right).$$

Thus by Slepian's inequality <u>1)</u> is proved.
<u>2)</u> For all u large

$$\begin{aligned}1-Cov\left(X_{j,k,u}(s),X_{j,k,u}(t)\right) &= 1-Cov\left(\overline{X}\left(d_k+\frac{jS+s}{u^{2/\alpha(d_k)}}\right),\overline{X}\left(d_k+\frac{jS+t}{}\right)\right)\\ &\leq (1+\nu)^{1/3}a\left|u^{-2/\alpha(d_k)}(s-t)\right|^{\alpha\left(d_k+u^{-2/\alpha(d_k)}(jS+t)\right)}.\end{aligned}$$

Following the argument analogous to that for the proof of <u>1)</u>, we obtain that for sufficiently large u, uniformly with respect to k, and $s,t\in[0,S]$

$$1-Cov\left(X_{j,k,u}(s),X_{j,k,u}(t)\right) \leq 1-Cov\left(Z_{\nu,u}(s),Z_{\nu,u}(t)\right).$$

Again the application of Slepian's inequality completes the proof. □

Lemma 5.3.2. *For $S>1$, $\nu\in(0,1)$, and $\lim_{u\to\infty}\frac{f(u)}{u}=1$, as $u\to\infty$, we have*
<u>1)</u> $\mathbb{P}\left\{\sup_{t\in[0,S]}Y_{\nu,u}(t)>f(u)\right\} = \mathcal{H}_\alpha\left[0,S((1-\nu)a)^{1/\alpha}\right]\Psi\left(f(u)\right)(1+o(1))$;
<u>2)</u> $\mathbb{P}\left\{\sup_{t\in[0,S]}Z_{\nu,u}(t)>f(u)\right\} = \mathcal{H}_\alpha\left[0,S((1+\nu)a)^{1/\alpha}\right]\Psi\left(f(u)\right)(1+o(1))$.

Proof of Lemma 5.3.2 We present the proof of <u>1)</u> and omit the proof of <u>2)</u> since it follows with similar arguments. Following the definition of $Y_{\nu,u}(t)$, for each $s,t\in[0,S]$

$$\begin{aligned}&\lim_{u\to\infty} f^2(u)\left[1-Cov\left(Y_{\nu,u}\left(t(a(1-\nu))^{-1/\alpha}\right),Y_{\nu,u}\left(s(a(1-\nu))^{-1/\alpha}\right)\right)\right]\\ &= \lim_{u\to\infty}(a(1-\nu))^{1-\left(\alpha+2b\delta^\beta(u)\right)/\alpha}|s-t|^{\alpha+2b\delta^\beta(u)} = |s-t|^\alpha.\end{aligned}$$

Moreover, for all $s, t \in [0, S]$, sufficiently large u and some constant $C > 0$

$$\begin{aligned} & f^2(u) \left[1 - Cov\left(Y_{\nu,u}\left(t(a(1-\nu))^{-1/\alpha}\right), Y_{\nu,u}\left(s(a(1-\nu))^{-1/\alpha}\right)\right)\right] \\ & \le (a(1-\nu))^{1-\left(\alpha+2b\delta^\beta(u)\right)/\alpha} |s-t|^{\alpha+2b\delta^\beta(u)} \le CT^{2\alpha}|s-t|^\alpha, \end{aligned}$$

where the last inequality follows from the fact that

$$|s-t|^{\alpha+2b\delta^\beta(u)} \le |s-t|^\alpha, \text{ if } |s-t| < 1,$$

and

$$|s-t|^{\alpha+2b\delta^\beta(u)} \le T^{2\alpha} \le T^{2\alpha}|s-t|^\alpha, \text{ if } 1 \le |s-t| \le T.$$

Hence, by [90][Lemma 7], we conclude that

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\in[0,S]} Y_{\nu,u}(t) > f(u)\right\} &= \mathbb{P}\left\{\sup_{t\in[0,((1-\nu)a)^{1/\alpha}S]} Y_{\nu,u}((a(1-\nu))^{-1/\alpha}t) > f(u)\right\} \\ &= \mathcal{H}_\alpha\left[0, ((1-\nu)a)^{1/\alpha}S\right]\Psi\left(f(u)\right)(1+o(1)), \end{aligned}$$

as $u \to \infty$. This completes the proof. □

Chapter 6

Extremes of Vector-Valued Gaussian Processes with Trend[1]

6.1 Introduction and Preliminaries

Motivated by various applied-oriented problems, the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in\mathcal{T}}(X(t)+h(t))>u\right\}, \tag{6.1}$$

as $u\to\infty$, for both $\mathcal{T}=[0,T]$ and $\mathcal{T}=[0,\infty)$, where $X(t)$ is a centered Gaussian process with continuous trajectories and $h(t)$ is a continuous function, attracted substantial interest in the literature; see e.g. [89, 47, 90, 63, 84, 59, 52, 51] and references therein for connections of (6.1) with problems considered, e.g., in risk theory or fluid queueing models. For example, in the setting of risk theory one usually supposes that $h(t)=-ct$, with $c>0$ and X has stationary increments. Then, using that $\mathbb{P}\{\sup_{t\in\mathcal{T}}(X(t)+h(t))>u\}=\mathbb{P}\{\inf_{t\in\mathcal{T}}(u-X(t)+ct)<0\}$, (6.1) represents *ruin probability*, with $X(t)$ modelling the accumulated claims amount in time interval $[0,t]$, c being the constant premium rate and u, the initial capital. The most celebrated model in this context is the Brownian risk model introduced in the seminal work by Iglehart [93], where X is a standard Brownian motion. Extensions to more general class of Gaussian processes with stationary increments, including fractional Brownian motions, was analyzed in, e.g., [113, 89, 90, 92, 91]. Recent interest in the analysis of risk models has turned to the investigation of multidimensional ruin problems, including investigation of *simultaneous ruin* probability of some number, say n, of independent risk processes

$$\mathbb{P}\left\{\exists_{t\in\mathcal{T}}\forall_{i=1,\dots,n}(u_i-X_i(t)+c_it)<0\right\},$$

see, e.g., [7] and [6]. Motivated by this sort of problems, in this paper we investigate multidimensional counterpart of (6.1), i.e., we are interested in the exact asymptotics of

$$\mathbb{P}\left\{\exists_{t\in[0,T]}\boldsymbol{X}(t)+\boldsymbol{h}(t)>u\boldsymbol{1}\right\}=\mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n}(X_i(t)+h_i(t))>u\right\}, \tag{6.2}$$

as $u\to\infty$, $T\in(0,\infty)$, where $\boldsymbol{X}(t)=(X_1(t),\dots,X_n(t)), t\in\mathcal{T}\subset\mathbb{R}$ is an $n-$dimensional centered Gaussian process with mutually independent coordinates and continuous trajectories and $\boldsymbol{h}(t)=(h_1(t),\dots,h_n(t)), t\in[0,T]$ is a vector-valued continuous function.

We note that (6.2) can also be viewed as the probability that the conjunction set $\mathcal{S}_{T,u}:=\{t\in[0,T]:\min_{1\le i\le n}(X_i(t)+h_i(t))>u\}$ is not empty in Gaussian conjunction problem, since

$$\mathbb{P}\left\{\mathcal{S}_{T,u}\neq\emptyset\right\}=\mathbb{P}\left\{\sup_{t\in[0,T]}\min_{1\le i\le n}(X_i(t)+h_i(t))>u\right\},$$

[1]This chapter is based on L. Bai, K. Dębicki and P. Liu, (2018): Extremes of Vector-Valued Gaussian Processes with Trend, published in the *Journal of Mathematical Analysis and Applications,* Volume 465, 47-74.

see, e.g., [137, 57] and references therein.

The main results of this contribution extend recent findings of [57], where the exact asymptotics of (6.2) for $h_i \equiv 0, 1 \leq i \leq n$ was analyzed; see also [61] where $\boldsymbol{X}(t)$ is a multidimensional Brownian motion, $h_i(t) = c_i t$ and $T = \infty$, and [56, 124] for LDP-type results. It appears that the presence of the drift function substantially increases difficulty of the problem when comparing it with the analysis given for the driftless case in [57]. More specifically, as advocated in Section 2, it requires to deal with

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \min_{1\leq i\leq n} X_{u,i}(t) > u\right\},$$

where $(X_{u,i}(t), t \in [0,T])_u$, $i = 1, ..., n$ are families (with respect to u) of centered threshold-dependent Gaussian processes; see Theorem 6.2.1.

In Section 6.3 we apply general results derived in Section 2 to two important families of Gaussian processes, i.e. i) to locally-stationary processes in the sense of Berman and ii) to processes with varying variance $\mathrm{var}(X_i(t))$, $t \in [0,T]$. Then, as an example to the derived theory, we analyze the probability of simultaneous ruin in Gaussian risk model. Complementary, we investigate the limit distribution of the *simultaneous ruin time*

$$\tau_u := \inf\{t \geq 0 : (\boldsymbol{X}(t) + \boldsymbol{h}(t)) > u\mathbf{1}\},$$

conditioned that $\tau_u \leq T$, as $u \to \infty$.

Organization of the rest of the paper: Section 2 is devoted to the main result of this contribution, concerning the extremes of the threshold-dependent centered Gaussian vector processes. In Section 3 we specify our result to locally-stationary vector-valued Gaussian processes with trend and non-stationary Gaussian vector-valued processes with trend. Detailed proofs of all the results are postponed to Section 4. Additionally, in Section 3 we analyze asymptotics of the simultaneous ruin probability.

6.2 Main Results

We begin with observation that, for sufficiently large u,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \min_{1\leq i\leq n} (X_i(t) + h_i(t)) > u\right\} = \mathbb{P}\left\{\exists_{t\in[0,T]} \boldsymbol{X}_u(t) > u\mathbf{1}\right\}, \tag{6.3}$$

where $\boldsymbol{X}_u(t) = \left(\frac{uX_1(t)}{u-h_1(t)}, \ldots, \frac{uX_n(t)}{u-h_n(t)}\right)$ is a family of centered vector-valued threshold-dependent Gaussian processes. Since the above rearrangement appears to be useful for the technique of the proof that we use in order to get the exact asymptotics of (6.2), then in this section we focus on asymptotics of extremes of threshold-dependent vector-valued Gaussian processes.

More specifically, let $\mathbf{X}_u(t) := (X_{u,1}(t), \ldots, X_{u,n}(t)), t \in E(u)$, with $0 \in E(u) = (x_1(u), x_2(u))$, be a family of centered n-dimensional vector-valued Gaussian processes with continuous trajectories. Let $\sigma^2_{u,i}(\cdot)$ and $r_{u,i}(\cdot,\cdot)$ be the variance function and the correlation function of $X_{u,i}(t)$, $1 \leq i \leq n$ respectively. Moreover, we tacitly assume that $X_{u,i}(t)$, $1 \leq i \leq n$ are mutually independent.

We shall impose the following assumptions on $\mathbf{X}_u(t)$:

A1: $\lim_{u\to\infty} \boldsymbol{\sigma}_u(0) = \boldsymbol{\sigma} > \mathbf{0}$.

A2: There exist $\lambda_i \in [0,\infty), 1 \leq i \leq n$ with $\max_{1\leq i\leq n} \lambda_i > 0$ and some continuous functions $f_i(\cdot), 1 \leq i \leq n$ with $f_i(0) = 0$ such that for any $\epsilon \in (0,1)$, as $u \to \infty$,

$$\left|\left(\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1\right) u^2 - f_i(u^{\lambda_i} t)\right| \leq \epsilon(\left|f_i(u^{\lambda_i} t)\right| + 1), \quad t \in E(u).$$

A3: There exist $\alpha_i \in (0,2]$ and $a_i > 0,\ 1 \le i \le n$ such that

$$\lim_{u\to\infty} \sup_{\substack{s,t\in E(u)\\ t\neq s}} \left| \frac{1-r_{u,i}(t,s)}{a_i|t-s|^{\alpha_i}} - 1 \right| = 0.$$

In the following we write $f \in \mathcal{R}_\alpha$ to denote that function f is regularly varying at ∞ with index α, see [69, 129, 136] for the definition and properties of regularly varying functions.
Let $\lambda := \max_{1\le i\le n} \lambda_i$, $\alpha := \min_{1\le i\le n} \alpha_i$, $\widetilde{\boldsymbol{f}}(t) := \left(\widetilde{f}_1(t), \ldots, \widetilde{f}_n(t)\right)$ with

$$\widetilde{f}_i(t) = f_i(t)\, \mathbb{I}_{\{\lambda_i=\lambda\}}$$

and suppose that $x_1(u) \in \mathcal{R}_{-\mu_1},\ x_2(u) \in \mathcal{R}_{-\mu_2}$ with $\mu_1, \mu_2 \ge \lambda$ and

$$\begin{aligned} &\lim_{u\to\infty} u^{\lambda} x_1(u) = x_1 \in [-\infty, \infty), \\ &\lim_{u\to\infty} u^{\lambda} x_2(u) = x_2 \in (-\infty, \infty],\ x_1 < x_2, \\ &\lim_{u\to\infty} u^{\lambda_j} x_i(u) = 0, i = 1, 2, \lambda_j < \lambda. \end{aligned} \tag{6.4}$$

If $|x_1| + |x_2| = \infty$, we additionally assume that

$$\liminf_{\substack{|t|\to\infty\\ t\in[x_1,x_2]}} \left(\sum_{i=1}^{n} \frac{\widetilde{f}_i(t)}{\sigma_i^2} \right) \Big/ \left(\sum_{i=1}^{n} \frac{|\widetilde{f}_i(t)|}{\sigma_i^2} \right) > 0. \tag{6.5}$$

Assumption (6.5) means that the negative components of $\frac{\widetilde{f}_i(t)}{\sigma_i^2}, 1 \le i \le n$ do not play a significant role to the sum in comparison with the positive components.
Moreover, we suppose that $0 \cdot \infty = 0$, $u^{-\infty} = 0$ for any $u > 0$ and introduce

$$[x_1, x_2] := \lim_{u\to\infty} f(u)[x_1(u), x_2(u)],$$

if $\lim_{u\to\infty} f(u)x_1(u) = x_1 \in [-\infty, \infty)$ and $\lim_{u\to\infty} f(u)x_2(u) = x_2 \in (-\infty, \infty]$ with $x_1 < x_2$.

Next we introduce some notation and definition of the vector-valued version Pickands-Piterbarg constants.

Throughout this paper, all the operations on vectors are meant componentwise, for instance, for any given $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$, we write $\boldsymbol{x} > \boldsymbol{y}$ if and only if $x_i > y_i$ for all $1 \le i \le n$, write $1/\boldsymbol{x} = (1/x_1, \cdots, 1/x_n)$ if $x_i \neq 0, 1 \le i \le n$, and write $\boldsymbol{xy} = (x_1y_1, \ldots, x_ny_n)$. Further we set $\boldsymbol{0} := (0, \ldots, 0) \in \mathbb{R}^n$ and $\boldsymbol{1} := (1, \ldots, 1) \in \mathbb{R}^n$.
Define for $S_1, S_2 \in \mathbb{R},\ S_1 < S_2$, $\boldsymbol{a} = (a_1, a_2, \ldots, a_n)$ with $a_i \ge 0,\ 1 \le i \le n$ and $\boldsymbol{f}(t) = (f_1(t), \ldots, f_n(t))$ with $f_i(t),\ 1 \le i \le n$ being continuous functions

$$\begin{aligned} \mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}[S_1, S_2] &:= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i} \mathbb{P}\left\{ \exists_{t\in[S_1,S_2]} \left(\sqrt{2\boldsymbol{a}}\boldsymbol{B}_\alpha(t) - \boldsymbol{a}|t|^\alpha - \boldsymbol{f}(t) \right) > \boldsymbol{w} \right\} d\boldsymbol{w} \\ &= \int_{\mathbb{R}^n} e^{\sum_{i=1}^n w_i} \mathbb{P}\left\{ \sup_{t\in[S_1,S_2]} \left(\min_{1\le i\le n} \sqrt{2a_i} B_{\alpha,i}(t) - a_i|t|^\alpha - f_i(t) - w_i \right) > 0 \right\} d\boldsymbol{w} \in (0, \infty), \end{aligned}$$

where $\boldsymbol{B}_\alpha(t), t \in \mathbb{R}$ is an n-dimensional vector-valued standard fractional Brownian motion (fBm) with mutually independent coordinates $B_{\alpha,i}(t)$ and common Hurst index $\alpha/2 \in (0, 1]$. Let

$$\mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}[0, \infty) := \lim_{S_2\to\infty} \mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}[0, S_2], \quad \mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}(-\infty, \infty) := \lim_{S_1\to-\infty, S_2\to\infty} \mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}[S_1, S_2].$$

Let, for $\boldsymbol{a} > 0$,

$$\mathcal{H}_{\alpha,\boldsymbol{a}} = \lim_{T\to\infty} \frac{1}{T} \mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{0}}[0, T].$$

Finiteness of $\mathcal{H}_{\alpha,\boldsymbol{a}}$, $\mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}[0, \infty)$ and $\mathcal{P}_{\alpha,\boldsymbol{a}}^{\boldsymbol{f}}(-\infty, \infty)$ is guaranteed under some restrictions on $\boldsymbol{f}(\cdot)$ which are satisfied in our setup; see [57, 12, 13]. We refer to, e.g., [13, 116, 118, 47, 63, 37, 65, 40, 121, 64, 44, 79, 58, 50] for properties of the above constants.

Let $\mathbf{I}_{\{\boldsymbol{a}=\boldsymbol{b}\}} := (\mathbb{I}_{\{a_1=b_1\}}, \dots, \mathbb{I}_{\{a_n=b_n\}})$.

Theorem 6.2.1. *Let $\mathbf{X}_u(t), t \in E(u)$ be a family of centered vector-valued Gaussian processes with continuous trajectories and independent coordinates satisfying* **A1-A3** *and (6.4)-(6.5) holds. Let further $\boldsymbol{m}_u$ be a vector function of u with $\lim_{u\to\infty} \frac{\boldsymbol{m}_u}{u} = \mathbf{1}$ and for $j \in \{1 \le i \le n : \lambda_i = \lambda\}$, $f_j(t)$ be regularly varying at $\pm\infty$ with positive index. Then we have*

$$\mathbb{P}\left\{\exists_{t\in E(u)} \boldsymbol{X}_u(t) > \boldsymbol{m}_u\right\} \sim u^{(\frac{2}{\alpha}-\lambda)_+} \prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right) \times \begin{cases} \mathcal{H}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}} \int_{x_1}^{x_2} e^{-\sum_{i=1}^{n} \frac{\widetilde{f}_i(t)}{\sigma_i^2}} dt, & \text{if } \lambda < 2/\alpha, \\ \mathcal{P}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}^{\frac{\widetilde{\boldsymbol{f}}}{\boldsymbol{\sigma}^2}} [x_1, x_2], & \text{if } \lambda = 2/\alpha, \\ \int_{\mathbb{R}^n} e^{\sum_{i=1}^{n} w_i} \mathbb{I}_{\left\{\exists_{t\in[x_1,x_2]} - \frac{\widetilde{\boldsymbol{f}}(t)}{\boldsymbol{\sigma}^2} > \boldsymbol{w}\right\}} d\boldsymbol{w}, & \text{if } \lambda > 2/\alpha. \end{cases}$$

6.3 Applications

In this section we apply Theorem 6.2.1 to the analysis of the exact asymptotics of

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\mathbf{1}\right\},$$

as $u \to \infty$. We distinguish two classes of processes $\boldsymbol{X}$: processes with non-stationary coordinates and processes with locally-stationary coordinates, including strictly stationary case.

6.3.1 Non-stationary coordinates

Let $\mathbf{X}(t), t \ge 0$ be a centered vector-valued Gaussian process with independent coordinates. Suppose that $\sigma_i(\cdot), 1 \le i \le n$ attains its maximum on $[0, T]$ at the unique point $t_0 \in [0, T]$, and further

$$\sigma_i(t) = \sigma_i(t_0) - b_i |t - t_0|^{\beta_i}(1 + o(1)), \quad t \to t_0 \tag{6.6}$$

with $b_i > 0, \beta_i > 0$, and

$$r_i(s, t) = 1 - a_i |t - s|^{\alpha_i}(1 + o(1)), \quad s, t \to t_0 \tag{6.7}$$

for some constants $a_i > 0$ and $\alpha_i \in (0, 2]$. We further assume that there exists $\mu_1 > 0$ such that

$$\max_{i=1,\dots,n} \sup_{s\ne t, s,t\in[0,T]} \frac{\mathbb{E}\left((X_i(t) - X_i(s))^2\right)}{|t - s|^{\mu_1}} < \infty. \tag{6.8}$$

Let $\boldsymbol{h}(t)$ be a continuous vector function over $[0, T]$ satisfying

$$h_i(t) = h_i(t_0) - c_i |t - t_0|^{\gamma_i}(1 + o(1)), \quad t \to t_0 \tag{6.9}$$

with $c_i < 0$ and $\gamma_i \ge \frac{\beta_i}{2}$; and $c_i \ge 0$ and $\gamma_i > 0$. Moreover, there exists $\mu_2 > 0$ such that

$$\max_{i=1,\dots,n} \sup_{s\ne t, s,t\in[0,T]} \frac{|h_i(t) - h_i(s)|}{|t - s|^{\mu_2}} < \infty. \tag{6.10}$$

Theorem 6.3.1. *Suppose that $\mathbf{X}(t), t \ge 0$ is a centered vector-valued Gaussian process with independent coordinates satisfying (6.6)-(6.8), and $\mathbf{h}(t),\ t \ge 0$ is a continuous vector function over $[0, T]$ satisfying (6.9)-(6.10). Then*

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\mathbf{1}\right\} \quad \sim \quad u^{(\frac{2}{\alpha}-\frac{2}{\beta})_+} \prod_{i=1}^{n} \Psi\left(\frac{u - h_i(t_0)}{\sigma_i(t_0)}\right)$$

$$\times \begin{cases} \mathcal{H}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}} \int_q^{\infty} e^{-\sum_{i=1}^n f_i(x)} dx, & \textit{if } \alpha < \beta, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}}[q, \infty), & \textit{if } \alpha = \beta, \\ 1, & \textit{if } \alpha > \beta, \end{cases}$$

where $\alpha = \min_{1 \le i \le n} \alpha_i$, $\beta = \min_{1 \le i \le n} \min(\beta_i, 2\gamma_i \mathbb{I}_{\{c_i \neq 0\}} + \infty \mathbb{I}_{\{c_i = 0\}})$, $\boldsymbol{a} = (a_1, \ldots, a_n)$, $\boldsymbol{\sigma}(t_0) = (\sigma_1(t_0), \ldots, \sigma_n(t_0))$, $\boldsymbol{f} = (f_1, \ldots, f_n)$ *with* $f_i(t) = \frac{b_i}{\sigma_i^3(t_0)} |t|^{\beta_i} \mathbb{I}_{\{\beta_i = \beta\}} + \frac{c_i}{\sigma_i^2(t_0)} |t|^{\gamma_i} \mathbb{I}_{\{2\gamma_i = \beta\}}$, *and*

$$q = \begin{cases} -\infty, & \textit{if } t_0 \in (0, T), \\ 0, & \textit{if } t_0 = 0 \textit{ or } t_0 = T. \end{cases} \tag{6.11}$$

Remark. If $n = 1$ and $h_1(t) \equiv 0$, then Theorem 6.3.1 covers the classical Piterbarg-Prisjažnjuk result; see [123].

In the following corollary we apply Theorem 6.3.1 for the analysis of exact asymptotics of $\tau_u = \inf\{t \ge 0 : (\boldsymbol{X}(t) + \boldsymbol{h}(t)) > u\mathbf{1}\}$, as $u \to \infty$, conditioned that $\tau_u \le T$.

Corollary 6.3.1. *Under the same assumptions as in Theorem 6.3.1 with* $t_0 = T$, *we have for* $x \in (0, \infty)$, *as* $u \to \infty$,

$$\mathbb{P}\left\{(T - \tau_u) u^{2/\beta} \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \int_0^x e^{-\sum_{i=1}^n f_i(t)} dt \Big/ \int_0^{\infty} e^{-\sum_{i=1}^n f_i(t)} dt, & \textit{if } \alpha < \beta, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}}[0, x] \Big/ \mathcal{P}^{\boldsymbol{f}}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}}[0, \infty), & \textit{if } \alpha = \beta, \\ 1, & \textit{if } \alpha > \beta. \end{cases} \tag{6.12}$$

We give a short proof of Corollary 6.3.1 in Appendix.

6.3.2 Locally-stationary coordinates

Suppose that for each $i = 1, ..., n$, X_i is a centered locally-stationary Gaussian process with continuous trajectories, that is process with unit variance and correlation function $r_i(\cdot, \cdot), 1 \le i \le n$ satisfying

$$r_i(t, t+s) = 1 - a_i(t)|s|^{\alpha_i} + o(|s|^{\alpha_i}), \quad s \to 0 \tag{6.13}$$

uniformly with respect to $t \in [0, T]$, where $\alpha_i \in (0, 2]$, and $a_i(t) \in (0, \infty)$ is a positive continuous function on $[0, T]$. Further, we suppose that

$$r_i(s, t) < 1, \ \forall s, t \in [0, T] \text{ and } s \neq t. \tag{6.14}$$

We refer to e.g., [16, 18, 87, 119] for the investigation of extremes of one-dimensional locally-stationary Gaussian processes under the above conditions.

Denote by

$$H = \bigcap_{i=1}^n \left\{ s \in [0, T] : h_i(s) = h_{m,i} := \max_{t \in [0, T]} h_i(t) \right\}.$$

Theorem 6.3.2. *Let* $\mathbf{X}(t), t \in [0, T]$ *be a locally stationary vector-valued Gaussian process satisfying (6.13) and (6.14). Moreover, assume that* $\boldsymbol{h}(t)$ *is a vector function satisfying (6.10) and* $\alpha = \min_{1 \le i \le n} \alpha_i$.
i) If $H = \{t_0\}$ *and (6.9) holds with* $c_i \ge 0$ *and* $\max_{1 \le i \le n} c_i > 0$, *then*

$$\mathbb{P}\left\{\exists_{t \in [0,T]} (\boldsymbol{X}(t) + \boldsymbol{h}(t)) > u\mathbf{1}\right\} \sim u^{(\frac{2}{\alpha} - \frac{1}{\gamma})_+} \prod_{i=1}^n \Psi(u - h_{m,i}) \begin{cases} \mathcal{H}_{\alpha, \boldsymbol{a}(t_0) \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}} \int_q^{\infty} e^{-\sum_{i=1}^n f_i(x)} dx, & \textit{if } \alpha < 2\gamma, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha, \boldsymbol{a}(t_0) \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}}[q, \infty), & \textit{if } \alpha = 2\gamma, \\ 1, & \textit{if } \alpha > 2\gamma, \end{cases}$$

where $\gamma = \min_{1 \le i \le n}(\gamma_i \mathbb{I}_{\{c_i \neq 0\}} + \infty \mathbb{I}_{\{c_i = 0\}})$, $f_i(t) = c_i |t|^{\gamma} \mathbb{I}_{\{\gamma_i = \gamma\}}$, *and* q *is given by (6.11).*
ii) If $H = [A, B] \subset [0, T]$ *with* $A > B$, *then*

$$\mathbb{P}\left\{\exists_{t \in [0,T]} (\boldsymbol{X}(t) + \boldsymbol{h}(t)) > u\mathbf{1}\right\} \sim \int_A^B \mathcal{H}_{\alpha, \boldsymbol{a}(t) \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha \mathbf{1}\}}} dt \, u^{\frac{2}{\alpha}} \prod_{i=1}^n \Psi(u - h_{m,i}).$$

Similarly to Corollary 6.3.1, we get the asymptotics of τ_u for locally-stationary coordinates of $\boldsymbol{X}$.

Corollary 6.3.2. *Under the same assumptions as in i) of Theorem 6.3.2, with $t_0 = T$, we have for $x \in (0,\infty)$, as $u \to \infty$,*

$$\mathbb{P}\left\{(T-\tau_u)u^{1/\gamma} \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \int_0^x e^{-\sum_{i=1}^n f_i(t)}dt \Big/ \int_0^\infty e^{-\sum_{i=1}^n f_i(t)}dt, & \text{if } \alpha < 2\gamma, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{a}(t_0)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,x] \Big/ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{a}(t_0)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,\infty), & \text{if } \alpha = 2\gamma, \\ 1, & \text{if } \alpha > 2\gamma. \end{cases} \tag{6.15}$$

6.3.3 A simultaneous ruin model

Consider portfolio $\boldsymbol{U}(t) = (U_1(t), \dots, U_n(t))$, where

$$\boldsymbol{U}(t) = u\boldsymbol{d} + \boldsymbol{c}t - \boldsymbol{B}_{\boldsymbol{\alpha}}(t), \quad t \ge 0,$$

with $\boldsymbol{c} = (c_1, \cdots, c_n) \in \mathbb{R}^n$, $\boldsymbol{d} = (d_1, \cdots, d_n) > \mathbf{0}$ and $B_{\alpha_i}(t),\ 1 \le i \le n$, independent standard fractional Brownian motions with variance $\mathrm{var}(B_{\alpha_i}(t)) = t^{\alpha_i}$ for $\alpha_i \in (0,2],\ 1 \le i \le n$, respectively. The corresponding simultaneous ruin probability over $[0,T]$ is defined as

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \boldsymbol{U}(t) < \mathbf{0}\right\}$$

and the simultaneous ruin time $\tau_u := \inf\{t \ge 0 : \boldsymbol{U}(t) < \mathbf{0}\}$. We refer to, e.g., [113] for theoretical justification of the use of fractional Brownian motion as the approximation of *the claim* process in risk theory.

In the following proposition we present exact asymptotics of the simultaneous ruin probability and the conditional simultaneous ruin time $\tau_u|\tau_u < T$, as $u \to \infty$.

Proposition 6.3.1. *For $T \in (0,\infty)$, $\alpha = \min_{1\le i\le n} \alpha_i$, $b_i = \frac{d_i^2}{2T^{2\alpha_i}}$ and $f_i(t) = \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}}t$, as $u \to \infty$, we have*

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \boldsymbol{U}(t) < \mathbf{0}\right\} \sim u^{(\frac{2}{\alpha}-2)_+} \prod_{i=1}^n \Psi\left(\frac{d_i u + c_i T}{T^{\alpha_i/2}}\right) \tag{6.16}$$

$$\times \begin{cases} \left(\sum_{i=1}^n \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}}\right)^{-1} \mathcal{H}_{\alpha,\boldsymbol{b}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}, & \text{if } \alpha < 1, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{b}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,\infty), & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha > 1 \end{cases}$$

and for $x \in (0,\infty)$

$$\mathbb{P}\left\{(T-\tau_u)u^2 \le x \middle| \tau_u \le T\right\} \sim \begin{cases} 1 - e^{-\left(\sum_{i=1}^n \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}}\right)x}, & \text{if } \alpha < 1, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{b}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,x] \Big/ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{b}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,\infty), & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha > 1. \end{cases} \tag{6.17}$$

Specifically, Proposition 6.3.1 allows us to get exact asymptotics for multidimensional counterpart of the classical Brownian risk model [93]. For simplicity we focus on 2-dimensional case. Let $\boldsymbol{B}(t) := (B^{(1)}(t), B^{(2)}(t))$, where $B^{(1)}(t)$ and $B^{(2)}(t)$ are two independent standard Brownian motions, $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2$ and $\boldsymbol{d} = (d_1, d_2) \in \mathbb{R}^2_+$. Then we have, as $u \to \infty$,

$$\mathbb{P}\left\{\exists_{t\in[0,T]} \begin{pmatrix} d_1 u + c_1 t - B^{(1)}(t) \\ d_2 u + c_2 t - B^{(2)}(t) \end{pmatrix} \le \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right\} \sim \mathcal{P}^{\boldsymbol{b}t}_{1,\boldsymbol{b}}[0,\infty)\Psi\left(\frac{d_1 u + c_1 T}{T^{1/2}}\right)\Psi\left(\frac{d_2 u + c_2 T}{T^{1/2}}\right)$$

and for $x \in (0,\infty)$

$$\mathbb{P}\left\{(T-\tau_u)u^2 \le x \middle| \tau_u \le T\right\} \sim \mathcal{P}^{\boldsymbol{b}t}_{1,\boldsymbol{b}}[0,x] \Big/ \mathcal{P}^{\boldsymbol{b}t}_{1,\boldsymbol{b}}[0,\infty),$$

where $\boldsymbol{b} = \left(\frac{d_1^2}{2T^2}, \frac{d_2^2}{2T^2}\right)$.

6.4 Proofs

Before proceeding to the proof of Theorem 6.2.1, we present two lemmas which play an important role in the proof of Theorem 6.2.1. The first one is a vector-valued version of the uniform Pickands-Piterbarg lemma while the second one gives an upper bound for the double maximum of vector-valued Gaussian process. Hereafter, we denote by $\mathbb{C}_l, l \in \mathbb{N}$ some positive constants that may differ from line to line. Moreover, the notation $f(u, S, \epsilon) \sim g(u)$ as $u \to \infty, S \to \infty, \epsilon \to 0$, means that $\lim_{\epsilon\to 0} \lim_{S\to\infty} \lim_{u\to\infty} \frac{f(u,S,\epsilon)}{g(u)} = 1$.
For $\boldsymbol{b} \geq \boldsymbol{0}$, $\lambda_i \in [0, \infty)$, and $-\infty < S_1 < S_2 < \infty$, define a vector-valued Gaussian process $\boldsymbol{Z}_u(t) = (Z_{u,1}(t), \dots, Z_{u,n}(t))$ by

$$Z_{u,i}(t) = \frac{\xi_i(t)}{1 + b_i u^{-2} f_i(u^{\lambda_i} t)}, \ t \in [S_1, S_2], \ i = 1, \dots, n, \tag{6.18}$$

where $\boldsymbol{\xi}(t) = (\xi_1(t), \dots, \xi_n(t)), t \in \mathbb{R}$ is a vector-valued Gaussian process with independent stationary coordinates, continuous sample paths, unit variance and correlation function $r_i(\cdot)$ on i-th coordinate, $1 \leq i \leq n$, satisfying

$$1 - r_i(t) = a_i |t|^{\alpha_i}(1 + o(1)), \tag{6.19}$$

for $a_i > 0$ and $\alpha_i \in (0, 2]$, and $f_i(t), 1 \leq i \leq n$ are some continuous functions. We suppose that the threshold vector $\boldsymbol{m}_u(k) = (m_{u,1}(k), \dots, m_{u,n}(k))$ satisfies

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \frac{1}{u} \boldsymbol{m}_u(k) - \boldsymbol{c} \right| = 0, \quad \boldsymbol{c} > \boldsymbol{0}, \tag{6.20}$$

with K_u a family of countable index sets.
Denote by

$$\alpha = \min_{1\leq i\leq n} \alpha_i, \quad \lambda = \max_{1\leq i\leq n} (\lambda_i \mathbb{I}_{\{b_i \neq 0\}}) > 0, \quad \widetilde{\boldsymbol{f}}(t) = (\widetilde{f}_1(t), \dots, \widetilde{f}_n(t)), \quad \text{with} \quad \widetilde{f}_i(t) = f_i(t) \mathbb{I}_{\{\lambda_i = \lambda\}}.$$

Lemma 6.4.1. *Let $\boldsymbol{Z}_u(t)$ be defined in (6.18) and $\boldsymbol{m}_u(k)$ satisfy (6.20).*
i) If $\lambda \leq 2/\alpha$, then

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \frac{\mathbb{P}\left\{\exists_{t\in[u^{-2/\alpha}S_1, u^{-2/\alpha}S_2]} \boldsymbol{Z}_u(t) > \boldsymbol{m}_u(k)\right\}}{\prod_{i=1}^n \Psi(m_{u,i}(k))} - \mathcal{R}_\lambda^f[S_1, S_2] \right| = 0,$$

where

$$\mathcal{R}_\lambda^f[S_1, S_2] = \begin{cases} \mathcal{P}_{\alpha, \boldsymbol{a}\boldsymbol{c}^2 \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}^{\boldsymbol{c}^2 \widetilde{\boldsymbol{f}}}[S_1, S_2], & \text{if } \lambda = 2/\alpha, \\ \mathcal{P}_{\alpha, \boldsymbol{a}\boldsymbol{c}^2 \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}^{\boldsymbol{c}^2 \widetilde{\boldsymbol{f}}(0)}[S_1, S_2], & \text{if } \lambda < 2/\alpha, \\ \mathcal{H}_{\alpha, \boldsymbol{a}\boldsymbol{c}^2 \mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[S_1, S_2], & \text{if } \boldsymbol{b} = \boldsymbol{0}. \end{cases}$$

ii) If $\lambda > 2/\alpha$, then

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \frac{\mathbb{P}\left\{\exists_{t\in[u^{-\lambda}S_1, u^{-\lambda}S_2]} \boldsymbol{Z}_u(t) > \boldsymbol{m}_u(k)\right\}}{\prod_{i=1}^n \Psi(m_{u,i}(k))} - \mathcal{P}_{\alpha, \boldsymbol{0}}^{\boldsymbol{c}^2 \widetilde{\boldsymbol{f}}}[S_1, S_2] \right| = 0.$$

Proof. i) Suppose that $\lambda \leq 2/\alpha$. Conditioning on$\left\{\boldsymbol{\xi}(0) = \boldsymbol{m}_u(k) - \frac{\boldsymbol{w}}{\boldsymbol{m}_u(k)}\right\}$, $\boldsymbol{w} \in \mathbb{R}^n$, we have for all u large enough

$$\begin{aligned} &\frac{\mathbb{P}\left\{\exists_{t\in[u^{-2/\alpha}S_1, u^{-2/\alpha}S_2]} \boldsymbol{Z}_u(t) > \boldsymbol{m}_u(k)\right\}}{\prod_{i=1}^n \Psi(m_{u,i}(k))} \\ &= \frac{1}{\prod_{i=1}^n \sqrt{2\pi} m_{u,i}(k) \Psi(m_{u,i}(k))} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_{i=1}^n \left(m_{u,i}(k) - \frac{w_i}{m_{u,i}(k)}\right)^2} \\ &\times \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{Z}_u(u^{-2/\alpha}t) > \boldsymbol{m}_u(k) \middle| \boldsymbol{\xi}(0) = \boldsymbol{m}_u(k) - \frac{\boldsymbol{w}}{\boldsymbol{m}_u(k)}\right\} d\boldsymbol{w} \end{aligned}$$

$$= \left(\prod_{i=1}^{n} \frac{e^{-\frac{(m_{u,i}(k))^2}{2}}}{\sqrt{2\pi} m_{u,i}(k)\Psi(m_{u,i}(k))} \right) \int_{\mathbb{R}^n} e^{\sum_{i=1}^n \left(w_i - \frac{w_i^2}{2(m_{u,i}(k))^2}\right)} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) > \boldsymbol{w}\right\} d\boldsymbol{w}$$
$$= \left(\prod_{i=1}^{n} \frac{e^{-\frac{(m_{u,i}(k))^2}{2}}}{\sqrt{2\pi} m_{u,i}(k)\Psi(m_{u,i}(k))} \right) I_{u,k},$$

where $\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) = (\mathcal{X}_{u,1}^w(t,k), \ldots, \mathcal{X}_{u,n}^w(t,k))$ with

$$\mathcal{X}_{u,i}^w(t,k) = m_{u,i}(k)(Z_{u,i}(u^{-2/\alpha}t) - m_{u,i}(k)) + w_i \Big| \xi_i(0) = m_{u,i}(k) - \frac{w_i}{m_{u,i}(k)}.$$

By (6.20), it follows that

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \left(\prod_{i=1}^{n} \frac{e^{-\frac{(m_{u,i}(k))^2}{2}}}{\sqrt{2\pi} m_{u,i}(k)\Psi(m_{u,i}(k))} \right) - 1 \right| = 0.$$

Thus in order to establish the proof, it suffices to prove that

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| I_{u,k} - \mathcal{R}_\lambda^f[S_1,S_2] \right| = 0. \tag{6.21}$$

It follows that, for each $W>0$, with $\widetilde{W}^n = [-W,W]^n$ and $\widetilde{W}_j^n = \{\boldsymbol{w}\in\mathbb{R}^n \big| w_j \in (-\infty,-W)\cup(W,\infty)\}$,

$$\begin{aligned}
&\sup_{k\in K_u} \left| I_{u,k} - \mathcal{R}_\lambda^f[S_1,S_2] \right| \\
&\le \sup_{k\in K_u} \left| \int_{\widetilde{W}^n} \left[e^{\sum_{i=1}^n \left(w_i - \frac{w_i^2}{2m_{u,i}^2(k)}\right)} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) > \boldsymbol{w}\right\} - e^{\sum_{i=1}^n w_i} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\zeta}(t) > \boldsymbol{w}\right\} \right] d\boldsymbol{w} \right| \\
&\quad + \sum_{j=1}^n \sup_{k\in K_u} \int_{\widetilde{W}_j^n} e^{\sum_{i=1}^n w_i} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) > \boldsymbol{w}\right\} d\boldsymbol{w} \\
&\quad + \sum_{j=1}^n \int_{\widetilde{W}_j^n} e^{\sum_{i=1}^n w_i} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\zeta}(t) > \boldsymbol{w}\right\} d\boldsymbol{w} \\
&:= I_1(u) + I_2(u) + I_3(u),
\end{aligned}$$

where $\boldsymbol{\zeta}(t) = (\boldsymbol{c}\sqrt{2\boldsymbol{a}}\boldsymbol{B}_\alpha - \boldsymbol{a}\boldsymbol{c}^2|t|^\alpha)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}} - \boldsymbol{c}^2\widetilde{\boldsymbol{f}}(t\mathbb{I}_{\{\lambda=2/\alpha\}})$.
Next, we give upper bounds for $I_i(u), i=1,2,3$. We begin with the weak convergence of process $\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k)$.
Weak convergence of $\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k)$. Direct calculation shows that

$$\begin{aligned}
\mathbb{E}\left\{(1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k)\right\} &= -m_{u,i}^2(k)\left(1 - r_i(u^{-2/\alpha}t) + b_iu^{-2}f_i(u^{\lambda_i-2/\alpha}t)\right) \\
&\quad + w_i\left(1 - r_i(u^{-2/\alpha}t) + b_iu^{-2}f_i(u^{\lambda_i-2/\alpha}t)\right),
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{var}\left((1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k) - (1+b_iu^{-2}f_i(u^{\lambda_i}t'))\mathcal{X}_{u,i}^w(t',k)\right) \\
&= m_{u,i}^2(k)\left(Var\left(\xi_i(u^{-2/\alpha}t) - \xi_i(u^{-2/\alpha}t')\right) - \left(r_i(u^{-2/\alpha}t) - r_i(u^{-2/\alpha}t')\right)^2\right).
\end{aligned}$$

By (6.19) and (6.20), it follows that

$$\mathbb{E}\left\{(1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k)\right\} \to -c_i^2a_i|t|^\alpha\mathbb{I}_{\{\alpha_i=\alpha\}} - c_i^2\left(\widetilde{f}_i(t\mathbb{I}_{\{\lambda=2/\alpha\}})\right), \tag{6.22}$$

as $u\to\infty$, uniformly with respect to $t\in[S_1,S_2], k\in K_u, w_i\in[-W,W]$. Moreover, for any $t,t'\in[S_1,S_2]$ uniformly with respect to $k\in K_u$, any $w_i\in\mathbb{R}$,

$$\mathrm{var}\left((1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k) - (1+b_iu^{-2}f_i(u^{\lambda_i}t'))\mathcal{X}_{u,i}^w(t',k)\right) \to 2c_i^2a_i|t-t'|^\alpha\mathbb{I}_{\{\alpha_i=\alpha\}}, \tag{6.23}$$

as $u \to \infty$. Combination of (6.22) and (6.23) shows that the finite-dimensional distributions of

$$\{(\mathbf{1} + \boldsymbol{b}u^{-2}\boldsymbol{f}(u^{\boldsymbol{\lambda}}t))\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k), t \in [S_1, S_2]\}$$

weakly converge to the finite-dimensional distributions of $\{\boldsymbol{\zeta}(t), t \in [S_1, S_2]\}$. Moreover, by (6.19) we have that there exists a constant $C > 0$ such that for all $t, t' \in [S_1, S_2]$ and all large u

$$\begin{aligned}
&\sup_{k \in K_u} \operatorname{var}\left((1 + b_i u^{-2} f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k) - (1 + b_i u^{-2} f_i(u^{\lambda_i}t'))\mathcal{X}_{u,i}^w(t',k)\right) \\
&\quad \le m_{u,i}^2(k) Var\left(\xi_i(u^{-2/\alpha}t) - \xi_i(u^{-2/\alpha}t')\right) \le C|t - t'|^\alpha,
\end{aligned} \tag{6.24}$$

which combined with (6.22) implies that the family of distributions

$$\mathbb{P}\left\{(\mathbf{1} + \boldsymbol{b}u^{-2}\boldsymbol{f}(u^{\boldsymbol{\lambda}}t))\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) \in (\cdot)\right\}$$

is uniformly tight with respect to $k \in K_u$ and $\boldsymbol{w}$ in a compact set of $\mathbb{R}^n$. Consequently,

$$\{(\mathbf{1} + \boldsymbol{b}u^{-2}\boldsymbol{f}(u^{\boldsymbol{\lambda}}t))\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k), t \in [S_1, S_2]\} \quad \text{weakly converges to} \quad \{\boldsymbol{\zeta}(t), t \in [S_1, S_2]\}.$$

Since

$$\lim_{u \to \infty} \max_{1 \le i \le n} \sup_{k \in K_u} \sup_{t \in [S_1, S_2]} \left|(1 + b_i u^{-2} f_i(u^{\lambda_i}t)) - 1\right| = 0,$$

we conclude that

$$\{\boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k), t \in [S_1, S_2]\} \quad \text{weakly converges to} \quad \{\boldsymbol{\zeta}(t), t \in [S_1, S_2]\}.$$

<u>Upper bound for $I_1(u)$</u>. We first show that

$$\begin{aligned}
c_u(\boldsymbol{w}) :&= \sup_{k \in K_u} \left|\mathbb{P}\left\{\exists_{t \in [S_1, S_2]} \boldsymbol{\mathcal{X}}_u^{\boldsymbol{w}}(t,k) > \boldsymbol{w}\right\} - \mathbb{P}\left\{\exists_{t \in [S_1, S_2]} \boldsymbol{\zeta}(t) > \boldsymbol{w}\right\}\right| \\
&= \sup_{k \in K_u} \left|\mathbb{P}\left\{\sup_{t \in [S_1, S_2]} \min_{1 \le i \le n} (\mathcal{X}_{u,i}^w(t,k) - w_i) > 0\right\} - \mathbb{P}\left\{\sup_{t \in [S_1, S_2]} \min_{1 \le i \le n} (\zeta_i(t) - w_i) > 0\right\}\right| \to 0,
\end{aligned}$$

for almost all $\boldsymbol{w} \in \mathbb{R}^n$. Let

$$\mathbb{A} := \left\{\boldsymbol{v} : \mathbb{P}\left\{\sup_{t \in [S_1, S_2]} \min_{1 \le i \le n} (\zeta_i(t) - v_i) > 0\right\} \text{ is continuous at } \boldsymbol{v}\right\}.$$

Note that if $\boldsymbol{w} \in \mathbb{A}$, then

$$\mathbb{P}\left\{\sup_{t \in [S_1, S_2]} \min_{1 \le i \le n} (\zeta_i(t) - w_i) > x\right\}$$

is continuous with respect to x at $x = 0$. Hence by the continuity of functional $\sup \min$, we have that

$$c_u(\boldsymbol{w}) \to 0,$$

for $\boldsymbol{w} \in \mathbb{A}$ and $mes(\mathbb{A}^c) = 0$. Thus in light of dominated convergence theorem, we have

$$I_1(u) \le e^{nW} \int_{\boldsymbol{w} \in \widetilde{W}^n \cap \mathbb{A}} c_u(\boldsymbol{w}) d\boldsymbol{w} + W^n e^{nW} \sup_{\boldsymbol{w} \in \widetilde{W}^n} \left|1 - e^{-\sum_{i=1}^n \frac{w_i^2}{2m_{u,i}^2(k)}}\right| \to 0, \quad u \to \infty.$$

<u>Upper bound for $I_2(u)$</u>. Using (6.22) and (6.23), for some $\delta \in (0, 1/2)$, $|w_i| > W$ with W sufficiently large and all u large we have

$$\sup_{k \in K_u, t \in [S_1, S_2]} \mathbb{E}\left\{(1 + b_i u^{-2} f_i(u^{\lambda_i}t))\mathcal{X}_{u,i}^w(t,k)\right\} \le \mathbb{C}_1 + \delta|w_i|$$

and

$$\sup_{k\in K_u, t\in[S_1,S_2]} \mathrm{var}\left((1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}^w_{u,i}(t,k)\right) \le \mathbb{C}_2.$$

Moreover, by the mutual independence of $\mathcal{X}^w_{u,i}(t,k), 1\le i\le n$

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[S_1,S_2]}\boldsymbol{\mathcal{X}}^{\boldsymbol{w}}_u(t,k) > \boldsymbol{w}\right\} &= \mathbb{P}\left\{\sup_{t\in[S_1,S_2]}\min_{1\le i\le n}\left(\mathcal{X}^w_{u,i}(t,k)-w_i\right)>0\right\}\\ &\le \mathbb{P}\left\{\min_{1\le i\le n}\left(\sup_{t\in[S_1,S_2]}\mathcal{X}^w_{u,i}(t,k)-w_i\right)>0\right\}\\ &= \prod_{i=1}^n \mathbb{P}\left\{\sup_{t\in[S_1,S_2]}\mathcal{X}^w_{u,i}(t,k)>w_i\right\}.\end{aligned}$$

Consequently, it follows that

$$\sup_{k\in K_u}\int_{\widetilde{W}^n_j} e^{\sum_{i=1}^n w_i}\mathbb{P}\left\{\exists_{t\in[S_1,S_2]}\boldsymbol{\mathcal{X}}^{\boldsymbol{w}}_u(t,k)>\boldsymbol{w}\right\}d\boldsymbol{w} \le J_1\times J_2,$$

where by (6.24) and Theorem 8.1 of [119]

$$\begin{aligned}J_1 &= \sup_{k\in K_u}\int_{|w_j|>W} e^{w_j}\mathbb{P}\left\{\sup_{t\in[S_1,S_2]}\mathcal{X}^w_{u,j}(t,k)>w_j\right\}dw_j\\ &\le \sup_{k\in K_u}\int_{|w_j|>W} e^{w_j}\mathbb{P}\left(\sup_{t\in[S_1,S_2]}\left((1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}^w_{u,j}(t,k)-\mathbb{E}\left\{(1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}^w_{u,j}(t,k)\right\}\right)\right.\\ &\qquad \left. >(1-\delta)|w_j|-\mathbb{C}_1\right)dw_j\\ &\le e^{-W}+\int_W^\infty e^{w_j}\mathbb{C}_3{w_j}^{2/\alpha}\Psi\left(\frac{(1-\delta)w_j-\mathbb{C}_1}{\mathbb{C}_2}\right)dw_j\\ &=: A_1(W)\to 0,\ W\to\infty,\end{aligned}$$

and

$$\begin{aligned}J_2 &= \sup_{k\in K_u}\prod_{\substack{i=1\\ i\ne j}}^n\left(\int_{\mathbb{R}} e^{w_i}\mathbb{P}\left\{\sup_{t\in[S_1,S_2]}\mathcal{X}^w_{u,i}(t,k)>w_i\right\}dw_i\right)\\ &\le \sup_{k\in K_u}\prod_{\substack{i=1\\ i\ne j}}^n\left(e^{W_1}+\right.\\ &\quad \left.+\int_{W_1}^\infty e^{w_i}\mathbb{P}\left(\sup_{t\in[S_1,S_2]}\left((1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}^w_{u,i}(t,k)-\mathbb{E}\left\{(1+b_iu^{-2}f_i(u^{\lambda_i}t))\mathcal{X}^w_{u,i}(t,k)\right\}\right)>(1-\delta)w_i-\mathbb{C}_1\right)dw_i\right)\\ &\le \prod_{\substack{i=1\\ i\ne j}}^n\left(e^{W_1}+\int_{W_1}^\infty e^{w_i}\mathbb{C}_4{w_i}^{2/\alpha}\Psi\left(\frac{(1-\delta)w_i-\mathbb{C}_1}{\mathbb{C}_2}\right)dw_i\right)\le\mathbb{C}_5,\end{aligned}$$

with W_1 some positive constant. Thus we have

$$I_2(u)\le n\mathbb{C}_5A_1(W)\to 0,\ W\to\infty.$$

Upper bound for $I_3(u)$. Borell-TIS inequality (see, e.g., [1]) implies that

$$I_3(u)\to 0,\ u,W\to\infty.$$

Hence (6.21) follows.

ii) Suppose that $\lambda>2/\alpha$. Observe that

$$\frac{\mathbb{P}\left\{\exists_{t\in[u^{-\lambda}S_1,u^{-\lambda}S_2]}\boldsymbol{Z}_u(t)>\boldsymbol{m}_u(k)\right\}}{\prod_{i=1}^n\Psi(m_{u,i}(k))}$$

$$= \left(\prod_{i=1}^n \frac{e^{-\frac{(m_{u,i}(k))^2}{2}}}{\sqrt{2\pi} m_{u,i}(k)\Psi(m_{u,i}(k))}\right) \int_{\mathbb{R}^n} e^{\sum_{i=1}^n \left(w_i - \frac{w_i^2}{2(m_{u,i}(k))^2}\right)} \mathbb{P}\left\{\exists_{t\in[S_1,S_2]} \boldsymbol{\mathfrak{X}}_u^{\boldsymbol{w}}(t,k) > \boldsymbol{w}\right\} d\boldsymbol{w},$$

where $\boldsymbol{\mathfrak{X}}_u^{\boldsymbol{w}}(t,k) = (\mathfrak{X}_{u,1}^w(t,k), \ldots, \mathfrak{X}_{u,n}^w(t,k))$ with

$$\mathfrak{X}_{u,i}^w(t,k) = m_{u,i}(k)(Z_{u,i}(u^{-\lambda}t) - m_{u,i}(k)) + w_i \Big| \xi_i(0) = m_{u,i}(k) - \frac{w_i}{m_{u,i}(k)}.$$

The rest of derivations for this case is the same as given in the proof for case $\lambda \le 2/\alpha$, with exception that

$$\mathbb{E}\left\{(1 + b_i u^{-2} f_i(u^{\lambda_i} t))\mathfrak{X}_{u,i}^w(t,k)\right\} \to -c_i^2 \widetilde{f}_i(t), \ u \to \infty,$$

and

$$\mathrm{var}\left(\mathfrak{X}_{u,i}^w(t,k) - \mathfrak{X}_{u,i}^w(t',k)\right) \to 0, \ u \to \infty.$$

Hence we omit the rest of the proof. □

Lemma 6.4.2. *Let $\boldsymbol{X}(t), (t) \in \mathbb{R}$ be a centered vector-valued stationary Gaussian process with independent coordinates X_i's. Suppose that for each $i = 1, ..., n$, $X_i(t)$ has continuous sample paths, unit variance and correlation function $r_i(\cdot)$, $1 \le i \le n$, satisfying*

$$0 < 1 - 2a_i|t|^{\alpha_i} \le r_i(t) \le 1 - \frac{a_i}{2}|t|^{\alpha_i}, \ a_i > 0, \ \alpha_i \in (0, 2], \tag{6.25}$$

for all $t \in [0, \varepsilon]$ with $0 < \varepsilon < 1$ small enough. Let K_u be a family of countable index sets. Then we have for any $\boldsymbol{m}_u(k)$, $\boldsymbol{w}_u(l)$ such that

$$\lim_{u\to\infty} \sup_{k\in K_u} \left|\frac{1}{u}\boldsymbol{m}_u(k) - \boldsymbol{c}\right| = 0, \ \lim_{u\to\infty} \sup_{l\in K_u} \left|\frac{1}{u}\boldsymbol{w}_u(l) - \boldsymbol{c}\right| = 0,$$

and any $T(k,l) > S > 1$ satisfying $\lim_{u\to\infty} \sup_{k,l\in K_u} \frac{T(k,l)}{u^{2/\alpha}} = 0$, that

$$\mathbb{P}\left\{\exists_{t\in[0,S]u^{-2/\alpha}} \boldsymbol{X}(t) > \boldsymbol{m}_u(k), \exists_{t\in[T(k,l),T(k,l)+S]u^{-2/\alpha}} \boldsymbol{X}(t) > \boldsymbol{w}_u(l)\right\}$$
$$\le F S^{2n} \exp(-G(T(k,l) - S)^\alpha) \prod_{i=1}^n \Psi\left(\frac{m_{u,i}(k) + w_{u,i}(l)}{2}\right)$$

holds uniformly for any $k, l \in K_u$ and all u large where $\alpha = \min_{1\le i\le n}(\alpha_i)$ and F, G are two positive constants.

Proof of Lemma 6.4.2 By the independence of X_i's, we have that

$$\mathbb{P}\left\{\exists_{t\in[0,S]u^{-2/\alpha}} \boldsymbol{X}(t) > \boldsymbol{m}_u(k), \exists_{t\in[T(k,l),T(k,l)+S]u^{-2/\alpha}} \boldsymbol{X}(t) > \boldsymbol{w}_u(l)\right\}$$
$$\le \mathbb{P}\left\{\bigcap_{i=1}^n \left\{\sup_{t\in[0,S]u^{-2/\alpha}} X_i(t) > m_{u,i}(k)\right\}, \bigcap_{i=1}^n \left\{\sup_{t\in[T(k,l),T(k,l)+S]u^{-2/\alpha}} X_i(t) > w_{u,i}(k)\right\}\right\}$$
$$\le \prod_{i=1}^n \mathbb{P}\left\{\sup_{t\in[0,S]u^{-2/\alpha}} X_i(t) > m_{u,i}(k), \sup_{t\in[T(k,l),T(k,l)+S]u^{-2/\alpha}} X_i(t) > w_{u,i}(k)\right\}.$$

Application of Lemma 6.3 in [119] (or Theorem 3.1 in [60]) for each term in the above product establishes the claim. □

Proof of Theorem 6.2.1 Let

$$\pi(u) := \mathbb{P}\left\{\exists_{t\in E(u)} \boldsymbol{X}_u(t) > \boldsymbol{m}_u\right\} = \mathbb{P}\left\{\exists_{t\in E(u)} \frac{\boldsymbol{X}_u(t)}{\boldsymbol{\sigma}_u(t)} \frac{\boldsymbol{\sigma}_u(t)}{\boldsymbol{\sigma}_u(0)} > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}.$$

In view of **A2-A3** and by Gordon inequality (see, e.g., Lemma 5.1 in [57]), we have that for $\epsilon \in (0,1)$ and u sufficiently large

$$\mathbb{P}\left\{\exists_{t\in E(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\le\pi(u)\le\mathbb{P}\left\{\exists_{t\in E(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}. \tag{6.26}$$

where

$$\boldsymbol{Z}_{u,\pm\varepsilon}(t)=\frac{\boldsymbol{Y}_{\pm\varepsilon}(t)}{\boldsymbol{w}_{u,\mp\varepsilon}(t)},\ t\in\mathbb{R},$$

with $\boldsymbol{Y}_{\pm\varepsilon}(t),t\in\mathbb{R}$ being homogeneous vector-valued Gaussian processes with independent coordinates $Y_{i,\pm\varepsilon}(t),t\in\mathbb{R}$ having continuous trajectories, unit variance and correlation function satisfying

$$r_{i,\pm\varepsilon}(t)=e^{-(1\pm\varepsilon)a_i|t|^{\alpha_i}},$$

and $\boldsymbol{w}_{u,\pm\varepsilon}(t)=(w_{u,1,\pm\varepsilon}(t),\dots,w_{u,n,\pm\varepsilon}(t))$ with

$$w_{u,i,\pm\varepsilon}(t)=1+u^{-2}\left(f_i(u^{\lambda_i}t)\pm\varepsilon|f_i(u^{\lambda_i}t)|\pm\varepsilon\right),\quad \epsilon\in(0,1).$$

Next, we use the double-sum method to derive an upper and a lower bound of (6.26) and then show that they are asymptotically tight. We distinguish three scenarios: $\lambda<2/\alpha$, $\lambda=2/\alpha$ and $\lambda<2/\alpha$.

◇ *Case $\lambda<2/\alpha$*. For any $S>0$, let

$$\begin{aligned} I_k(u) &= [ku^{-2/\alpha}S,(k+1)u^{-2/\alpha}S],\quad k\in\mathbb{Z},\ N_1(u)=\left\lfloor\frac{x_1(u)}{Su^{-2/\alpha}}\right\rfloor-\mathbb{I}_{\{x_1\le0\}},\\ N_2(u) &= \left\lfloor\frac{x_2(u)}{Su^{-2/\alpha}}\right\rfloor+\mathbb{I}_{\{x_2\le0\}},\quad \boldsymbol{v}_{u,\pm\varepsilon}(k)=(v_{u,1,\pm\varepsilon}(k),\dots,v_{u,n,\pm\varepsilon}(k)), \end{aligned} \tag{6.27}$$

with

$$v_{u,i,+\varepsilon}(k)=\frac{m_{u,i}}{\sigma_{u,i}(0)}\sup_{s\in I_k(u)}w_{u,i,+\varepsilon}(s),\quad v_{u,i,-\varepsilon}(k)=\frac{m_{u,i}}{\sigma_{u,i}(0)}\inf_{s\in I_k(u)}w_{u,i,-\varepsilon}(s).$$

For u large enough, in view of (6.26) we have

$$\begin{aligned}\pi(u)\le\mathbb{P}\left\{\exists_{t\in E(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}&\le\sum_{k=N_1(u)}^{N_2(u)}\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\},\\ \pi(u)\ge\mathbb{P}\left\{\exists_{t\in E(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}&\ge\sum_{k=N_1(u)+1}^{N_2(u)-1}\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}-\sum_{i=1}^{2}\Lambda_i(u),\end{aligned}$$

where

$$\Lambda_1(u)=\sum_{k=N_1(u)}^{N_2(u)}\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)},\exists_{t\in I_{k+1}(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\},$$

and

$$\Lambda_2(u)=\sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2}\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)},\exists_{t\in I_l(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}.$$

Asymptotics of $\pi(u)$. By stationarity of $\boldsymbol{Y}_{+\epsilon}$ and Lemma 6.4.1, we have that

$$\begin{aligned}\pi(u)&\le\sum_{k=N_1(u)}^{N_2(u)}\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Y}_{+\varepsilon}(t)>\boldsymbol{v}_{u,-\varepsilon}(k)\right\}\\ &\le\sum_{k=N_1(u)}^{N_2(u)}\mathbb{P}\left\{\exists_{t\in I_0(u)}\boldsymbol{Y}_{+\varepsilon}(t)>\boldsymbol{v}_{u,-\varepsilon}(k)\right\}\\ &\sim\mathcal{H}_{\alpha,(1+\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\alpha=\alpha1\}}}[0,S]\sum_{k=N_1(u)}^{N_2(u)}\prod_{i=1}^{n}\Psi(v_{u,i,-\varepsilon}(k)),\ u\to\infty.\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\sum_{k=N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \Psi(v_{u,i,-\varepsilon}(k)) \\
&\sim \sum_{k=N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \left(\frac{1}{\sqrt{2\pi} v_{u,i,-\varepsilon}(k)} \exp\left(-\frac{v^2_{u,i,-\varepsilon}(k)}{2} \right) \right) \\
&\sim \left(\prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k=N_1(u)}^{N_2(u)} \exp\left(-\sum_{i=1}^{n} \frac{m^2_{u,i} u^{-2} \inf_{s\in I_k(u)} \left(f_i(u^{\lambda_i} s) - \varepsilon |f_i(u^{\lambda_i} s)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} \right) \\
&\sim \left(\prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) \sum_{k=N_1(u)}^{N_2(u)} \exp\left(-\sum_{i=1}^{n} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} \right) \\
&\le \left(\prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)} \right) \right) S^{-1} u^{2/\alpha-\lambda} \int_{x_1}^{x_2} \exp\left(-\sum_{i=1}^{n} \frac{\widetilde{f}^{\varepsilon}_i(t)}{\sigma^2_i} \right) dt, \qquad (6.28)
\end{aligned}$$

where $\widetilde{f}^{\varepsilon}_i(t) = \widetilde{f}_i(t) - \varepsilon|\widetilde{f}_i(t)| - \varepsilon$. In order to prove (6.28), we note that for $-\infty < x_1 < x_2 < \infty$,

$$\begin{aligned}
&\sum_{k=N_1(u)}^{N_2(u)} \exp\left(-\sum_{i=1}^{n} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} \right) \\
&\sim S^{-1} u^{2/\alpha-\lambda} \int_{x_1}^{x_2} \exp\left(-\sum_{i=1}^{n} \frac{\widetilde{f}^{\varepsilon}_i(t)}{\sigma^2_i} \right) dt, \quad u\to\infty,
\end{aligned}$$

which implies that (6.28) holds for $-\infty < x_1 < x_2 < \infty$. Next we assume that $-\infty < x_1 < x_2 = \infty$. Let y be a positive constant satisfying $x_1 < y < \infty$ and $N(u,y) = \left[\frac{yu^{2/\alpha-\lambda}}{S}\right]$. Then it follows that

$$\begin{aligned}
&\sum_{k=N_1(u)}^{N(u,y)} \exp\left(-\sum_{i=1}^{n} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} \right) \\
&\sim S^{-1} u^{2/\alpha-\lambda} \int_{x_1}^{y} \exp\left(-\sum_{i=1}^{n} \frac{\widetilde{f}^{\varepsilon}_i(t)}{\sigma^2_i} \right) dt, \quad u\to\infty. \qquad (6.29)
\end{aligned}$$

By Potter's Theorem (Theorem 1.5.6 in [19]) and the fact that for $j \in \{1 \le i \le n : \lambda_i = \lambda\}$, $f_j(t)$ is regularly varying at ∞ with positive index, we have that for any $\eta > 0$ and sufficiently large y and u

$$\left| \frac{\sigma^2_j}{\sigma^2_{u,j}(0)} \frac{m^2_{u,j} u^{-2} \inf_{s\in [k,k+1]} \left(f_j(u^{\lambda - \frac{2}{\alpha}} Ss) - \varepsilon |f_j(u^{\lambda-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\widetilde{f}^{\varepsilon}_j(u^{\lambda-\frac{2}{\alpha}} Sk)} - 1 \right| < \eta$$

holds for all $k > N(u,y)$. Then we have that for $k > N(u,y)$

$$\left| \sum_{\lambda_i=\lambda} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} - \sum_{\lambda_i=\lambda} \frac{\widetilde{f}^{\varepsilon}_i(u^{\lambda-\frac{2}{\alpha}} Sk)}{\sigma^2_i} \right| \le \eta \sum_{\lambda_i=\lambda} \frac{|\widetilde{f}^{\varepsilon}_i(u^{\lambda-\frac{2}{\alpha}} Sk)|}{\sigma^2_i}$$

Using (6.4), it follows that

$$\lim_{u\to\infty} \sup_{N_1(u)\le k\le N_2(u)} \left| \sum_{\lambda_i<\lambda} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} - \sum_{\lambda_i<\lambda} \frac{\widetilde{f}^{\varepsilon}_i(u^{\lambda_i-\frac{2}{\alpha}} Sk)}{\sigma^2_i} \right| = 0.$$

Hence, for sufficiently large y and u we have that

$$\sum_{i=1}^{n} \frac{m^2_{u,i} u^{-2} \inf_{s\in [k,k+1]} \left(f_i(u^{\lambda_i - \frac{2}{\alpha}} Ss) - \varepsilon |f_i(u^{\lambda_i-\frac{2}{\alpha}} Ss)| - \varepsilon \right)}{\sigma^2_{u,i}(0)} \ge \sum_{i=1}^{n} \frac{\widetilde{f}^{\varepsilon}_i(u^{\lambda-\frac{2}{\alpha}} Sk)}{\sigma^2_i} - \eta \sum_{i=1}^{n} \frac{|\widetilde{f}^{\varepsilon}_i(u^{\lambda-\frac{2}{\alpha}} Sk)|}{\sigma^2_i}$$

holds for $k > N(u,y)$. Combining the above with (6.5) implies that

$$\begin{aligned}
&\sum_{k=N(u,y)+1}^{N_2(u)} \exp\left(-\sum_{i=1}^{n} \frac{m_{u,i}^2 u^{-2} \inf_{s\in[k,k+1]}\left(f_i(u^{\lambda_i-\frac{2}{\alpha}}Ss)-\varepsilon|f_i(u^{\lambda_i-\frac{2}{\alpha}}Ss)|-\varepsilon\right)}{\sigma_{u,i}^2(0)}\right)\\
&\le \sum_{k=N(u,y)+1}^{N_2(u)} \exp\left(-\sum_{i=1}^{n}\frac{\widetilde{f}_i^{\varepsilon}(u^{\lambda-\frac{2}{\alpha}}Sk)}{\sigma_i^2}+\eta\sum_{i=1}^{n}\frac{|\widetilde{f}_i^{\varepsilon}(u^{\lambda-\frac{2}{\alpha}}Sk)|}{\sigma_i^2}\right)\\
&\le u^{\frac{2}{\alpha}-\lambda}S^{-1}\int_y^{\infty}\exp\left(-\sum_{i=1}^{n}\frac{\widetilde{f}_i^{\varepsilon}(t)}{\sigma_i^2}+\eta\sum_{i=1}^{n}\frac{|\widetilde{f}_i^{\varepsilon}(t)|}{\sigma_i^2}\right)dt,
\end{aligned}$$

which together with (6.29) and the arbitrariness of $\eta > 0$ confirms that (6.28) holds. For other cases of x_1 and x_2, we can similarly show that (6.28) is satisfied. By (6.4) and (6.5), we have that

$$\int_{x_1}^{x_2} \exp\left(-\sum_{i=1}^{n}\frac{\widetilde{f}_i^{\varepsilon}(t)}{\sigma_i^2}\right)dt < \infty.$$

Consequently,

$$\pi(u) \le \mathcal{H}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}} u^{2/\alpha-\lambda}\int_{x_1}^{x_2}\exp\left(-\sum_{i=1}^{n}\frac{\widetilde{f}_i(t)}{\sigma_i^2}\right)dt\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right), \tag{6.30}$$

as $u\to\infty,\ S\to\infty,\ \varepsilon\to 0$. Analogously, we have

$$\begin{aligned}
&\sum_{k=N_1(u)+1}^{N_2(u)-1} \mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\\
&\qquad \ge \mathcal{H}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}} u^{2/\alpha-\lambda}\int_{x_1}^{x_2}\exp\left(-\sum_{i=1}^{n}\frac{\widetilde{f}_i(t)}{\sigma_i^2}\right)dt\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right),
\end{aligned}$$

as $u\to\infty,\ S\to\infty,\ \varepsilon\to 0$.

<u>*Upper bound for $\Lambda_1(u)$*</u>. It follows that

$$\begin{aligned}
\Lambda_1(u) &= \sum_{k=N_1(u)}^{N_2(u)}\left(\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}+\mathbb{P}\left\{\exists_{t\in I_{k+1}(u)}\boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\right.\\
&\quad \left.-\mathbb{P}\left\{\exists_{t\in I_k(u)\cup I_{k+1}(u)}\boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\right)\\
&\le \sum_{k=N_1(u)}^{N_2(u)}\left(\mathbb{P}\left\{\exists_{t\in I_k(u)}\boldsymbol{Y}_{-\varepsilon}(t) > \widehat{\boldsymbol{v}}_{u,+\varepsilon}(k)\right\}+\mathbb{P}\left\{\exists_{t\in I_{k+1}(u)}\boldsymbol{Y}_{-\varepsilon}(t) > \widehat{\boldsymbol{v}}_{u,+\varepsilon}(k)\right\}\right.\\
&\quad \left.-\mathbb{P}\left\{\exists_{t\in I_k(u)\cup I_{k+1}(u)}\boldsymbol{Y}_{-\varepsilon}(t) > \widetilde{\boldsymbol{v}}_{u,+\varepsilon}(k)\right\}\right)\\
&\sim \left(2\mathcal{H}_{\alpha,(1-\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,S]-\mathcal{H}_{\alpha,(1-\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,2S]\right)\sum_{k=N_1(u)}^{N_2(u)}\left(\prod_{i=1}^{n}\Psi(v_{u,i,+\varepsilon}(k))\right)\\
&= o\left(u^{2/\alpha-\lambda}\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right),\ u\to\infty, S\to\infty, \varepsilon\to 0,
\end{aligned} \tag{6.31}$$

where

$$\widehat{v}_{u,i,+\varepsilon}(k) = \min\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\inf_{s\in I_k(u)} w_{u,i,+\varepsilon}(s), \frac{m_{u,i}}{\sigma_{u,i}(0)}\inf_{s\in I_{k+1}(u)} w_{u,i,+\varepsilon}(s)\right)$$

and

$$\widetilde{v}_{u,i,+\varepsilon}(k) = \max\left(v_{u,i,+\varepsilon}(k), v_{u,i,+\varepsilon}(k+1)\right).$$

<u>*Upper bound for* $\Lambda_2(u)$</u>. In light of Lemma 6.4.2, we have that

$$\begin{aligned}
\Lambda_2(u) &= \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{P}\left\{\exists_{t\in I_k(u)} \boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}, \exists_{t\in I_l(u)} \boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{P}\left\{\exists_{t\in I_k(u)} \boldsymbol{Y}_{-\varepsilon}(t) > \overline{\boldsymbol{v}}_{u,+\varepsilon}(k), \exists_{t\in I_l(u)} \boldsymbol{Y}_{-\varepsilon}(t) > \overline{\boldsymbol{v}}_{u,+\varepsilon}(l)\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{P}\left\{\exists_{t\in I_0(u)} \boldsymbol{Y}_{-\varepsilon}(t) > \overline{\boldsymbol{v}}_{u,+\varepsilon}(k), \exists_{t\in I_{l-k}(u)} \boldsymbol{Y}_{-\varepsilon}(t) > \overline{\boldsymbol{v}}_{u,+\varepsilon}(l)\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{C}_1 S^{2n} \exp(-\mathbb{C}_2((l-k-1)S)^{\alpha}) \prod_{i=1}^{n} \Psi\left(\frac{\overline{v}_{u,i,-\varepsilon}(k)+\overline{v}_{u,i,-\varepsilon}(l)}{2}\right)\\
&\le 2\sum_{l=1}^{\infty} \mathbb{C}_1 S^{2n} \exp(-\mathbb{C}_2(lS)^{\alpha}) \sum_{k=N_1(u)}^{N_2(u)} \prod_{i=1}^{n} \Psi\left(\overline{v}_{u,i,-\varepsilon}(k)\right)\\
&\le S^{2n}\exp(-\mathbb{C}_3 S^{\alpha}) u^{2/\alpha-\lambda} \prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\\
&= o\left(u^{2/\alpha-\lambda} \prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right), \quad u\to\infty, S\to\infty,
\end{aligned} \tag{6.32}$$

where

$$\overline{v}_{u,i,+\varepsilon}(k) = \frac{m_{u,i}}{\sigma_{u,i}(0)} \inf_{s\in I_k(u)} w_{u,i,+\varepsilon}(s).$$

Combination of (6.28)-(6.32) leads to

$$\pi(u) \sim \mathcal{H}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}} u^{2/\alpha-\lambda} \int_{x_1}^{x_2} \exp\left(-\sum_{i=1}^{n} \frac{\widetilde{f}_i(t)}{\sigma_i^2}\right) dt \left(\prod_{i=1}^{n} \Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right), \quad u\to\infty.$$

$\diamond$ <u>*Case* $\lambda = 2/\alpha$</u>. Without loss of generality we assume that $x_1 = -\infty$ and $x_2 = \infty$. The cases $x_1 > -\infty$ and $x_2 < \infty$ can be dealt with analogously. In what follows, we use notation introduced in (6.27) and set $\widetilde{I}(u) = I_0(u) \cup I_{-1}(u)$. Observe that for large u

$$\pi(u) \ge \mathbb{P}\left\{\exists_{t\in\widetilde{I}(u)} \boldsymbol{Z}_{u,-\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}, \tag{6.33}$$

$$\pi(u) \le \mathbb{P}\left\{\exists_{t\in\widetilde{I}(u)} \boldsymbol{Z}_{u,+\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\} + \sum_{\substack{k=N_1(u)\\k\ne -1,0}}^{N_2(u)} \mathbb{P}\left\{\exists_{t\in I_k(u)} \boldsymbol{Z}_{u,+\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}. \tag{6.34}$$

Lemma 6.4.1 yields that

$$\mathbb{P}\left\{\exists_{t\in\widetilde{I}(u)} \boldsymbol{Z}_{u,\pm\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\} \sim \mathcal{P}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}^{\frac{\widetilde{\boldsymbol{f}}}{\boldsymbol{\sigma}^2}}[-S,S] \prod_{i=1}^{n} \Psi\left(\frac{m_{i,u}}{\sigma_{i,u}(0)}\right), \tag{6.35}$$

as $u\to\infty, \varepsilon\to 0$. Moreover, in light of Lemma 6.4.1 and (6.5) we have

$$\begin{aligned}
&\sum_{\substack{k=N_1(u)\\k\ne -1,0}}^{N_2(u)} \mathbb{P}\left\{\exists_{t\in I_k(u)} \boldsymbol{Z}_{u,+\varepsilon}(t) > \frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\\
&\quad\le \sum_{\substack{k=N_1(u)\\k\ne -1,0}}^{N_2(u)} \mathbb{P}\left\{\exists_{t\in I_0(u)} \boldsymbol{Y}_{+\varepsilon}(t) > \boldsymbol{v}_{u,-\varepsilon}(k)\right\}\\
&\quad\sim \mathcal{H}_{\alpha,(1+\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}[0,S] \sum_{\substack{k=N_1(u)\\k\ne -1,0}}^{N_2(u)} \prod_{i=1}^{n} \Psi(v_{u,i,-\varepsilon}(k))
\end{aligned}$$

$$\sim \mathcal{H}_{\alpha,(1+\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}[0,S]\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right)\sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)}\exp\left(-\sum_{i=1}^{n}\frac{m_{u,i}^2u^{-2}\inf_{s\in[k,k+1]}\widetilde{f}_i^{\varepsilon}(Ss)}{\sigma_{u,i}^2(0)}\right)$$

$$\sim \mathcal{H}_{\alpha,(1+\varepsilon)\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}[0,S]\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right)\sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)}\exp\left(-\sum_{i=1}^{n}\frac{\inf_{s\in[k,k+1]}\widetilde{f}_i^{\varepsilon}(Ss)}{\sigma_i^2}\right)$$

$$\leq \mathbb{C}_4\mathcal{H}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right)Se^{-\eta\ln S}=o\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right), \tag{6.36}$$

as $u\to\infty,\varepsilon\to 0, S\to\infty$, where $\eta\in(1,\infty)$ is a constant. Inserting (6.35)-(6.36) into (6.33)-(6.34) and letting $S\to\infty$, we obtain that

$$\pi(u)\sim \mathcal{P}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha_1\}}}^{\frac{\tilde{\boldsymbol{f}}}{\boldsymbol{\sigma}^2}}\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right),\ u\to\infty.$$

This establishes the claim.

$\diamond$ *Case* $\lambda>\frac{2}{\alpha}$. Without loss of generality we assume that $x_1=-\infty$ and $x_2=\infty$. For any $S>0$, define

$$J_k(u)=[ku^{-\lambda}S,(k+1)u^{-\lambda}S], k\in\mathbb{Z},\quad \widetilde{J}(u)=J_0(u)\cup J_{-1}(u),$$

$$K_1(u)=\left\lfloor\frac{x_1(u)}{Su^{-\lambda}}\right\rfloor-\mathbb{I}_{\{x_1\leq 0\}},\ K_2(u)=\left\lfloor\frac{x_2(u)}{Su^{-\lambda}}\right\rfloor+\mathbb{I}_{\{x_2\leq 0\}},\quad \boldsymbol{v}_{u,\pm\varepsilon}(k)=(v_{u,1,+\varepsilon}(k),\dots,v_{u,n,+\varepsilon}(k)),$$

with

$$v_{u,i,+\varepsilon}(k)=\frac{m_{u,i}}{\sigma_{u,i}(0)}\sup_{s\in J_k(u)}w_{u,i,+\varepsilon}(s),\quad v_{u,i,-\varepsilon}(k)=\frac{m_{u,i}}{\sigma_{u,i}(0)}\inf_{s\in J_k(u)}w_{u,i,-\varepsilon}(s).$$

Then for u large enough, we have

$$\pi(u)\ \geq\ \mathbb{P}\left\{\exists_{t\in\widetilde{J}(u)}\boldsymbol{Z}_{u,-\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}, \tag{6.37}$$

$$\pi(u)\ \leq\ \mathbb{P}\left\{\exists_{t\in\widetilde{J}(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}+\sum_{\substack{k=K_1(u)\\k\neq 0,-1}}^{K_2(u)}\mathbb{P}\left\{\exists_{t\in J_k(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}. \tag{6.38}$$

It follows from Lemma 6.4.1 that

$$\mathbb{P}\left\{\exists_{t\in\widetilde{J}(u)}\boldsymbol{Z}_{u,\pm\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\sim\int_{\mathbb{R}^n}e^{\sum_{i=1}^n w_i}\mathbb{I}_{\left\{\exists_{t\in[-S,S]}-\frac{\tilde{\boldsymbol{f}}(t)}{\boldsymbol{\sigma}^2}>\boldsymbol{w}\right\}}d\boldsymbol{w}\prod_{i=1}^{n}\Psi\left(\frac{m_{i,u}}{\sigma_{i,u}(0)}\right), \tag{6.39}$$

as $u\to\infty,\ \varepsilon\to 0$. Moreover, similarly to (6.36), we have that

$$\sum_{\substack{k=K_1(u)\\k\neq -1,0}}^{K_2(u)}\mathbb{P}\left\{\exists_{t\in J_k(u)}\boldsymbol{Z}_{u,+\varepsilon}(t)>\frac{\boldsymbol{m}_u}{\boldsymbol{\sigma}_u(0)}\right\}\ \leq\ \mathbb{C}_6\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right)e^{-\eta\ln S}$$

$$=\ o\left(\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right)\right),\ u\to\infty, S\to\infty. \tag{6.40}$$

Inserting (6.39)-(6.40) into (6.37)-(6.38) and letting $S\to\infty$ and $\epsilon\to 0$ we derive that

$$\pi(u)\sim\left(\int_{\mathbb{R}^n}e^{\sum_{i=1}^n w_i}\mathbb{I}_{\left\{\exists_{t\in(-\infty,\infty)}-\frac{\tilde{\boldsymbol{f}}(t)}{\boldsymbol{\sigma}^2}>\boldsymbol{w}\right\}}d\boldsymbol{w}\right)\prod_{i=1}^{n}\Psi\left(\frac{m_{u,i}}{\sigma_{u,i}(0)}\right),\ u\to\infty.$$

This completes the proof. □

Proof of Theorem 6.3.1 We first focus on the case of $t_0 \in (0, T)$. Set

$$E(u) = [-\delta(u), \delta(u)], \quad D(u) := [t_0 - \theta, t_0 + \theta] \setminus (t_0 + E(u)),$$

where $\theta \in (0, \frac{1}{2})$ is a small constant and $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^{2/\beta}$ with $q > 1$, $\beta = \min_{1 \le i \le n} \beta_i^*$ and

$$\beta_i^* = \min\left(\beta_i, 2\gamma_i \mathbb{I}_{\{c_i \neq 0\}} + \infty \mathbb{I}_{\{c_i = 0\}}\right).$$

Then it follows that

$$\Pi_1(u) \le \mathbb{P}\left\{\exists_{t \in [0,T]} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\boldsymbol{1}\right\} \le \Pi_1(u) + \Pi_2(u) + \Pi_3(u),$$

where

$$\begin{aligned} \Pi_1(u) &= \mathbb{P}\left\{\exists_{t \in E(u)} \left(\boldsymbol{X}(t_0 + t) + \boldsymbol{h}(t_0 + t)\right) > u\boldsymbol{1}\right\}, \quad \Pi_2(u) = \mathbb{P}\left\{\exists_{t \in D(u)} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\boldsymbol{1}\right\}, \\ \Pi_3(u) &= \mathbb{P}\left\{\exists_{t \in [0,T] \setminus [t_0 - \theta, t_0 + \theta]} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\boldsymbol{1}\right\}. \end{aligned}$$

Asymptotics of $\Pi_1(u)$. In order to derive the asymptotics of $\Pi_1(u)$, we check the assumptions in Theorem 6.2.1. For this purpose, rewrite

$$\Pi_1(u) = \mathbb{P}\left\{\exists_{t \in E(u)} \boldsymbol{X}_u(t) > u\boldsymbol{1}\right\}, \quad \text{with} \quad \boldsymbol{X}_u(t) = \frac{\boldsymbol{X}(t_0 + t)}{\boldsymbol{1} - \boldsymbol{h}(t_0 + t)/u}.$$

It follows straightforwardly that $\boldsymbol{\sigma}_u(t) = \frac{\boldsymbol{\sigma}(t_0+t)}{\boldsymbol{1} - \boldsymbol{h}(t_0+t)/u}$ satisfies $\lim_{u \to \infty} \boldsymbol{\sigma}_u(0) = \boldsymbol{\sigma}(t_0) > \boldsymbol{0}$ implying that **A1** holds. Next we verify **A2**. Direct calculation shows that

$$\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1 = \frac{1}{\sigma_i(t_0 + t)}(\sigma_i(t_0) - \sigma_i(t_0 + t)) + \frac{1}{u - h_i(t_0)} \frac{\sigma_i(t_0)}{\sigma_i(t_0 + t)}(h_i(t_0) - h_i(t_0 + t)).$$

Thus by (6.6) and (6.9) we have that for all u large

$$\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} = 1 + \left(\frac{b_i}{\sigma_i(t_0)}|t|^{\beta_i} + \frac{c_i}{u - h_i(t_0)}|t|^{\gamma_i}\right)(1 + o(1)), \; t \to 0. \tag{6.41}$$

Denote by $\widetilde{f}_i(t) = \frac{b_i}{\sigma_i(t_0)}|t|^{\beta_i}\mathbb{I}_{\{\beta_i = \beta_i^*\}} + c_i|t|^{\gamma_i}\mathbb{I}_{\{\beta_i^* = 2\gamma_i\}}$. Then we have

$$\lim_{u \to \infty} \sup_{t \in E(u)} \left| \frac{\left(\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)} - 1\right) u^2 - \widetilde{f}_i(u^{2/\beta_i^*} t)}{|\widetilde{f}_i(u^{2/\beta_i^*} t)| + 1} \right| = 0, \tag{6.42}$$

which confirms that **A2** is satisfied. Apparently, **A3** follows by (6.7). Thus we conclude that **A1-A3** are satisfied. Also, (6.4) holds with $x_1 = -\infty$ and $x_2 = \infty$. Therefore, in light of Theorem 6.2.1, we have, as $u \to \infty$,

$$\Pi_1(u) \sim u^{(\frac{2}{\alpha} - \frac{2}{\beta})_+} \prod_{i=1}^n \Psi\left(\frac{u - h_i(t_0)}{\sigma_i(t_0)}\right) \begin{cases} \mathcal{H}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\alpha = \alpha_1\}}} \int_{-\infty}^{\infty} e^{-\sum_{i=1}^n f_i(x)} dx, & \text{if } \alpha < \beta, \\ \mathcal{P}^{\boldsymbol{f}}_{\alpha, \frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)} \mathbf{I}_{\{\alpha = \alpha_1\}}}(-\infty, \infty), & \text{if } \alpha = \beta, \\ 1, & \text{if } \alpha > \beta, \end{cases} \tag{6.43}$$

where $f_i(t) = \frac{b_i}{\sigma_i^3(t_0)}|t|^{\beta_i}\mathbb{I}_{\{\beta_i = \beta\}} + \frac{c_i}{\sigma_i^2(t_0)}|t|^{\gamma_i}\mathbb{I}_{\{2\gamma_i = \beta\}}, 1 \le i \le n$.
Upper bound for $\Pi_2(u)$. Observe that

$$\Pi_2(u) = \mathbb{P}\left\{\exists_{t \in D(u)} \left(\boldsymbol{X}(t) + \boldsymbol{h}(t)\right) > u\boldsymbol{1}\right\} \le \mathbb{P}\left\{\sup_{t \in [-\theta, \theta] \setminus E(u)} Y_u(t) > u\right\}, \tag{6.44}$$

where

$$Y_u(t) = \sum_{i=1}^{n} G_{u,i}(t) X_i(t_0+t),\ t \in [-t_0, T-t_0], \tag{6.45}$$

with

$$G_{u,i}(t) := \left(\frac{\prod_{j=1, j\neq i}^{n} \frac{\sigma_j^2(t_0+t)}{(1-h_j(t_0+t)/u)^2}}{A_u(t_0+t)} \right) \frac{1}{1-h_i(t_0+t)/u},\quad t \in [-t_0, T-t_0],$$
$$A_u(t) = \sum_{k=1}^{n} \left(\prod_{j=1, j\neq k}^{n} \frac{\sigma_j^2(t)}{(1-h_j(t)/u)^2} \right),\quad t \in [0,T].$$

In order to analyze the variance of Y_u, we introduce $g_u(t) = \sum_{i=1}^{n} \frac{1}{\sigma_{u,i}^2(t)}$. Using (6.41) we have that

$$\begin{aligned} g_u(t) - g_u(0) &= \sum_{i=1}^{n} \frac{1}{\sigma_{u,i}^2(t)} - \sum_{i=1}^{n} \frac{1}{\sigma_{u,i}^2(0)} \\ &= \sum_{i=1}^{n} \frac{(\sigma_{u,i}(0) - \sigma_{u,i}(t))(\sigma_{u,i}(0) + \sigma_{u,i}(t))}{\sigma_{u,i}^2(t)\sigma_{u,i}^2(0)} \\ &\geq \mathbb{C}_0 \sum_{i=1}^{n} \frac{1}{\sigma^2(t_0)} \left(\frac{b_i}{\sigma_i(t_0)} |t|^{\beta_i} + \frac{c_i}{u} |t|^{\gamma_i} \right) \\ &\geq C \frac{(\ln u)^q}{u^2} \end{aligned} \tag{6.46}$$

holds for all $t \in [-\theta, \theta] \setminus E(u)$ with a positive constant C. Consequently,

$$\sup_{t\in[-\theta,\theta]\setminus E(u)} \mathrm{var}(Y_u(t)) = \sup_{t\in[-\theta,\theta]\setminus E(u)} \left(\sum_{i=1}^{n} \frac{(1-h_i(t_0+t)/u)^2}{\sigma_i^2(t_0+t)} \right)^{-1} = \sup_{t\in[-\theta,\theta]\setminus E(u)} \frac{1}{g_u(t)} \leq \frac{1}{g_u(0) + \frac{C(\ln u)^q}{u^2}}.$$

By (6.10) and the fact that in view of (6.8),

$$(\sigma_i(t) - \sigma_i(s))^2 \leq \mathbb{E}\left\{ (X_i(t) - X_i(s))^2 \right\} \leq \mathbb{C}_1 |t-s|^{\mu_1},\quad s,t \in [0,T],$$

we have that there exists $\mu_3 > 0$ such that

$$\max_{i=1,\ldots,n} (G_{u,i}(t) - G_{u,i}(s))^2 \leq \mathbb{C}_2 |t-s|^{\mu_3},\ s,t \in [0,T],$$

which together with (6.8) implies that

$$\begin{aligned} \mathbb{E}\left(Y_u(t) - Y_u(s)\right)^2 &= \mathbb{E}\left(\sum_{i=1}^{n} G_{u,i}(t) X_i(t) - \sum_{i=1}^{n} G_{u,i}(s) X_i(s) \right)^2 \\ &= \sum_{i=1}^{n} \mathbb{E}\left(G_{u,i}(t) X_i(t) - G_{u,i}(s) X_i(s)\right)^2 \\ &\leq 2 \sum_{i=1}^{n} \sigma_i^2(t) \left(G_{u,i}(t) - G_{u,i}(s)\right)^2 + 2 \sum_{i=1}^{n} G_{u,i}^2(s) \mathbb{E}\left(X_i(t) - X_i(s)\right)^2 \\ &\leq \mathbb{C}_3 |t-s|^{\mu_4},\quad s,t \in [0,T] \end{aligned} \tag{6.47}$$

with $\mu_4 > 0$. Consequently Piterbarg inequality (Theorem 8.1 in [119]) gives that

$$\begin{aligned} \Pi_2(u) &\leq \mathbb{P}\left\{ \sup_{t\in[-\theta,\theta]\setminus E(u)} Y_u(t) > u \right\} \\ &\leq \mathbb{C}_4 u^{2/\mu_4} \Psi\left(\sqrt{u^2 g_u(0) + C(\ln u)^q} \right) \end{aligned}$$

$$= o\left(u^{(\frac{2}{\alpha}-\frac{2}{\beta})_+}\prod_{i=1}^{n}\Psi\left(\frac{u-h_i(t_0)}{\sigma_i(t_0)}\right)\right), \quad u\to\infty.$$

Upper bound for $\Pi_3(u)$. Note that there exists $\epsilon\in(0,1)$ such that

$$\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])}\sigma_i(t)\le(1-\epsilon)\sigma_i(t_0),\ 1\le i\le n.$$

Thus

$$\sup_{t\in[0,T]\setminus[-\theta,\theta]}\mathrm{var}(Y_u(t))=\left(\inf_{t\in[0,T]\setminus[-\theta,\theta]}g_u(t)\right)^{-1}\le(1-\epsilon/2)^{-2}\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2(t_0)}\right)^{-1},$$

which together with (6.47) and Piterbarg inequality (Theorem 8.1 in [119]) implies that

$$\begin{aligned}\Pi_3(u) &= \mathbb{P}\left\{\exists_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right)>u\mathbf{1}\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])}Y_u(t)>u\right\}\\ &\le \mathbb{C}_5u^{2/\mu_4}\Psi\left((1-\epsilon/2)\left(\sum_{i=1}^{n}\frac{1}{\sigma_i^2(t_0)}\right)^{1/2}u\right)\\ &= o(\Pi_1(u)),\quad u\to\infty.\end{aligned}$$

Therefore, we conclude that

$$\mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right)>u\mathbf{1}\right\}\sim\Pi_1(u),\quad u\to\infty,$$

which combined with (6.43) establishes the claim.
The case of $t_0=0$ ($t_0=T$) can be dealt with using the same argument as above with the only difference that one has to substitute $E(u)$ by $[0,\delta(u)]$ (or by $[-\delta(u),0]$).
Thus the proof is complete. □

Proof of Theorem 6.3.2 i) We provide the proof only for case $t_0\in(0,T)$, since cases $t_0=0$ and $t_0=T$ can be established analogously. Let $E(u)=[-\delta(u),\delta(u)]$, where $\delta(u)=\left(\frac{(\ln u)^q}{u}\right)^{1/\gamma}$ with $q>1$. It follows that

$$\Pi(u)\le\mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right)>u\mathbf{1}\right\}\le\Pi(u)+\Pi_1(u),$$

where

$$\Pi(u)=\mathbb{P}\left\{\exists_{t\in E(u)}\left(\boldsymbol{X}(t_0+t)+\boldsymbol{h}(t_0+t)\right)>u\mathbf{1}\right\},\quad \Pi_1(u)=\mathbb{P}\left\{\exists_{t\in[0,T]\setminus(t_0+E(u))}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right)>u\mathbf{1}\right\}.$$

In order to derive the asymptotics of $\Pi(u)$ we apply Theorem 6.2.1 by checking conditions **A1-A3**. Set $\sigma_{u,i}(t)=\frac{1}{1-h_i(t_0+t)/u}$ and then $\lim_{u\to\infty}\sigma_{u,i}(0)=1$, which indicates that **A1** holds. By the fact that

$$\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)}-1=\frac{h_i(t_0)-h_i(t_0+t)}{u-h_i(t_0)},$$

and (6.9), we have

$$\lim_{u\to\infty}\sup_{\substack{t\in E(u)\\ t\neq0}}\left|\frac{\left(\frac{\sigma_{u,i}(0)}{\sigma_{u,i}(t)}-1\right)u^2-c_i|u^{\frac{1}{\gamma_i}}t|^{\gamma_i}}{c_i|u^{\frac{1}{\gamma_i}}t|^{\gamma_i}+1}\right|=0.$$

This confirms that **A2** is satisfied. Moreover, (6.13) implies that

$$\lim_{u\to\infty}\sup_{\substack{t\in E(u),s\in E(u)\\ t\neq s}}\left|\frac{1-r_i(t_0+t,t_0+s)}{a_i(t_0)|t-s|^{\alpha_i}}-1\right|=0,$$

which means that **A3** holds. Also, we have that (6.4) holds with $x_1=-\infty$ and $x_2=\infty$. Therefore, by Theorem 6.2.1

$$\Pi(u)\sim u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\prod_{i=1}^n\Psi\left(u-h_{m,i}\right)\begin{cases}\mathcal{H}_{\alpha,\boldsymbol{a}_0\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}\int_{-\infty}^{\infty}e^{-\sum_{i=1}^n f_i(x)}dx, & \text{if } \alpha<2\gamma,\\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{a}_0\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}(-\infty,\infty), & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma,\end{cases}$$

where $\gamma=\min_{1\le i\le n}(\gamma_i\mathbb{I}_{\{c_i\neq 0\}}+\infty\mathbb{I}_{\{c_i=0\}})$, $f_i(t)=c_i|t|^{\gamma}\mathbb{I}_{\{\gamma_i=\gamma\}}, 1\le i\le n$. Next we show that $\Pi_1(u)=o(\Pi(u)), u\to\infty$. Observe that

$$\Pi_1(u)=\mathbb{P}\left\{\exists_{t\in[0,T]\setminus(t_0+E(u))}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right)>u\mathbf{1}\right\}\le\mathbb{P}\left\{\sup_{t\in[0,T]\setminus(t_0+E(u))}Y_u(t)>u\right\},$$

where

$$Y_u(t)=\sum_{i=1}^n G_{u,i}(t)X_i(t_0+t),\ t\in[-t_0,T-t_0], \tag{6.48}$$

with

$$G_{u,i}(t):=\left(\frac{\prod_{j=1,j\neq i}^n\frac{1}{(1-h_j(t_0+t)/u)^2}}{A_u(t_0+t)}\right)\frac{1}{1-h_i(t_0+t)/u},\quad t\in[-t_0,T-t_0],$$
$$A_u(t)=\sum_{k=1}^n\left(\prod_{j=1,j\neq k}^n\frac{1}{(1-h_j(t)/u)^2}\right),\quad t\in[0,T].$$

Let

$$g_u(t)=\sum_{i=1}^n\frac{1}{\sigma^2_{u,i}(t)}=\sum_{i=1}^n(1-h_i(t_0+t)/u)^2. \tag{6.49}$$

Then by (6.9) and the fact that $\min_{1\le i\le n}c_i>0$, we have for $\theta>0$ sufficiently small and u sufficiently large

$$\begin{aligned}g_u(t)-g_u(0)&=\sum_{i=1}^n(1-h_i(t_0+t)/u)^2-\sum_{i=1}^n(1-h_i(t_0)/u)^2\\&\ge\sum_{i=1}^n\frac{h_i(t_0)-h_i(t_0+t)}{u}\\&\ge\mathbb{C}_1\frac{|t|^\gamma}{u}\ge\mathbb{C}_1\frac{(\ln u)^q}{u^2},\quad t\in[t_0-\theta,t_0+\theta]\setminus(t_0+E(u)).\end{aligned}$$

Consequently, there exists $C>0$ such that

$$\sup_{t\in[t_0-\theta,t_0+\theta]\setminus(t_0+E(u))}\mathrm{var}(Y_u(t))=\sup_{t\in[t_0-\theta,t_0+\theta]\setminus(t_0+E(u))}\frac{1}{g_u(t)}\le\frac{1}{g_u(0)+\frac{C(\ln u)^q}{u^2}}.$$

Moreover, for $\theta>0$ sufficiently small and u sufficiently large

$$\begin{aligned}g_u(t)-g_u(0)&\ge\frac{\sum_{i=1}^n h_i(t_0)-\sum_{i=1}^n h_i(t_0+t)}{u}\\&\ge\frac{\mathbb{C}_2}{u},\quad t\in[0,T]\setminus[t_0-\theta,t_0+\theta].\end{aligned}\tag{6.50}$$

Thus there exists $C_1>0$ such that

$$\sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]}\mathrm{var}(Y_u(t))=\sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]}\frac{1}{g_u(t)}\le\frac{1}{g_u(0)+\frac{C_1}{u}}. \tag{6.51}$$

Consequently,

$$\sup_{t\in[t_0-\theta,t_0+\theta]\setminus(t_0+E(u))}\mathrm{var}(Y_u(t))\le\frac{1}{g_u(0)+\frac{C_2(\ln u)^q}{u^2}},\quad[0,T]\setminus(t_0+E(u)),$$

with $C_2 > 0$. Moreover, in light of (6.10) and (6.13), we have that

$$\mathbb{E}\left(Y_u(t) - Y_u(s)\right)^2 \le \mathbb{C}_3 |t-s|^{\mu}, \quad s,t \in [0,T] \tag{6.52}$$

for $\mu > 0$. Piterbarg inequality (Theorem 8.1 in [119]) leads to

$$\begin{aligned}\Pi_1(u) &\le \mathbb{P}\left\{\sup_{t\in[0,T]\setminus(t_0+E(u))} Y_u(t) > u\right\}\\ &\le \mathbb{C}_3 u^{2/\mu}\Psi\left(u\sqrt{g_u(0)+\frac{C_2(\ln u)^q}{u^2}}\right)\\ &= o(\Pi(u)), \quad u\to\infty.\end{aligned}$$

This establishes the claim.

ii) Without loss of generality, we assume that $0 < A < B < T$. Then for $\epsilon > 0$ sufficiently small

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[A,B]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(A)\right) > u\mathbf{1}\right\} &\le \mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}\\ &\le \mathbb{P}\left\{\exists_{t\in[0,A-\epsilon]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\} + \mathbb{P}\left\{\exists_{t\in[A-\epsilon,B+\epsilon]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(A)\right) > u\mathbf{1}\right\}\\ &\quad +\mathbb{P}\left\{\exists_{t\in[B+\epsilon,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}.\end{aligned}$$

In view of (6.13) and (6.14) and by Theorem 4.1 in [57], we have that for any $0 \le x < y \le T$

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[x,y]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(A)\right) > u\mathbf{1}\right\} &= \mathbb{P}\left\{\exists_{t\in[x,y]}\boldsymbol{X}(t) > u\mathbf{1}-\boldsymbol{h}(A)\right\}\\ &\sim u^{\frac{2}{\alpha}}\int_x^y \mathcal{H}_{\alpha,\boldsymbol{a}(t)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}dt\prod_{i=1}^n \Psi\left(u-h_{m,i}\right), \quad u\to\infty,\end{aligned}$$

where $\int_x^y \mathcal{H}_{\alpha,\boldsymbol{a}(t)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}dt$ is a finite and positive constant (see [57]). Next we show that $\mathbb{P}\left\{\exists_{t\in[0,A-\epsilon]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}$ is negligible. Rewrite

$$\mathbb{P}\left\{\exists_{t\in[0,A-\epsilon]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\} = \mathbb{P}\left\{\exists_{t\in[0,A-\epsilon]}Y_u(t) > u\right\},$$

where Y_u is defined in (6.48). Note that (6.51) still holds in the case considered with $[0, A-\epsilon]$ instead of $[0,T]\setminus[t_0-\theta, t_0+\theta]$. Therefore, in view of (6.52), by Piterbarg inequality we have that

$$\mathbb{P}\left\{\exists_{t\in[0,A-\epsilon]}Y_u(t) > u\right\} \le \mathbb{C}_4 u^{2/\mu}\Psi\left(u\sqrt{g_u(0)+\frac{C_1}{u}}\right) = o\left(u^{\frac{2}{\alpha}}\prod_{i=1}^n\Psi\left(u-h_{m,i}\right)\right), \quad u\to\infty.$$

Analogously,

$$\mathbb{P}\left\{\exists_{t\in[B+\epsilon,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\} = o\left(u^{\frac{2}{\alpha}}\prod_{i=1}^n\Psi\left(u-h_{m,i}\right)\right), \quad u\to\infty.$$

Therefore, we conclude that as $u\to\infty$

$$\begin{aligned}u^{\frac{2}{\alpha}}\int_A^B \mathcal{H}_{\alpha,\boldsymbol{a}(t)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}dt\prod_{i=1}^n\Psi\left(u-h_{m,i}\right) &\le \mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}\\ &\le u^{\frac{2}{\alpha}}\int_{A-\epsilon}^{B+\epsilon} \mathcal{H}_{\alpha,\boldsymbol{a}(t)\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}dt\prod_{i=1}^n\Psi\left(u-h_{m,i}\right).\end{aligned}$$

We establish the claim by letting $\epsilon\to 0$ in the above inequalities. This completes the proof. □

Proof of Theorem 6.3.1 We notice that

$$p(u) = \mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{B_\alpha}(t)-\boldsymbol{c}t\right) > u\boldsymbol{d}\right\} = \mathbb{P}\left\{\exists_{t\in[0,T]}\left(\frac{1}{\boldsymbol{d}}\boldsymbol{B_\alpha}(t)-\frac{\boldsymbol{c}t}{\boldsymbol{d}}\right) > u\mathbf{1}\right\},$$

and the variance function $\sigma_i^2(t)$ and correlation function $r_i(s,t)$ of $\frac{B_{\alpha_i}(t)}{d_i}$ satisfy

$$r_i(s,t) = 1 - \frac{1}{2T^{\alpha_i}}|t-s|^{\alpha_i}(1+o(1)), s,t \to T,$$
$$\sigma_i(t) = \frac{T^{\alpha_i/2}}{d_i o} - \frac{\alpha_i}{2d_i}T^{\alpha_i/2-1}(T-t)(1+o(1)), t \to T,$$

where T is the unique maximum point of $\sigma_i(t), 1 \le i \le n$ over $[0,T]$. Moreover,

$$-\frac{c_i t}{d_i} = -\frac{c_i T}{d_i} + \frac{c_i}{d_i}|T-t|, \quad t \to T.$$

Therefore, in light of Theorem 6.3.1 and Corollary 6.3.1, we have that

$$\mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{B}_{\boldsymbol{\alpha}}(t) - \boldsymbol{c}t\right) > u\boldsymbol{d}\right\} \sim u^{(\frac{2}{\alpha}-2)_+}\prod_{i=1}^n \Psi\left(\frac{d_i u + c_i T}{T^{\alpha_i/2}}\right)\begin{cases} \mathcal{H}_{\alpha,\boldsymbol{\varsigma}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}\int_0^\infty e^{-\sum_{i=1}^n f_i(t)}dt, & \text{if } \alpha<1,\\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{\varsigma}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,\infty), & \text{if } \alpha=1,\\ 1, & \text{if } \alpha>1,\end{cases}$$

and

$$\mathbb{P}\left\{(T-\tau_u)u^2 \le x \middle| \tau_u \le T\right\} \sim \begin{cases} 1 - e^{-\left(\sum_{i=1}^n \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}}\right)x}, & \text{if } \alpha<1,\\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{\varsigma}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,x] \Big/ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\boldsymbol{\varsigma}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[0,\infty), & \text{if } \alpha=1,\\ 1, & \text{if } \alpha>1,\end{cases}$$

where $\alpha = \min_{1\le i\le n}\alpha_i$, $\boldsymbol{\varsigma} = (\varsigma_1,\ldots,\varsigma_n)$ with $\varsigma_i = \frac{d_i^2}{2T^{2\alpha_i}}$ and $f_i(t) = \frac{\alpha_i d_i^2}{2T^{\alpha_i+1}}|t|$. □

Proof of Corollary 6.3.1 By definition,

$$\mathbb{P}\left\{(T-\tau_u)u^{2/\beta} \le x \middle| \tau_u \le T\right\} = \frac{\mathbb{P}\left\{\exists_{t\in[T-u^{-2/\beta}x,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}}{\mathbb{P}\left\{\exists_{t\in[0,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\}} \tag{6.53}$$

The asymptotics of denominator in (6.53) follows by Theorem 6.3.1. In order to get the asymptotics of nominator of (6.53) we follow the same argument as in the proof of Theorem 6.3.1 (part related with the asymptotics of $\Pi_1(u)$), which leads to

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[T-u^{-2/\beta}x,T]}\left(\boldsymbol{X}(t)+\boldsymbol{h}(t)\right) > u\mathbf{1}\right\} \quad &\sim \quad u^{(\frac{2}{\alpha}-\frac{2}{\beta})_+}\prod_{i=1}^n \Psi\left(\frac{u-h_i(t_0)}{\sigma_i(t_0)}\right)\\ &\times \begin{cases} \mathcal{H}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}\int_{-x}^0 e^{-\sum_{i=1}^n f_i(x)}dx, & \text{if } \alpha<\beta,\\ \mathcal{P}^{\boldsymbol{f}}_{\alpha,\frac{\boldsymbol{a}}{\boldsymbol{\sigma}^2(t_0)}\mathbf{I}_{\{\boldsymbol{\alpha}=\alpha\mathbf{1}\}}}[-x,0], & \text{if } \alpha=\beta,\\ 1, & \text{if } \alpha>\beta,\end{cases}\end{aligned} \tag{6.54}$$

which completes the proof. □

Chapter 7

Extremes of L^p-Norm of Vector-Valued Gaussian Processes with Trend[1]

7.1 Introduction

In engineering sciences, extreme values of non-linear functions of multivariate Gaussian processes are of interest in dealing with the safety of structures, see [104] and the references therein. Probabilistic structural analysis to answer the question is: what is the probability that a certain mechanical (or other) structure will survive when it is subject to a random load. The load is then usually defined by some n-dimensional vector process $\boldsymbol{Y}(t) = (Y_1(t), \ldots, Y_n(t)),\ n \geq 1,\ t \in [0, T]$, and one seeks the probability that $\boldsymbol{Y}$ exceeds some more or less well-defined safe region, which is specific for the structure as

$$\mathbb{P}\left\{\boldsymbol{Y}(t) \notin \boldsymbol{S}_u(t), \text{for some } t \in [0, T]\right\}, \tag{7.1}$$

where the time-dependent safety region $\boldsymbol{S}_u(t)$ is defined by

$$\boldsymbol{S}_u(t) = \{(x_1, \cdots, x_n) \in \mathbb{R}^n : \|\boldsymbol{x}\|_p \leq h(t, u)\}$$

with $h(t, u),\ t, u \geq 0$ some continuous function and $|| \cdot ||_p$, $p \in [1, \infty]$ the L^p norm, i.e.,

$$||\boldsymbol{x}||_p = \begin{cases} \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, & p \in [1, \infty), \\ \max(|x_1|, \ldots, |x_n|), & p = \infty, \end{cases}$$

in the space $L_n^p = \{\boldsymbol{x} = (x_1, \ldots, x_n) : ||\boldsymbol{x}||_p < \infty\}$.
Assume that $\boldsymbol{X}(t) = (X_1(t), \ldots, X_n(t))$ where $X_i's$ are independent copies of $X(t)$ a centered Gaussian process which has continuous trajectories, variance function $\sigma^2(\cdot)$ and correlation function $r(\cdot, \cdot)$ and

$$\boldsymbol{d} = (d_1, \ldots, d_n), \quad 1 = d_1 = \cdots = d_m > d_{m+1} \geq d_{m+2} \geq \cdots \geq d_n > 0, \quad 1 \leq m \leq n. \tag{7.2}$$

In the framework of (7.1), set $\boldsymbol{Y}(t) = \boldsymbol{d} * \boldsymbol{X}(t) := (d_1 X_1(t), \cdots, d_n X_n(t))$, then we can rewrite (7.1) as

$$\mathbb{P}\left\{\exists_{t \in [0,T]} Z(t) > h(t, u)\right\}$$

where

$$Z(t) := Z_p(t) := \|\boldsymbol{X}(t) * \boldsymbol{d}\|_p, \tag{7.3}$$

[1] This chapter is based on L. Bai (2018): Extremes of L^p-Norm of Vector-Valued Gaussian Processes with Trend, published in the *Stochastics: An International Journal of Probability and Stochastic Processes,* to appear.

and hereafter, we call $Z_p(t)$ the *L^p norm process.*
When $p = 2$, for a positive constant c, as in the convention $Z_2^c(t) = (Z_2(t))^c$ is called the *chi process* when $c = 1$ and the *chi-square process* when $c = 2$.
Further, as the Gaussian processes, we can introduce the *stationary, locally-stationary, and non-stationary L^p norm processes* according to the stationary, locally-stationary, and non-stationary properties of $X(t)$, respectively.
The investigate of

$$\mathbb{P}\left\{\exists_{t\in[0,T]} Z_2(t) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]} Z_2(t) > u\right\}, \quad \text{as } u \to \infty$$

is initiated by the studies of high excursions of envelope of a Gaussian process, see e.g., [15] and generalized in [104–106]. When $X(t)$ is stationary with $\sigma(t) \equiv 1$ and $r(\cdot)$ satisfies (2.77), [2, 3] develop the Berman's approach in [17] to obtain an asymptotic behavior of large deviation probabilities of the stationary chi-square processes.
Further, if there exists unique $t_0 \in [0, T]$ satisfies $\sigma(t_0) = \sup_{t\in[0,T]} \sigma(t)$ and

$$\sigma(t) = 1 - b(t_0)|t - t_0|^2 + o(|t - t_0|^2), \quad r(s,t) = 1 - a(t_0)|t - s|^2 + o(|t - s|^2), \quad s, t \to t_0,$$

where $b(t_0)$ and $a(t_0)$ are positive constants related to t_0, the tail asymptotic behavior of the non-stationary $Z_2^2(t)$ and $Z_p(t), p \in (1, 2) \cup (2, \infty)$ are investigated in [126] and [72], respectively, under the application of the so-called "double-sum method" in [119].
Some recent contributions are focused on more general scenarios of chi process and chi-square process with $h(t, u) = u - g(t)$, i.e.,

$$\mathbb{P}\left\{\exists_{t\in[0,T]} Z_2^c(t) > h(t,u)\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]} (Z_2^c(t) + g(t)) > u\right\}, \ c = 1, 2,$$

where the continuous function $g(t)$ is generally considered as a trend or a drift.
When $X_i, i = 1, \ldots, n$ are *non-stationary Gaussian processes*, $Z_2(t) + g(t)$, the non-stationary chi processes with trend, and $Z_2^2(t) - wt^\beta, w, \beta > 0$, the non-stationary chi-square processes with trend, are studied in [82] and [107], respectively.
When $X_i, i = 1, \ldots, n$ are *locally-stationary Gaussian processes*, [108] obtains the extreme of the supremum of $Z_2^2(t)$ with trend, see, e.g., [18, 87] for more details about locally stationary Gaussian processes.
Considering both the locally stationary and non-stationary L^p norm processes, the contribution of this paper concerns an exact asymptotic behavior of large deviation probabilities for $Z_p^c(t) + g(t)$ with $p \in [1, \infty]$, constant $c \in (0, \infty)$ and $g(t),\ t \in [0, T]$ a continuous function, which contains the aforementioned results.
Organisation of the rest of the paper: In Section 2, the notation and some preliminaries are given. Our main results are displayed in Section 3. Finally, we present the proofs in Section 4 and several lemmas in Section 5.

7.2 Preliminaries

For the L^p norm process $Z(t)$ in (7.3) and a continuous function $g(t), t \in \mathbb{R}$, we shall investigate the asymptotics of

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (Z^c(t) + g(t)) > u\right\}, \quad u \to \infty, \tag{7.4}$$

with $c > 0$ a constant. As in [72, 126], for $p \in [1, \infty]$, using the duality property of L^p norm we find

$$\mathbb{P}\left\{\sup_{t\in[0,T]} Z^c(t) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,T]} Z(t) > u^{1/c}\right\} = \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[0,T]\times\mathcal{S}_q} Y(t,\boldsymbol{v}) > u^{1/c}\right\},$$

where $Y(t, \boldsymbol{v}) = \sum_{i=1}^n d_i v_i X_i(t)$ is a centered Gaussian field defined on cylinder $[0, T] \times \mathcal{S}_q$ with

$$\mathcal{S}_q = \{\boldsymbol{v} \in \mathbb{R}^n : ||\boldsymbol{v}||_q = 1\}, \tag{7.5}$$

where $\frac{1}{p}+\frac{1}{q}=1$ if $q\in(1,\infty)$, $q=\infty$ if $p=1$ and $q=1$ if $p=\infty$.

Lemma 7.2.1. *On $\mathcal{S}_q$, $\sum_{i=1}^n d_i^2 v_i^2$ attains its maximum d^2 at:*
(i) for $p\in(2,\infty]$ at $2m$ points $\boldsymbol{v}_+^i, \boldsymbol{v}_-^i, i=1,\dots,m$, where $\boldsymbol{v}_+^i=(0,\dots,0,1,0,\dots,0)$ (1 stands at the i-th position), $\boldsymbol{v}_-^i=(0,\dots,0,-1,0,\dots,0)$ (-1 stands at the i-th position), $d=1$;
(ii) for $p=2$ at points on $\{\boldsymbol{v},\boldsymbol{v}\in\mathcal{S}_q, v_i=0, m+1\le i\le n\}$, $d=1$;
(iii) for $p\in[1,2)$ at 2^n points $\boldsymbol{z}$, where

$$\boldsymbol{z}=(z_1,\dots,z_n),\ \ z_i=\pm(d_i/d)^{2/(q-2)},\quad d=\left[\sum_{i=1}^n d_i^{2p/(2-p)}\right]^{(2-p)/2p},$$

(we take all possible 2^n combinations of signs "+" and "-"), where $z_i=\pm(d_i/d)^0=\pm1$.

The proof can be easily carried out by method of Lagrangian multipliers or referring to [72] [Lemma 3.1]. Next by [99], we have the following lemma.

Lemma 7.2.2. *For the L^p norm process $Z(t)$ in (7.3), if $\sigma^2(t_0)=var(X_i(t_0))=1,\ i=1,\dots,n$ for some $t_0\in[0,\infty)$, then we have that as $u\to\infty$*

$$\mathbb{P}\left\{Z^c(t_0)>u\right\}\sim\Psi\left(\frac{u^{1/c}}{d}\right)\begin{cases}2^n(2-p)^{(1-n)/2}, & \text{if } p\in[1,2),\\ \frac{\sqrt{2\pi}2^{\frac{(2-m)}{2}}u^{\frac{m-1}{c}}}{\Gamma(m/2)}\prod_{i=m+1}^n(1-d_i^2)^{-\frac12}, & \text{if } p=2,\\ 2m, & \text{if } p\in(2,\infty],\end{cases}$$

with the convention $\prod_{i=n+1}^n(1-d_i^2)^{-\frac12}=1$ and d the same as in Lemma 7.2.1.

7.3 Extremes of L^p norm processes with trend

In this section, recall that $Z(t)$ in (7.3) is the L^p *norm process* and $X_i(t)$'s are independent copies of $X(t)$ with continuous trajectories, variance functions $\sigma^2(\cdot)$ and correlation functions $r(\cdot,\cdot)$.

7.3.1 Extremes of non-stationary L^p norm processes with trend

As in [12], if $X(t)$ is non-stationary, we introduce the following assumptions:

(i) $\sigma(\cdot)$ attains its maximum on $[0,T]$ at the unique point $t_0\in[0,T]$ and

$$\sigma(t)=1-b|t-t_0|^\beta+o(|t-t_0|^\beta),\qquad t\to t_0$$

for some positive constants b,β.

(ii) $r(s,t)=1-a|t-s|^\alpha+o(|t-s|^\alpha),\ s,t\to t_0$ for some constants $a>0$ and $\alpha\in(0,2]$.

Further, we introduce a bounded measurable trend function $g(t)$ which satisfies

(iii) $g(t)\sim -w|t-t_0|^\gamma,\quad t\to t_0$ for some constants $\gamma>0$ and $w\ge0$.

Theorem 7.3.1. *If assumptions (i)-(iii) are satisfied, then for $\boldsymbol{d}$ in (7.2) and d in Lemma 7.2.1, we have as $u\to\infty$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(Z^c(t)+g(t))>u\right\}\sim\mathbb{P}\left\{Z^c(t_0)>u\right\}\begin{cases}u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}}a^{1/\alpha}d^{-2/\alpha}\mathcal{H}_\alpha\int_Q^\infty e^{-f(t)}dt, & \text{if } \alpha^*<\beta^*,\\ \mathcal{P}_{\alpha,ad^{-2}}^{f(t)}[Q,\infty), & \text{if } \alpha^*=\beta^*,\\ 1, & \text{if } \alpha^*>\beta^*,\end{cases}$$

where $\alpha^*=\alpha c$, $\beta^*=\min(\beta c,\frac{2\gamma c}{2-c})\mathbb{I}_{\{c<2\}}+\beta c\mathbb{I}_{\{c\ge2\}}$, $f(t)=\frac{b|t|^\beta}{d^2}\mathbb{I}_{\{\beta^*=\beta c\}}+\frac{w}{cd^2}|t|^\gamma\mathbb{I}_{\{\beta^*=\frac{2\gamma c}{2-c}\}}$, and $Q=-\infty$ if $t_0\in(0,T)$, $Q=0$ if $t_0\in\{0,T\}$.

Remark. In Theorem 7.3.1, if we assume that $w = 0$, we get the extremes of centered non-stationary L^p norm processes i.e.,

$$\mathbb{P}\left\{\sup_{t\in[0,T]} Z^c(t) > u\right\}, \ u \to \infty.$$

7.3.2 Extremes of locally stationary L^p norm processes with trend

Before giving the scenarios with trend, we consider the extremes of the centered locally stationary L^p norm processes.

Theorem 7.3.2. *Assume that $\sigma(t) \equiv 1$, i.e., unit variance and covariance functions $r(\cdot,\cdot)$ satisfying assumptions (2.17) and (2.18). Then we have for $c > 0$*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} Z^c(t) > u\right\} \sim \int_0^T (a(t))^{\frac{1}{\alpha}} dt d^{-\frac{2}{\alpha}} \mathcal{H}_\alpha u^{\frac{2}{\alpha c}} \mathbb{P}\left\{Z^c(0) > u\right\}, \ u \to \infty,$$

where d is the same as in Lemma 7.2.1.

Theorem 7.3.3. *Assume that $\sigma(t) \equiv 1$, i.e., unit variance and correlation function $r(\cdot,\cdot)$ satisfies assumptions (2.17) and (2.17). Assume that $g(t)$ $t \in [0,T]$ is a continuous function which attains its maximum at a unique point $t_0 \in [0,S]$ satisfying assumption (iii) for some constants $w, \gamma > 0$. Further, set $\alpha^* = \alpha c$, $\beta^* = \frac{2\gamma c}{2-c}\mathbb{I}_{\{c<2\}}$ and $f(t) = \frac{w|t|^\gamma}{cd^2}$ and d is the same as in Lemma 7.2.1.*
If $c \in (0,2)$, then we have as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (Z^c(t) + g(t)) > u\right\} \sim u^{(\frac{2}{\alpha^*} - \frac{2}{\beta^*})_+} \mathbb{P}\left\{Z^c(0) > u\right\} \begin{cases} a^{\frac{1}{\alpha}} d^{-\frac{2}{\alpha}} \mathcal{H}_\alpha \int_Q^\infty e^{-f(t)} dt, & \text{if } \alpha^* < \beta^*, \\ \mathcal{P}^{f(t)}_{\alpha, ad^{-2}}[Q,\infty), & \text{if } \alpha^* = \beta^*, \\ 1, & \text{if } \alpha^* > \beta^*, \end{cases}$$

where $a = a(t_0)$ and $Q = -\infty$ if $t_0 \in (0,T)$, $Q = 0$ if $t_0 \in \{0,T\}$.
If $c = 2$, then we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (Z^c(t) + g(t)) > u\right\} \sim \int_0^T (a(t))^{\frac{1}{\alpha}} e^{\frac{g(t)}{2d^2}} dt d^{-\frac{2}{\alpha}} \mathcal{H}_\alpha u^{\frac{2}{\alpha^*}} \mathbb{P}\left\{Z^c(0) > u\right\}, \ u \to \infty.$$

If $c > 2$, then we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (Z^c(t) + g(t)) > u\right\} \sim \int_0^T (a(t))^{\frac{1}{\alpha}} dt d^{-\frac{2}{\alpha}} \mathcal{H}_\alpha u^{\frac{2}{\alpha^*}} \mathbb{P}\left\{Z^c(0) > u\right\}, \ u \to \infty.$$

Remark. By the proof, we notice that for the case $c = 2$ in Theorem 7.3.3, the result always holds for any continuous function $g(t), t \in [0,1]$. When $c > 0$, the result holds for any bounded function $g(t), t \in [0,1]$.

Example 7.3.1. For $Z(t)$ in (7.3) with $X_i(t) = B^i_\alpha(t), i = 1,\dots,n$ the independent fractional Brownian motions, we have as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in[0,1]} Z(t) - \sqrt{1-t} > u\right\} = \mathbb{P}\left\{Z(1) > u\right\} \begin{cases} u^{\frac{2}{\alpha}-2} \left(\frac{1}{2d^2}\right)^{1/\alpha} \mathcal{H}_\alpha \int_0^\infty e^{-f(t)} dt, & \text{if } \alpha < 1, \\ \mathcal{P}^{f(t)}_{\alpha, d^{-2}/2}[0,\infty), & \text{if } \alpha = 1, \\ 1, & \text{if } \alpha > 1, \end{cases}$$

where $f(t) = \frac{\alpha}{2d^2} t + \frac{1}{d^2} t^{\frac{1}{2}}$ and d is the same as in Lemma 7.2.1

Following example is a special case of Theorem 7.3.3, which is corresponded with [108] [Theorem 2.1].

Example 7.3.2. In Theorem 7.3.3, assume that $p = 2$, $c = 2$ and $g(t), t \in [0,T]$ is a continuous function, then we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} \left(Z_2^2(t) + g(t)\right) > u\right\} \sim \int_0^T (a(t))^{\frac{1}{\alpha}} e^{\frac{g(t)}{2}} dt \mathcal{H}_\alpha \frac{2^{1-m/2} \prod_{i=m+1}^n (1-d_i^2)^{-\frac{1}{2}}}{\Gamma(m/2)} u^{\frac{m-1}{2}+\frac{1}{\alpha}} e^{-\frac{u}{2}}, \ u \to \infty.$$

7.4 Proofs

During the following proofs, $\mathbb{Q}_i, i \in \mathbb{N}$ are some positive constants which can be different from line by line and for interval $\Delta_1, \Delta_2 \subseteq [0, \infty)$ we denote

$$\mathcal{L}_u(\Delta_1) := \mathbb{P}\left\{\sup_{t\in\Delta_1} (Z^c(t)+g(t)) > u\right\}, \quad \mathcal{L}_u(\Delta_1,\Delta_2) := \mathbb{P}\left\{\sup_{t\in\Delta_1} (Z^c(t)+g(t)) > u, \sup_{t\in\Delta_2} (Z^c(t)+g(t)) > u\right\},$$

and

$$\mathcal{K}_u(\Delta_1) := \mathbb{P}\left\{\sup_{t\in\Delta_1} Z^c(t) > u\right\}, \quad \mathcal{K}_u(\Delta_1,\Delta_2) := \mathbb{P}\left\{\sup_{t\in\Delta_1} Z^c(t) > u, \sup_{t\in\Delta_2} Z^c(t) > u\right\}.$$

Proof of Theorem 7.3.1 We first present the proof for the case $t_0 = 0$.
Set $\beta^* = \min(\beta c, \frac{2\gamma c}{2-c})\mathbb{I}_{\{c<2\}} + \beta c\mathbb{I}_{\{c\ge 2\}}$, $\alpha^* = \alpha c$, $\delta(u) = \frac{(\ln u)^\rho}{u^{2/\beta^*}}$ with $\rho > \max\left(\frac{1}{\beta}, \frac{1}{\gamma}\right)$ and for u large enough

$$Y(t,\boldsymbol{v}) = \sum_{i=1}^n d_i v_i X_i(t), \quad (t,\boldsymbol{v}) \in \mathbb{R}\times\mathcal{S}_q$$

with $\mathcal{S}_q$ the same as in (7.5) which is a centered Gaussian field.
We have for some small $\theta > 0$ and u large enough

$$\mathcal{L}_u([0,\delta(u)]) \le \mathcal{L}_u([0,T]) \le \mathcal{L}_u([0,\delta(u)]) + \mathcal{L}_u([\delta(u),\theta]) + \mathcal{L}_u([\theta,T]). \tag{7.6}$$

We first give the upper bounds of $\mathcal{L}_u([\delta(u),\theta])$ and $\mathcal{L}_u([\theta,T])$.
Set $\sigma_\theta := \sup_{t\in[\theta,T]}\sigma(t) < 1$ and $g_m = \sup_{t\in[0,T]} g(t) < \infty$. Then by Borell inequality as in [1] and Lemma 7.2.2 for large u

$$\begin{aligned}
\mathcal{L}_u([\theta,T]) &\le \mathbb{P}\left\{\sup_{t\in[\theta,T]} Z(t) > (u-g_m)^{1/c}\right\}\\
&\le \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[\theta,T]\times\mathcal{S}_q} Y(t,\boldsymbol{v}) > (u-g_m)^{1/c}\right\}\\
&\le \exp\left(-\frac{\left((u-g_m)^{1/c}-\mathbb{Q}_1\right)^2}{2V_Y^*}\right)\\
&= o\left(\mathbb{P}\left\{Z^c(0) > u\right\}\right), \ u\to\infty,
\end{aligned} \tag{7.7}$$

where $\mathbb{Q}_1 := \mathbb{E}\left\{\sup_{(t,\boldsymbol{v})\in[\theta,T]\times\mathcal{S}_q} Y(t,\boldsymbol{v})\right\} < \infty$ and

$$V_Y^* := \sup_{(t,\boldsymbol{v})\in[\theta,T]\times\mathcal{S}_q} \mathrm{var}\left(Y(t,\boldsymbol{v})\right) \le \left(\sup_{t\in[\theta,T]}\sigma^2(t)\right) d^2 = \sigma_\theta^2 d^2 < d^2.$$

By assumptions (i) and (iii), we know that for some $\varepsilon_1 \in (0,1)$

$$\frac{u-g(t)}{\sigma^c(t)} \ge (u + w(1-\varepsilon_1)|t|^\gamma)(1+(1-\varepsilon_1)bc|t|^\beta) \ge u\left(1+\frac{w(1-\varepsilon_1)}{u}|t|^\gamma + (1-\varepsilon_1)bc|t|^\beta\right), \tag{7.8}$$

$$\frac{u-g(t)}{\sigma^c(t)} \le (u + w(1+\varepsilon_1)|t|^\gamma)(1+(1+\varepsilon_1)bc|t|^\beta) \le u\left(1+\frac{w(1+\varepsilon_1)}{u}|t|^\gamma + (1+\varepsilon_1)bc|t|^\beta\right) \tag{7.9}$$

hold for $t \in [0,\theta]$ when θ small enough, then

$$\begin{aligned}
\inf_{t\in[\delta(u),\theta]} \frac{(u-g(t))^{2/c}}{\sigma^2(t)} &\ge \inf_{t\in[\delta(u),\theta]} u^{2/c}\left(1+\frac{w(1-\varepsilon_1)}{u}|t|^\gamma + (1-\varepsilon_1)bc|t|^\beta\right)^{2/c}\\
&\ge u^{2/c} + \mathbb{Q}_2(\ln u)^{(\rho\beta)\vee(\rho\gamma)}.
\end{aligned} \tag{7.10}$$

Denote $\overline{\boldsymbol{X}}(t) = (\overline{X}_1(t), \ldots, \overline{X}_n(t))$ with $\overline{X}_i(t) = \frac{X_i(t)}{\sigma(t)}$, $t \in [0, \theta]$. By assumption (ii), we have that

$$\begin{aligned}
\mathbb{E}\left\{\left(\left(\sum_{i=1}^n d_i v_i \overline{X}_i(t)\right) - \left(\sum_{i=1}^n d_i v_i' \overline{X}_i(s)\right)\right)^2\right\} &\le 2\mathbb{E}\left\{\left(\left(\sum_{i=1}^n d_i v_i \overline{X}_i(t)\right) - \left(\sum_{i=1}^n d_i v_i \overline{X}_i(s)\right)\right)^2\right\} \\
&\quad + 2\mathbb{E}\left\{\left(\left(\sum_{i=1}^n d_i v_i \overline{X}_i(s)\right) - \left(\sum_{i=1}^n d_i v_i' \overline{X}_i(s)\right)\right)^2\right\} \\
&\le 4\mathbb{E}\left\{\sum_{i=1}^n \left(\overline{X}_i(t) - \overline{X}_i(s)\right)^2\right\} + 4\mathbb{E}\left\{\sum_{i=1}^n (v_i - v_i')^2 \left(\overline{X}_i(s)\right)^2\right\} \\
&\le \mathbb{Q}_3 |s-t|^\alpha + \mathbb{Q}_4 \sum_{i=1}^n |v_i - v_i'|^2 \\
&\le \mathbb{Q}_5 \left(|s-t|^\alpha + \sum_{i=1}^n |v_i - v_i'|^\alpha\right)
\end{aligned}$$

holds for $s, t \in [0, \theta]$ and $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{S}_q$. Thus it follows from [119] [Theorem 8.1], (7.10) and Lemma 7.2.2 that

$$\begin{aligned}
\mathcal{L}_u([\delta(u), \theta]) &\le \mathbb{P}\left\{\sup_{(t,\boldsymbol{v}) \in [\delta(u),\theta] \times \mathcal{S}_q} \sum_{i=1}^n d_i v_i \overline{X}_i(t) > \inf_{s \in [\delta(u),\theta]} \frac{(u - g(s))^{1/c}}{\sigma(s)}\right\} \\
&\le \mathbb{Q}_6 u^{\frac{2(n+1)}{\alpha}} \Psi\left(\inf_{s \in [\delta(u),\theta]} \frac{(u-g(s))^{1/c}}{d\sigma(s)}\right) \\
&\le \frac{\mathbb{Q}_6}{d\sqrt{2\pi}} u^{\frac{2(n+1)}{\alpha} - \frac{2}{c}} \exp\left(-\frac{1}{2d}\left(u^{2/c} + \mathbb{Q}_2 (\ln u)^{(\rho\beta)\vee(\rho\gamma)}\right)\right) \\
&= o\left(\mathbb{P}\left\{Z^c(0) > u\right\}\right), \ u \to \infty. \qquad (7.11)
\end{aligned}$$

Thus by (7.7), (7.11) and the fact that $\mathcal{L}_u([0, \delta(u)]) \ge \mathbb{P}\left\{Z^c(0) > u\right\}$ for u positive, we have

$$\mathcal{L}_u([\delta(u), \theta]) = o\left(\mathcal{L}_u([0, \delta(u)])\right), \quad \mathcal{L}_u([\theta, T]) = o\left(\mathcal{L}_u([0, \delta(u)])\right), \ u \to \infty, \qquad (7.12)$$

which combined with (7.6) imply

$$\mathcal{L}_u([0, T]) \sim \mathcal{L}_u([0, \delta(u)]), \ u \to \infty. \qquad (7.13)$$

Now we focus on the asymptotic of $\mathcal{L}_u([0, \delta(u)])$, as $u \to \infty$.
Denote for any $\lambda > 0$ and some $\varepsilon \in (0, 1)$

$$\begin{aligned}
I_k(u) &= [ku^{-2/\alpha^*}\lambda, (k+1)u^{-2/\alpha^*}\lambda], \quad k \in \mathbb{N}, \quad N(u) = \left\lfloor (\ln u)^{\frac{2q}{\beta^*}} u^{\frac{2}{\alpha^*} - \frac{2}{\beta^*}} \lambda^{-1} \right\rfloor, \\
\mathcal{G}_{u,+\varepsilon}(k) &= u\left(1 + \frac{w(1+\varepsilon)}{u}|(k+1)u^{-2/\alpha^*}\lambda|^\gamma + (1+\varepsilon)bc|(k+1)u^{-2/\alpha^*}\lambda|^\beta\right), \\
\mathcal{G}_{u,-\varepsilon}(k) &= u\left(1 + \frac{w(1-\varepsilon)}{u}|ku^{-2/\alpha^*}\lambda|^\gamma + (1-\varepsilon)bc|ku^{-2/\alpha^*}\lambda|^\beta\right).
\end{aligned}$$

Case 1: $\beta^* > \alpha^*$. For u large enough, we have

$$\sum_{k=0}^{N(u)-1} \mathcal{L}_u(I_k(u)) - \sum_{i=1}^2 \Lambda_i(u) \le \mathcal{L}_u([0, \delta(u)]) \le \sum_{k=0}^{N(u)} \mathcal{L}_u(I_k(u)), \qquad (7.14)$$

where

$$\Lambda_1(u) = \sum_{k=0}^{N(u)} \mathcal{L}_u(I_k(u), I_{k+1}(u)), \quad \Lambda_2(u) = \sum_{0 \le k, l \le N(u), l \ge k+2} \mathcal{L}_u(I_k(u), I_l(u)).$$

In the view of Lemma 7.5.2 and (7.8), we have that for some $\epsilon \in [0,1)$,

$$\begin{aligned}
\sum_{k=0}^{N(u)} \mathcal{L}_u(I_k(u)) &\le \sum_{k=0}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&\sim \mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda] \sum_{k=0}^{N(u)} \mathbb{P}\left\{Z^c(0) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&\sim \mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda]\mathbb{P}\left\{Z^c(0) > u\right\} \\
&\quad \times \sum_{k=0}^{N(u)} \exp\left(-(1-\varepsilon-\epsilon)\frac{w}{cd^2}u^{\frac{2-c}{c}}|kSu^{-\frac{2}{\alpha^*}}|^\gamma - (1-\varepsilon-\epsilon)\frac{b}{d^2}u^{2/c}|kSu^{-\frac{2}{\alpha^*}}|^\beta\right) \\
&\sim \mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda]\mathbb{P}\left\{Z^c(0) > u\right\} \sum_{k=0}^{N(u)} \exp\left(-(1-\varepsilon-\epsilon)f(u^{\frac{2}{\beta^*}}kSu^{-\frac{2}{\alpha^*}})\right) \\
&\sim \mathbb{P}\left\{Z^c(0) > u\right\} \frac{\mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda]}{\lambda} u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}} \int_0^\infty \exp\left(-(1-\varepsilon-\epsilon)f(t)\right)dt \\
&\sim \mathbb{P}\left\{Z^c(0) > u\right\} a^{1/\alpha}d^{-2/\alpha}\mathcal{H}_\alpha u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}} \int_0^\infty e^{-f(t)}dt, \qquad (7.15)
\end{aligned}$$

as $u \to \infty,\ \lambda \to \infty,\ \varepsilon \to 0,\ \epsilon \to 0$ where $f(t) = \frac{b|t|^\beta}{d^2}\mathbb{I}_{\{\beta^*=\beta c\}} + \frac{w}{cd^2}|t|^\gamma \mathbb{I}_{\{\beta^*=\frac{2\gamma c}{2-c}\}}$. Similarly, we derive that

$$\sum_{k=0}^{N(u)-1} \mathcal{L}_u(I_k(u)) \ge \mathbb{P}\left\{Z^c(0) > u\right\} a^{1/\alpha}d^{-2/\alpha}\mathcal{H}_\alpha u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}} \int_0^\infty e^{-f(t)}dt,\ u \to \infty,\ \lambda \to \infty. \qquad (7.16)$$

Moreover,

$$\begin{aligned}
\Lambda_1(u) &\le \sum_{k=0}^{N(u)} \left(\mathcal{L}_u(I_k(u)) + \mathcal{L}_u(I_{k+1}(u)) - \mathcal{L}_u(I_k(u)\cup I_{k+1}(u))\right) \\
&\le \sum_{k=0}^{N(u)} \left(\mathbb{P}\left\{\sup_{t\in I_k(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k)\right\} + \mathbb{P}\left\{\sup_{t\in I_{k+1}(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k)\right\}\right. \\
&\quad \left. -\mathbb{P}\left\{\sup_{t\in((I_k(u)\cup I_{k+1}(u)))} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\}\right) \\
&\le \left(2\mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda] - \mathcal{H}_\alpha[0, 2a^{1/\alpha}d^{-2/\alpha}\lambda]\right) \sum_{k=0}^{N(u)} \mathbb{P}\left\{Z^c(0) > \widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\} \\
&\sim \frac{2\mathcal{H}_\alpha[0, a^{1/\alpha}d^{-2/\alpha}\lambda] - \mathcal{H}_\alpha[0, 2a^{1/\alpha}d^{-2/\alpha}\lambda]}{\lambda} \int_0^\infty \exp\left(-(1-\varepsilon-\epsilon)f(t)\right)dt \\
&\quad \times u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}}\mathbb{P}\left\{Z^c(0) > u\right\} \\
&= o\left(u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}}\mathbb{P}\left\{Z^c(0) > u\right\}\right),\ u \to \infty, \lambda \to \infty, \varepsilon \to 0, \epsilon \to 0, \qquad (7.17)
\end{aligned}$$

where $\widehat{\mathcal{G}}_{u,-\varepsilon}(k) = \min(\mathcal{G}_{u,-\varepsilon}(k), \mathcal{G}_{u,-\varepsilon}(k+1))$. By Lemma 7.5.3, we have

$$\begin{aligned}
\Lambda_2(u) &\le \sum_{0\le k,l\le N(u), l\ge k+2} \mathbb{P}\left\{\sup_{t\in I_k(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k), \sup_{t\in I_l(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(l)\right\} \\
&\le \sum_{0\le k\le N(u)} \sum_{l=2}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k), \sup_{t\in I_{k+l}(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&\le \mathbb{Q}_7 \left(\sum_{k=0}^{N(u)} \mathbb{P}\left\{Z^c(0) > \mathcal{G}_{u,-\varepsilon}(k)\right\}\right) \sum_{l=1}^{\infty} \exp\left(-(l\lambda)^\alpha/8\right) \\
&\le \mathbb{Q}_8 \mathbb{P}\left\{Z^c(0) > u\right\} u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}} \lambda \sum_{l=1}^{\infty} \exp\left(-(l\lambda)^\alpha/8\right) \\
&= o\left(\mathbb{P}\left\{Z^c(0) > u\right\} u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}}\right),\ u \to \infty,\ \lambda \to \infty. \qquad (7.18)
\end{aligned}$$

Combing (7.15)-(7.18) with (7.14), we obtain

$$\mathcal{L}_u([0,\delta(u)]) \sim \mathbb{P}\left\{Z^c(0)>u\right\} a^{1/\alpha} d^{-2/\alpha} \mathcal{H}_\alpha u^{\frac{2}{\alpha^*}-\frac{2}{\beta^*}} \int_0^\infty e^{-f(t)} dt, \ u \to \infty. \tag{7.19}$$

Case 2: $\beta^*=\alpha^*$. We consider that for u large enough,

$$\mathcal{L}_u(I_0(u)) \le \mathcal{L}_u([0,\delta(u)]) \le \sum_{k=0}^{N(u)} \mathcal{L}_u(I_k(u)). \tag{7.20}$$

Using (7.28) of Lemma 7.5.2 with u replaced by $u^{1/c}$ and (7.9), we have that

$$\begin{aligned}
\mathcal{L}_u(I_0(u)) &= \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \left(Z^c(tu^{-2/\alpha^*})+g(tu^{-2/\alpha^*})\right)>u\right\}\\
&\ge \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \frac{\left\|\overline{\boldsymbol{X}}(tu^{-2/\alpha^*})*\boldsymbol{d}\right\|_p^c}{1+\frac{w(1+\varepsilon)}{u}|tu^{-2/\alpha^*}|^\gamma+(1+\varepsilon)bc|tu^{-2/\alpha^*}|^\beta}>u\right\}\\
&\ge \mathbb{P}\left\{\sup_{t\in[0,\lambda]} \frac{\left\|\overline{\boldsymbol{X}}(tu^{-2/\alpha^*})*\boldsymbol{d}\right\|_p}{1+(1+\varepsilon+\epsilon)u^{-2/c}d^2 f(t)}>u^{1/c}\right\}\\
&\sim \mathbb{E}\left\{\sup_{t\in[0,\lambda]} \exp\left(\sqrt{\frac{2a}{d^2}}B_\alpha(t)-\frac{a}{d^2}|t|^\alpha-(1+\varepsilon+\epsilon)f(t)\right)\right\}\mathbb{P}\left\{Z^c(0)>u\right\}\\
&\sim \mathcal{P}^f_{\alpha,\frac{a}{d^2}}[0,\infty)\mathbb{P}\left\{Z^c(0)>u\right\}, \ u\to\infty, \varepsilon\to 0, \epsilon\to 0, \lambda\to\infty.
\end{aligned} \tag{7.21}$$

Similarly,

$$\mathcal{L}_u(I_0(u)) \le \mathcal{P}^f_{\alpha,\frac{a}{d^2}}[0,\infty)\mathbb{P}\left\{Z^c(0)>u\right\}, \ u\to\infty, \lambda\to\infty. \tag{7.22}$$

Moreover, by Lemma 7.5.2,

$$\begin{aligned}
\sum_{k=1}^{N(u)} \mathcal{L}_u(I_k(u)) &\le \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > \mathcal{G}_{u,-\varepsilon}(k)\right\}\\
&\sim \mathcal{H}_\alpha[0,a^{1/\alpha}d^{-2/\alpha}\lambda] \sum_{k=1}^{N(u)} \mathbb{P}\left\{Z^c(0)>\mathcal{G}_{u,-\varepsilon}(k)\right\}\\
&\le \mathcal{H}_\alpha[0,a^{1/\alpha}d^{-2/\alpha}\lambda]\mathbb{P}\left\{Z^c(0)>u\right\} \sum_{k=1}^{N(u)} \exp\left(-(1-\varepsilon-\epsilon)f(k\lambda)\right)\\
&\le \mathcal{H}_\alpha[0,a^{1/\alpha}d^{-2/\alpha}\lambda]\mathbb{P}\left\{Z^c(0)>u\right\} \sum_{k=1}^{\infty} \exp\left(-\mathbb{Q}_9(k\lambda)^{\gamma\wedge\beta}\right)\\
&\sim \mathbb{Q}_{10}\mathbb{P}\left\{Z^c(0)>u\right\}\lambda\exp\left(-\mathbb{Q}_{11}\lambda^{\gamma\wedge\beta}\right)\\
&= o\left(\mathbb{P}\left\{Z^c(0)>u\right\}\right), \ u\to\infty, \ \lambda\to\infty.
\end{aligned} \tag{7.23}$$

Inserting (7.21), (7.22), and (7.23) into (7.20), we have

$$\mathcal{L}_u([0,\delta(u)]) \sim \mathcal{P}^f_{\alpha,\frac{a}{d^2}}[0,\infty)\mathbb{P}\left\{Z^c(0)>u\right\}, u\to\infty. \tag{7.24}$$

Case 3: $\beta^*<\alpha^*$. Obviously,

$$\mathcal{L}_u([0,\delta(u)]) \ge \mathbb{P}\left\{Z^c(0)>u\right\}. \tag{7.25}$$

For any $\varepsilon_2\in(0,1)$, $[0,\delta(u)]\subseteq[0,u^{-2/\alpha^*}\varepsilon_2]$ when u large enough. By Lemma 7.5.2 and the fact that $\sup_{t\in[0,\delta(u)]} g(t)\le 0$, we obtain

$$\mathcal{L}_u([0,\delta(u)]) \le \mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha^*}\varepsilon_2]} \left\|\overline{\boldsymbol{X}}(t)*\boldsymbol{d}\right\|_p^c > u\right\}$$

$$\begin{aligned}&\sim \mathcal{H}_\alpha[0, a^{1/\alpha} d^{-2/\alpha}\varepsilon_2]\mathbb{P}\left\{Z^c(0) > u\right\}\\&\sim \mathbb{P}\left\{Z^c(0) > u\right\},\ u \to \infty, \varepsilon_2 \to 0.\end{aligned}$$

Together with (7.25), we get

$$\mathcal{L}_u([0, \delta(u)]) \sim \mathbb{P}\left\{Z^c(0) > u\right\},\ u \to \infty. \tag{7.26}$$

Consequently, we have the results according to (7.13), (7.19), (7.24) and (7.26).
For $t_0 \in (0, T)$ and $t_0 = T$, we just need to replace $[0, \delta(u)]$ as $[t_0 - \delta(u), t_0 + \delta(u)]$ and $[T - \delta(u), T]$. Thus we complete the proof. □

Proof of Theorem 7.3.2 For any $\theta > 0$ and $\lambda > 0$, set $\alpha^* = \alpha c$

$$\begin{aligned}&I_k(\theta) = [k\theta, (k+1)\theta], \quad a_k = a(k\theta), \quad k \in \mathbb{N}, \quad N(\theta) = \left\lfloor \frac{T}{\theta} \right\rfloor,\\&J_l^k(u) = \left[k\theta + l u^{-2/\alpha^*}\lambda, k\theta + (l+1)u^{-2/\alpha^*}\lambda\right], \quad M(u) = \left\lfloor \frac{\theta u^{2/\alpha^*}}{\lambda} \right\rfloor.\end{aligned}$$

We have

$$\sum_{k=0}^{N(\theta)-1}\left(\sum_{l=0}^{M(u)-1} \mathcal{K}_u(J_l^k(u))\right) - \sum_{i=1}^{4} \mathcal{A}_i(u) \le \mathcal{K}_u([0, T]) \le \sum_{k=0}^{N(\theta)} \mathcal{K}_u(I_k(\theta)) \le \sum_{k=0}^{N(\theta)}\left(\sum_{l=0}^{M(u)} \mathcal{K}_u(J_l^k(u))\right),$$

where

$$\mathcal{A}_i(u) = \sum_{(k_1,l_1,k_2,l_2)\in\mathcal{L}_i} \mathcal{K}_u(J_{l_1}^{k_1}, J_{l_2}^{k_2}),\ i = 1, 2, 3, 4,$$

with

$$\begin{aligned}\mathcal{L}_1 &= \left\{0 \le k_1 = k_2 \le N(\theta) - 1, 0 \le l_1 + 1 = l_2 \le M(u) - 1\right\},\\\mathcal{L}_2 &= \left\{0 \le k_1 + 1 = k_2 \le N(\theta) - 1, l_1 = M(u), l_2 = 0\right\},\\\mathcal{L}_3 &= \left\{0 \le k_1 + 1 < k_2 \le N(\theta) - 1, 0 \le l_1, l_2 \le M(u) - 1\right\},\\\mathcal{L}_4 &= \left\{0 \le k_1 \le k_2 \le N(\theta) - 1, k_2 - k_1 \le 1, 0 \le l_1, l_2 \le M(u) - 1\right\} \setminus (\mathcal{L}_1 \cup \mathcal{L}_2).\end{aligned}$$

By Lemma 7.5.2

$$\begin{aligned}\sum_{k=0}^{N(\theta)}\left(\sum_{l=0}^{M(u)} \mathcal{K}_u(J_l^k(u))\right) &= \sum_{k=0}^{N(\theta)}\left(\sum_{l=0}^{M(u)} \mathbb{P}\left\{\sup_{t\in[0,\lambda]} Z^c(k\theta + l u^{-2/\alpha^*}\lambda + u^{-2/\alpha^*}t) > u\right\}\right)\\&\le \sum_{k=0}^{N(\theta)}\left(\sum_{l=0}^{M(u)} (a_k + \varepsilon_\theta)^{\frac{1}{\alpha}} d^{-2/\alpha}\mathcal{H}_\alpha \lambda \mathbb{P}\left\{Z^c(0) > u\right\}\right)\\&\sim \left(\sum_{k=0}^{N(\theta)} (a_k + \varepsilon_\theta \theta)^{\frac{1}{\alpha}}\right) d^{-2/\alpha}\mathcal{H}_\alpha u^{2/\alpha^*}\mathbb{P}\left\{Z^c(0) > u\right\}\\&\sim \int_0^T (a(t))^{1/\alpha} dt u^{2/\alpha^*} d^{-2/\alpha}\mathcal{H}_\alpha \mathbb{P}\left\{Z^c(0) > u\right\},\ u \to \infty,\ \lambda \to \infty,\ \theta \to 0.\end{aligned}$$

Similarly,

$$\sum_{k=0}^{N(\theta)-1}\left(\sum_{l=0}^{M(u)-1} \mathcal{K}_u(J_l^k(u))\right) \ge \int_0^T (a(t))^{1/\alpha} dt u^{2/\alpha^*} d^{-2/\alpha}\mathcal{H}_\alpha \mathbb{P}\left\{Z^c(0) > u\right\},\ u \to \infty,\ \lambda \to \infty,\ \theta \to 0.$$

Further, by Lemma 7.5.2

$$
\begin{aligned}
\mathcal{A}_1(u) &= \sum_{k=0}^{N(\theta)-1}\left(\sum_{l=0}^{M(u)-1}\left(\mathcal{K}_u(J_l^k(u))+\mathcal{K}_u(J_{l+1}^k(u))-\mathcal{K}_u(J_l^k(u)\cup J_{l+1}^k(u))\right)\right)\\
&\sim \sum_{k=0}^{N(\theta)-1}\left(\left(\mathcal{H}_\alpha[0,(a_k+\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda]+\mathcal{H}_\alpha[0,(a_k+\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda]-\mathcal{H}_\alpha[0,2(a_k-\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda]\right)\right.\\
&\quad\left.\times\sum_{l=0}^{M(u)-1}\mathbb{P}\left\{Z^c(0)>u\right\}\right)\\
&\le \mathbb{Q}_1\left(\sum_{k=0}^{N(\theta)-1}\left((a_k+\varepsilon_\theta)^{\frac{1}{\alpha}}-(a_k-\varepsilon_\theta)^{\frac{1}{\alpha}}\right)\theta\right)u^{2/\alpha^*}\mathbb{P}\left\{Z^c(t)>u\right\}\\
&= o\left(u^{2/\alpha^*}\mathbb{P}\left\{Z^c(t)>u\right\}\right),\ u\to\infty,\quad \lambda\to\infty,\theta\to 0.
\end{aligned}
$$

Similarly, by Lemma 7.5.2

$$
\begin{aligned}
\mathcal{A}_2(u) &= \sum_{k=0}^{N(\theta)-1}\mathcal{K}_u(J_{M(u)-1}^k(u),J_0^{k+1}(u))\\
&\le \sum_{k=0}^{N(\theta)-1}\mathbb{P}\left\{\sup_{t\in[0,2\lambda]}Z^c((k+1)\theta-u^{-2/\alpha^*}t)>u,\ \sup_{t\in[0,2\lambda]}Z^c((k+1)\theta+u^{-2/\alpha^*}t)>u\right\}\\
&= \sum_{k=0}^{N(\theta)-1}\left(\mathbb{P}\left\{\sup_{t\in[0,2\lambda]}Z^c((k+1)\theta-u^{-2/\alpha^*}t)>u\right\}+\mathbb{P}\left\{\sup_{t\in[0,2\lambda]}Z^c((k+1)\theta+u^{-2/\alpha^*}t)>u\right\}\right.\\
&\quad\left.-\mathbb{P}\left\{\sup_{t\in[-2\lambda,2\lambda]}Z^c((k+1)\theta-u^{-2/\alpha^*}t)>u\right\}\right)\\
&\sim \sum_{k=0}^{N(\theta)-1}\left(\left(2\mathcal{H}_\alpha[0,2(a_{k+1}+\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda]-\mathcal{H}_\alpha[-2(a_k-\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda,2(a_k-\varepsilon_\theta)^{\frac{1}{\alpha}}d^{-1/\alpha}\lambda]\right)\right.\\
&\quad\left.\times\sum_{l=0}^{M(u)-1}\mathbb{P}\left\{Z^c(0)>u\right\}\right)\\
&\le \mathbb{Q}_2\left(\sum_{k=0}^{N(\theta)-1}\left((a_k+\varepsilon_\theta)^{\frac{1}{\alpha}}-(a_k-\varepsilon_\theta)^{\frac{1}{\alpha}}\right)\theta\right)u^{2/\alpha^*}\mathbb{P}\left\{Z^c(0)>u\right\}\\
&= o\left(u^{2/\alpha^*}\mathbb{P}\left\{Z^c(0)>u\right\}\right),\ u\to\infty,\quad \lambda\to\infty,\theta\to 0.
\end{aligned}
$$

For any $\theta>0$

$$
\mathbb{E}\left\{X_i(t)X_i(s)\right\}=r(s,t)\le 1-\delta(\theta)
$$

for $(s,t)\in J_{l_1}^{k_1}(u)\times J_{l_2}^{k_2}(u),(j_1,k_1,j_2,k_2)\in\mathcal{L}_3$ where $\delta(\theta)>0$ is related to θ. Then by Lemma 7.5.1

$$
\begin{aligned}
\mathcal{A}_3(u) &\le N(\theta)M(u)2\Psi\left(\frac{2\,(u)^{\frac{1}{c}}-\mathbb{Q}_3}{d\sqrt{4-\delta(\theta)}}\right)\\
&\le \frac{T}{\lambda}u^{2/\alpha^*}2\Psi\left(\frac{2u^{\frac{1}{c}}-\mathbb{Q}_3}{d\sqrt{4-\delta(\theta)}}\right)\\
&= o\left(u^{2/\alpha^*}\mathbb{P}\left\{Z^c(0)>u\right\}\right),\ u\to\infty,\lambda\to\infty,\theta\to 0.
\end{aligned}
$$

where $\mathbb{Q}_3$ is a large constant. Finally by Lemma 7.5.3 for u large enough and θ small enough

$$
\mathcal{A}_4(u) \le \sum_{k=0}^{N(\theta)-1}\left(\sum_{l=0}^{2M(u)}\sum_{i=2}^{2M(u)}\mathcal{K}_u(J_l^k(u),J_{l+i}^k(u))\right)
$$

$$
\begin{aligned}
&\le \sum_{k=0}^{N(\theta)-1}\sum_{l=0}^{2M(u)} \mathbb{P}\left\{Z^c(0)>u\right\}\left(\sum_{i=1}^{\infty}\mathbb{Q}_4\exp\left(-\frac{\mathbb{Q}_5}{8}|i\lambda|^{\alpha}\right)\right)\\
&\le \mathbb{Q}_6\frac{T}{\lambda}u^{-2/\alpha^*}\mathbb{P}\left\{Z^c(0)>u\right\}\left(\sum_{i=1}^{\infty}\exp\left(-\frac{\mathbb{Q}_5}{8}|i\lambda|^{\alpha}\right)\right)\\
&= o\left(u^{2/\alpha^*}\mathbb{P}\left\{Z^c(0)>u\right\}\right),\ u\to\infty,\lambda\to\infty,\theta\to 0.
\end{aligned}
$$

Thus the claim follows. □

Proof of Theorem 7.3.3 Through this proof, denote $\mathcal{L}_u(\Delta_1)$ and $\mathcal{L}_u(\Delta_1,\Delta_2)$ the same as in the proof of Theorem 7.3.1.

When $c\in(0,2)$, in the proof of Theorem 7.3.1, if we take $\beta^*=\frac{2\gamma c}{2-c}$ and $f(t)=\frac{w|t|^{\gamma}}{cd^2}$, then all argumentations still hold and the results follow.

When $c=2$, for any constant $\theta>0$, we define

$$
I_k=[k\theta,(k+1)\theta],\ k\in\mathbb{N},\ N(\theta)=\left\lfloor\frac{T}{\theta}\right\rfloor,
$$

and

$$
M_1(k)=\sup_{t\in I_k} g(t),\quad M_2(k)=\inf_{t\in I_k} g(t).
$$

Then

$$
\mathcal{L}_u([0,T])\ge\sum_{k=0}^{N(\theta)-1}\mathcal{L}_u(I_k)-\sum_{j=1}^{2}\Lambda_j,
$$

where

$$
\Lambda_1=\sum_{k=0}^{N(\theta)}\mathcal{L}_u(I_k,I_{k+1}),\quad \Lambda_2=\sum_{\substack{k=0\\ j>k+1}}^{N(\theta)}\mathcal{L}_u(I_k,I_j),
$$

and by Theorem 7.3.2

$$
\begin{aligned}
\mathcal{L}_u([0,T]) &\le \sum_{k=0}^{N(\theta)}\mathcal{L}_u(I_k)\\
&\le \sum_{k=0}^{N(\theta)}\mathbb{P}\left\{\sup_{t\in I_k}Z^c(t)>u-M_1(k)\right\}\\
&\sim \sum_{k=0}^{N(\theta)}(a(k\theta))^{\frac{1}{\alpha}}\left(u-M_1(k)\right)^{\frac{1}{\alpha}}d^{-2/\alpha}\mathcal{H}_{\alpha}\theta\mathbb{P}\left\{Z^c(0)>u-M_1(k)\right\}\\
&\sim u^{\frac{1}{\alpha}}d^{-2/\alpha}\mathcal{H}_{\alpha}\mathbb{P}\left\{Z^c(0)>u\right\}\theta\sum_{k=0}^{N(\theta)}(a(k\theta))^{\frac{1}{\alpha}}e^{\frac{M_1(k)}{2d^2}}\\
&\sim u^{\frac{1}{\alpha}}d^{-2/\alpha}\mathcal{H}_{\alpha}\mathbb{P}\left\{Z^c(0)>u\right\}\int_0^T(a(t))^{1/\alpha}e^{\frac{g(t)}{2d^2}}dt,\ u\to\infty,\ \theta\to 0.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{k=0}^{N(\theta)-1}\mathcal{L}_u(I_k) &\ge \sum_{k=0}^{N(\theta)-1}\mathbb{P}\left\{\sup_{t\in I_k}Z^c(t)>u-M_2(k)\right\}\\
&\sim u^{\frac{1}{\alpha}}d^{-2/\alpha}\mathcal{H}_{\alpha}\mathbb{P}\left\{Z^c(0)>u\right\}\int_0^T(a(t))^{1/\alpha}e^{\frac{g(t)}{2d^2}}dt,\ u\to\infty,\ \theta\to 0.
\end{aligned}
$$

Further, we have

$$\begin{aligned}
\Lambda_1 &\le \sum_{k=0}^{N(\theta)} \left(\mathcal{L}_u(I_k)+\mathcal{L}_u(I_{k+1})-\mathcal{L}_u(I_k\cup I_{k+1})\right)\\
&\le \sum_{k=0}^{N(\theta)} \left(\mathbb{P}\left\{\sup_{t\in I_k} Z^c(t)>u-\widetilde{M}_1(k)\right\}+\mathbb{P}\left\{\sup_{t\in I_{k+1}} Z^c(t)>u-\widetilde{M}_1(k)\right\}\right.\\
&\quad \left. -\mathbb{P}\left\{\sup_{t\in I_k\cup I_{k+1}} Z^c(t)>u-\widetilde{M}_1(k)\right\}\right)\\
&\sim \sum_{k=0}^{N(\theta)} \left((a(k\theta))^{1/\alpha}+(a((k+1)\theta))^{1/\alpha}-2(a(k\theta))^{1/\alpha}\right)\theta u^{\frac{1}{\alpha}} d^{-2/\alpha}\mathcal{H}_\alpha e^{\frac{\widetilde{M}_1(k)}{2d^2}}\mathbb{P}\left\{Z^c(0)>u\right\}\\
&= o\left(u^{1/\alpha}\mathbb{P}\left\{Z^c(0)>u\right\}\right),\ u\to\infty,\ \theta\to 0,
\end{aligned}$$

where $\widetilde{M}_1(k)=\max(M_1(k),M_1(k+1))$.
Then for $g_m=\sup_{t\in[0,T]} g(t)$ by Lemma 7.5.1

$$\begin{aligned}
\Lambda_2 &\le \sum_{\substack{k=0\\ j>k+1}}^{N(\theta)} \mathbb{P}\left\{\sup_{t\in I_k} Z^c(t)>u-g_m, \sup_{t\in I_j} Z^c(t)>u-g_m\right\}\\
&\le \sum_{\substack{k=0\\ j>k+1}}^{N(\theta)} 2\Psi\left(\frac{2(u-g_m)^{1/c}-\mathbb{Q}_1}{d\sqrt{4-\delta(\theta)}}\right)\\
&= o\left(\mathbb{P}\left\{Z^c(0)>u\right\}\right),\ u\to\infty,\theta\to 0.
\end{aligned}$$

Thus, we have

$$\mathcal{L}_u([0,T])\sim\int_0^T (a(t))^{1/\alpha}e^{\frac{g(t)}{2d^2}}dt\mathcal{H}_\alpha d^{-2/\alpha}u^{\frac{1}{\alpha}}\mathbb{P}\left\{Z^c(0)>u\right\},\ u\to\infty.$$

When $c\in(2,\infty)$, set $M_1=\inf_{t\in[0,T]} g(t)$ and $M_2=\sup_{t\in[0,T]} g(t)$. Since $g(t)$ is a continuous function, we have $-\infty<M_1\le M_2<\infty$. Further, since when $c\in(2,\infty)$,

$$\mathbb{P}\left\{Z^c(0)>u+\mathbb{Q}_2\right\}\sim\mathbb{P}\left\{Z^c(0)>u\right\}$$

holds for any $\mathbb{Q}_2>0$. Hence, by Theorem 7.3.2

$$\begin{aligned}
\mathcal{L}_u([0,T]) &\ge \mathbb{P}\left\{\sup_{t\in[0,T]} Z^c(t)>u-M_1\right\}\\
&\sim \int_0^T (a(t))^{\frac{1}{\alpha}}dt d^{-\frac{2}{\alpha}}\mathcal{H}_\alpha u^{\frac{2}{\alpha c}}\mathbb{P}\left\{Z^c(0)>u-M_1\right\}\\
&\sim \int_0^T (a(t))^{\frac{1}{\alpha}}dt d^{-\frac{2}{\alpha}}\mathcal{H}_\alpha u^{\frac{2}{\alpha c}}\mathbb{P}\left\{Z^c(0)>u\right\},u\to\infty,
\end{aligned}$$

and

$$\mathcal{L}_u([0,T])\le\mathbb{P}\left\{\sup_{t\in[0,T]} Z^c(t)>u-M_2\right\}\sim\int_0^T (a(t))^{\frac{1}{\alpha}}dt d^{-\frac{2}{\alpha}}\mathcal{H}_\alpha u^{\frac{2}{\alpha c}}\mathbb{P}\left\{Z^c(0)>u\right\},u\to\infty.$$

The result follows. □

7.5 Some technical results

In this section, we give several lemmas which are used in the proofs of the theorems.

Lemma 7.5.1. *let $\boldsymbol{X}(t) = (X_1(t)\dots, X_n(t)),\ t \in [0,T], n \geq 1$ be an centered $\mathbb{R}^n$-valued vector process with independent marginals, which have continuous samples, unit variances and correlation functions satisfying (2.18). Then for $0 < t_1 < t_2 < t_3 < \infty$ and u large enough*

$$\mathbb{P}\left\{\sup_{t\in[0,t_1]} Z^c(t) > u,\ \sup_{t\in[t_2,t_3]} Z^c(t) > u\right\} \leq 2\Psi\left(\frac{2u^{1/c} - D}{d\sqrt{4-\delta}}\right),$$

where D, δ are some constant.

Proof of Lemma 7.5.1 By (2.18) and the continuity of $r(t)$, for some $\delta > 0$ we have

$$\mathbb{E}\left\{X_i(t)X_i(s)\right\} = r(s,t) \leq 1 - \frac{\delta}{2}, i = 1, 2, \dots, n,$$

holds for any $(s,t) \in [0,t_1] \times [t_2,t_3]$. Set $\widetilde{Y}(t,\boldsymbol{v},s,\boldsymbol{w}) = \sum_{i=1}^n X_i(t)d_i v_i + \sum_{i=1}^n X_i(s)d_i w_i$ where $\boldsymbol{v},\boldsymbol{w} \in \mathcal{S}_q$ with $\mathcal{S}_q = \{\boldsymbol{v} \in \mathbb{R}^n : ||\boldsymbol{v}||_q = 1\}$. Since $\widetilde{Y}(t,\boldsymbol{v},s,\boldsymbol{w})$ is a center Gaussian fields, we have further

$$\begin{aligned}
\mathrm{var}\left(\widetilde{Y}(t,\boldsymbol{v},s,\boldsymbol{w})\right) &= \sum_{i=1}^n (v_i^2 + w_i^2 + 2r(s,t)v_i w_i)d_i^2 \\
&\leq 2d^2 + 2r(s,t)\sum_{i=1}^n (v_i^2 + w_i^2)d_i^2 \\
&= 2d^2 + 2d^2 r(s,t) \\
&\leq d^2(4-\delta),
\end{aligned}$$

for any $(t,\boldsymbol{v},s,\boldsymbol{w}) \in [0,t_1] \times \mathcal{S}_q \times [t_2,t_3] \times \mathcal{S}_q$. By Borell inequality,

$$\begin{aligned}
\mathbb{P}\left\{\sup_{t\in[0,t_1]} Z^c(t) > u,\ \sup_{t\in[t_2,t_3]} Z^c(t) > u\right\} &= \mathbb{P}\left\{\sup_{t\in[0,t_1]} Z(t) > u^{1/c},\ \sup_{t\in[t_2,t_3]} Z(t) > u^{1/c}\right\} \\
&\leq \mathbb{P}\left\{\sup_{(t,\boldsymbol{v},s,\boldsymbol{w})\in[0,t_1]\times\mathcal{S}_q\times[t_2,t_3]\times\mathcal{S}_q} \widetilde{Y}(t,\boldsymbol{v},s,\boldsymbol{w}) > 2u^{1/c}\right\} \\
&\leq 2\Psi\left(\frac{2u^{1/c} - D}{d\sqrt{4-\delta}}\right),
\end{aligned}$$

where D is some constant such that

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v},s,\boldsymbol{w})\in[0,t_1]\times\mathcal{S}_q\times[t_2,t_3]\times\mathcal{S}_q} \widetilde{Y}(t,\boldsymbol{v},s,\boldsymbol{w}) > D\right\} \leq \frac{1}{2},$$

hence the claim follows. □

Lemma 7.5.2. *let $\boldsymbol{X}(t) = (X_1(t)\dots, X_n(t)),\ t \in \mathbb{R}, n \geq 1$ be an centered $\mathbb{R}^n$-valued vector process with independent marginals, which have continuous samples, unit variances and correlation functions satisfying (2.17). Set $a := a(t_0),\ t_0 \in \mathbb{R}$, and K_u a family of countable index sets and u_k satisfying that*

$$\lim_{u\to\infty} \sup_{k\in K_u} \left|\frac{u_k}{u} - 1\right| = 0. \tag{7.27}$$

If $f(t)$ is a nonnegative continuous function with $f(0) = 0, f(t) > 0, t \neq 0$ and $\boldsymbol{d}$ is the same as in (7.2), then we have that for some constants $S_1, S_2 \geq 0$ and $\max(S_1, S_2) > 0$

$$\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]} \frac{Z(u^{-2/\alpha}t + t_0)}{1 + u^{-2}f(t)} > u\right\} \sim \mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2]\mathbb{P}\left\{Z(t_0) > u\right\},\ u \to \infty, \tag{7.28}$$

and

$$\mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2] = \mathbb{E}\left\{\exp\left(\sup_{t\in[-S_1,S_2]} \sqrt{\frac{2a}{d^2}}B_\alpha(t) - \frac{a}{d^2}|t|^\alpha - \frac{1}{d^2}f(t)\right)\right\}.$$

If $\lim_{u\to\infty}\sup_{k\in K_u}|ku^{-2/\alpha}|\le\theta$ *for some small enough* $\theta\ge 0$*, we have for some constant* $S>0$

$$\begin{aligned}\mathcal{H}_\alpha[-(a-\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}S_1,(a-\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}S_2] &\le \lim_{u\to\infty}\forall_{k\in K_u}\frac{\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}Z(u^{-\frac{2}{\alpha}}(t+kS)+t_0)>u_k\right\}}{\mathbb{P}\left\{Z(t_0)>u_k\right\}}\\ &\le \mathcal{H}_\alpha[-(a+\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}S_1,(a+\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}S_2], \quad (7.29)\end{aligned}$$

where $\varepsilon_\theta\to 0$*, as* $\theta\to 0$*.*
Specially, if $\theta=0$*, we have*

$$\lim_{u\to\infty}\sup_{k\in K_u}\left|\frac{\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}Z(u^{-\frac{2}{\alpha}}(t+kS)+t_0)>u_k\right\}}{\mathbb{P}\left\{Z(t_0)>u_k\right\}}-\mathcal{H}_\alpha[-a^{\frac{1}{\alpha}}d^{-\frac{2}{\alpha}}S_1,a^{\frac{1}{\alpha}}d^{-\frac{2}{\alpha}}S_2]\right|=0, \quad (7.30)$$

and

$$\mathcal{H}_\alpha[-a^{1/\alpha}d^{-2/\alpha}S_1,a^{1/\alpha}d^{-2/\alpha}S_2]=\mathbb{E}\left\{\exp\left(\sup_{t\in[-a^{1/\alpha}d^{-2/\alpha}S_1,a^{1/\alpha}d^{-2/\alpha}S_2]}\sqrt{2}B_\alpha(t)-|t|^\alpha\right)\right\}.$$

Proof of Lemma 7.5.2 **Step 1:** First we give the proof of (7.28). When $p=1$, set $\mathcal{W}=\{\boldsymbol{w}=(w_1,\cdots,w_n):w_i=\pm 1,\ i=1,\cdots,n\}$. Then we have

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{Z(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\}\\ &=\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n|d_iX_i(u^{-2/\alpha}t+t_0)|}{1+u^{-2}f(t)}>u\right\}\\ &=\sum_{\boldsymbol{w}\in\mathcal{W}}\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\}\\ &\quad-\sum_{\substack{\boldsymbol{w},\boldsymbol{w}'\in\mathcal{W}\\ \boldsymbol{w}\neq\boldsymbol{w}'}}\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u,\ \sup_{s\in[-S_1,S_2]}\frac{\sum_{i=1}^n w_i'd_iX_i(u^{-2/\alpha}s+t_0)}{1+u^{-2}f(s)}>u\right\}\\ &=2^n\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n d_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\}\\ &\quad-\sum_{\substack{\boldsymbol{w},\boldsymbol{w}'\in\mathcal{W}\\ \boldsymbol{w}\neq\boldsymbol{w}'}}\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u,\ \sup_{s\in[-S_1,S_2]}\frac{\sum_{i=1}^n w_i'd_iX_i(u^{-2/\alpha}s+t_0)}{1+u^{-2}f(s)}>u\right\}.\end{aligned}$$

By [12] [Lemma 4.1], we have

$$\begin{aligned}2^n\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n d_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\} &= 2^n\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\sum_{i=1}^n d_iX_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\}\\ &= 2^n\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}\frac{\left(\sum_{i=1}^n d_i^2\right)^{1/2}X_1(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}>u\right\}\\ &\sim 2^n\mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2]\Psi\left(\frac{u}{d}\right)\\ &\sim \mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2]\mathbb{P}\left\{Z(t_0)>u\right\},\ u\to\infty.\end{aligned}$$

Since for any $\boldsymbol{w}\neq\boldsymbol{w}'$

$$\begin{aligned}V_1^2: &= \mathbb{E}\left\{\left(\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}t+t_0)+\sum_{i=1}^n w_i'd_iX_i(u^{-2/\alpha}s+t_0)\right)^2\right\}\\ &= 2\sum_{i=1}^n d_i^2+2\sum_{i=1}^n w_iw_i'd_i^2r(u^{-2/\alpha}t+t_0,u^{-2/\alpha}s+t_0)\end{aligned}$$

$$< \quad 4\sum_{i=1}^{n} d_i^2 = 4d^2,$$

then by Borell inequality, we have

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]} \frac{\sum_{i=1}^n w_i d_i X_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)} > u, \sup_{s\in[-S_1,S_2]} \frac{\sum_{i=1}^n w_i' d_i X_i(u^{-2/\alpha}s+t_0)}{1+u^{-2}f(s)} > u\right\}\\
&\le \mathbb{P}\left\{\sup_{(t,s)\in[-S_1,S_2]\times[S_1,S_2]} \left(\sum_{i=1}^n w_i d_i X_i(u^{-2/\alpha}t+t_0) + \sum_{i=1}^n w_i' d_i X_i(u^{-2/\alpha}s+t_0)\right) > 2u\right\}\\
&\le \exp\left(-\frac{(2u-\mathbb{Q})^2}{2V_1^2}\right)\\
&= o\left(\mathbb{P}\left\{Z(t_0)>u\right\}\right), \ u\to\infty.
\end{aligned}$$

Then (7.28) with $p=1$ is follow.
When $p\in(1,\infty]$, set $Y(t,\boldsymbol{v}) = \sum_{i=1}^n d_i v_i X_i(t), (t,\boldsymbol{v})\in\mathbb{R}\times\mathbb{S}_q$ which is a centered Gaussian field. Then we have

$$\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]} \frac{Z(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)} > u\right\} = \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathbb{S}_q} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\}.$$

Set $\mathbb{S}_q^\delta = \left\{\boldsymbol{v}\in\mathbb{S}_q : d^2 - \sum_{i=1}^n d_i^2 v_i^2 \le \delta\right\}, \delta>0$. Next we prove that as $u\to\infty$

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathbb{S}_q} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\} \sim \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathbb{S}_q^\delta} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\}.$$

Since

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathbb{S}_q^\delta} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\} \ge \mathbb{P}\left\{\sup_{\boldsymbol{v}\in\mathbb{S}_q^\delta} Y(t_0,\boldsymbol{v}) > u\right\} = \mathbb{P}\left\{Z(t_0)>u\right\},$$

we just need to show as $u\to\infty$

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\} = o\left(\mathbb{P}\left\{Z(t_0)>u\right\}\right).$$

In fact, since

$$\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} \mathrm{var}(Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})) = \sup_{\boldsymbol{v}\in(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} \left(\sum_{i=1}^n d_i^2 v_i^2\right) \le d^2-\delta,$$

by Borell inequality, we have

$$\begin{aligned}
\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} \frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)} > u\right\} \quad &\le \quad \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} Y(u^{-2/\alpha}t+t_0,\boldsymbol{v}) > u\right\}\\
&\le \quad \exp\left(-\frac{(u-\mathbb{Q}_1)^2}{2(d^2-\delta)}\right)\\
&= \quad o\left(\mathbb{P}\left\{Z(t_0)>u\right\}\right), \ u\to\infty,
\end{aligned}$$

where $\mathbb{Q}_1 := \mathbb{E}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times(\mathbb{S}_q\setminus\mathbb{S}_q^\delta)} Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})\right\} < \infty$.
When $p\in(1,2)\cup(2,\infty]$, by Lemma 7.2.1, we know $\sigma_1^2(t,\boldsymbol{v}) := \mathrm{var}\left(\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}\right)$ attains the maximum over $[-S_1,S_2]\times\mathbb{S}_q$ at several discrete points, so we can choose δ small enough such that $\mathcal{D}_\delta^i = [-S_1,S_2]\times\mathbb{S}_q^\delta(i)$ with $\mathbb{S}_q^\delta(i)$ the union of non-overlapping compact neighborhoods of $\boldsymbol{v}_+^i, \boldsymbol{v}_-^i$ or $\boldsymbol{z}$ in Lemma 7.2.1. Then as mentioned in [119] or

[71][Lemma 2.1]

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathcal{S}_q}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\}\sim\sum_{i=1}^{M}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\mathcal{D}_\delta^i}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\},\ u\to\infty, \tag{7.31}$$

where M is the number of the maximum point of $\sigma_1^2(t,\boldsymbol{v})$.
Case 1) $p\in(1,2)$ and $M=2^n$. It is enough to find the asymptotics of single term in (7.31), for instance, for a point $(0,\boldsymbol{z}),\ z_i=(d_i/d)^{2/q-2}$. In a neighborhood $\mathcal{S}_q^\delta(1)$ of $\boldsymbol{z}$, we have

$$v_n=\left(1-\sum_{i=1}^{n-1}v_i^q\right)^{1/q},$$

hence the fields $\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}$ can be represented as

$$Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})=\sum_{i=1}^{n-1}v_id_i\frac{X_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}+\left(1-\sum_{i=1}^{n-1}v_i^q\right)^{1/q}d_n\frac{X_n(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)},\widetilde{\boldsymbol{v}}=(v_1,\cdots,v_{n-1}),$$

which is defined in $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q^\delta(1)$ where

$$\widetilde{\mathcal{S}}_q^\delta(1)=\left\{\widetilde{\boldsymbol{v}}:\left(v_1,\cdots,v_{n-1},\left(1-\sum_{i=1}^{n-1}v_i^q\right)^{1/q}\right)\in\mathcal{S}_q^\delta(1)\right\},$$

is a small neighborhood of $\widetilde{\boldsymbol{z}}=(z_1,\cdots,z_{n-1})$. On $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q^\delta(1)$, the variance

$$\sigma_1^2(t,\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\sigma_1^2(\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\left[\sum_{i=1}^{n-1}d_i^2v_i^2+d_n^2\left(1-\sum_{i=1}^{n-1}v_i^q\right)^{2/q}\right]$$

of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$ attains its maximum d^2 at $(0,\widetilde{\boldsymbol{z}})$ where $\widetilde{\boldsymbol{z}}$ is a interior point of a set $\widetilde{\mathcal{S}}_q^\delta(1)$. We can write the following Taylor expansion for $\sigma_1(t,\widetilde{\boldsymbol{v}})$

$$\sigma_1(t,\widetilde{\boldsymbol{v}})=\frac{d}{1+u^{-2}f(t)}-\frac{q-2}{2d}(\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}})\Lambda(\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}})^T+o(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2),\widetilde{\boldsymbol{v}}\to\widetilde{\boldsymbol{z}},\ u\to\infty,$$

where $\Lambda=(\lambda_{i,j})_{i,j=1,\cdots,n-1}$ is a non-negative define matrix with elements

$$\lambda_{i,j}=-(2(q-2))^{-1}\frac{\partial^2}{\partial v_i\partial v_j}\left[\sum_{i=1}^{n-1}d_i^2v_i^2+d_n^2\left(1-\sum_{i=1}^{n-1}v_i^q\right)^{2/q}\right]|_{\widetilde{\boldsymbol{v}}=\widetilde{\boldsymbol{z}}},i,j=1,\cdots,n-1.$$

We have the following expansion for the correlation function $r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')$ of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')=1-u^{-2}a(t-s)^\alpha-\frac{1}{2d}(\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{v}}')\Lambda(\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{v}}')^T+o(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{v}}'|^2),\qquad\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\boldsymbol{z}},u\to\infty.$$

There exists a non-singular matrix Q such that $Q\Lambda Q^T$ is diagonal, and set the diagonal is $(c_1,\cdots,c_{n-1})$. Then

$$\sigma_1(t,Q\widetilde{\boldsymbol{v}})=d-du^{-2}f(t)-\frac{q-2}{2d}\sum_{i=1}^{n-1}c_i(v_i-z_i)^2+o(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2),\widetilde{\boldsymbol{v}}\to\widetilde{\boldsymbol{z}},\ u\to\infty,$$

and

$$r_1(t,Q\widetilde{\boldsymbol{v}},s,Q\widetilde{\boldsymbol{v}}')=1-u^{-2}a(t-s)^\alpha-\frac{1}{2d}\sum_{i=1}^{n-1}c_i(v_i-z_i)^2+o(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2),\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\boldsymbol{z}},u\to\infty.$$

Then set $Y_2(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})=Y_1(u^{-2/\alpha}t+t_0,Q\widetilde{\boldsymbol{v}})$, defined on a set $[-S_1,S_2]\times(Q^{-1}\widetilde{\mathcal{S}}_q^\delta(1))$. We know that the point

$Q\widetilde{z}$ is a interior point of $Q^{-1}\widetilde{\mathcal{S}}_q^\delta(1)$. Then the proof follows by similar arguments as in the proof of [119] [Theorem 8.2]. Consequently, we get

$$\begin{aligned}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\mathcal{D}_\delta^1}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\} &= \mathbb{P}\left\{\sup_{(t,\widetilde{\boldsymbol{v}})\in[-S_1,S_2]\times(Q^{-1}\widetilde{\mathcal{S}}_q^\delta(1))} Y_2(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})>u\right\}\\ &\sim \mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2]\left(\prod_{i=1}^{n-1}\mathcal{P}_{2,1}^{(q-2)t^2}(-\infty,\infty)\right)\Psi\left(\frac{u}{d}\right)\\ &= \mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2](2-p)^{(1-n)/2}\Psi\left(\frac{u}{d}\right),\ u\to\infty,\end{aligned}$$

where we use the fact in [98] that

$$\mathcal{P}_{2,1}^{(q-2)t^2}(-\infty,\infty)=\sqrt{1+\frac{1}{q-2}}=(2-p)^{-1/2},$$

and

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathcal{S}_q}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\}\sim 2^n\mathcal{P}_{\alpha,ad^{-2}}^{\frac{1}{d^2}f(t)}[-S_1,S_2](2-p)^{(1-n)/2}\Psi\left(\frac{u}{d}\right),\ u\to\infty.$$

Case 2) $p\in(2,\infty]$ and $M=2m$. Again we need to find the asymptotics of single term in (7.31), to wish namely for a maximum point $(0,\boldsymbol{v}_+^1)$, $\boldsymbol{v}_+^1=(1,0,\cdots,0)$ of variance $\sigma_1^2(t,\boldsymbol{v})$. hence the fields $\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}$ can be represented as

$$Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})=\sum_{i=2}^n v_id_i\frac{X_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}+\left(1-\sum_{i=2}^n|v_i|^q\right)^{1/q}d_1\frac{X_1(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)},\widetilde{\boldsymbol{v}}=(v_2,\cdots,v_n),$$

which is defined in $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q^\delta(1)$ where

$$\widetilde{\mathcal{S}}_q^\delta(1)=\left\{\widetilde{\boldsymbol{v}}:\left(\left(1-\sum_{i=2}^n|v_i|^q\right)^{1/q},v_2,\cdots,v_n,\right)\in\mathcal{S}_q^\delta(1)\right\},$$

is a small neighborhood of $\widetilde{\mathbf{0}}:=(0,\cdots,0)\in\mathbb{R}^{n-1}$. On $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q^\delta(1)$, the variance

$$\sigma_1^2(t,\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\sigma_1^2(\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\left[\sum_{i=2}^n d_i^2v_i^2+d_n^2\left(1-\sum_{i=2}^n|v_i|^q\right)^{2/q}\right]$$

of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$ attains its maximum 1 at $(0,\widetilde{\mathbf{0}})$ where $\widetilde{\mathbf{0}}$ is a interior point of a set $\widetilde{\mathcal{S}}_q^\delta(1)$. We can write the following Taylor expansion for $\sigma_1(t,\widetilde{\boldsymbol{v}})$

$$\sigma_1(t,\widetilde{\boldsymbol{v}})=1-u^{-2}f(t)-\frac{1}{q}\sum_{i=2}^n|v_i|^q+o\left(\sum_{i=2}^n|v_i|^q\right),\widetilde{\boldsymbol{v}}\to\widetilde{\mathbf{0}},\ u\to\infty,$$

and the following expansion for the correlation function $r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')$ of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')=1-u^{-2}a(t-s)^\alpha-\frac{1}{2}\sum_{i=2}^n d_i^2(v_i-v_i')^2+o\left(\sum_{i=2}^n d_i^2(v_i-v_i')^2\right),\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\mathbf{0}},u\to\infty.$$

Then the proof again follows by similar arguments as in the proof of [119] [Theorem 8.2]. Consequently, we get

$$\begin{aligned}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\mathcal{D}_\delta^1}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\} &= \mathbb{P}\left\{\sup_{(t,\widetilde{\boldsymbol{v}})\in[-S_1,S_2]\times(\widetilde{\mathcal{S}}_q^\delta(1))} Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})>u\right\}\\ &\sim \mathcal{P}_{\alpha,a}^{f(t)}[-S_1,S_2]\Psi(u),\end{aligned}$$

and

$$\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathcal{S}_q}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\}\sim 2m\mathcal{P}_{\alpha,a}^{f(t)}[-S_1,S_2]\Psi(u),\ u\to\infty.$$

Case 3) $p=2$. By Lemma 7.2.1, we know that $\sigma_1^2(t,\boldsymbol{v})$ attains its maximum (equal to 1) over $[-S_1,S_2]\times\mathcal{S}_q$ only at points on $\{(0,\boldsymbol{v}),\boldsymbol{v}\in\mathcal{S}_q,v_i=0,m+1\le i\le n\}$. The fields $\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}$ again can be represented as

$$Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})=\sum_{i=2}^{n}v_id_i\frac{X_i(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)}+\left(1-\sum_{i=2}^{n}v_i^q\right)^{1/q}d_1\frac{X_1(u^{-2/\alpha}t+t_0)}{1+u^{-2}f(t)},\widetilde{\boldsymbol{v}}=(v_2,\cdots,v_n),$$

which is defined in $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q$ where

$$\widetilde{\mathcal{S}}_q=\left\{\widetilde{\boldsymbol{v}}:\left(\left(1-\sum_{i=2}^{n}v_i^q\right)^{1/q},v_2,\cdots,v_n,\right)\in\mathcal{S}_q\right\}.$$

On $[-S_1,S_2]\times\widetilde{\mathcal{S}}_q$, the variance

$$\sigma_1^2(t,\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\sigma_1^2(\widetilde{\boldsymbol{v}}):=\frac{1}{(1+u^{-2}f(t))^2}\left[\sum_{i=2}^{n}d_i^2v_i^2+d_n^2\left(1-\sum_{i=2}^{n}v_i^q\right)^{2/q}\right]$$

of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$ attains its maximum 1 at $\left\{(0,\widetilde{\boldsymbol{v}}),\widetilde{\boldsymbol{v}}\in\widetilde{\mathcal{S}}_q,v_i=0,m+1\le i\le n\right\}$. Furthermore, following the arguments as in [126] we conclude that $\sigma_1(t,\widetilde{\boldsymbol{v}})$ and the correlation function $r_1(t,\boldsymbol{v},s,\widetilde{\boldsymbol{v}})$ of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$ have the following asymptotic expansions:

$$\sigma_1(t,\widetilde{\boldsymbol{v}})=1-u^{-2}f(t)-\sum_{i=m+1}^{n}\frac{1-d_i^2}{2}|v_i|^2+o\left(\sum_{i=m+1}^{n}\frac{1-d_i^2}{2}|v_i|^2+u^{-2}\right),\widetilde{\boldsymbol{v}}\to\widetilde{\mathbf{0}},\ u\to\infty,$$

and the following expansion for the correlation function $r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')$ of $Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})$

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}')=1-u^{-2}a(t-s)^{\alpha}-\frac{1}{2}\sum_{i=2}^{n}d_i^2(v_i-v_i')^2+o\left(\sum_{i=2}^{n}d_i^2(v_i-v_i')^2+u^{-2}\right),\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\mathbf{0}},u\to\infty.$$

Then the proof follows by similar arguments as in the proof of [107] [Theorem 6.1] with the case $\mu=\nu$. Consequently, we get

$$\begin{aligned}\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[-S_1,S_2]\times\mathcal{S}_q}\frac{Y(u^{-2/\alpha}t+t_0,\boldsymbol{v})}{1+u^{-2}f(t)}>u\right\} &= \mathbb{P}\left\{\sup_{(t,\widetilde{\boldsymbol{v}})\in[-S_1,S_2]\times(\widetilde{\mathcal{S}}_q)}Y_1(u^{-2/\alpha}t+t_0,\widetilde{\boldsymbol{v}})>u\right\}\\ &\sim \mathcal{P}_{\alpha,a}^{f(t)}[-S_1,S_2]\frac{\sqrt{2\pi}2^{\frac{(2-m)}{2}}u^{m-3}}{\Gamma(m/2)}\left(\prod_{i=m+1}^{n}(1-d_i^2)^{-\frac{1}{2}}\right)\Psi(u).\end{aligned}$$

Step 2: Next we proceed to the proof of (7.29). Setting $a_{u,k}=(a(ku^{-2/\alpha}S+t_0))^{1/\alpha}$, then for any $k\in K_u$ with $\lim_{u\to\infty}\sup_{k\in K_u}|ku^{-2/\alpha}|\le\theta$ and $t\in[-S_1,S_2]$ when u large enough

$$(a-\varepsilon_\theta)^{1/\alpha}\le a_{u,k}\le(a+\varepsilon_\theta)^{1/\alpha}$$

holds for some $\varepsilon_\theta\in(0,a)$.
Then we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[-S_1,S_2]}Z(u^{-\frac{2}{\alpha}}(t+kS)+t_0)>u_k\right\} &= \mathbb{P}\left\{\sup_{t\in[-a_{u,k}S_1,a_{u,k}S_2]}Z(u^{-\frac{2}{\alpha}}(a_{u,k}^{-1}t+kS)+t_0)>u_k\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in[-(a+\varepsilon_\theta)^{1/\alpha}S_1,(a+\varepsilon_\theta)^{1/\alpha}S_2]}Z(u^{-\frac{2}{\alpha}}(a_{u,k}^{-1}t+kS)+t_0)>u_k\right\}\end{aligned}$$

$$= \; : \Pi^+(u)$$

and

$$\mathbb{P}\left\{ \sup_{t\in[-S_1,S_2]} Z(u^{-\frac{2}{\alpha}}(t+kS)+t_0) > u_k \right\} \geq \mathbb{P}\left\{ \sup_{t\in[-(a-\varepsilon_\theta)^{1/\alpha}S_1,(a-\varepsilon_\theta)^{1/\alpha}S_2]} Z(u^{-\frac{2}{\alpha}}(a_{u,k}^{-1}t+kS)+t_0) > u_k \right\}$$
$$= \; : \Pi^-(u).$$

We notice that by assumption (iv)

$$\begin{aligned} Cov(X(u^{-\frac{2}{\alpha}}(a_{u,k}^{-1}t+kS)+t_0), X(u^{-\frac{2}{\alpha}}kS+t_0)) &\sim 1-a(u^{-\frac{2}{\alpha}}kS+t_0)|u^{-\frac{2}{\alpha}}a_{u,k}^{-1}t|^\alpha \\ &= 1-u^{-2}|t|^\alpha, \; u\to\infty. \end{aligned}$$

For $\Pi^+(u)$ and $\Pi^-(u)$, when $p=1$, (7.29) follows with the same arguments as in **Step 1**.
When $p\in(1,\infty]$, for $\Pi^+(u)$ and $\Pi^-(u)$ we use the similar arguments as in in **Step 1** with $Y_1(u^{-2/\alpha}(a_{u,k}^{-1}t+kS)+t_0,\widetilde{\boldsymbol{v}}) = Y(u^{-2/\alpha}(a_{u,k}^{-1}t+kS)+t_0,\widetilde{\boldsymbol{v}})$.
When $p\in(1,2)$,

$$\sigma_1(t,Q\widetilde{\boldsymbol{v}}) = d - \frac{q-2}{2d}\sum_{i=1}^{n-1} c_i(v_i-z_i)^2 + o\left(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2\right), \widetilde{\boldsymbol{v}}\to\widetilde{\boldsymbol{z}},$$

and

$$r_1(t,Q\widetilde{\boldsymbol{v}},s,Q\widetilde{\boldsymbol{v}}') = 1-u^{-2}(t-s)^\alpha - \frac{1}{2d}\sum_{i=1}^{n-1} c_i(v_i-z_i)^2 + o\left(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2+u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\boldsymbol{z}}, u\to\infty.$$

When $p\in(2,\infty]$,

$$\sigma_1(t,\widetilde{\boldsymbol{v}}) = 1-\frac{1}{q}\sum_{i=2}^{n}|v_i|^q + o\left(\sum_{i=2}^{n}|v_i|^q\right), \widetilde{\boldsymbol{v}}\to\widetilde{\boldsymbol{0}},$$

and

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}') = 1-u^{-2}(t-s)^\alpha - \frac{1}{2}\sum_{i=2}^{n} d_i^2(v_i-v_i')^2 + o\left(\sum_{i=2}^{n} d_i^2(v_i-v_i')^2+u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\boldsymbol{0}}, u\to\infty.$$

When $p=2$,

$$\sigma_1(t,\widetilde{\boldsymbol{v}}) = 1-\sum_{i=m+1}^{n}\frac{1-d_i^2}{2}|v_i|^2 + o\left(\sum_{i=m+1}^{n}\frac{1-d_i^2}{2}|v_i|^2\right), \widetilde{\boldsymbol{v}}\to\widetilde{\boldsymbol{0}},$$

and

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}') = 1-u^{-2}(t-s)^\alpha - \frac{1}{2}\sum_{i=2}^{n} d_i^2(v_i-v_i')^2 + o\left(\sum_{i=2}^{n} d_i^2(v_i-v_i')^2+u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}'\to\widetilde{\boldsymbol{0}}, u\to\infty.$$

We get that as $u\to\infty$

$$\Pi^+(u) \sim \mathcal{H}_\alpha[-S_1(a+\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}, S_2(a+\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}]\mathbb{P}\left\{Z(t_0))>u_k\right\},$$
$$\Pi^-(u) \sim \mathcal{H}_\alpha[-S_1(a-\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}, S_2(a-\varepsilon_\theta)^{1/\alpha}d^{-2/\alpha}]\mathbb{P}\left\{Z(t_0))>u_k\right\}.$$

Thus (7.29) follows.
Further, if letting $\theta\to0$ in (7.29), we get (7.30).

□

Lemma 7.5.3. *Assume that Gaussian vector process $\boldsymbol{X}(t)$ with independent marginals which have unit variances,* *correlation functions $r(t)$ is the same as in Lemma 7.5.2. Further, set K_u a family of countable index sets and u_k satisfying (7.27). Let ε_0 be such that for all $s, t \in [t_0-\varepsilon_0, t_0+\varepsilon_0]$,*

$$\frac{a}{2}|t-s|^\alpha \le 1-r(s,t) \le 2a|t-s|^\alpha.$$

Then we can find a constant $\mathbb{C}$ such that for all $S>0$ and $T_2-T_1>S$,

$$\limsup_{u\to\infty} \sup_{k\in K_u} \frac{\mathbb{P}\left\{\mathcal{A}_1(u_k), \mathcal{A}_2(u_k)\right\}}{\mathbb{P}\left\{Z(t_0)>u_k\right\}} \le \mathbb{C}\exp\left(-\frac{a}{8}|T_2-T_1-S|^\alpha\right),$$

where $\mathcal{A}_i(u_k) = \{\sup_{t\in[T_i,T_i+S]} Z(u^{-2/\alpha}(t+kS)+t_0)>u_k\},\ i=1,2,$ and

$$\lim_{u\to\infty}\sup_{k\in K_u}|u^{-2/\alpha}kS| \le \varepsilon_0.$$

Proof of Lemma 7.5.3 Through this proof, $\mathbb{C}_i, i\in\mathbb{N}$ are some positive constant.
When $p=1$, set $\mathcal{W}=\{\boldsymbol{w}=(w_1,\cdots,w_n): w_i=\pm 1,\ i=1,\cdots,n\}$. We have by [60][Theorem 3.1] for u large enough

$$\begin{aligned}
&\mathbb{P}\left\{\mathcal{A}_1(u_k), \mathcal{A}_2(u_k)\right\}\\
&= \mathbb{P}\left\{\sup_{t\in[T_1,T_1+S]}\sum_{i=1}^n |d_iX_i(u^{-2/\alpha}(t+kS)+t_0)|>u_k,\ \sup_{s\in[T_2,T_2+S]}\sum_{i=1}^n |d_iX_i(u^{-2/\alpha}(s+kS)+t_0)|>u_k\right\}\\
&\le \sum_{\boldsymbol{w}\in\mathcal{W}}\mathbb{P}\left\{\sup_{t\in[T_1,T_1+S]}\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}(t+kS)+t_0)>u_k,\ \sup_{s\in[T_2,T_2+S]}\sum_{i=1}^n w_id_iX_i(u^{-2/\alpha}(s+kS)+t_0)>u_k\right\}\\
&= 2^n\mathbb{P}\left\{\sup_{t\in[T_1,T_1+S]}\sum_{i=1}^n d_iX_i(u^{-2/\alpha}(t+kS)+t_0)>u_k,\ \sup_{s\in[T_2,T_2+S]}\sum_{i=1}^n d_iX_i(u^{-2/\alpha}(s+kS)+t_0)>u_k\right\}\\
&= 2^n\mathbb{P}\left\{\sup_{t\in[T_1,T_1+S]}\left(\sum_{i=1}^n d_i^2\right)^{1/2} X_1(u^{-2/\alpha}(t+kS)+t_0)>u_k,\ \sup_{s\in[T_2,T_2+S]}\left(\sum_{i=1}^n d_i^2\right)^{1/2} X_1(u^{-2/\alpha}(s+kS)+t_0)>u_k\right\}\\
&\le \mathbb{C}_0\exp\left(-\frac{a}{8}|T_2-T_1-S|^\alpha\right)\mathbb{P}\left\{Z(t_0)>u_k\right\}.
\end{aligned}$$

When $p\in(1,\infty]$, set $Y_u(t,\boldsymbol{v})=\sum_{i=1}^n d_iv_iX_i(u^{-2/\alpha^*}(t+kS)+t_0), (t,\boldsymbol{v})\in\mathbb{R}\times\mathcal{S}_q$ which is a centered Gaussian field and $\mathcal{S}_q^\delta=\{\boldsymbol{v}\in\mathcal{S}_q: d^2-\sum_{i=1}^n d_i^2v_i^2\le\delta\}, \delta>0$.
Below for $\Delta_1,\Delta_2\subseteq\mathbb{R}^{n+1}$, denote

$$\mathcal{Y}_u(\Delta_1,\Delta_2)=\mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in\Delta_1}Y_u(t,\boldsymbol{v})>u_k,\ \sup_{(t,\boldsymbol{v})\in\Delta_2}Y_u(t,\boldsymbol{v})>u_k\right\}.$$

We have

$$\begin{aligned}
\mathbb{P}\left\{\mathcal{A}_1(u_k),\mathcal{A}_2(u_k)\right\} &\ge \mathcal{Y}_u([T_1,T_1+S]\times\mathcal{S}_q^\delta,[T_2,T_2+S]\times\mathcal{S}_q^\delta),\\
\mathbb{P}\left\{\mathcal{A}_1(u_k),\mathcal{A}_2(u_k)\right\} &\le \mathcal{Y}_u([T_1,T_1+S]\times\mathcal{S}_q^\delta,[T_2,T_2+S]\times\mathcal{S}_q^\delta)+\mathcal{Y}_u([T_1,T_1+S]\times\mathcal{S}_q^\delta,[T_2,T_2+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta))\\
&\quad +\mathcal{Y}_u([T_1,T_1+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta),[T_2,T_2+S]\times\mathcal{S}_q^\delta),
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{Y}_u([T_1,T_1+S]\times\mathcal{S}_q^\delta,[T_2,T_2+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta)) &\le \mathbb{P}\left\{\sup_{(t,\boldsymbol{v})\in[T_2,T_2+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta)}Y_u(t,\boldsymbol{v})>u_k\right\}\\
&\le \exp\left(-\frac{(u_k-\mathbb{C}_1)^2}{2(d^2-\delta)}\right)\\
&= o\left(\mathbb{P}\left\{Z(t_0)>u_k\right\}\right),
\end{aligned}$$

as $u\to\infty$ where the last second inequality follows from Borell inequality and the fact that

$$\sup_{(t,\boldsymbol{v})\in[T_2,T_2+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta)}\mathrm{var}(Y_u(t,\boldsymbol{v}))=\sup_{\boldsymbol{v}\in(\mathcal{S}_q\setminus\mathcal{S}_q^\delta)}\left(\sum_{i=1}^n d_i^2v_i^2\right)\le d^2-\delta.$$

Similarly, we have

$$\mathcal{Y}_u([T_1,T_1+S]\times(\mathcal{S}_q\setminus\mathcal{S}_q^\delta),[T_2,T_2+S]\times\mathcal{S}_q^\delta)=o\left(\mathbb{P}\left\{Z(t_0)>u_k\right\}\right),\ u\to\infty.$$

Then we just need to focus on

$$\Pi(u):=\mathcal{Y}_u([T_1,T_1+S]\times\mathcal{S}_q^\delta,[T_2,T_2+S]\times\mathcal{S}_q^\delta).$$

We split $\mathcal{S}_q^\delta$ into sets of small diameters $\{\partial\mathcal{S}_i,0\le i\le\mathcal{N}^*\}$, where

$$\mathcal{N}^*=\sharp\{\partial\mathcal{S}_i\}<\infty.$$

Further, we see that $\Pi(u)\le\Pi_1(u)+\Pi_2(u)$ with

$$\Pi_1(u)=\sum_{\substack{0\le i,l\le\mathcal{N}^*\\ \partial\mathcal{S}_i\cap\partial\mathcal{S}_l=\emptyset}}\mathcal{Y}_u([T_1,T_1+S]\times\partial\mathcal{S}_i,[T_2,T_2+S]\times\partial\mathcal{S}_l),$$

$$\Pi_2(u)=\sum_{\substack{0\le i,l\le\mathcal{N}^*\\ \partial\mathcal{S}_i\cap\partial\mathcal{S}_l\neq\emptyset}}\mathcal{Y}_u([T_1,T_1+S]\times\partial\mathcal{S}_i,[T_2,T_2+S]\times\partial\mathcal{S}_l),$$

where $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l\neq\emptyset$ means $\partial\mathcal{S}_i,\partial\mathcal{S}_l$ are identical or adjacent, and $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l=\emptyset$ means $\partial\mathcal{S}_i,\partial\mathcal{S}_l$ are neither identical nor adjacent. Denote the distance of two set $\boldsymbol{A},\boldsymbol{B}\in\mathbb{R}^n$ as

$$\rho(\boldsymbol{A},\boldsymbol{B})=\inf_{\boldsymbol{x}\in\boldsymbol{A},\boldsymbol{y}\in\boldsymbol{B}}\|\boldsymbol{x}-\boldsymbol{y}\|_2.$$

if $\partial\mathcal{S}_i\cap\partial\mathcal{S}_l=\emptyset$, then there exists some small positive constant ρ_0 (independent of i,l) such that $\rho(\partial\mathcal{S}_i,\partial\mathcal{S}_l)>\rho_0$. Next we estimate $\Pi_1(u)$. For any $u\ge0$

$$\Pi_1(u)\le\mathbb{P}\left\{\sup_{\substack{(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S]\\ \boldsymbol{v}\in\partial\mathcal{S}_i,\boldsymbol{w}\in\partial\mathcal{S}_i}}Z_u(t,\boldsymbol{v},s,\boldsymbol{w})>2u_k\right\},$$

where $Z_u(t,\boldsymbol{v},s,\boldsymbol{w})=Y_u(t,\boldsymbol{v})+Y_u(s,\boldsymbol{w}),\ t,s\ge0,\boldsymbol{v},\boldsymbol{w}\in\mathbb{R}^n$.
When u is sufficiently large for $(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S],\boldsymbol{v}\in\partial\mathcal{S}_i\subset[-2,2]^n,\boldsymbol{w}\in\partial\mathcal{S}_i\subset[-2,2]^n$, with $\rho(\partial\mathcal{S}_i,\partial\mathcal{S}_l)>\rho_0$ we have

$$\begin{aligned}
Var(Z_u(t,\boldsymbol{v},s,\boldsymbol{w})) &\le \sum_{i=1}^n(v_i^2+w_i^2+2v_iw_i)d_i^2\\
&\le 4d^2-2\sum_{i=1}^n(v_i-w_i)^2d_i^2\\
&= 4d^2-2d_n^2\rho_0\\
&\le d^2(4-\delta_0),
\end{aligned}$$

for some $\delta_0>0$. Therefore, it follows from the Borell inequality that

$$\Pi_1(u)\le\mathbb{C}_2\mathcal{N}^*\exp\left(-\frac{(2u_k-\mathbb{C}_3)^2}{2d^2(4-\delta_0)}\right)\quad=\quad o\left(\mathbb{P}\left\{Z(t_0)>u_k\right\}\right),\ u\to\infty,$$

with

$$\mathbb{C}_3 = \mathbb{E}\left\{\sup_{\substack{(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S]\\ (\boldsymbol{v},\boldsymbol{w})\in[-2,2]^2n}} Z_u(t,\boldsymbol{v},s,\boldsymbol{w})\right\} < \infty.$$

Now we consider $\Pi_2(u)$. Similar to the argumentation as in **Step1** of the proof of Lemma 7.5.2. we set $\widetilde{Y}_u(t,\widetilde{\boldsymbol{v}}) = Y_u(t,Q\widetilde{\boldsymbol{v}})$ and $\widetilde{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}) = \widetilde{Y}_u(t,\widetilde{\boldsymbol{v}}) + \widetilde{Y}_u(s,\widetilde{\boldsymbol{w}})$ with $\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}} \in \mathbb{R}^{n-1}$. Since for $(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S], \widetilde{\boldsymbol{v}} \in [-2,2]^{n-1}, \widetilde{\boldsymbol{w}} \in [-2,2]^{n-1}$, we have

$$\begin{aligned}
2d^2 \le Var(\widetilde{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}})) &\le \sum_{i=1}^n (v_i^2 + w_i^2 + 2r(u^{-2/\alpha}(t+kS)+t_0, u^{-2/\alpha}(s+kS)+t_0)v_iw_i)d_i^2\\
&\le 2d^2 + 2\left(1 - \frac{a}{2}u^{-2}|t-s|^\alpha\right)\sum_{i=1}^n v_iw_id_i^2\\
&\le 4d^2 - d^2au^{-2}|t-s|^\alpha\\
&\le 4d^2 - d^2au^{-2}|T_2-T_1-S|^\alpha.
\end{aligned}$$

Set

$$\overline{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}) = \frac{\widetilde{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}})}{Var(Z_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}))}.$$

Borrowing the arguments of the proof in [119] [Lemma 6.3] we show that

$$\mathbb{E}\left\{\left(\overline{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}) - \overline{Z}_u(t',\widetilde{\boldsymbol{v}}',s',\widetilde{\boldsymbol{w}'})\right)\right\} \le 4\left(\mathbb{E}\left\{(\widetilde{Y}_u(t,\widetilde{\boldsymbol{v}}) - \widetilde{Y}_u(t',\widetilde{\boldsymbol{v}'}))^2\right\} + \mathbb{E}\left\{(Y_u(s,\widetilde{\boldsymbol{w}}) - Y_u(s',\widetilde{\boldsymbol{w}'}))^2\right\}\right).$$

Moreover, since when $p\in(1,2)$,

$$r_1(t,Q\widetilde{\boldsymbol{v}},s,Q\widetilde{\boldsymbol{v}}') = 1 - u^{-2}a(t-s)^\alpha - \frac{1}{2d}\sum_{i=1}^{n-1} c_i(v_i-z_i)^2 + o\left(|\widetilde{\boldsymbol{v}}-\widetilde{\boldsymbol{z}}|^2 + u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}' \to \widetilde{\boldsymbol{z}}, u\to\infty.$$

When $p\in(2,\infty)$,

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}') = 1 - u^{-2}a(t-s)^\alpha - \frac{1}{2}\sum_{i=2}^n d_i^2(v_i-v_i')^2 + o\left(\sum_{i=2}^n d_i^2(v_i-v_i')^2 + u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}' \to \widetilde{\boldsymbol{0}}, u\to\infty.$$

When $p=2$,

$$r_1(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{v}}') = 1 - u^{-2}a(t-s)^\alpha - \frac{1}{2}\sum_{i=2}^n d_i^2(v_i-v_i')^2 + o\left(\sum_{i=2}^n d_i^2(v_i-v_i')^2 + u^{-2}\right), \widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{v}}' \to \widetilde{\boldsymbol{0}}, u\to\infty.$$

Then we have

$$\mathbb{E}\left\{(Y_u(t,\widetilde{\boldsymbol{v}}) - Y_u(t',\widetilde{\boldsymbol{v}'}))^2\right\} \le 4d^2au^{-2}|t-t'|^\alpha + 2\sum_{i=2}^n (v_i-v_i')^2.$$

Therefore

$$\mathbb{E}\left\{\left(\overline{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}) - \overline{Z}_u(t',\widetilde{\boldsymbol{v}'},s',\widetilde{\boldsymbol{w}'})\right)\right\} \le 16d^2au^{-2}|t-t'|^\alpha + 16d^2au^{-2}|s-s'|^\alpha + 8\sum_{i=2}^n(v_i-v_i')^2 + 8\sum_{i=2}^n(w_i-w_i')^2.$$

Set $\zeta(t,s,\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}}), t,s\ge 0, \boldsymbol{v},\boldsymbol{w}\in\mathbb{R}^{n-1}$ is a stationary Gaussian field with unit variance and correlation function

$$r_\zeta(t,s,\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}}) = \exp\left(-9d^2at^\alpha - 9d^2as^\alpha - 5\sum_{i=2}^n v_i^2 - 5\sum_{i=2}^n w_i^2\right).$$

Then

$$\begin{aligned}\Pi_2(u) &\le \mathbb{P}\left\{\sup_{\substack{(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S]\\ \widetilde{\boldsymbol{v}}\in Q^{-1}\mathcal{S}_q,\widetilde{\boldsymbol{w}}\in Q^{-1}\mathcal{S}_q}} \widetilde{Z}_u(t,\widetilde{\boldsymbol{v}},s,\widetilde{\boldsymbol{w}}) > 2u_k\right\}\\ &\le \mathbb{P}\left\{\sup_{\substack{(t,s)\in[T_1,T_1+S]\times[T_2,T_2+S]\\ \widetilde{\boldsymbol{v}}\in Q^{-1}\mathcal{S}_q,\widetilde{\boldsymbol{w}}\in Q^{-1}\mathcal{S}_q}} \zeta(u^{-2/\alpha}t,u^{-2/\alpha}s,\widetilde{\boldsymbol{v}},\widetilde{\boldsymbol{w}}) > \frac{2u_k}{\sqrt{4d^2-d^2au^{-2}|T_2-T_1-S|^\alpha}}\right\}.\end{aligned}$$

Then following the similar argumentation as in [82], we have

$$\Pi_2(u) \le \mathbb{C}_4 u_k^{M-2}\exp\left(-\frac{u_k^2}{2d^2}-\frac{a}{8}|T_2-T_1-S|^\alpha\right)$$

where $M=0$ when $p\in(1,2)\cup(2,\infty]$ and $M=m$ when $p=2$. Thus we have

$$\limsup_{u\to\infty}\frac{\Pi_2(u)}{\mathbb{P}\left\{Z(t_0)>u_k\right\}} \le \mathbb{C}_5\exp\left(-\frac{a}{8}|T_2-T_1-S|^\alpha\right).$$

Thus we complete the proof.

□

Proof of Eaxmple 7.3.1: We notice that $B_\alpha(t)$ attain its maximum over $[0,1]$ at $t=1$ and

$$\sigma(t)\sim 1-\frac{\alpha}{2}(1-t),\quad r(s,t)\sim 1-\frac{1}{2}|s-t|^\alpha,\ \ s,t\uparrow 1.$$

For $g(t)=-(1-t)^{1/2},\ t\in[0,1]$, by Theorem 7.3.1 with $c=1$ we get the results.

□

Chapter 8

Drawdown and Drawup for Fractional Brownian Motion with Trend[1]

8.1 Introduction and Preliminaries

Drawdown, defined as the distance of present value away from its historical running maximum, is an important indicator of downside risks in financial risk management. For instance, the drawdown and the maximum drawdown have been customarily used as risk measures in finance where they measure the current drop of a stock price, an index or the value of a portfolio from its running maximum; see, e.g., [75, 138]. Instead of Value-at-Risk, the Maximum Drawdown-at-Risk has been proposed to capture the cumulative losses; see [94]. Moreover, maximum drawdown and maximum drawup also appear in the portfolio sensitivities of underlying asset; see [127]. They can also be deployed in the context of portfolio optimization as constrains; see, e.g.,[25, 97]. Drawdown processes also appear in other applications, such as applied probability and queueing theory; see, e.g., [111, 39, 14, 101]. Complementary, drawup, the dual of drawdown, which is the distance of current value from its historical running minimum, has been encountered in many financial applications; see, e.g., [128, 138].

In the literature, e.g., [67, 135], the stock price S can be modeled by the so-called geometric fractional Brownian motion, i.e.,

$$S_t = S_0 \exp\left(\mu t + \sigma B_H(t) - \frac{1}{2}\sigma^2 t^{2H}\right), \tag{8.1}$$

where $\sigma > 0, \mu \in \mathbb{R}$ and B_H is a fractional Brownian motion (fBm) with index $H \in (0,1)$ and covariance function satisfying

$$Cov(B_H(s), B_H(t)) = \frac{|s|^{2H} + |t|^{2H} - |s-t|^{2H}}{2}, s, t \geq 0.$$

Note that S_t is reduced to geometric Brownian motion if $H = 1/2$ which has massive applications in Finance. To facilitate our analysis, we shall work with the log-prices. This motivates us to consider the drawdown and drawup for fBm with trend. Let $X_t = \sigma B_H(t) - \frac{1}{2}\sigma^2 t^{2H} + \mu t$, $\mu \in \mathbb{R}$. For simplicity, we assume that $\sigma = 1$. The drawdown and drawup processes of X are defined, respectively, by

$$D_t = \overline{X}_t - X_t, \quad U_t = X_t - \underline{X}_t,$$

where $\overline{X}_t = \sup_{0 \leq s \leq t} X_s$ and $\underline{X}_t = \inf_{0 \leq s \leq t} X_s$. For some fixed $T \in (0, \infty)$, we are interested in, for any $u > 0$,

$$\mathbb{P}\left\{\sup_{0 \leq t \leq T} D_t > u\right\} \quad \text{and} \quad \mathbb{P}\left\{\sup_{0 \leq t \leq T} U_t > u\right\}. \tag{8.2}$$

Notice that the maximum of drawdown over $[0, T]$ has the interpretation as the largest log-loss up to time T and

[1] This chapter is based on L. Bai and P. Liu, (2018): Drawdown and drawup for fractional Brownian motion with trend, published in the *Journal of Theoretical Probability,* to appear.

accordingly, the maximum of drawup can be viewed as the largest log-return; see e.g., [14]. Additionally, for $H = \frac{1}{2}$, in context of queueing theory, D_t is the transient queue length process starting at 0 and the corresponding probability in (8.2) represents the overload probability over $[0, T]$; see, e.g., [111, 39].
Note that for the special case $H = 1/2$, the exact expressions of (8.2) were obtained in [66, 109]; see also [131] concerning the joint distribution of maximum drawdown and maximum drawup up to an independent exponential time. Due to the fact that fBm is neither a semi-martingale nor a Markov process, the exact expressions for $H \neq \frac{1}{2}$ are not available in literature. Hence in this paper we focus on the asymptotics of (8.2) as $u \to \infty$.
It is worthwhile to mention that infinite series representation of (8.2) in [66, 109] for $H = \frac{1}{2}$ is quite complicated. In contrast, we get concise asymptotics for $H = 1/2$ in this paper. Theorems 8.2.1 and 8.2.2 in section 2 shows that, for $H = \frac{1}{2}$, as $u \to \infty$,

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim 4\mathbb{P}\left\{B_{1/2}(1) > \frac{u + (\mu - \frac{1}{2}T)}{\sqrt{T}}\right\}, \quad \mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} \sim 4\mathbb{P}\left\{B_{1/2}(1) > \frac{u + (\frac{1}{2}T - \mu)}{\sqrt{T}}\right\}.$$

The technique used in this paper is *uniform double-sum method* in [60], which is the development of the so-called *double-sum method* widely applied in extreme value theory of Gaussian processes and random fields; see, e.g., [119]. As it is shown in Theorem 8.2.1 in section 2, the special trend renders the asymptotics for drawdown quite different from those of non-centered Gaussian random fields related to fBm in literature (see, e.g., [120, 84, 52, 45]), leading to new scenarios of asymptotics according to the value of H.
Our results can be applied to calculate the Maximum Drawdown-at-Risk and the probability of stock market crashes and rallies for (8.1); see [94] and [75].
In this chapter, in order to unity the definition of fBm, we redefine the Pickands constant, which is

$$\mathcal{H}_H = \lim_{b\to\infty} \frac{1}{b}\mathcal{H}_H([0,b]) \quad \text{with } \mathcal{H}_H([a,b]) = \mathbb{E}\left\{\sup_{t\in[a,b]} e^{\sqrt{2}B_H(t) - |t|^{2H}}\right\}, \quad a < b.$$

Further, Piterbarg constant is given by, for $\nu > 0$,

$$\mathcal{P}_H^{\nu} = \lim_{b\to\infty} \mathcal{P}_H^{\nu}([0,b]) \quad \text{with} \quad \mathcal{P}_H^{\nu}([0,b]) = \mathbb{E}\left\{\sup_{t\in[0,b]} e^{\sqrt{2}B_H(t) - (1+\nu)|t|^{2H}}\right\}, \quad b > 0.$$

We can refer to [119, 1, 47, 90, 63, 44] for the definition, properties and extensions of Pickands and Piterbarg constants, to [55, 65, 64, 13, 77] for the bounds and simulations of Pickands and Piterbarg constants. In particular, by [55], we have that

$$\mathcal{P}_{1/2}^{\nu} = 1 + \frac{1}{\nu}, \quad \nu > 0. \tag{8.3}$$

The organization of paper is as follows. In section 2, the main results are displayed. Section 3 is devoted to the proofs of main theorems in section 2. Proofs of lemmas in section 3 is postponed in Appendix A, followed by some useful lemmas in Appendix B.

8.2 Main Results

In this section, we present our main results concerning the asymptotics of (8.2) as $u \to \infty$. In contrast to the infinite series representation in [66, 109], the asymptotic expressions in the following theorems are quite concise, which allows us to readily understand the asymptotic behavior of the probability that maximum drawdown (maximum drawup) exceeds a threshold over finite-time horizon. Then we have the following results.

Theorem 8.2.1. *Assume that* $0 < T < \infty$.
If $H > 1/2$, *then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim \Psi\left(\frac{u + \mu T - \frac{1}{2}T^{2H}}{T^H}\right).$$

If $H = 1/2$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim 4\Psi\left(\frac{u+\mu T-\frac{1}{2}T^{2H}}{T^H}\right).$$

If $1/4 < H < 1/2$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2 u^{\frac{2}{H}-4}\Psi\left(\frac{u+\mu T-\frac{1}{2}T^{2H}}{T^H}\right).$$

If $H = 1/4$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim \left(\mathcal{H}_{\frac{1}{4}}\right)^2 T^{-1}\int_0^\infty e^{-x-T^{\frac{1}{4}}x^{\frac{1}{2}}}dx u^4\Psi\left(\frac{u+\mu T-\frac{1}{2}T^{2H}}{T^H}\right).$$

If $0 < H < 1/4$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \sim H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)(\mathcal{H}_H)^2 u^{\frac{3}{2H}-2}\Psi\left(\frac{u+\mu T-\frac{1}{2}T^{2H}}{T^H}\right).$$

Theorem 8.2.2. *Assume that* $0 < T < \infty$*.*
If $H > 1/2$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} \sim \Psi\left(\frac{u-\mu T+\frac{1}{2}T^{2H}}{T^H}\right).$$

If $H = 1/2$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} \sim 4\Psi\left(\frac{u-\mu T+\frac{1}{2}T^{2H}}{T^H}\right).$$

If $0 < H < 1/2$*, then*

$$\mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} \sim 2^{-\frac{1}{H}-\frac{1}{2}}T^{3H}\sqrt{\frac{\pi}{H^3(H-1)}}(\mathcal{H}_H)^2 u^{\frac{2}{H}-3}\Psi\left(\inf_{0\le s\le T}\frac{u-\mu(T-s)+\frac{1}{2}(T^{2H}-s^{2H})}{(T-s)^H}\right).$$

Remark. i) In the extremes of Gaussian processes and random fields associated with fBm for finite-time horizon, e.g.,[120, 84, 52, 45], we usually have three different types of asymptotics according to H: $H > 1/2, H = 1/2$ and $H < 1/2$. However, Theorem 8.2.1 gives more types of asymptotics due to the complexity of the trend that is the combination of linear function (μt) and power function ($-\frac{1}{2}|t|^{2H}$). As we can see from the proof of Theorem 8.2.1, for $1/4 < H < 1/2$ only the linear trend contribute to the power part of the asymptotics; for $H = 1/4$, both linear trend and power trend affect the power part; whereas, for $0 < H < 1/4$, the power trend has the major influence on the power part of the asymptotics. However, this phenomena does not appear in Theorem 8.2.2, where both of linear trend and power trend contribute to the power part of the asymptotics for $0 < H < 1/2$.
ii) We here interpret that the analysis of drawdown and drawup for the case $T = \infty$ is meaningless. Let $T = \infty$ and $\widetilde{B}_H = -B_H$. Then

$$\begin{aligned}
\sup_{0\le t<\infty} D_t &= \sup_{0\le s\le t<\infty}\left(B_H(s)-B_H(t)+\frac{1}{2}(t^{2H}-s^{2H})-\mu(t-s)\right)\\
&= \sup_{0\le s\le t<\infty}\left(\widetilde{B}_H(t)-\widetilde{B}_H(s)+\frac{1}{2}(t^{2H}-s^{2H})-\mu(t-s)\right)\\
&\ge \sup_{0\le s\le t<\infty}\left(\widetilde{B}_H(t)-\widetilde{B}_H(s)-(|\mu|+1)(t-s)\right)\\
&= \sup_{s\ge 0} Q(s),
\end{aligned}$$

where

$$Q(s) = \sup_{t\ge s}\left(\widetilde{B}_H(t)-\widetilde{B}_H(s)-(|\mu|+1)(t-s)\right).$$

Corollary 1 in [36, 100] shows that for $H \in (0,1)$

$$\limsup_{s\to\infty} \frac{Q(s)}{(\log s)^{\frac{1}{2(1-H)}}} = C > 0 \quad a.s..$$

Therefore we have that for $H \in (0,1)$

$$\sup_{0\le t<\infty} D_t \ge \sup_{s\ge 0} Q(s) = \infty \quad a.s..$$

Note that for $t \ge s \ge 1$ and $H \in (0,1/2]$, there exists $C_1 > 0$ such that

$$t^{2H} - s^{2H} \le C_1(t-s).$$

Hence we can analogously show that for $H \in (0,1/2]$

$$\begin{aligned}\sup_{0\le t<\infty} U_t &= \sup_{0\le s\le t<\infty} \left(B_H(t) - B_H(s) - \frac{1}{2}(t^{2H} - s^{2H}) + \mu(t-s)\right)\\ &\ge \sup_{1\le s\le t<\infty} (B_H(t) - B_H(s) - C_2(t-s)) = \infty \quad a.s.,\end{aligned}$$

where C_2 is a positive constant. We conjecture that for $H > 1/2$,

$$\sup_{0\le t<\infty} U_t = \infty \quad a.s.$$

also holds, which needs more technical analysis similarly to [36, 100].

8.3 Proofs

In this section we give the proof of Theorems 8.2.1-8.2.2. In order to prove the aforementioned theorems, we first present several lemmas related to the local behaviors of variance and correlation functions of the underlying Gaussian random fields. In rest of the paper, denote by $Q, Q_i, i = 1, 2, \ldots$ some positive constants that may differ from line to line. Moreover,

$$f(u,S,\epsilon) \sim h(u), \quad u\to\infty, \epsilon\to 0, S\to\infty,$$

means that

$$\lim_{S\to\infty}\lim_{\epsilon\to 0}\lim_{u\to\infty} \frac{f(u,S,\epsilon)}{h(u)} = 1.$$

Let

$$\sigma_u^{\pm}(s,t) = \frac{|t-s|^H}{u \mp \mu(t-s) \pm \frac{1}{2}(t^{2H} - s^{2H})}, \quad 0 \le s \le t \le T.$$

Lemma 8.3.1. *For u sufficiently large $(0,T) = \arg\sup_{0\le s\le t\le T} \sigma_u^-(s,t)$ is unique and for any $\delta_u > 0$ and $\lim_{u\to\infty}\delta_u = 0$*

$$\lim_{u\to\infty} \sup_{(s,t)\in[0,\delta_u]\times[T-\delta_u,T]} \left| \frac{1 - \frac{\sigma_u^-(s,t)}{\sigma_u^-(0,T)}}{\frac{H(T-t)}{T} + \frac{H}{T}s + \frac{1}{2u}s^{2H}} - 1 \right| = 0.$$

Lemma 8.3.2. *i) For $H \ge \frac{1}{2}$ and u sufficiently large $(0,T) = \arg\sup_{0\le s\le t\le T} \sigma_u^+(s,t)$ is unique and for any $\delta_u > 0$ and $\lim_{u\to\infty}\delta_u = 0$*

$$\lim_{u\to\infty} \sup_{(s,t)\in[0,\delta_u]\times[T-\delta_u,T]} \left| \frac{1 - \frac{\sigma_u^+(s,t)}{\sigma_u^+(0,T)}}{\frac{H(T-t)}{T} + \frac{H}{T}s} - 1 \right| = 0.$$

ii) For $0 < H < \frac{1}{2}$ and u sufficiently large $(s_u, T) = \arg\sup_{0\le s\le t\le T} \sigma_u^+(s,t)$ is unique and $s_u \sim T^{\frac{1}{1-2H}} u^{-\frac{1}{1-2H}}$.

Moreover, for any $\delta_u > 0$ and $\lim_{u\to\infty} \delta_u = 0$

$$\lim_{u\to\infty} \sup_{(s,t)\in[0,s_u+\delta_u]\times[T-\delta_u,T]} \left| \frac{1-\frac{\sigma_u^+(s,t)}{\sigma_u^+(s_u,T)}}{\frac{H(T-t)}{T}+\frac{H(1-H)}{2T^2}(s-s_u)^2} - 1 \right| = 0.$$

Lemma 8.3.3. *For any $\delta_u > 0$ and $\lim_{u\to\infty} \delta_u = 0$*

$$\lim_{u\to\infty} \sup_{(s,t),(s',t')\in[0,\delta_u]\times[T-\delta_u,T]} \left| \frac{1-Corr\left(B_H(t)-B_H(s), B_H(t')-B_H(s')\right)}{\frac{|s-s'|^{2H}+|t-t'|^{2H}}{2T^{2H}}} - 1 \right| = 0.$$

Proof of Theorem 8.2.1 Observe that

$$\begin{aligned}
\mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} &= \mathbb{P}\left\{\sup_{(s,t)\in A} (X_s - X_t) > u\right\} \\
&= \mathbb{P}\left\{\sup_{(s,t)\in A} \left(B_H(s) - B_H(t) + \mu(s-t) - \frac{1}{2}(s^{2H} - t^{2H})\right) > u\right\} \\
&= \mathbb{P}\left\{\sup_{(s,t)\in A} Z_u(s,t) > m(u)\right\},
\end{aligned}$$

where

$$Z_u(s,t) = \frac{B_H(s) - B_H(t)}{u + \mu(t-s) + \frac{1}{2}(s^{2H} - t^{2H})} m(u), \quad m(u) = \frac{u + \mu T - \frac{1}{2}T^{2H}}{T^H}, \quad A = \{(s,t) : 0 \le s \le t \le T\}.$$

Thus we have that

$$\begin{aligned}
\mathbb{P}\left\{\sup_{(s,t)\in E_u} Z_u(s,t) > m(u)\right\} &\le \mathbb{P}\left\{\sup_{0\le t\le T} D_t > u\right\} \\
&\le \mathbb{P}\left\{\sup_{(s,t)\in E_u} Z_u(s,t) > m(u)\right\} + \mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_u} Z_u(s,t) > m(u)\right\}, \qquad (8.4)
\end{aligned}$$

where $E_u = [0, (\ln m(u))^2/m^2(u)] \times [T - (\ln m(u))^2/m^2(u), T]$. In light of Lemma 8.3.1, it follows that for u sufficiently large, $\sqrt{Var\left(Z_u(s,t)\right)} = \frac{\sigma_u^-(s,t)}{\sigma_u^-(0,T)}$ attains its maximum over $0 \le s \le t \le T$ at unique point $(0,T)$ and there exists a positive constant Q such that

$$\sup_{(s,t)\in A\setminus E_u} \sqrt{Var\left(Z_u(s,t)\right)} \le 1 - Q\left(\frac{\ln m(u)}{m(u)}\right)^2.$$

Moreover,

$$\mathbb{E}\left((Z_u(s,t) - Z_u(s',t'))^2\right) \le Q_1\left(|s-s'|^{2H} + |t-s'|^{2H}\right), \quad (s,t),(s',t') \in A,$$

with Q_1 a positive constant. Hence by Piterbarg Theorem (Theorem 8.1 in [119]), we have for u sufficiently large

$$\mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_u} Z_u(s,t) > m(u)\right\} \le Q_2(m(u))^{\frac{2}{H}} \Psi\left(\frac{m(u)}{1 - Q\left(\frac{\ln m(u)}{m(u)}\right)^2}\right). \qquad (8.5)$$

Next we analyze $\mathbb{P}\left\{\sup_{(s,t)\in E_u} Z_u(s,t) > u\right\}$. Let

$$\Delta(u) = 2^{\frac{1}{2H}} T(m(u))^{-\frac{1}{H}}, \quad E_{u,1} = [0, (\ln m(u))^2/(m^2(u)\Delta(u))]^2.$$

Then rewrite

$$\mathbb{P}\left\{\sup_{(s,t)\in E_u} Z_u(s,t) > u\right\} = \mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}} Z_u(\Delta(u)s, T - \Delta(u)t) > u\right\}.$$

We distinguish between $H > \frac{1}{2}$, $H = \frac{1}{2}$, $\frac{1}{4} < H < \frac{1}{2}$, $H = \frac{1}{4}$ and $0 < H < \frac{1}{4}$.

Case $H > \frac{1}{2}$. In order to apply Lemma 8.4.1 in Appendix, we need to check conditions. By Lemmas 8.3.1 and 8.3.3, we have

$$\lim_{u\to\infty}\sup_{(s,t)\in E_{u,1}}\left|\frac{1-\sqrt{Var(Z_u(\Delta(u)s,T-\Delta(u)t))}}{\Delta(u)\left(\frac{H}{T}t+\frac{H}{T}s\right)}-1\right|=0, \tag{8.6}$$

$$\lim_{u\to\infty}\sup_{(s,t),(s',t')\in E_{u,1}}\left|m^2(u)\frac{1-Corr\left(Z_u(\Delta(u)s,T-\Delta(u)t),Z_u(\Delta(u)s',T-\Delta(u)t')\right)}{|s-s'|^{2H}+|t-t'|^{2H}}-1\right|=0. \tag{8.7}$$

These imply that (8.26) and (8.27) hold. Following the notation in Lemma 8.4.1, we have that Using the fact that

$$\nu_i=\lim_{u\to\infty}(m(u))^2\frac{H}{T}\Delta(u)=\infty, i=1,2.$$

Noting that $(0,0)\in E_{u,1}$ and by case iii) in Lemma 8.4.1 in Appendix, we have

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\}\sim\Psi(m(u)),$$

which together with (8.4) and (8.5) establishes the claim.
Case $H = \frac{1}{2}$. Note that (8.6) and (8.7) still hold for $H=\frac{1}{2}$. Following the notation in Lemma 8.4.1, we have for $i=1,2$,

$$\nu_i=\lim_{u\to\infty}(m(u))^2\frac{H}{T}\Delta(u)=2^{\frac{1}{2H}}H=1,\quad \lim_{u\to\infty}a_i(u)=0,\quad \lim_{u\to\infty}b_i(u)=\lim_{u\to\infty}(\ln m(u))^2/(m^2(u)\Delta(u))=\infty.$$

Thus by case ii) in Lemma 8.4.1 in Appendix, we have

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\}\sim\left(\mathcal{P}^1_{1/2}\right)^2\Psi(m(u)),$$

which combined with (8.4), (8.5) and (8.3) establishes the claim.
Case $\frac{1}{4} < H < \frac{1}{2}$. Let

$$I_{k,l}=[kS,(k+1)S]\times[lS,(l+1)S], k,l\ge 0,\quad N(u)=\left[\frac{(\ln m(u))^2}{m^2(u)\Delta(u)S}\right],$$

$$\Lambda_1(u)=\{(k,l,k',l'):0\le k,l,k',l'\le N(u)+1, I_{k,l}\cap I_{k',l'}\neq\emptyset,(k,l)\neq(k',l')\},$$

$$\Lambda_2(u)=\{(k,l,k',l'):0\le k,l,k',l'\le N(u)+1, I_{k,l}\cap I_{k',l'}=\emptyset\}.$$

Bonferroni inequality gives that

$$\Sigma^-(u)-\Sigma\Sigma_1(u)-\Sigma\Sigma_2(u)\le\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\}\le\Sigma^+(u), \tag{8.8}$$

where

$$\Sigma^\pm(u) = \sum_{k,l=0}^{N(u)\pm1}\mathbb{P}\left\{\sup_{(s,t)\in I_{k,l}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\},$$

$$\Sigma\Sigma_i(u) = \sum_{(k,l,k',l')\in\Lambda_i}\mathbb{P}\left\{\sup_{(s,t)\in I_{k,l}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u),\sup_{(s,t)\in I_{k',l'}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\}, i=1,2.$$

Upper or Lower bounds for $\Sigma^\pm(u)$. By Lemma 8.3.1, we have

$$\lim_{u\to\infty}\sup_{(s,t)\in E_{u,1}}\left|\frac{1-\sqrt{Var(Z_u(\Delta(u)s,T-\Delta(u)t))}}{\Delta(u)\frac{H}{T}t+\Delta(u)\frac{H}{T}s+\frac{(\Delta(u))^{2H}}{2u}s^{2H}}-1\right|=0. \tag{8.9}$$

Thus for any $0 < \epsilon < 1$, let

$$m_{k,l}^{\pm\epsilon}(u) = m(u)\left(1 + (1 \pm \epsilon)\left(\Delta(u)\frac{H}{T}(l \pm 1)S + \Delta(u)\frac{H}{T}(k \pm 1)S + \frac{(\Delta(u))^{2H}}{2u}(k \pm 1)^{2H}S^{2H}\right)\right).$$

Moreover, denote by

$$Z_{u,k,l}(s,t) = \frac{Z_u(\Delta(u)(kS+s), T - \Delta(u)(lS+t))}{\sqrt{Var\left(Z_u(\Delta(u)(kS+s), T - \Delta(u)(lS+t))\right)}}.$$

Then we have

$$\begin{aligned}\Sigma^+(u) &\leq \sum_{k,l=0}^{N(u)+1} \mathbb{P}\left\{\sup_{(s,t)\in[0,S]^2} Z_{u,k,l}(s,t) > m_{k,l}^{-\epsilon}(u)\right\},\\ \Sigma^-(u) &\geq \sum_{k,l=0}^{N(u)-1} \mathbb{P}\left\{\sup_{(s,t)\in[0,S]^2} Z_{u,k,l}(s,t) > m_{k,l}^{+\epsilon}(u)\right\}.\end{aligned}$$

Note that (8.7) implies that

$$\lim_{u\to\infty} \sup_{(s,t)\in[0,S]^2} \left|(m_{k,l}^{\pm\epsilon}(u))^2 \frac{1 - Corr\left(Z_{u,k,l}(s,t), Z_{u,k,l}(s',t')\right)}{|s-s'|^{2H} + |t-t'|^{2H}} - 1\right| = 0. \tag{8.10}$$

Thus by Lemma 8.4.2, we have that

$$\lim_{u\to\infty} \sup_{0\leq k,l\leq N(u)+1} \left|\frac{\mathbb{P}\left\{\sup_{(s,t)\in[0,S]^2} Z_{u,k,l}(s,t) > m_{k,l}^{\pm\epsilon}(u)\right\}}{\Psi(m_{k,l}^{\pm\epsilon}(u))} - (\mathcal{H}_H([0,S]))^2\right| = 0.$$

This implies that

$$\begin{aligned}\Sigma^+(u) &\leq (\mathcal{H}_H([0,S]))^2 \sum_{k,l=0}^{N(u)+1} \Psi(m_{k,l}^{-\epsilon}(u))\\ &\leq (\mathcal{H}_H([0,S]))^2 \Psi(m(u)) \sum_{k,l=0}^{N(u)+1} e^{-(1-\epsilon)\left(m^2(u)\Delta(u)\frac{H}{T}(l-1)S + m^2(u)\Delta(u)\frac{H}{T}(k-1)S + m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)}\\ &=: \left(\frac{\mathcal{H}_H([0,S])}{S}\right)^2 \Psi(m(u))\Theta^-(u,S,\epsilon).\end{aligned} \tag{8.11}$$

and

$$\begin{aligned}\Sigma^-(u) &\geq (\mathcal{H}_H([0,S]))^2 \sum_{k,l=0}^{N(u)-1} \Psi(m_{k,l}^{+\epsilon}(u))\\ &\geq (\mathcal{H}_H([0,S]))^2 \Psi(m(u)) \sum_{k,l=0}^{N(u)+1} e^{-(1+\epsilon)\left(m^2(u)\Delta(u)\frac{H}{T}(l+1)S + m^2(u)\Delta(u)\frac{H}{T}(k+1)S + m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k+1)^{2H}S^{2H}\right)}\\ &=: \left(\frac{\mathcal{H}_H([0,S])}{S}\right)^2 \Psi(m(u))\Theta^+(u,S,\epsilon).\end{aligned} \tag{8.12}$$

Next we analyze $\Theta^\pm(u,S,\epsilon)$. Note that

$$\sup_{0\leq k\leq N(u)+1} m^2(u)\frac{(\Delta(u))^{2H}}{2u}|k-1|^{2H}S^{2H} \leq Q(m(u))^{2-4H}\frac{(\ln m(u))^{4H}}{u} \leq Qu^{1-4H}(\ln u)^{4H} \to 0.$$

Hence, setting

$$\upsilon(u,\epsilon) = (1-\epsilon)m^2(u)\Delta(u)\frac{H}{T}, \tag{8.13}$$

it follows that

$$\begin{aligned}\Theta^-(u,S,\epsilon) &\leq S^2\sum_{k,l=0}^{N(u)+1} e^{-(\upsilon(u,\epsilon)(l-1)S+\upsilon(u,\epsilon)(k-1)S)}\\ &= (\upsilon(u,\epsilon))^{-2}\left(\sum_{l=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(l-1)S}\upsilon(u,\epsilon)S\right)\left(\sum_{k=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(k-1)S}\upsilon(u,\epsilon)S\right)\\ &\leq (\upsilon(u,\epsilon))^{-2}\left(\int_0^\infty e^{-t}dt\right)^2\\ &\sim (m^2(u)\Delta(u))^{-2}\left(\frac{T}{H}\right)^2=\left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\right)^2u^{\frac{2}{H}-4},\quad u\to\infty,\epsilon\to 0,S\to\infty,\end{aligned}\tag{8.14}$$

which together with the fact that

$$\lim_{S\to\infty}\frac{\mathcal{H}_H([0,S])}{S}=\mathcal{H}_H$$

leads to

$$\Sigma^+(u)\leq\left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2u^{\frac{2}{H}-4}\Psi(m(u)),\quad u\to\infty.\tag{8.15}$$

Similarly, we can show that

$$\Theta^+(u,S,\epsilon)\geq\left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\right)^2u^{\frac{2}{H}-4},\quad u\to\infty,\epsilon\to 0,S\to\infty.$$

Hence

$$\Sigma^-(u)\geq\left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2u^{\frac{2}{H}-4}\Psi(m(u)),\quad u\to\infty.\tag{8.16}$$

Upper bounds of $\Sigma\Sigma_i(u), i=1,2$. For $(k,l,k',l')\in\Lambda_1$, without loss of generality, we assume that $k'=k+1$. Then denote by

$$I_{k',l'}^{(1)}=[(k+1)S,(k+1)S+\sqrt{S}]\times[l'S,(l'+1)S],\quad I_{k',l'}^{(2)}=[(k+1)S+\sqrt{S},(k+2)S,]\times[l'S,(l'+1)S].$$

Hence, for $(k,l,k',l')\in\Lambda_1$,

$$\begin{aligned}&\mathbb{P}\left\{\sup_{(s,t)\in I_{k,l}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u),\sup_{(s,t)\in I_{k',l'}}Z_u(\Delta(u)s,T-\Delta(u)t)>m(u)\right\}\\ &\leq\mathbb{P}\left\{\sup_{(s,t)\in I_{k,l}}\overline{Z}_u(\Delta(u)s,T-\Delta(u)t)>m_{k,l}^{-\epsilon}(u),\sup_{(s,t)\in I_{k',l'}^{(2)}}\overline{Z}_u(\Delta(u)s,T-\Delta(u)t)>m_{k',l'}^{-\epsilon}(u)\right\}\\ &\quad+\mathbb{P}\left\{\sup_{(s,t)\in I_{k',l'}^{(1)}}\overline{Z}_u(\Delta(u)s,T-\Delta(u)t)>m_{k',l'}^{-\epsilon}(u)\right\},\end{aligned}$$

where

$$\overline{Z}_u(\Delta(u)s,T-\Delta(u)t)=\frac{Z_u(\Delta(u)s,T-\Delta(u)t)}{\sqrt{Var(Z_u(\Delta(u)s,T-\Delta(u)t))}}.$$

Noting that (8.10) holds and

$$\mathbb{P}\left\{\sup_{(s,t)\in I_{k',l'}^{(1)}}\overline{Z}_u(\Delta(u)s,T-\Delta(u)t)>m_{k',l'}^{-\epsilon}(u)\right\}=\mathbb{P}\left\{\sup_{(s,t)\in[0,\sqrt{S}]\times[0,S]}Z_{u,k',l'}(s,t)>m_{k',l'}^{-\epsilon}(u)\right\},$$

by Lemma 8.4.2 in Appendix, we have that

$$\lim_{u\to\infty}\sup_{0\leq k',l'\leq N(u)+1}\left|\frac{\mathbb{P}\left\{\sup_{(s,t)\in[0,\sqrt{S}]\times[0,S]}Z_{u,k',l'}(s,t)>m_{k,l}^{\pm\epsilon}(u)\right\}}{\Psi(m_{k,l}^{-\epsilon}(u))}-\mathcal{H}_H([0,\sqrt{S}])\mathcal{H}_H([0,S])\right|=0.$$

Using also the fact that $I_{k,l}$ has at most 8 neighborhoods and

$$\lim_{S\to\infty} \frac{\mathcal{H}_H([0,\sqrt{S}])}{S} = \lim_{S\to\infty} \frac{\mathcal{H}_H([0,\sqrt{S}])}{\sqrt{S}} \lim_{S\to\infty} S^{-\frac{1}{2}} = \mathcal{H}_H \lim_{S\to\infty} S^{-\frac{1}{2}} = 0,$$

in light of (8.11) and (8.14), we have

$$\begin{aligned}
&\sum_{(k,l,k',l')\in\Lambda_1} \mathbb{P}\left\{ \sup_{(s,t)\in I^{(1)}_{k',l'}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k',l'}(u) \right\} \\
&\le 8 \sum_{k',l'=0}^{N(u)+1} \mathcal{H}_H([0,\sqrt{S}])\mathcal{H}_H([0,S])\Psi(m^{-\epsilon}_{k,l}(u)) \\
&\le 8 \frac{\mathcal{H}([0,\sqrt{S}])}{S} \frac{\mathcal{H}([0,S])}{S} \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\right)^2 u^{\frac{2}{H}-4}\Psi(m(u)) \\
&= o\left(u^{\frac{2}{H}-4}\Psi(m(u))\right), \quad u\to\infty, S\to\infty.
\end{aligned} \tag{8.17}$$

Lemma 8.3.3 shows that for u sufficiently large and $(s,t),(s',t')\in E_{u,1}$

$$Corr\left(\overline{Z}_u(\Delta(u)s, T-\Delta(u)t), \overline{Z}_u(\Delta(u)s', T-\Delta(u)t')\right) > 0$$

and

$$\lim_{u\to\infty} \sup_{(s,t)\neq(s',t'),(s,t),(s',t')\in E_{u,1}} \left| (m(u))^2 \frac{1-Corr\left(\overline{Z}_u(\Delta(u)s, T-\Delta(u)t), \overline{Z}_u(\Delta(u)s', T-\Delta(u)t')\right)}{|s-s'|^{2H}+|t-t'|^{2H}} - 1 \right| = 0.$$

Hence by Lemma 8.4.3 in Appendix, there exists constants $\mathcal{C},\mathcal{C}_1>0$ such that for $(k,l,k',l')\in\Lambda_1$ and u sufficiently large

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{(s,t)\in I_{k,l}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k,l}(u), \sup_{(s,t)\in I^{(2)}_{k',l'}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k',l'}(u) \right\} \\
&\le \mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}} \Psi\left(m^{-\epsilon}_{k,l,k',l'}(u)\right);
\end{aligned}$$

and for $(k,l,k',l')\in\Lambda_2$ and u sufficiently large

$$\begin{aligned}
&\mathbb{P}\left\{ \sup_{(s,t)\in I_{k,l}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k,l}(u), \sup_{(s,t)\in I_{k',l'}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k',l'}(u) \right\} \\
&\le \mathcal{C}S^4 e^{-\mathcal{C}_1(|k-k'|^2+|l-l'|^2)^{\frac{H}{2}} S^H} \Psi\left(m^{-\epsilon}_{k,l,k',l'}(u)\right),
\end{aligned} \tag{8.18}$$

where

$$m^{-\epsilon}_{k,l,k',l'}(u) = \min(m^{-\epsilon}_{k,l}(u), m^{-\epsilon}_{k',l'}(u)).$$

Consequently, noting that $I_{k,l}$ has at most 8 neighborhoods and in light of (8.11) and (8.14)

$$\begin{aligned}
&\sum_{(k,l,k',l')\in\Lambda_1} \mathbb{P}\left\{ \sup_{(s,t)\in I_{k,l}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k,l}(u), \sup_{(s,t)\in I^{(2)}_{k',l'}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k',l'}(u) \right\} \\
&\le \sum_{(k,l,k',l')\in\Lambda_1} \mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}} \Psi\left(m^{-\epsilon}_{k,l,k',l'}(u)\right) \\
&\le \sum_{(k,l,k',l')\in\Lambda_1} \mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}} \left(\Psi\left(m^{-\epsilon}_{k,l}(u)\right) + \Psi\left(m^{-\epsilon}_{k',l'}(u)\right)\right) \\
&\le \sum_{k,l=0}^{N(u)+1} 16\mathcal{C}S^4 e^{-\mathcal{C}_1 S^{\frac{H}{2}}} \Psi\left(m^{-\epsilon}_{k,l}(u)\right) \\
&\le QS^2 e^{-\mathcal{C}_1 S^{\frac{H}{2}}} u^{\frac{2}{H}-4}\Psi(m(u)) = o\left(u^{\frac{2}{H}-4}\Psi(m(u))\right), \quad u\to\infty, S\to\infty.
\end{aligned}$$

Therefore, we can conclude that

$$\Sigma\Sigma_1(u) = o\left(u^{\frac{2}{H}-4}\Psi(m(u))\right), \quad u \to \infty, S \to \infty. \tag{8.19}$$

Moreover, by (8.18) and (8.11)-(8.14)

$$\begin{aligned}
\Sigma\Sigma_2(u) &\le \sum_{(k,l,k',l')\in\Lambda_2} \mathbb{P}\left\{\sup_{(s,t)\in I_{k,l}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k,l}(u), \sup_{(s,t)\in I_{k',l'}} \overline{Z}_u(\Delta(u)s, T-\Delta(u)t) > m^{-\epsilon}_{k',l'}(u)\right\}\\
&\le \sum_{(k,l,k',l')\in\Lambda_2} \mathcal{C}S^4 e^{-\mathcal{C}_1(|k-k'|^2+|l-l'|^2)^{\frac{H}{2}}S^H}\Psi\left(m^{-\epsilon}_{k,l,k',l'}(u)\right)\\
&\le \sum_{k,l=0}^{N(u)+1}\Psi\left(m^{-\epsilon}_{k,l}(u)\right)2\mathcal{C}S^4 \sum_{k',l'\ge 0, k'+l'\neq 0} e^{-\mathcal{C}_1(|k-k'|^2+|l-l'|^2)^{\frac{H}{2}}S^H}\\
&\le \sum_{k,l=0}^{N(u)+1} QS^4 e^{-Q_1 S^H}\Psi\left(m^{-\epsilon}_{k,l}(u)\right)\\
&\le QS^2 e^{-Q_1 S^H} u^{\frac{2}{H}-4}\Psi(m(u)) = o\left(u^{\frac{2}{H}-4}\Psi(m(u))\right), \quad u \to \infty, S \to \infty.
\end{aligned} \tag{8.20}$$

Inserting (8.15)-(8.16) and (8.19)-(8.20) into (8.8), we derive that

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}} Z_u(\Delta(u)s, T-\Delta(u)t) > m(u)\right\} \sim \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2 u^{\frac{2}{H}-4}\Psi(m(u)), \quad u \to \infty,$$

which together with (8.4) and (8.5) establishes the claim.
Case $H = \frac{1}{4}$. Note that (8.8)-(8.12) still hold for $H = \frac{1}{4}$. We next focus on $\Theta^{\pm}(u, S, \epsilon)$. Recalling that

$$\upsilon(u,\epsilon) = (1-\epsilon)m^2(u)\Delta(u)\frac{H}{T},$$

it follows that

$$\begin{aligned}
\Theta^-(u,S,\epsilon) &= S^2\sum_{k,l=0}^{N(u)+1} e^{-\left(\upsilon(u,\epsilon)(l-1)S+\upsilon(u,\epsilon)(k-1)S+m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)}\\
&= \sum_{l=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(l-1)S}S\sum_{k=0}^{N(u)+1} e^{-\left(\upsilon(u,\epsilon)(k-1)S+(1-\epsilon)m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)}S.
\end{aligned}$$

The first sum satisfies

$$\begin{aligned}
\sum_{l=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(l-1)S}S &= (\upsilon(u,\epsilon))^{-1}\sum_{l=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(l-1)S}\upsilon(u,\epsilon)S\\
&\le (\upsilon(u,\epsilon))^{-1}\int_0^\infty e^{-t}dt \sim \left(m^2(u)\Delta(u)\frac{H}{T}\right)^{-1}, \quad u\to\infty, \epsilon\to 0.
\end{aligned} \tag{8.21}$$

For the second one

$$\begin{aligned}
&\sum_{k=0}^{N(u)+1} e^{-\left(\upsilon(u,\epsilon)(k-1)S+(1-\epsilon)m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)}S\\
&= (\upsilon(u,\epsilon))^{-1}\sum_{k=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(k-1)S+\left(\frac{(1-\epsilon)^{\frac{1}{2H}}2^{-\frac{1}{2H}}u^{-\frac{1}{2H}}(m(u))^{\frac{1}{H}}\Delta(u)}{\upsilon(u,\epsilon)}\upsilon(u,\epsilon)(k-1)S\right)^{2H}}\upsilon(u,\epsilon)S.
\end{aligned}$$

Note that for $H = \frac{1}{4}$,

$$\frac{(1-\epsilon)^{\frac{1}{2H}}2^{-\frac{1}{2H}}u^{-\frac{1}{2H}}(m(u))^{\frac{1}{H}}\Delta(u)}{\upsilon(u,\epsilon)} \sim \sqrt{T}(1-\epsilon), \quad u\to\infty.$$

Thus

$$\sum_{k=0}^{N(u)+1} e^{-\left(\upsilon(u,\epsilon)(k-1)S+(1-\epsilon)m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)}S$$
$$\sim \left(m^2(u)\Delta(u)\frac{H}{T}\right)^{-1}\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dx, \quad u\to\infty, \epsilon\to 0.$$

Consequently,

$$\Theta^-(u,S,\epsilon) \le \left(m^2(u)\Delta(u)\frac{H}{T}\right)^{-2}\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dx, \quad u\to\infty, \epsilon\to 0.$$

Similarly,

$$\Theta^+(u,S,\epsilon) \ge \left(m^2(u)\Delta(u)\frac{H}{T}\right)^{-2}\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dx, \quad u\to\infty, \epsilon\to 0.$$

In light of (8.11) and (8.12), we have that

$$\begin{aligned}
\Sigma^-(u) &\le \left(\frac{\mathcal{H}_H([0,S])}{S}\right)^2\left(m^2(u)\Delta(u)\frac{H}{T}\right)^{-2}\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dx\Psi(m(u)) \\
&\le \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dxu^{\frac{2}{H}-4}\Psi(m(u)), \quad u\to\infty, S\to\infty, \\
\Sigma^+(u) &\ge \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dxu^{\frac{2}{H}-4}\Psi(m(u)), \quad u\to\infty, S\to\infty.
\end{aligned}$$

The negligibility of $\Sigma\Sigma_i(u), i=1,2$ holds due to the fact that (8.17)-(8.20) are also valid for $H=\frac{1}{4}$. Therefore we have

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}} Z_u(\Delta(u)s, T-\Delta(u)t) > m(u)\right\} \sim \left(H^{-1}2^{-\frac{1}{2H}}T^{2H-1}\mathcal{H}_H\right)^2\int_0^\infty e^{-x-T^{\frac{1}{4}}\sqrt{x}}dxu^{\frac{2}{H}-4}\Psi(m(u)), \quad u\to\infty,$$

which combined with (8.4) and (8.5) establishes the claim.
Case $0<H<\frac{1}{4}$. For $0<H<\frac{1}{4}$, (8.8)-(8.12) are satisfied. In order to get the upper or lower bounds of $\Sigma^\pm(u)$, it suffices to analyze $\Theta^\pm(u,S,\epsilon)$. Denote by

$$\upsilon'(u,\epsilon) = (1-\epsilon)^{\frac{1}{2H}}2^{-\frac{1}{2H}}u^{-\frac{1}{2H}}(m(u))^{\frac{1}{H}}\Delta(u),$$

it follows that

$$\begin{aligned}
\Theta^-(u,S,\epsilon) &= S^2\sum_{k,l=0}^{N(u)+1} e^{-(1-\epsilon)\left(m^2(u)\Delta(u)\frac{H}{T}(l-1)S+m^2(u)\Delta(u)\frac{H}{T}(k-1)S+m^2(u)\frac{(\Delta(u))^{2H}}{2u}(k-1)^{2H}S^{2H}\right)} \\
&= \sum_{l=0}^{N(u)+1} e^{-\upsilon(u,\epsilon)(l-1)S}S\sum_{k=0}^{N(u)+1} e^{-\left(\upsilon(u,\epsilon)(k-1)S+\left(\upsilon'(u,\epsilon)(k-1)S\right)^{2H}\right)}S,
\end{aligned}$$

where $\upsilon(u,\epsilon)$ is defined in (8.13). The first sum satisfies (8.21) with $0<H<1/4$. For the second sum

$$\sum_{k=0}^{N(u)+1} e^{-(\upsilon(u,\epsilon)(k-1)S+\left(\upsilon'(u,\epsilon)(k-1)S\right)^{2H})}S$$
$$= (\upsilon'(u,\epsilon))^{-1}\sum_{k=0}^{N(u)+1} e^{-\gamma(u)\upsilon'(u,\epsilon)(k-1)S+\left(\upsilon'(u,\epsilon)(k-1)S\right)^{2H}}\upsilon'(u,\epsilon)S,$$

where

$$\gamma(u) = \frac{\upsilon(u,\epsilon)}{\upsilon'(u,\epsilon)}$$

$$
\begin{aligned}
&= \frac{(1-\epsilon)m^2(u)\Delta(u)\frac{H}{T}}{(1-\epsilon)^{\frac{1}{2H}}2^{-\frac{1}{2H}}u^{-\frac{1}{2H}}(m(u))^{\frac{1}{H}}\Delta(u)} \\
&\sim Qu^{2-\frac{1}{2H}} \to 0, \quad u \to \infty.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\sum_{k=0}^{N(u)+1} e^{-(v(u,\epsilon)(k-1)S+\left(v'(u,\epsilon)(k-1)S\right)^{2H})}S \\
&\sim (v'(u,\epsilon))^{-1}\int_0^\infty e^{-x^{2H}}dx \\
&\sim \Gamma\left(\frac{1}{2H}+1\right)T^{-1}u^{\frac{1}{2H}}, \quad u\to\infty, \epsilon\to 0.
\end{aligned}
$$

Consequently,

$$
\Theta^-(u,S,\epsilon) \le H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)u^{\frac{3}{2H}-2}, \quad u\to\infty, \epsilon\to 0.
$$

Similarly,

$$
\Theta^+(u,S,\epsilon) \ge H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)u^{\frac{3}{2H}-2}, \quad u\to\infty, \epsilon\to 0.
$$

In light of (8.11) and (8.12), we have that, as $u\to\infty, S\to\infty$,

$$
\begin{aligned}
\Sigma^-(u) &\le H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)\left(\mathcal{H}_H\right)^2u^{\frac{3}{2H}-2}\Psi(m(u)), \\
\Sigma^+(u) &\ge H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)\left(\mathcal{H}_H\right)^2u^{\frac{3}{2H}-2}\Psi(m(u)).
\end{aligned}
$$

Following line by line the same as (8.17)-(8.20), we can show that for $i=1,2$

$$
\Sigma\Sigma_i(u) = o\left(u^{\frac{3}{2H}-2}\Psi(m(u))\right), \quad u\to\infty, S\to\infty.
$$

Therefore, we conclude that

$$
\mathbb{P}\left\{\sup_{(s,t)\in E_{u,1}} Z_u(\Delta(u)s, T-\Delta(u)t) > m(u)\right\} \sim H^{-1}2^{-\frac{1}{2H}}T^{2H-2}\Gamma\left(\frac{1}{2H}+1\right)\left(\mathcal{H}_H\right)^2u^{\frac{3}{2H}-2}\Psi(m(u)), \quad u\to\infty,
$$

which establishes the claim with aid of (8.4) and (8.5). This completes the proof. □

Proof of Theorem 8.2.2 We distinguish between $H\ge\frac{1}{2}$ and $H<\frac{1}{2}$.

Case $H\ge\frac{1}{2}$. We have that

$$
\begin{aligned}
\mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} &= \mathbb{P}\left\{\sup_{(s,t)\in A}(X_t-X_s)>u\right\} \\
&= \mathbb{P}\left\{\sup_{(s,t)\in A}\left(B_H(t)-B_H(s)-\frac{1}{2}(t^{2H}-s^{2H})+\mu(t-s)\right)>u\right\} \\
&= \mathbb{P}\left\{\sup_{(s,t)\in A} Z_{u,1}(s,t) > m_1(u)\right\},
\end{aligned}
$$

where

$$
Z_{u,1}(s,t) = \frac{B_H(t)-B_H(s)}{u-\mu(t-s)+\frac{1}{2}(t^{2H}-s^{2H})}m_1(u), \quad m_1(u) = \frac{u-\mu T+\frac{1}{2}T^{2H}}{T^H}, \quad A=\{(s,t), 0\le s\le t\le T\}.
$$

Furthermore,

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\} \le \mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\}$$
$$\le \mathbb{P}\left\{\sup_{(s,t)\in E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\} + \mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\}, \quad (8.22)$$

where

$$E_{u,2} = [0, (\ln m_1(u))^2/(m_1(u))^2] \times [T - (\ln m_1(u))^2/(m_1(u))^2, T].$$

In light of Lemma 8.3.2, it follows that for u sufficiently large

$$\sup_{(s,t)\in A\setminus E_{u,2}} \sqrt{Var(Z_{u,1}(s,t))} \le 1 - Q\left(\frac{\ln m_1(u)}{m_1(u)}\right)^2.$$

Moreover, direct calculation shows that

$$\mathbb{E}\left\{(Z_{u,1}(s,t) - Z_{u,1}(s',t'))^2\right\} \le Q_1(|t-t'|^{2H} + |s-s'|^{2H}), \quad (s,t),(s',t') \in A.$$

Using Piterbarg Theorem (Theorem 8.1 in [119]), we have for u sufficiently large

$$\mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\} \le Q_2(m_1(u))^{\frac{2}{H}}\Psi\left(\frac{m_1(u)}{1 - Q\left(\frac{\ln m_1(u)}{m_1(u)}\right)^2}\right). \quad (8.23)$$

Next we focus on $\mathbb{P}\left\{\sup_{(s,t)\in E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\}$. Lemmas 8.3.1 and 8.3.3 lead to

$$\lim_{u\to\infty}\sup_{(s,t)\in E_{u,2}}\left|\frac{1-\sqrt{Var(Z_{u,1}(s,t))}}{\frac{H(T-t)}{T} + \frac{H}{T}s} - 1\right| = 0, \quad \lim_{u\to\infty}\sup_{(s,t),(s',t')\in E_{u,2}}\left|\frac{1 - Corr\left(Z_{u,1}(s,t), Z_{u,1}(s',t')\right)}{\frac{|s-s'|^{2H}+|t-t'|^{2H}}{2T^{2H}}} - 1\right| = 0,$$

which coincide with the local variance and correlation behavior of $Z_u(s,t)$ in proof of Theorem 8.2.1 for case $H \ge \frac{1}{2}$. Similarly as in proof of Theorem 8.2.1, we derive that for $H > \frac{1}{2}$

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\} \sim \Psi\left(m_1(u)\right), \quad u \to \infty;$$

and for $H = \frac{1}{2}$

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,2}} Z_{u,1}(s,t) > m_1(u)\right\} \sim \left(\mathcal{P}^1_{1/2}\right)^2 \Psi\left(m_1(u)\right), \quad u \to \infty.$$

Inserting the above asymptotics and (8.23), (8.3) in (8.22), we establish the claim.
Case $0 < H < \frac{1}{2}$. Observe that

$$\begin{aligned}\mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\} &= \mathbb{P}\left\{\sup_{(s,t)\in A} (X_t - X_s) > u\right\}\\ &= \mathbb{P}\left\{\sup_{(s,t)\in A}\left(B_H(t) - B_H(s) - \frac{1}{2}(t^{2H} - s^{2H}) + \mu(t-s)\right) > u\right\}\\ &= \mathbb{P}\left\{\sup_{(s,t)\in A} Z_{u,2}(s,t) > m_2(u)\right\},\end{aligned}$$

where

$$Z_{u,2}(s,t) = \frac{B_H(t) - B_H(s)}{u - \mu(t-s) + \frac{1}{2}(t^{2H} - s^{2H})} m_2(u), \quad m_2(u) = \inf_{0\le s\le T}\frac{u - \mu(T-s) + \frac{1}{2}(T^{2H} - s^{2H})}{(T-s)^H}.$$

Thus we have

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\} \le \mathbb{P}\left\{\sup_{0\le t\le T} U_t > u\right\}$$
$$\le \mathbb{P}\left\{\sup_{(s,t)\in E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\} + \mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\}, \quad (8.24)$$

where

$$E_{u,3} = [0, s_u + (\ln m_2(u))/m_2(u)] \times [T - (\ln m_2(u))^2/(m_2(u))^2, T].$$

In light of Lemma 8.3.2, it follows that for u sufficiently large

$$\sup_{(s,t)\in A\setminus E_{u,3}} \sqrt{Var(Z_{u,2}(s,t))} \le 1 - Q_3\left(\frac{\ln m_2(u)}{m_2(u)}\right)^2,$$

and direct calculation shows that

$$\mathbb{E}\left((Z_{u,2}(s,t) - Z_{u,2}(s',t'))^2\right) \le Q_4(|t-t'|^{2H} + |s-s'|^{2H}), \quad (s,t),(s',t') \in A.$$

By Piterbarg Theorem, we have for u sufficiently large

$$\mathbb{P}\left\{\sup_{(s,t)\in A\setminus E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\} \le Q_5(m_2(u))^{\frac{2}{H}}\Psi\left(\frac{m_2(u)}{1 - Q\left(\frac{\ln m_2(u)}{m_2(u)}\right)^2}\right). \quad (8.25)$$

Next we consider $\mathbb{P}\left\{\sup_{(s,t)\in E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\}$. Rewrite

$$\mathbb{P}\left\{\sup_{(s,t)\in E_{u,3}} Z_{u,2}(s,t) > m_2(u)\right\} = \mathbb{P}\left\{\sup_{(s,t)\in E_{u,4}} Z_{u,2}(s_u + \Delta_1(u)s, T - \Delta_1(u)t) > m_2(u)\right\}$$

where

$$E_{u,4} = [-s_u/\Delta_1(u), (\ln m_2(u))/(m_2(u)\Delta_1(u))] \times [0, (\ln m_2(u))^2/((m_2(u))^2\Delta_1(u))], \quad \Delta_1(u) = 2^{\frac{1}{2H}}T(m_2(u))^{-\frac{1}{H}},$$

and s_u is defined in Lemma 8.3.2. Lemmas 8.3.2 and 8.3.3 lead to

$$\lim_{u\to\infty}\sup_{(s,t)\in E_{u,4}}\left|\frac{1-\sqrt{Var(Z_{u,2}(s_u+\Delta_1(u)s, T-\Delta_1(u)t))}}{\frac{H(1-H)}{2T^2}(\Delta_1(u))^2 s^2 + \frac{H}{T}\Delta_1(u)t} - 1\right| = 0,$$

and

$$\lim_{u\to\infty}\sup_{(s,t),(s',t')\in E_{u,4}}\left|(m_2(u))^2\frac{1 - Corr\left(Z_{u,2}(s_u+\Delta_1(u)s, T-\Delta_1(u)t)), Z_{u,2}(s_u+\Delta_1(u)s, T-\Delta_1(u)t))\right)}{|s-s'|^{2H} + |t-t'|^{2H}} - 1\right| = 0.$$

Next we check the conditions of Lemma 8.4.1 in Appendix. Following the same notation as in Lemma 8.4.1, we have that

$$\nu_1 = \lim_{u\to\infty}(m_2(u))^2\frac{H}{T}\Delta_1(u) = 2^{\frac{1}{2H}}H\lim_{u\to\infty}(m_2(u))^{2-\frac{1}{H}} = 0, \quad \nu_2 = \lim_{u\to\infty}(m_2(u))^2\frac{H(1-H)}{2T^2}(\Delta_1(u))^2 = 0,$$

$$y_{1,2} = \lim_{u\to\infty} m_2(u)\sqrt{\frac{H(1-H)}{2T^2}}\Delta_1(u)(\ln m_2(u))/(m_2(u)\Delta_1(u)) = \infty,$$

$$y_{2,1} = 0, \quad y_{2,2} = \lim_{u\to\infty}(m_2(u))^2\frac{H}{T}\Delta_1(u)(\ln m_2(u))^2/((m_2(u))^2\Delta_1(u)) = \infty.$$

Moreover, by Lemma 8.3.2, $s_u \sim T^{\frac{1}{1-2H}}u^{-\frac{1}{1-2H}}$, which implies that

$$y_{1,1} = -\lim_{u\to\infty} m_2(u)\sqrt{\frac{H(1-H)}{2T^2}}\Delta_1(u)s_u/\Delta_1(u) = -Q\lim_{u\to\infty}u^{1-\frac{1}{1-2H}} = 0.$$

Thus by case i) in Lemma 8.4.1, we have that

$$\begin{aligned}
&\mathbb{P}\left\{\sup_{(s,t)\in E_{u,4}} Z_{u,2}(s_u+\Delta_1(u)s, T-\Delta_1(u)t) > m_2(u)\right\}\\
&\sim (\mathcal{H}_H)^2\sqrt{\frac{2T^2}{H(1-H)}}\frac{T}{H}\int_0^\infty e^{-t^2}dt\int_0^\infty e^{-s}ds(m_2(u))^{-3}(\Delta_1(u))^{-2}\Psi(m_2(u))\\
&\sim 2^{-\frac{1}{H}-\frac{1}{2}}T^{3H}\sqrt{\frac{\pi}{H^3(H-1)}}(\mathcal{H}_H)^2u^{\frac{2}{H}-3}\Psi(m_2(u)).
\end{aligned}$$

Inserting the above asymptotics and (8.25) into (8.24) establishes the claim. This completes the proof. □

8.4 Appendix

8.4.1 Appendix A

This subsection is devoted to the proofs of Lemma 8.3.1-8.3.2.

PROOF OF LEMMA 8.3.1 Note that for any $\delta>0$ and u sufficiently large, the maximum of $\sigma_u^-(s,t)$ over $0\le s\le t\le T$ is only obtained in $[0,\delta]\times[T-\delta,T]$. Next we consider the variance function $\sigma_u^-(s,t)$ over $[0,\delta]\times[T-\delta,T]$. It follows that

$$\begin{aligned}
1-\frac{\sigma_u^-(s,t)}{\sigma_u^-(0,T)} &= 1-\frac{|t-s|^H}{u+\mu(t-s)-\frac{1}{2}(t^{2H}-s^{2H})}\frac{u+\mu T-\frac{1}{2}T^{2H}}{T^H}\\
&= 1-\frac{\frac{|t-s|^H}{T^H}}{\frac{u+\mu(t-s)-\frac{1}{2}(t^{2H}-s^{2H})}{u+\mu T-\frac{1}{2}T^{2H}}}\\
&= \left(1-\frac{|t-s|^H}{T^H}\right)(1+o(1))+\left(\frac{u+\mu(t-s)-\frac{1}{2}(t^{2H}-s^{2H})}{u+\mu T-\frac{1}{2}T^{2H}}-1\right)(1+o(1))\\
&= \frac{H}{T}(T-t+s)(1+o(1))+\frac{-\mu(T-t+s)+\frac{1}{2}(2HT^{2H-1}(T-t)+s^{2H})}{u+\mu T-\frac{1}{2}T^{2H}}(1+o(1))\\
&= \left(\frac{H}{T}(T-t)+\frac{H}{T}s+\frac{1}{2u}s^{2H}\right)(1+a(\delta,u)),\quad (s,t)\in[0,\delta]\times[T-\delta,T],
\end{aligned}$$

as δ sufficiently small and u sufficiently large, where $\lim_{\delta\to0,u\to\infty}a(\delta,u)=0$. The fact that

$$\frac{H}{T}(T-t)+\frac{H}{T}s+\frac{1}{2u}s^{2H}>0$$

for $(s,t)\in([0,\delta]\times[T-\delta,T])\setminus\{(0,T)\}$ implies that the maximum point of $\sigma_u^-(s,t)$ over $0\le s\le t\le T$ is unique and is $(0,T)$. This completes the proof. □

PROOF OF LEMMA 8.3.2 For any $\delta>0$ and u sufficiently large, the maximum of $\sigma_u^+(s,t)$ over $0\le s\le t\le T$ is only obtained in $[0,\delta]\times[T-\delta,T]$. Next we focus on $\sigma_u^+(s,t)$ over $[0,\delta]\times[T-\delta,T]$. For $\delta>0$ sufficiently small and u sufficiently large,

$$\begin{aligned}
1-\frac{\sigma_u^+(s,t)}{\sigma_u^+(0,T)} &= 1-\frac{|t-s|^H}{u-\mu(t-s)+\frac{1}{2}(t^{2H}-s^{2H})}\frac{u-\mu T+\frac{1}{2}T^{2H}}{T^H}\\
&= 1-\frac{\frac{|t-s|^H}{T^H}}{\frac{u-\mu(t-s)+\frac{1}{2}(t^{2H}-s^{2H})}{u-\mu T+\frac{1}{2}T^{2H}}}\\
&= \left(1-\frac{|t-s|^H}{T^H}\right)(1+o(1))+\left(\frac{u-\mu(t-s)+\frac{1}{2}(t^{2H}-s^{2H})}{u-\mu T+\frac{1}{2}T^{2H}}-1\right)(1+o(1))\\
&= \frac{H}{T}(T-t+s)(1+o(1))+\frac{\mu(T-t+s)-\frac{1}{2}(2HT^{2H-1}(T-t)+s^{2H})}{u-\mu T+\frac{1}{2}T^{2H}}(1+o(1))\\
&= \left(\frac{H}{T}(T-t)+\frac{H}{T}s\right)(1+a_1(\delta,u))-\frac{1}{2u}s^{2H}(1+a_2(\delta,u)),\quad (s,t)\in[0,\delta]\times[T-\delta,T],
\end{aligned}$$

where $\lim_{\delta\to 0, u\to\infty} a_i(\delta,u)=0, i=1,2$. If $H\geq \frac{1}{2}$, then

$$1-\frac{\sigma_u^+(s,t)}{\sigma_u^+(0,T)}=\left(\frac{H}{T}(T-t)+\frac{H}{T}s\right)(1+a_1(\delta,u)),\quad (s,t)\in[0,\delta]\times[T-\delta,T],$$

which implies that the maximum point of $\sigma_u^+(s,t)$ is obtained at $(0,T)$ and is unique. For $0<H<\frac{1}{2}$,

$$\begin{aligned}1-\frac{\sigma_u^+(s,T)}{\sigma_u^+(0,T)} &= \frac{H}{T}s(1+a_1(\delta,u))-\frac{1}{2u}s^{2H}(1+a_2(\delta,u))\\ &= \frac{H}{T}s^{2H}\left(s^{1-2H}(1+a_1(\delta,u))-\frac{1}{2u}(1+a_2(\delta,u))\right)<0,\end{aligned}$$

as $s<\left(\frac{(1+a_2(\delta,u))}{2u(1+a_1(\delta,u))}\right)^{\frac{1}{1-2H}}\sim(2u)^{-\frac{1}{1-2H}}$. This implies that the maximum of $\sigma_u^+(s,T)$ over $[0,T]$ is attained over $(0,\delta)$ for $\delta>0$ sufficiently small and u sufficiently large. We denote this point by s_u. Using the fact that

$$\frac{\partial\sigma_u^+(s_u,T)}{\partial s}=\frac{-H(T-s_u)^{H-1}\left(u-\mu(T-s_u)+\frac{1}{2}(T^{2H}-s_u^{2H})\right)-(T-s_u)^H(\mu-Hs_u^{2H-1})}{(u-\mu(T-s_u)+\frac{1}{2}(T^{2H}-s_u^{2H}))^2}=0,$$

we have that

$$s_u=\left(\frac{u}{T}+\frac{1}{2}T^{2H-1}+\frac{\mu(1-H)}{H}+\frac{1}{2T}s_u^{2H}-\frac{\mu(1-H)}{TH}s_u\right)^{\frac{1}{2H-1}}\sim T^{\frac{1}{1-2H}}u^{-\frac{1}{1-2H}}.$$

Next we show that the maximizer of $\sigma_u^+(s,t)$ is (s_u,T) for $0<H<\frac{1}{2}$ and u sufficiently large. Observe that

$$1-\frac{\sigma_u^+(s,t)}{\sigma_u^+(s_u,T)}=-\frac{\sigma_u^+(s,T)-\sigma_u^+(s_u,T)}{\sigma_u^+(s_u,T)}+\frac{\sigma_u^+(s,T)-\sigma_u^+(s,t)}{\sigma_u^+(s_u,T)}.$$

Direct calculation gives that, as $u\to\infty$,

$$\begin{aligned}\sigma_u^+(s_u,T) &\sim \frac{T^H}{u},\\ \sigma_u^+(s,T)-\sigma_u^+(s_u,T) &= \frac{1}{2}\frac{\partial^2\sigma_u^+(s_u,T)}{\partial^2 s}(s-s_u)^2(1+o(1))\sim\frac{H(H-1)T^{H-2}}{2u}(s-s_u)^2,\\ \sigma_u^+(s,T)-\sigma_u^+(s,t) &= \frac{\partial\sigma_u^+(s,T)}{\partial t}(T-t)(1+o(1))\sim\frac{HT^{H-1}}{u}(T-t),\quad t\to T.\end{aligned}$$

Thus we have

$$1-\frac{\sigma_u^+(s,t)}{\sigma_u^+(s_u,T)}=\frac{H(1-H)}{2T^2}(s-s_u)^2(1+o(1))+\frac{H}{T}(T-t)(1+o(1)),\quad u\to\infty, |s-s_u|, T-t\to 0.$$

The above local behavior implies that the maximizer of $\sigma_u^+(s,t)$ is (s_u,T) for u large and is unique. This completes the proof. □

Proof of Lemma 8.3.3 Let $\sigma_H(s,t):=\sqrt{\mathrm{var}(B_H(t)-B_H(s))}$. Observe that

$$\sigma_H(s,t)=|t-s|^H,$$

and

$$\begin{aligned}&1-Corr(B_H(t)-B_H(s),B_H(t')-B_H(s'))\\ &=\frac{\mathbb{E}\left\{((B_H(t)-B_H(s))-(B_H(t')-B_H(s')))^2\right\}-(\sigma_H(s,t)-\sigma_H(s',t'))^2}{2\sigma_H(s,t)\sigma_H(s',t')}\\ &=\frac{\mathbb{E}\left\{((B_H(t)-B_H(t'))-(B_H(s)-B_H(s')))^2\right\}-(|t-s|^H-|t'-s'|^H)^2}{2|t-s|^H|t'-s'|^H}\\ &=\frac{|t-t'|^{2H}+|s-s'|^{2H}+(|t-s|^{2H}+|t'-s'|^{2H}-|t-s'|^{2H}-|t'-s|^{2H})-(|t-s|^H-|t'-s'|^H)^2}{2|t-s|^H|t'-s'|^H}.\end{aligned}$$

Using Taylor formula, we have that for $(s,t)\in[0,\delta_u]\times[T-\delta_u,T]$, with $\lim_{u\to\infty}\delta_u=0$ and u sufficiently large

$$\begin{aligned}|t-s|^{2H}-|t-s'|^{2H}-(|t'-s|^{2H}-|t'-s'|^{2H}) &= 2H(|\theta_1-s|^{2H-1}-|\theta_1-s'|^{2H-1})(t-t')\\ &= 2H(2H-1)(\theta_1-\theta_2)^{2H-2}(s-s')(t-t'),\\ (|t-s|^H-|t'-s'|^H)^2 &= (H\theta_3(t-t'-s+s'))^2,\end{aligned}$$

where $\theta_1\in(t,t')$, $\theta_2\in(s,s')$ and $\theta_3\in(t-s,t'-s')$. Moreover,

$$\lim_{u\to\infty}\lim_{s,t\in[0,\delta_u]\times[T-\delta_u,T]}\left||t-s|^H-T^H\right|=0.$$

Consequently, for $\lim_{u\to\infty}\delta_u=0$

$$\lim_{u\to\infty}\sup_{(s,t),(s',t')\in[0,\delta_u]\times[T-\delta_u,T]}\left|\frac{1-Corr\,(B_H(t)-B_H(s),B_H(t')-B_H(s'))}{\frac{|s-s'|^{2H}+|t-t'|^{2H}}{2T^{2H}}}-1\right|=0.$$

8.4.2 Appendix B

In this subsection we present some useful results derived in [60]. First we give an accommodated to our needs version of Theorem 3.2 in [60]. Let $X_u(s,t),(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]$ with $0\in\prod_{i=1,2}[a_i(u),b_i(u)]$, be a family of centered continuous Gaussian random fields with variance function $\sigma_u(s,t)$ satisfying,

$$\sigma_u(0,0)=1,\text{and }\lim_{u\to\infty}\sup_{(s,t)\neq(0,0),(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]}\left|\frac{1-\sigma_u(s,t)}{\frac{|s|^{\beta_1}}{g_1(u)}+\frac{|t|^{\beta_2}}{g_2(u)}}-1\right|=0 \tag{8.26}$$

with $\beta_i>0,i=1,2$, $\lim_{u\to\infty}g_i(u)=\infty,i=1,2$, $\lim_{u\to\infty}\frac{|a_i(u)|^{\beta_1}}{g_1(u)}+\frac{+|b_i(u)|^{\beta_2}}{g_2(u)}=0,i=1,2$, and correlation function satisfying

$$\lim_{u\to\infty}\sup_{(s,t),(s',t')\in\prod_{i=1,2}[a_i(u),b_i(u)],(s,t)\neq(s',t')}\left|n^2(u)\frac{1-Corr(X_u(s,t),X_u(s',t'))}{|s-s'|^\alpha+|t-t'|^\alpha}-1\right|=0, \tag{8.27}$$

with $\alpha\in(0,2]$ and $\lim_{u\to\infty}n(u)=\infty$.
We suppose that $\lim_{u\to\infty}\frac{n^2(u)}{g_i(u)}=\nu_i\in[0,\infty],i=1,2.$

Lemma 8.4.1. *Let $X_u(s,t),(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]$ with $0\in\prod_{i=1,2}[a_i(u),b_i(u)]$ be a family of centered continuous Gaussian random fields satisfying (8.26) and (8.27).*
i) If $\nu_i=0,i=1,2$ and for $i=1,2$,

$$\lim_{u\to\infty}\frac{(n(u))^{2/\beta_i}a_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,1},\quad \lim_{u\to\infty}\frac{(n(u))^{2/\beta_i}b_i(u)}{(g_i(u))^{1/\beta_i}}=y_{i,2},\quad \lim_{u\to\infty}\frac{(n(u))^{2/\beta_i}(a_i^2(u)+b_i^2(u))}{(g_i(u))^{2/\beta_i}}=0,$$

with $-\infty\le y_{i,1}<y_{i,2}\le\infty$, then

$$\mathbb{P}\left\{\sup_{(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]}X_u(s,t)>n(u)\right\}\sim\left(\mathcal{H}_{\alpha/2}\right)^2\prod_{i=1}^2\int_{y_{i,1}}^{y_{i,2}}e^{-|s|^{\beta_i}}ds\prod_{i=1}^2\left(\frac{g_i(u)}{n^2(u)}\right)^{1/\beta_i}\Psi(n(u)).$$

ii) If $\nu_i\in(0,\infty)$ and further $\lim_{u\to\infty}a_i(u)=a_i\in[-\infty,0],\lim_{u\to\infty}b_i(u)=b_i\in[0,\infty]$, then

$$\mathbb{P}\left\{\sup_{(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]}X_u(s,t)>n(u)\right\}\sim\prod_{i=1}^2\mathcal{P}_{\alpha/2}^{v_i,\beta_i}([a_i,b_i])\Psi(n(u)),$$

where

$$\mathcal{P}_{\alpha/2}^{v_i,\beta_i}([a_i,b_i])=\mathbb{E}\left\{\sup_{t\in[a_i,b_i]}e^{\sqrt{2}B_{\alpha/2}(t)-|t|^\alpha-\nu_i|t|^{\beta_i}}\right\}\in(0,\infty),\ \ i=1,2.$$

iii) If $\nu_i = \infty, i = 1, 2,$ *then*

$$\mathbb{P}\left\{\sup_{(s,t)\in\prod_{i=1,2}[a_i(u),b_i(u)]} X_u(s,t) > n(u)\right\} \sim \Psi(n(u)).$$

Next we give a simpler version of Proposition 2.2 in [60]. Denote by $\Lambda(u)$ a series of index sets depending on u and by $[a_1, a_2] \times [b_1, b_2]$ a rectangle with $a_1 < a_2$ and $b_1 < b_2$. Let $X_{u,k,l}(s,t), (s,t) \in [a_1, a_2] \times [b_1, b_2], (k,l) \in \Lambda(u)$ be a family of two-dimensional continuous Gaussian random fields with mean 0 and variance function 1. There exists $n_{k,l}(u), (k,l) \in \Lambda(u)$ satisfying

$$\lim_{u\to\infty} \sup_{(k,l),(k',l')\in\Lambda(u)} \left|\frac{n_{k,l}(u)}{n_{k'l'}(u)} - 1\right| = 0, \lim_{u\to\infty} \inf_{(k,l)\in\Lambda(u)} n_{k,l} = \infty, \tag{8.28}$$

such that the correlation function satisfies

$$\lim_{u\to\infty} \sup_{(k,l)\in\Lambda(u)} \sup_{(s,t)\neq(s',t'),(s,t),(s',t')\in[a_1,a_2]\times[b_1,b_2]} \left|(n_{k,l}(u))^2 \frac{1 - Corr\left(X_{u,k,l}(s,t), X_{u,k,l}(s',t')\right)}{|s-s'|^{\alpha_1} + |t-t'|^{\alpha_2}} - 1\right| = 0, \tag{8.29}$$

where $\alpha_i \in (0, 2], i = 1, 2$.
Then Proposition 2.2 in [60] leads to the following result.

Lemma 8.4.2. *Let* $X_{u,k,l}(s,t), (s,t) \in E, (k,l) \in \Lambda(u)$ *be a family of centered two-dimensional continuous Gaussian random fields with variance function* 1*. Assume further that (8.28)-(8.29) hold. Then*

$$\lim_{u\to\infty} \sup_{(k,l)\in\Lambda(u)} \left|\frac{\mathbb{P}\left\{\sup_{(s,t)\in[a_1,a_2]\times[b_1,b_2]} X_{u,k,l}(s,t) > n_{k,l}(u)\right\}}{\Psi\left(n_{k,l}(u)\right)} - \mathcal{H}_{\frac{\alpha_1}{2}}([a_1,a_2])\mathcal{H}_{\frac{\alpha_1}{2}}([b_1,b_2])\right| = 0$$

Finally, we display a lemma concerning the uniform double maximum, a simpler version of Corollary 3.2 in [60]. Let E_u be a family of non-empty compact subset of $\mathbb{R}^2$ and $A_i \subset [0, S]^2, i = 1, 2$ be two non-empty compact subsets of $\mathbb{R}^2$. Denote by $\Lambda_0(u) = \{(k_1, l_1, k_2, l_2) : (k_i, l_i) + A_i \subset E_u, i = 1, 2\}$. Let $n(u)$ and $n_{k_i,l_i}(u), (k_i, l_i) + A_i \subset E_u$ be a family of positive functions such that

$$\lim_{u\to\infty} \sup_{(k_i,l_i)+A_i\in E_u} \left|\frac{n_{k_i,l_i}(u)}{n(u)} - 1\right| = 0, i = 1, 2, \quad \lim_{u\to\infty} n(u) = \infty. \tag{8.30}$$

Lemma 8.4.3. *Let* $X_u(s,t), (s,t) \in E_u$ *be a family of centered Gaussian random variance* 1 *and correlation function satisfying*

$$\lim_{u\to\infty} \sup_{(s,t)\neq(s',t'),(s,t),(s',t')\in E_u} \left|(n(u))^2 \frac{1 - Corr(X_u(s,t), X_u(s',t'))}{|s-s'|^{\alpha_1} + |t-t'|^{\alpha_2}} - 1\right| = 0$$

Moreover, there exists $\delta > 0$ *such that for* u *large enough*

$$Corr(X_u(s,t), X_u(s',t')) > \delta - 1, (s,t), (s',t') \in E_u.$$

If further (8.30) is satisfied, then there exits $\mathcal{C} > 0, \mathcal{C}_1 > 0$ *such that for all* u *large*

$$\sup_{(k_1,l_1,k_2,l_2)\in\Lambda_0(u),A_i\subset[0,S]^2,A_i\neq\emptyset,i=1,2} \frac{\mathbb{P}\left\{\sup_{(s,t)\in(k_1,l_1)+A_1} X_u(s,t) > n_{k_1,l_1}(u), \sup_{(s,t)\in(k_2,l_2)+A_2} X_u(s,t) > n_{k_2,l_2}(u)\right\}}{e^{-\mathcal{C}_1(F((k_1,l_1)+A_1,(k_2,l_2)+A_2))^{\frac{1}{2}\min(\alpha_1,\alpha_2)}} S^4 \Psi(n_{k_1,l_1,k_2,l_2}(u))} \le \mathcal{C},$$

where

$$F(A,B) = \inf_{s\in A,t\in B} ||s-t||, \quad n_{k_1,l_1,k_2,l_2}(u) = \min(n_{k_1,l_1}(u), n_{k_2,l_2}(u)),$$

and $\mathcal{C}$ *and* $\mathcal{C}_1$ *are independent of* u *and* S*.*

Chapter 9

On Generalised Piterbarg Constants.[1]

9.1 Introduction

Let $X(t), t \ge 0$ be a centered Gaussian process with continuous sample paths and unit variance. Pickands' theorem (see [117, 17, 18, 119, 121, 118]) shows that for any $\delta \ge 0, T > 0$ (set $\delta\mathbb{Z} = \mathbb{R}$ if $\delta = 0$),

$$\mathbb{P}\left\{ \sup_{t \in u^{-2/\alpha}\delta\mathbb{Z}\cap[0,T]} X(t) > u \right\} \sim T\mathcal{H}_{\alpha,\delta} u^{2/\alpha} \mathbb{P}\{X(0) > u\}, \quad u \to \infty \tag{9.1}$$

is valid, provided that the correlation r satisfies the Pickands condition

$$1 - r(t) \sim |t|^\alpha, \alpha \in (0, 2], \quad t \to 0, \quad r(t) < 1, \ \forall t > 0. \tag{9.2}$$

Here the *Pickands constant* $\mathcal{H}_{\alpha,\delta}$ is given by the following limit

$$\mathcal{H}_{\alpha,\delta} = \lim_{T\to\infty} T^{-1}\mathbb{E}\left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)} \right\} \in (0,\infty), \quad W(t) = \sqrt{2}B_\alpha(t) - |t|^\alpha, \tag{9.3}$$

where $\{B_\alpha(t), t \ge 0\}$ is a standard fractional Brownian motion with Hurst index $\alpha/2 \in (0, 1]$, i.e., a mean zero Gaussian process with continuous sample paths and covariance function $\mathrm{Cov}(B_\alpha(s), B_\alpha(t)) = \frac{1}{2}\Big(|t|^\alpha + |s|^\alpha - |t-s|^\alpha\Big), s, t \ge 0$. In the current literature, the only known values of $\mathcal{H}_{\alpha,\delta}$ are for $\alpha = 1, 2$ if $\delta = 0$. Numerous papers have investigated the problem of calculation of Pickands constants, with particular focus on the case of $\delta = 0$; see for instance [134, 89, 80, 55, 48, 70, 35, 76, 70, 77, 64, 44].

Let us consider a non-stationary centered Gaussian process $Y(t) = (1 - t^\alpha)X(t), t \in [0, 1]$. In view of Piterbarg's theorem (see [119, 121]) we have that under (9.2) and for $\delta = 0$

$$\mathbb{P}\left\{ \sup_{t\in[0,1]} Y(t) > u \right\} \sim \mathcal{P}^h_{\alpha,0}\mathbb{P}\{X(0) > u\}, \quad u \to \infty, \tag{9.4}$$

with $h(t) = t^\alpha$, where for any $\delta \ge 0$

$$\mathcal{P}^h_{\alpha,\delta} = \lim_{T\to\infty} \mathbb{E}\left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)-h(t)} \right\} \in (0,\infty). \tag{9.5}$$

is the so-called *Piterbarg constant*; see also [88] for expressions analogous to (9.5) in the context of Pickands constants. Due to the fact that $\mathcal{P}^{h(t)}_{\alpha,\delta} = e^{-h(0)}\mathcal{P}^{h(t)-h(0)}_{\alpha,\delta}$, in the following analysis we focus only on the case that $h(0) = 0$.

So far in the literature only the case $\delta = 0$ has been considered. In particular, by [38], we have

$$\mathcal{P}^{Rt}_{1,0} = 1 + \frac{1}{R},$$

[1] This chapter is based on L. Bai, K. Dębicki, E. Hashorva, and L. Luo (2018): On generalised Piterbarg constants, published in the *Methodology and Computing in Applied Probability,* Volume 20, 137-164.

whereas in view of [98]

$$\mathcal{P}_{2,0}^{Rt^2} = \frac{1}{2}\left(1+\sqrt{1+\frac{1}{R}}\right).$$

Besides the case $h(t) = Rt^{\alpha}$, Piterbarg constants have been introduced also for $h(t) = Rt^{\alpha/2}$; see [82].
In this contribution we show that for a general class of functions h and $\delta \geq 0$ constants $\mathcal{P}_{\alpha,\delta}^h$ appear naturally in the tail asymptotics of extremes of nonhomogeneous Gaussian processes (see Theorem 9.2.1) and provide regularity conditions for h that guarantee finiteness of $\mathcal{P}_{\alpha,\delta}^h$. Then we investigate $\mathcal{P}_{\alpha,\delta}^h$. As summarized in the following result and shown in Section 4, for particular functions h one can derive the exact value of $\mathcal{P}_{2,0}^h$. Hereafter $\Phi(\cdot)$ denotes the distribution function of an $N(0,1)$ random variable and $\Gamma(\cdot)$ stands for the Euler Gamma function.

Proposition 9.1.1. We have
i) $\mathcal{P}_{2,0}^{Rt} = \frac{1}{\sqrt{\pi}R}\exp\left(-\frac{R^2}{4}\right) + \Phi\left(\frac{R}{\sqrt{2}}\right)$;
ii) $\mathcal{P}_{2,0}^{Rt^{\frac{3}{2}}} = \frac{1}{2} + \frac{1}{\sqrt{\pi}}\exp\left(-\frac{27}{1024}R^4\right)\int_0^\infty \exp\left(\frac{9}{16}R^2t^2 - Rt^3\right)dt^2$;
iii) $\mathcal{P}_{2,0}^{Rt^3} = \frac{1}{2} + \frac{1}{6R\sqrt{\pi}}\exp\left(\frac{1}{108R^2}\right)\int_0^\infty \exp\left(-\frac{t^4}{36R^2} + \frac{2t^3}{27R^2} - \frac{t^2}{18R^2}\right)dt^2$;
iv) for $h(t) = -t^2 + Rt^{\lambda}, \lambda \in (2,\infty), R > 0$

$$\mathcal{P}_{2,0}^h = \frac{1}{\sqrt{2\pi}}\int_0^\infty \exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)\left(\frac{\sqrt{2}x}{\lambda}\right)^{\frac{\lambda}{\lambda-1}} - \frac{x^2}{2}\right)dx + \frac{1}{2}.$$

For the case of $\delta > 0$, general $\alpha \in (0,2]$ or more general h is too difficult to derive $\mathcal{P}_{\alpha,\delta}^h$ explicitly. Therefore, in Section 2 we shall focus on upper and lower estimates for $\mathcal{P}_{\alpha,\delta}^h$. Interestingly, we have the following relation between $\mathcal{P}_{\alpha,\delta}^h$ and Pickands constant:

Proposition 9.1.2. We have

$$\mathcal{P}_{\alpha,\delta}^{Rt^{\lambda}} \geq (e\lambda R)^{-1/\lambda}\mathcal{H}_{\alpha,\delta}, \quad \forall\, \alpha \in (0,2], R > 0. \tag{9.6}$$

Brief organisation of the rest of the paper. We present our main results in Section 2 followed then by the proofs in Section 3. In Section 4 we display the proofs of Propositions 9.1.1 and 9.1.2. Section 5 gives additional bounds for $\mathcal{P}_{2,0}^{Rt^{\lambda}}$ and includes several illustrative graphs on the bounds of generalised Piterbarg constants.

9.2 Main Results

In this section we are concerned with two questions: Q1) what are the basic properties of generalised Piterbarg constants $\mathcal{P}_{\alpha,\delta}^h$, and Q2) do these constants appear in some asymptotic settings in analogy with the corresponding generalised Pickands constants?

We begin with demonstration that generalised Piterbarg constants appear in the context of extreme values of non-stationary Gaussian processes. We recall that following our notation $\delta\mathbb{Z} = \mathbb{R}$ if $\delta = 0$.

Theorem 9.2.1. *Let $\{X(t), t \geq 0\}$ be a centered stationary Gaussian process with continuous trajectories, unit variance and correlation function $r(\cdot)$ satisfying (9.2). Suppose that h is a continuous function such that $h(0) = 0$ and*

$$\lim_{t\to\infty}\frac{h(t)}{t^{\epsilon_1}} = \infty, \quad \lim_{t\to\infty}\frac{h(t)}{t^{\epsilon_2}} = 0 \tag{9.7}$$

for some $\epsilon_2 > \epsilon_1 > 0$. For any $\delta \geq 0$ as $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)}\frac{X(t)}{1+u^{-2}h(u^{2/\alpha}t)} > u\right\} \sim \mathcal{P}_{\alpha,\delta}^h\mathbb{P}\{X(0) > u\}, \tag{9.8}$$

where $0 < \mathcal{P}_{\alpha,\delta}^h < \infty$ and $\Delta(u) = u^{-2/\alpha}\{\delta\mathbb{Z}\cap[0,N_u]\}$, provided that $\lim_{u\to\infty}N_u = \infty$ and $N_u = o(u^c)$ with $c > 0$ such that $c\epsilon_2 \leq 2$.

Next we investigate properties of generalised Piterbarg constants $\mathcal{P}_{\alpha,\delta}^h$. It turns out that the finiteness of $\mathcal{P}_{\alpha,\delta}^h$ in the case that $\delta > 0$ is established under weaker conditions on the function h compared to the case $\delta = 0$. In the following proposition we present upper and lower bounds for $\mathcal{P}_{\alpha,\delta}^h$ for some general h, which in particular, provides a sufficient condition for finiteness of $\mathcal{P}_{\alpha,\delta}^h$.

Theorem 9.2.2. *Let h be an increasing continuous function such that $\lim_{x\to\infty} \frac{h(x)}{\ln x} = l \in (1, \infty]$.*
i) If $\delta \in (0, \infty)$, then

$$\mathcal{P}_{\alpha,\delta}^h \le e^{-h(0)} + + \frac{1}{\delta} \int_0^\infty e^{-h(x)} dx < \infty.$$

ii) If $\delta > 4^{\frac{1}{\alpha}} \Gamma(\frac{1}{\alpha} + 1)$, then

$$\mathcal{P}_{\alpha,\delta}^h \ge \frac{1}{\delta} \left(1 - \frac{1}{\delta} \int_0^\infty e^{-\frac{x^\alpha}{4}} dx\right) \int_\delta^\infty e^{-h(x)} dx.$$

iii) If $\delta = 0$, then

$$\mathcal{P}_{\alpha,0}^h \ge \frac{1}{4^{1/\alpha+1}\Gamma\left(\frac{1}{\alpha}+1\right)} \int_{2^{2/\alpha+1}\Gamma\left(\frac{1}{\alpha}+1\right)}^\infty e^{-h(x)} dx.$$

In the case when $h(t) = Rt^\lambda$ more precise upper and lower bounds are available as displayed by the next result, see also Appendix.

Theorem 9.2.3. *Suppose that $h(t) = Rt^\lambda$.*
i) If $0 < \lambda \le \alpha \le 1$, then

$$\mathcal{P}_{\alpha,0}^h \ge 1 + \max_{y\ge 0} \left(\frac{\alpha}{R\lambda} y^{1-\lambda/\alpha} e^{R(\lambda/\alpha - 1) y^{\lambda/\alpha}(1 + R\frac{\lambda}{\alpha} y^{\lambda/\alpha - 1})}\right)$$

and in particular $\mathcal{P}_{\alpha,0}^h \ge 1 + \frac{1}{R}$ for $\lambda = \alpha \le 1$.
ii) If $\lambda \ge \alpha \ge 1$, then

$$\mathcal{P}_{\alpha,0}^h \le \min_{y\ge 0} \left(\left(1 + \frac{\alpha}{R\lambda} y^{1-\lambda/\alpha}\right) e^{R(\lambda/\alpha - 1) y^{\lambda/\alpha}}\right)$$

and $\mathcal{P}_{\alpha,0}^h \le 1 + \frac{1}{R}$ for $\lambda = \alpha \ge 1$.

Remarks 9.2.1. In the [44] the generalised Pickands constants are discussed. These constants are defined by

$$\mathcal{H}_{\widetilde{W},\delta} = \lim_{T\to\infty} \frac{1}{T} \mathbb{E} \left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{\widetilde{W}(t)} \right\},$$

with $\widetilde{W}(t), t \in \mathbb{R}$ a stochastic process which determines an appropriately defined stationary Brown-Resnick process. For a large class of Brown-Resnick stationary processes, we have $\widetilde{W}(t) = X(t) - \sigma^2(t)/2, t \in \mathbb{R}$ where X is a centered Gaussian process with continuous sample paths, stationary increments, $X(0) = 0$ and variance function $\sigma^2(t), t \in \mathbb{R}$, see e.g., [96, 95].
The main challenge when dealing with $\mathcal{H}_{\widetilde{W},\delta}$ is to show that it is positive. In contrast, for generalised Piterbarg constants the main challenge is to show that they are finite.
Some extensions of the above results are possible by replacing $W(t)$ with a stochastic process $\widetilde{W}(t)$, which determines the corresponding Brown-Resnick stationary process, and thus redefining the Piterbarg constant as

$$\mathcal{P}_{\widetilde{W},\delta}^h = \lim_{T\to\infty} \mathbb{E} \left\{ \sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{\widetilde{W}(t) - h(t)} \right\}.$$

9.3 Proofs

PROOF OF THEOREM 9.2.2 The proof is similar to that of Proposition 3.2 in [57].
i) Since for any $i \in \mathbb{Z}$

$$\mathbb{E}\left\{\exp(\sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha)\right\} = 1 \tag{9.9}$$

we obtain

$$\begin{aligned}
\mathbb{E}\left\{\sup_{i\in\{0,1,\cdots,N\}} e^{\sqrt{2}B_\alpha(\delta i)-(\delta i)^\alpha - h(\delta i)}\right\} &\le \sum_{i=0}^{N} \mathbb{E}\left\{e^{\sqrt{2}B_\alpha(\delta i)-(\delta i)^\alpha - h(\delta i)}\right\} \\
&= \sum_{i=0}^{N} e^{-h(\delta i)} \\
&\le e^{-h(0)} + \frac{1}{\delta}\int_0^\infty e^{-h(x)}dx < \infty,
\end{aligned}$$

where the last inequality follows by the assumption $\lim_{x\to\infty}\frac{h(x)}{\ln x} = l \in (1,\infty]$.
ii) From (9.9) and the fact that for any $i, k \in \mathbb{N}$ such that $k > i$

$$\mathbb{E}\left\{\exp\left(\frac{\sqrt{2}(B_\alpha(\delta i) + B_\alpha(\delta k)) - (\delta i)^\alpha - (\delta k)^\alpha}{2}\right)\right\} = e^{-\frac{\delta^\alpha (k-i)^\alpha}{4}},$$

by Bonferroni's inequality

$$\begin{aligned}
&\mathbb{E}\left\{\sup_{i\in\{0,1,\cdots,N\}} e^{\sqrt{2}B_\alpha(\delta i)-(\delta i)^\alpha - h(\delta i)}\right\} \\
&= \int_{\mathbb{R}} e^s \mathbb{P}\left\{\exists_{i\in\{0,1,\cdots,N\}} \sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha - h(\delta i) > s\right\} ds \\
&\ge \sum_{i=0}^{N}\int_{\mathbb{R}} e^s \mathbb{P}\left\{\sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha - h(\delta i) > s\right\} ds \\
&\quad - \sum_{i=0}^{N-1}\sum_{k=i+1}^{N}\int_{\mathbb{R}} e^s \mathbb{P}\left\{\sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha - h(\delta i) > s, \sqrt{2}B_\alpha(\delta k) - (\delta k)^\alpha - h(\delta k) > s\right\} ds \\
&\ge \sum_{i=0}^{N}\int_{\mathbb{R}} e^s \mathbb{P}\left\{\sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha - h(\delta i) > s\right\} ds \\
&\quad - \sum_{i=0}^{N-1}\sum_{k=i+1}^{N}\int_{\mathbb{R}} e^s \mathbb{P}\left\{\sqrt{2}(B_\alpha(\delta i) + B_\alpha(\delta k)) - (\delta i)^\alpha - (\delta k)^\alpha - h(\delta i) - h(\delta k) > 2s\right\} ds \\
&= \sum_{i=0}^{N}\mathbb{E}\left\{\exp(\sqrt{2}B_\alpha(\delta i) - (\delta i)^\alpha - h(\delta i))\right\} \\
&\quad - \sum_{i=0}^{N-1}\sum_{k=i+1}^{N}\mathbb{E}\left\{e^{\frac{\sqrt{2}(B_\alpha(\delta i)+B_\alpha(\delta k)) - (\delta i)^\alpha - (\delta k)^\alpha - h(\delta i) - h(\delta k)}{2}}\right\} \\
&= \sum_{i=0}^{N} e^{-h(\delta i)} - \sum_{i=0}^{N-1}\sum_{k=i+1}^{N} e^{-\frac{\delta^\alpha (k-i)^\alpha}{4}} e^{-\frac{h(\delta i)+h(\delta k)}{2}} \\
&\ge \sum_{i=0}^{N} e^{-h(\delta i)} - \left(\sum_{i=0}^{N} e^{-h(\delta i)}\right)\left(\sum_{j=1}^{N} e^{-\frac{\delta^\alpha j^\alpha}{4}}\right) \qquad (9.10)\\
&\ge \frac{1}{\delta}\left(1 - \frac{1}{\delta}\int_0^\infty e^{-\frac{x^\alpha}{4}}dx\right)\int_\delta^{\delta N} e^{-h(x)}dx,
\end{aligned}$$

where in (9.10) we set $j = k - i$ and use the fact that $h(t)$ is an increasing function. Further, letting $N \to \infty$, we get the lower bound.

iii) For any $\delta \in (0,\infty)$, we have

$$\mathcal{P}^h_{\alpha,0} \ge \mathcal{P}^h_{\alpha,\delta} \ge \frac{1}{\delta}\left(1-\frac{1}{\delta}\int_0^\infty e^{-\frac{x^\alpha}{4}}dx\right)\int_\delta^\infty e^{-h(x)}dx$$
$$= \frac{1}{\delta}\left(1-\frac{1}{\delta}\Gamma\left(\frac{1}{\alpha}+1\right)4^{\frac{1}{\alpha}}\right)\int_\delta^\infty e^{-h(x)}dx.$$

In order to optimize the above, we note that $\frac{1}{\delta}\left(1-\frac{1}{\delta}4^{\frac{1}{\alpha}}\Gamma\left(\frac{1}{\alpha}+1\right)\right)$ attains its maximum at $\delta^* = 2^{\frac{2}{\alpha}+1}\Gamma\left(\frac{1}{\alpha}+1\right)$ which is equal to $\frac{1}{4^{1/\alpha+1}\Gamma\left(\frac{1}{\alpha}+1\right)}$ implying

$$\mathcal{P}^h_{\alpha,0} \ge \frac{1}{4^{1/\alpha+1}\Gamma\left(\frac{1}{\alpha}+1\right)}\int_{2^{2/\alpha+1}\Gamma\left(\frac{1}{\alpha}+1\right)}^\infty e^{-h(x)}dx$$

establishing the proof. □

Proof of Theorem 9.2.3 We define next

$$g(t) := \frac{\sqrt{2}}{2}t + \frac{\sqrt{2}}{2}Rt^{\frac{\lambda}{\alpha}}, \quad t>0$$

and put $K_y(t) = g'(y)t + g(y) - g'(y)y$ for the tangent function to g at y $(y \ge 0)$. i) Since $g(t)$ is concave for $0<\lambda\le\alpha\le 1$, then $K_y(t)\ge g(t)$ for any $y\ge 0$. Using a geometric approach as in [46], we have (set $Z(t)=\sqrt{2}B_\alpha(t)-t^\alpha-Rt^\lambda$)

$$\begin{aligned}
\mathcal{P}^{Rt^\lambda}_{\alpha,0} &= \mathbb{E}\left\{\sup_{t\in[0,\infty)} e^{Z(t)}\right\}\\
&= \int_{-\infty}^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} Z(t)>x\right\}dx\\
&= 1+\int_0^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} Z(t)>x\right\}dx\\
&\ge 1+\int_0^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} \left(\sqrt{2}B(t^\alpha)-t^\alpha-Rt^\lambda\right)>x\right\}dx \qquad (9.11)\\
&= 1+\int_0^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t)-g(t))>\frac{\sqrt{2}}{2}x\right\}dx\\
&\ge 1+\int_0^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t)-K_y(t))>\frac{\sqrt{2}}{2}x\right\}dx\\
&= 1+\int_0^\infty e^x\mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t)-g'(y)t)>g(y)-g'(y)y+\frac{\sqrt{2}}{2}x\right\}dx\\
&= 1+\int_0^\infty e^x\exp\left(-2g'(y)\left(g(y)-g'(y)y+\frac{\sqrt{2}}{2}x\right)\right)dx\\
&= 1+\frac{1}{\sqrt{2}g'(y)-1}e^{-2g'(y)(g(y)-g'(y)y)}\\
&= 1+\frac{\alpha}{R\lambda}y^{1-\lambda/\alpha}e^{R(\lambda/\alpha-1)y^{\lambda/\alpha}(1+R\frac{\lambda}{\alpha}y^{\lambda/\alpha-1})}, \qquad (9.12)
\end{aligned}$$

where (9.11) follows by Slepian inequality (see e.g., [1] and note in passing that a remarkable extension of this inequality for stable processes is obtained in [133]) and the fact that for any $\alpha\in(0,1]$

$$\mathrm{Cov}(B(t^\alpha),B(s^\alpha)) \ge \mathrm{Cov}(B_\alpha(t),B_\alpha(s)).$$

Then for $\lambda=\alpha$, $\mathcal{P}^{Rt^\lambda}_{\alpha,0}\ge 1+\frac{1}{R}=\mathcal{P}^{Rt}_{1,0}$, and for $\lambda<\alpha$ we have

$$\mathcal{P}^{h(t)}_{\alpha,0} \ge 1+\max_{y\ge 0} m(y), \quad m(y)=\frac{\alpha}{R\lambda}y^{1-\lambda/\alpha}e^{R(\lambda/\alpha-1)y^{\lambda/\alpha}(1+R\lambda/\alpha y^{\lambda/\alpha-1})},$$

where we used here that (9.12) holds for all $y\ge 0$.

ii) Since g is convex for $\lambda \geq \alpha \geq 1$, we have that $K_y(t) \leq g(t)$ for any $y \geq 0$. Using the same reasoning as i), we have

$$\begin{aligned}
\mathcal{P}_{\alpha,0}^{Rt^\lambda} &= 1+\int_0^\infty e^x \mathbb{P}\left\{\sup_{t\in[0,\infty)} Z(t) > x\right\} dx \\
&\leq 1+\int_0^\infty e^x \mathbb{P}\left\{\sup_{t\in[0,\infty)} (\sqrt{2}B(t^\alpha) - t^\alpha - Rt^\lambda) > x\right\} dx \qquad (9.13)\\
&= 1+\int_0^\infty e^x \mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t) - g(t)) > \frac{\sqrt{2}}{2}x\right\} dx \\
&\leq 1+\int_0^\infty e^x \mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t) - K_y(t)) > \frac{\sqrt{2}}{2}x\right\} dx \\
&= 1+\int_0^\infty e^x \mathbb{P}\left\{\sup_{t\in[0,\infty)} (B(t) - g'(y)t > g(y) - g'(y)y + \frac{\sqrt{2}}{2}x\right\} dx \\
&= 1+\int_0^{\sqrt{2}(g'(y)y-g(y))} e^x dx + \int_{\sqrt{2}(g'(y)y-g(y))}^\infty e^x \exp\left(-2g'(y)\left(g(y)-g'(y)y+\frac{\sqrt{2}}{2}x\right)\right) dx \\
&= e^{\sqrt{2}(g'(y)y-g(y))} + \frac{1}{\sqrt{2}g'(y)-1} e^{-2g'(y)(g(y)-g'(y)y)-\sqrt{2}(\sqrt{2}g'(y)-1)(g'(y)y-g(y))} \\
&= \frac{\sqrt{2}g'(y)}{\sqrt{2}g'(y)-1} e^{\sqrt{2}(g'(y)y-g(y))} \\
&= \left(1+\frac{\alpha}{R\lambda} y^{1-\lambda/\alpha}\right) e^{R(\lambda/\alpha-1)y^{\lambda/\alpha}}, \qquad (9.14)
\end{aligned}$$

where (9.13) follows by Slepian inequality and the fact that

$$\mathrm{Cov}(B(t^\alpha), B(s^\alpha)) \leq \mathrm{Cov}(B_\alpha(t), B_\alpha(s))$$

for $\alpha \geq 1$. Then for $\lambda = \alpha$, $\mathcal{P}_{\alpha,0}^{Rt^\lambda} \leq 1 + \frac{1}{R} = \mathcal{P}_{1,0}^{Rt}$, and for $\lambda > \alpha$ we have

$$\mathcal{P}_{\alpha,0}^{h(t)} \leq \min_{y\geq 0} f(y), \quad f(y) = \left(1+\frac{\alpha}{R\lambda} y^{1-\lambda/\alpha}\right) e^{R(\lambda/\alpha-1)y^{\lambda/\alpha}},$$

where we used that (9.14) holds for all $y \geq 0$. □

For notational simplicity we shall denote in the following

$$\mathcal{P}_{\alpha,\delta}^h(K) = \mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap K} e^{W(t)-h(t)}\right\},$$

where h is a continuous function, K is a compact set and $\delta \geq 0$. Analogously, let

$$\mathcal{H}_{\alpha,\delta}(K) = \mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap K} e^{W(t)}\right\}.$$

It is straightforward that $\mathcal{P}_{\alpha,\delta}^h(K),\ \mathcal{H}_{\alpha,\delta}(K) \in (0,\infty)$.

The next result is crucial for the proof of Theorem 9.2.1. It slightly extends Theorem 2.1 in [60] for the case that the functional is the supremum.

Theorem 9.3.1. *Let $\{\xi(t), t\in\mathbb{R}\}$ be a zero-mean stationary Gaussian process with continuous sample paths, unit variance and correlation function $r(\cdot)$ satisfying (9.2). Let $h(t)$ be a continuous function with $h(0)=0$ and $S_u, u>0$ be some countable index set parameterised by u. If $M_k(u), k\in S_u, u>0$ is such that*

$$\lim_{u\to\infty}\sup_{k\in S_u}\left|\frac{M_k(u)}{u} - 1\right| = 0, \qquad (9.15)$$

then for $b=\{0,1\}$ and any compact set $K \ni 0$, we have

$$\lim_{u\to\infty}\sup_{k\in S_u}\left|\sqrt{2\pi}M_k(u)e^{M_k^2(u)/2}\mathbb{P}\left\{\sup_{t\in K}\frac{\xi(u^{-2/\alpha}t)}{1+bu^{-2}h(t)} > M_k(u)\right\} - \mathcal{R}_b^h(K)\right| = 0,$$

with

$$\mathcal{R}_b^h(K) = \mathbb{E}\left\{\sup_{t\in\cap K} e^{W(t)-bh(t)}\right\} = \begin{cases} \mathcal{P}_{\alpha,0}^h(K) & \textit{if } b=1, \\ \mathcal{H}_{\alpha,0}(K) & \textit{if } b=0. \end{cases}$$

Proof of Theorem 9.3.1 The proof following the same ideas as in Lemma 6.4.1. In fact here is a special one dimensional case of Lemma 6.4.1 with $\lambda = 2/\alpha$. □

Proof of Theorem 9.2.1 Below $S, \mathbb{Q}_i, i\ge 1$ are positive constants. Set $\sigma_u(t) = (1+h(u^{2/\alpha}t)u^{-2})^{-1}$ and recall

$$\Delta(u) = u^{-2/\alpha}\{\delta\mathbb{Z}\cap[0,N_u]\}.$$

Further for $u>0$ we define

$$\pi(u) := \mathbb{P}\left\{\sup_{t\in\Delta(u)} X(t)\sigma_u(t) > u\right\},$$
$$I_k(u) = [kS,(k+1)S]u^{-2/\alpha}, \quad k\in\mathbb{N},$$
$$N(u) = \left\lfloor \frac{N_u}{S}\right\rfloor + 1.$$

Then for all u large $I_0(u)\subset\Delta(u)\subset\bigcup_{k=0}^{N(u)} I_k(u)$. First, note that for any $S_1>0$ and u large enough

$$\pi(u) \ge \mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha}S_1]\cap\Delta(u)} X(t)\sigma_u(t) > u\right\}, \tag{9.16}$$

$$\pi(u) \le \mathbb{P}\left\{\sup_{t\in I_0(u)\cap\Delta(u)} X(t)\sigma_u(t) > u\right\} + \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X(t)\sigma_u(t) > u\right\}. \tag{9.17}$$

Using Theorem 9.3.1, we obtain for u large enough

$$\begin{aligned} \mathbb{P}\left\{\sup_{t\in[0,u^{-2/\alpha}S_1]\cap\Delta(u)} X(t)\sigma_u(t) > u\right\} &= \mathbb{P}\left\{\sup_{t\in[0,S_1]\cap\delta\mathbb{Z}} \frac{X(u^{-2/\alpha}t)}{1+u^{-2}h(t)} > u\right\} \\ &\sim \mathbb{E}\left\{\sup_{t\in[0,S_1]\cap\delta\mathbb{Z}} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha-h(t)}\right\}\Psi(u) \\ &= \mathcal{P}_{\alpha,\delta}^h([0,S_1])\Psi(u), \ u\to\infty, \end{aligned}$$

and similarly as $u\to\infty$,

$$\mathbb{P}\left\{\sup_{t\in I_0(u)\cap\Delta(u)} X(t)\sigma_u(t) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,S]\cap\delta\mathbb{Z}} \frac{X(u^{-2/\alpha}t)}{1+u^{-2}h(t)} > u\right\} \sim \mathcal{P}_{\alpha,\delta}^h([0,S])\Psi(u). \tag{9.18}$$

By (9.7), we have for all S large $t^{\epsilon_1}\le h(t)\le t^{\epsilon_2}, t\in[S,\infty)$. Further for all u large

$$\begin{aligned} \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X(t)\sigma_u(t) > u\right\} &= \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \frac{X(t)}{1+u^{-2}h(u^{2/\alpha}t)} > u\right\} \\ &\le \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} X(t) > \mathcal{G}_u(k)\right\} \\ &= \sum_{k=1}^{N(u)} \mathbb{P}\left\{\sup_{t\in[0,S]} X(u^{-2/\alpha}t) > \mathcal{G}_u(k)\right\}, \end{aligned}$$

where $\mathcal{G}_u(k) = u(1+u^{-2}\inf_{s\in I_k(u)} h(u^{2/\alpha}s)), \ k\in\mathbb{N}$. We have that

$$\inf_{1\le k\le N(u)} \frac{\mathcal{G}_u(k)}{u} \ \ge \ 1,$$

$$\begin{aligned}
\sup_{1\le k\le N(u)} \frac{\mathcal{G}_u(k)}{u} &\le 1+u^{-2}\sup_{1\le k\le N(u)}\inf_{s\in I_k(u)} h(u^{2/\alpha}s)\\
&= 1+u^{-2}\sup_{1\le k\le N(u)}\inf_{s\in[kS,(k+1)S]} h(s)\\
&\le 1+u^{-2}\sup_{s\in[S,N_u+2S]} s^{\epsilon_2}\\
&\le 1+u^{-2}(N_u+2S)^{\epsilon_2}\to 1,\ u\to\infty.
\end{aligned}$$

Consequently, $\lim_{u\to\infty}\sup_{1\le k\le N(u)}\left|\frac{\mathcal{G}_u(k)}{u}-1\right|=0$, and thus we can apply Theorem 9.3.1 which yields

$$\sum_{k=1}^{N(u)}\mathbb{P}\left\{\sup_{t\in I_k(u)} X(t)\sigma_u(t)>u\right\}\le \mathbb{E}\left\{\sup_{t\in[0,S]} e^{W(t)}\right\}\left(\sum_{k=1}^{N(u)}\frac{1}{\sqrt{2\pi}\mathcal{G}_u(k)}\exp\left(-\frac{\mathcal{G}_u^2(k)}{2}\right)\right)(1+o(1)), u\to\infty.$$

Further $\mathcal{G}_u^2(k)\ge u^2+2\left(\inf_{s\in I_k(u)} h(u^{2/\alpha}s)\right)$, and

$$\begin{aligned}
\sum_{k=1}^{N(u)}\exp\left(-\inf_{s\in I_k(u)} h(u^{2/\alpha}s)\right) &= \sum_{k=1}^{N(u)}\exp\left(-\inf_{s\in[kS,(k+1)S]} h(s)\right)\\
&\le \sum_{k=1}^{N(u)}\exp\left(-\inf_{s\in[kS,(k+1)S]} |s|^{\epsilon_1}\right)\\
&\le \sum_{k=1}^{\infty}\exp\left(-|kS|^{\epsilon_1}\right)\\
&\le \mathbb{Q}_1 e^{-\mathbb{Q}_2 S^{\epsilon_1}}.
\end{aligned}$$

Hence for u sufficiently large

$$\begin{aligned}
\sum_{k=1}^{N(u)}\mathbb{P}\left\{\sup_{t\in I_k(u)} X(t)\sigma_u(t)>u\right\} &\le \mathcal{H}_{\alpha,0}([0,S])\frac{1}{\sqrt{2\pi}u}e^{-\frac{u^2}{2}}\sum_{k=1}^{N(u)}\exp\left(-\inf_{s\in I_k(u)} h(u^{2/\alpha}|s|)\right)\\
&\le \mathbb{Q}_1\mathcal{H}_{\alpha,0}\Psi(u)Se^{-\mathbb{Q}_2 S^{\epsilon_1}}.
\end{aligned}\tag{9.19}$$

Inserting (9.18), (9.19) into (9.17) yields

$$\limsup_{u\to\infty}\frac{\pi(u)}{\Psi(u)}\le \mathcal{P}_{\alpha,\delta}^h([0,S])+\mathbb{Q}_1\mathcal{H}_{\alpha,0}Se^{-\mathbb{Q}_2 S^{\epsilon_1}}<\infty,$$

and by (9.16), we have

$$\liminf_{u\to\infty}\frac{\pi(u)}{\Psi(u)}\ge \mathcal{P}_{\alpha,\delta}^h([0,S_1])>0.$$

Letting $S_1\to\infty$, $S\to\infty$ we conclude that $\mathcal{P}_{\alpha,\delta}^h\in(0,\infty)$ and $\pi(u)\sim\mathcal{P}_{\alpha,\delta}^h\Psi(u)$.
Hence the proof is complete.

□

9.4 Appendix

9.4.1 Proof of Proposition 9.1.1

i) For ζ an $N(0,1)$ random variable and any $T>0$, we have

$$\begin{aligned}
\mathcal{P}_{2,0}^{Rt}[0,T] &= \mathbb{E}\left\{\sup_{t\in[0,T]}\exp\left(\sqrt{2}t\zeta-t^2-Rt\right)\right\}\\
&= \mathbb{E}\left\{\exp\left(\frac{(\sqrt{2}\zeta-R)^2}{4}\right)\Big|0<\frac{\sqrt{2}\zeta-R}{2}\le T\right\}\mathbb{P}\left\{0<\frac{\sqrt{2}\zeta-R}{2}\le T\right\}
\end{aligned}$$

$$
\begin{aligned}
& +\mathbb{E}\left\{\exp(0)\Big|\frac{\sqrt{2}\zeta - R}{2} \le 0\right\}\mathbb{P}\left\{\frac{\sqrt{2}\zeta - R}{2} \le 0\right\} \\
& +\mathbb{E}\left\{\exp(\sqrt{2}\zeta T - RT - T^2)\Big|\frac{\sqrt{2}\zeta - R}{2} > T\right\}\mathbb{P}\left\{\frac{\sqrt{2}\zeta - R}{2} > T\right\} \\
=: \;& I_1 + I_2 + I_3.
\end{aligned}
$$

Letting $T \to \infty$, we obtain

$$
\begin{aligned}
\lim_{T\to\infty} I_1 &= \int_{\frac{R}{\sqrt{2}}}^{\infty} \exp\left(\frac{(\sqrt{2}x - R)^2}{4}\right)\exp\left(-\frac{x^2}{2}\right)\frac{1}{\sqrt{2\pi}}dx = \frac{1}{\sqrt{\pi}R}\exp\left(-\frac{R^2}{4}\right), \\
\lim_{T\to\infty} I_2 &= \Phi\left(\frac{R}{\sqrt{2}}\right), \\
\lim_{T\to\infty} I_3 &= \lim_{T\to\infty}\frac{e^{-RT}}{\sqrt{2\pi}}\int_{\frac{R\sqrt{2}}{2}}^{\infty}\exp\left(-\frac{x^2}{2}\right)dx = 0.
\end{aligned}
$$

Hence the claim follows.

ii) First we note that the solution of equation $\sqrt{2}\zeta - 2t - \frac{3}{2}Rt^{\frac{1}{2}} = 0$ for $t \in [0,T]$ is $t = \frac{\sqrt{2}}{2}\zeta + \frac{9}{32}R^2 - \frac{3}{4}R\sqrt{\frac{9}{64}R^2 + \frac{\sqrt{2}}{2}\zeta}$. Define next

$$
F(\zeta, R) := \frac{\zeta^2}{2} + \frac{27}{512}R^4 + \frac{9\sqrt{2}}{32}\zeta R^2 - \frac{\sqrt{2}}{2}R\zeta\sqrt{\frac{9}{64}R^2 + \frac{\sqrt{2}}{2}\zeta} - \frac{9R^3}{64}\sqrt{\frac{9}{64}R^2 + \frac{\sqrt{2}}{2}\zeta}.
$$

Since $0 < t < T$, then $0 < \zeta < \sqrt{2}\left(T + \frac{3}{4}R\sqrt{T}\right)$ implying that

$$
\begin{aligned}
& \mathcal{P}_{2,0}^{Rt^{\frac{3}{2}}}[0,T] \\
= \;& \mathbb{E}\left\{\exp\left(F(\zeta,R)\right)\Big|0 < \zeta < \sqrt{2}(T + \frac{3}{4}R\sqrt{T})\right\}\mathbb{P}\left\{0 < \zeta < \sqrt{2}(T + \frac{3}{4}R\sqrt{T})\right\} + \mathbb{P}\left\{\zeta \le 0\right\} \\
& +\mathbb{E}\left\{\exp\left(\sqrt{2}\zeta T - T^2 - RT^{\frac{3}{2}}\right)\Big|\zeta \ge \sqrt{2}(T + \frac{3}{4}R\sqrt{T})\right\}\mathbb{P}\left\{\zeta \ge \sqrt{2}(T + \frac{3}{4}R\sqrt{T})\right\} \\
=: \;& I_1 + I_2 + I_3.
\end{aligned}
$$

Further, we have

$$
\begin{aligned}
\lim_{T\to\infty} I_1 &= \frac{1}{\sqrt{\pi}}\exp\left(-\frac{27}{1024}R^4\right)\int_0^{\infty}\exp\left(\frac{9}{16}R^2t^2 - Rt^3\right)dt^2, \\
\lim_{T\to\infty} I_2 &= \frac{1}{2}, \\
\lim_{T\to\infty} I_3 &= \lim_{T\to\infty}\frac{1}{\sqrt{2\pi}}e^{-RT^{\frac{3}{2}}}\int_{\sqrt{2}(T+\frac{3}{4}R\sqrt{T})}^{\infty}\exp\left(-\frac{(x-\sqrt{2}T)^2}{2}\right)dx = 0,
\end{aligned}
$$

which establishes the claim.

iii) The function $f(t) = \sqrt{2}t\zeta - t^2 - Rt^3$ attains its maximum at $t^* = \frac{\sqrt{1+3\sqrt{2}R\zeta}-1}{3R}$, with

$$
G(\zeta, R) := f(t^*) = -\frac{2}{27R^2} - \frac{\sqrt{2}\zeta}{3R} + \frac{2\sqrt{1+3\sqrt{2}R\zeta}}{27R^2} + \frac{2\zeta\sqrt{2+6\sqrt{2}R\zeta}}{9R}.
$$

Since $0 < t < T$, then we can consider $0 < \zeta < \frac{3\sqrt{2}RT^2}{2} + \sqrt{2}T$ hence for any $T > 0$

$$
\begin{aligned}
& \mathcal{P}_{2,0}^{Rt^3}[0,T] \\
= \;& \mathbb{E}\left\{\exp\left(G(\zeta,R)\right)\Big|0 < \zeta < \frac{3\sqrt{2}RT^2}{2} + \sqrt{2}T\right\}\mathbb{P}\left\{0 < \zeta < \frac{3\sqrt{2}RT^2}{2} + \sqrt{2}T\right\} + \mathbb{P}\left\{\zeta \le 0\right\} \\
& +\mathbb{E}\left\{\exp\left(\sqrt{2}T\zeta - T^2 - RT^3\right)\Big|\zeta \ge \frac{3\sqrt{2}RT^2}{2} + \sqrt{2}T\right\}\mathbb{P}\left\{\zeta \ge \frac{3\sqrt{2}RT^2}{2} + \sqrt{2}T\right\} \\
=: \;& I_1 + I_2 + I_3.
\end{aligned}
$$

Since further Letting now $T \to \infty$, we have

$$\begin{aligned}
\lim_{T\to\infty} I_1 &= \frac{1}{6R\sqrt{\pi}} \exp\left(\frac{1}{108R^2}\right) \int_0^\infty \exp\left(-\frac{t^4}{36R^2} + \frac{2t^3}{27R^2} - \frac{t^2}{18R^2}\right) dt^2,\\
\lim_{T\to\infty} I_2 &= \frac{1}{2},\\
\lim_{T\to\infty} I_3 &= \lim_{T\to\infty} \frac{1}{\sqrt{2\pi}} e^{-RT^3} \int_{\frac{3\sqrt{2}RT^2}{2}}^\infty \exp\left(-\frac{x^2}{2}\right) dx\\
&= 0,
\end{aligned}$$

which completes the proof.

iv) Let ζ be an $N(0,1)$ random variable. Suppose for a while that $0 < \zeta < \frac{\sqrt{2}}{2} T^{\lambda-1}\lambda R$. Then for $\lambda \in (2,\infty)$ the function $f(t) = \sqrt{2}\zeta t - Rt^\lambda$ attains its maximum at point $t^* = \left(\frac{\sqrt{2}\zeta}{\lambda R}\right)^{\frac{1}{\lambda-1}}$ and $f(t^*) = R^{\frac{1}{1-\lambda}}(\lambda-1)\left(\frac{\sqrt{2}\zeta}{\lambda}\right)^{\frac{\lambda}{\lambda-1}}$ implying

$$\begin{aligned}
\mathcal{P}^h_{2,0}[0,T] &= \mathbb{E}\left\{\exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)\left(\frac{\sqrt{2}\zeta}{\lambda}\right)^{\frac{\lambda}{\lambda-1}}\right) \Big| 0 < \zeta \le \frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R\right\}\\
&\quad \times \mathbb{P}\left\{0 < \zeta \le \frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R\right\} + \mathbb{P}\{\zeta \le 0\}\\
&\quad + \mathbb{E}\left\{\exp\left(\sqrt{2}T\zeta - RT^\lambda\right) \Big| \zeta > \frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R\right\} \mathbb{P}\left\{\zeta > \frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R\right\}\\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R} \exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)\left(\frac{\sqrt{2}x}{\lambda}\right)^{\frac{\lambda}{\lambda-1}} - \frac{x^2}{2}\right) dx\\
&\quad + \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_{\frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R - \sqrt{2}T}^\infty \exp\left(-\frac{x^2}{2} + T^2 - RT^\lambda\right) dx\\
&= \frac{1}{\sqrt{2\pi}} \int_0^{\frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R} \exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)\left(\frac{\sqrt{2}x}{\lambda}\right)^{\frac{\lambda}{\lambda-1}} - \frac{x^2}{2}\right) dx\\
&\quad + \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \exp\left(T^2 - RT^\lambda\right)\left(1 - \Phi\left(\frac{\sqrt{2}}{2}T^{\lambda-1}\lambda R - \sqrt{2}T\right)\right).
\end{aligned}$$

Since

$$\frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)(\frac{\sqrt{2}x}{\lambda})^{\frac{\lambda}{\lambda-1}} - \frac{x^2}{2}\right) dx$$

is finite and $\lim_{T\to\infty} \exp\left(T^2 - RT^\lambda\right) = 0$, then

$$\lim_{T\to\infty} \mathcal{P}^h_{2,0}[0,T] = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp\left(R^{\frac{1}{1-\lambda}}(\lambda-1)(\frac{\sqrt{2}x}{\lambda})^{\frac{\lambda}{\lambda-1}} - \frac{x^2}{2}\right) dx + \frac{1}{2}.$$

Hence the proof is complete. □

9.4.2 Proof of Proposition 9.1.2

The proof is similar to that of Proposition 3.4 in [57], therefore we give only main steps of argumentation. For all $T > 0$, we have

$$\begin{aligned}
\mathcal{P}^{Rt^\lambda}_{\alpha,\delta} \ge \mathcal{P}^{Rt^\lambda}_{\alpha,\delta}([0,T]) &= \mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)-Rt^\lambda}\right\}\\
&\ge \mathbb{E}\left\{\sup_{t\in\delta\mathbb{Z}\cap[0,T]} e^{W(t)-RT^\lambda}\right\}\\
&= \mathcal{H}_{\alpha,\delta}([0,T])e^{-RT^\lambda}.
\end{aligned}$$

Consequently,

$$\begin{aligned}\mathcal{P}_{\alpha,\delta}^{Rt^\lambda} &\geq \sup_{T>0}\left(\frac{\mathcal{H}_{\alpha,\delta}([0,T])}{T}Te^{-RT^\lambda}\right)\\ &\geq \inf_{T>0}\frac{\mathcal{H}_{\alpha,\delta}([0,T])}{T}\sup_{T>0}Te^{-RT^\lambda}\\ &\geq \mathcal{H}_{\alpha,\delta}\sup_{T>0}Te^{-RT^\lambda},\end{aligned}$$

where the last inequality follows from the subadditivity of $\mathcal{H}_{\alpha,\delta}([0,T])$.
Since $\sup_{x>0}xe^{-Rx^\lambda}=(e\lambda R)^{-\frac{1}{\lambda}}$ the proof follows easily. □

9.5 Appendix: Bounds for $\mathcal{P}_{2,0}^{Rt^\lambda}$ and Graphical illustrations

This section is dedicated to the special case when $\alpha=2$, $\delta=0$, and $h(t)=Rt^\lambda$. Although it does not seem possible to have tractable formulas for $\mathcal{P}_{2,0}^{Rt^\lambda}$, nonetheless we derive several upper and lower bounds for $\mathcal{P}_{2,0}^{Rt^\lambda}$.

Theorem 9.5.1. *i) For all $\lambda\in(0,1)$ and $R>0$*

$$\mathcal{P}_{2,0}^{Rt^\lambda}\geq\Phi\left(\frac{1+R}{\sqrt{2}}\right)+e^{-R}\left[\Phi\left(\frac{\sqrt{2}}{2}R\right)-\Phi\left(\frac{R-1}{\sqrt{2}}\right)\right]+\frac{1}{R\sqrt{\pi}}e^{-\frac{R^2}{4}-R}.$$

ii) For all $\lambda\in(1,2)$ and $R>0$

$$\begin{aligned}\mathcal{P}_{2,0}^{Rt^\lambda} &\geq \frac{1}{\sqrt{2\pi}}\int_0^{\frac{(1+R)\lambda}{\sqrt{2}}}e^{A(x)}dx+\frac{1}{2}+e^{-R}\left[\Phi\left(\sqrt{2}R\right)-\Phi\left(\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}\right)\right]\\ &\quad+\sqrt{\frac{R+1}{R}}\left[1-\Phi\left(\sqrt{2R(1+R)}\right)\right],\\ \mathcal{P}_{2,0}^{Rt^\lambda} &\leq \frac{1}{2}+\sqrt{\frac{1+R}{R}}\left[\Phi\left(\frac{\sqrt{R(1+R)}+\sqrt{R}}{\sqrt{2}}\right)-\frac{1}{2}\right]+\frac{1}{R\sqrt{\pi}}e^{-\frac{R}{2}(1+\frac{1}{2}R+\sqrt{1+R})},\end{aligned}$$

where $A(x)=((1+R)\lambda)^{\frac{1}{1-\lambda}}\left(1-\frac{1}{\lambda}\right)(\sqrt{2}x)^{\frac{\lambda}{\lambda-1}}-\frac{x^2}{2}$.
iii) For all $\lambda\in(2,\infty)$ and $R>0$

$$\begin{aligned}\mathcal{P}_{2,0}^{Rt^\lambda} &\geq \frac{1}{2}+\sqrt{\frac{1+R}{R}}\left(\Phi\left(\sqrt{2R(1+R)}\right)-\frac{1}{2}\right)+e^{-R}\left[\Phi\left(\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}\right)-\Phi\left(\sqrt{2}R\right)\right]\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_{\frac{(1+R)\lambda}{\sqrt{2}}}^{\infty}e^{A(x)}dx,\\ \mathcal{P}_{2,0}^{Rt^\lambda} &\leq \frac{1}{2}+\sqrt{\frac{1+R}{R}}\left[1-\Phi\left(2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}\sqrt{R(1+R)}\right)\right]\\ &\quad+\frac{1}{\sqrt{2\pi}}\int_0^{(1+R)2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}}e^{A(x)}dx.\end{aligned}$$

Proof of Theorem 9.5.1: Recall that

$$\mathcal{P}_{2,0}^{Rt^\lambda}([0,T])=\mathbb{E}\left\{\sup_{t\in[0,T]}e^{\sqrt{2}B_2(t)-t^2-Rt^\lambda}\right\}\to\mathcal{P}_{2,0}^{Rt^\lambda},\quad T\to\infty.$$

For any $u>0,T>1$ and ζ an $N(0,1)$ random variable, we have that $\sup_{t\in[0,T]}\left(\sqrt{2}B_2(t)-t^2-Rt^\lambda\right)$ has the same distribution as

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right)=\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right)\right).$$

i) If $0<\lambda<1$ and $T>1$, then

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right)\geq \max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t-Rt^\lambda\right), \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt\right)\right).$$

For $m(t)=\sqrt{2}\zeta t-t-Rt^\lambda$, we have $m'(t)=\sqrt{2}\zeta-\lambda Rt^{\lambda-1}-1=0$. Hence $m(t)$ decreases in $(-\infty,t_1]$ and increases in (t_1,∞) with

$$t_1=\left(\frac{\sqrt{2}\zeta-1}{R\lambda}\right)^{\frac{1}{\lambda-1}}.$$

It follows that $h(t)=\sqrt{2}\zeta t-t^2-Rt$ has a unique maximizer $t_2=\frac{\sqrt{2}\zeta-R}{2}$ on $[0,T]$. Further, for $T>1$

$$\begin{aligned}
&\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t-Rt^\lambda\right), \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt\right)\right)\\
&=\begin{cases} 0, & \text{if } \zeta\in(-\infty,\frac{1+R}{\sqrt{2}}],\\ \sqrt{2}\zeta-(1+R), & \text{if } \zeta\in(\frac{1+R}{\sqrt{2}},\sqrt{2}+\frac{\sqrt{2}}{2}R],\\ \frac{(\sqrt{2}\zeta-R)^2}{4}, & \text{if } \zeta\in(\sqrt{2}+\frac{\sqrt{2}}{2}R,\sqrt{2}(T+\frac{R}{2})],\\ \sqrt{2}\zeta T-RT-T^2, & \text{if } \zeta\in(\sqrt{2}(T+\frac{R}{2}),\infty)\end{cases}
\end{aligned}$$

implying that for any $T>1$

$$\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \geq \mathbb{E}\left\{e^{\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t-Rt^\lambda\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt\right)\right)}\right\}=: I_{11}+I_{12}+I_{13}+I_{14},$$

where

$$\begin{aligned}
I_{11} &= \mathbb{E}\left\{\exp(0);\zeta\leq\frac{1+R}{\sqrt{2}}\right\}=\Phi\left(\frac{1+R}{\sqrt{2}}\right),\\
I_{12} &= \mathbb{E}\left\{\exp\left(\sqrt{2}\zeta-(1+R)\right);\frac{1+R}{\sqrt{2}}<\zeta\leq\sqrt{2}+\frac{\sqrt{2}}{2}R\right\}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{1+R}{\sqrt{2}}}^{\sqrt{2}+\frac{\sqrt{2}}{2}R} e^{-R}\exp\left(-\frac{(x-\sqrt{2})^2}{2}\right)dx=e^{-R}\left[\Phi\left(\frac{\sqrt{2}}{2}R\right)-\Phi\left(\frac{R-1}{\sqrt{2}}\right)\right],\\
I_{13} &= \mathbb{E}\left\{\exp\left(\frac{(\sqrt{2}\zeta-R)^2}{4}\right);\sqrt{2}+\frac{\sqrt{2}}{2}R<\zeta\leq\sqrt{2}(T+\frac{R}{2})\right\}\\
&= \frac{1}{\sqrt{\pi}R}e^{\frac{R^2}{4}}\left[e^{-(\frac{R^2}{2}+R)}-e^{-(T+\frac{R}{2})R}\right],\\
I_{14} &= \mathbb{E}\left\{\exp(\sqrt{2}\zeta T-RT-T^2);\sqrt{2}(T+\frac{R}{2})<\zeta\right\}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}(T+\frac{R}{2})}^{\infty}\exp\left(-\frac{1}{2}(x-\sqrt{2}T)^2-RT\right)dx=e^{-RT}\left(1-\Phi\left(\frac{\sqrt{2}}{2}R\right)\right).
\end{aligned}$$

Therefore, for any $0<\lambda<1$ and $T>1$, we get

$$\begin{aligned}
\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \geq\ & \Phi\left(\frac{1+R}{\sqrt{2}}\right)+e^{-R}\left[\Phi\left(\frac{\sqrt{2}}{2}R\right)-\Phi\left(\frac{R-1}{\sqrt{2}}\right)\right]+\frac{1}{R\sqrt{\pi}}\left(e^{-\frac{R^2}{4}-R}-e^{-RT-\frac{R^2}{4}}\right)\\
&+e^{-RT}\left(1-\Phi\left(\frac{\sqrt{2}}{2}R\right)\right).
\end{aligned}$$

Letting $T\to\infty$ establishes the proof.

ii) If $1<\lambda<2$ and $T>1$, then we make use of the following bounds

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right)\geq \max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right), \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right),$$

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right)\le\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt\right)\right).$$

First, we calculate the lower bound of $\mathcal{P}_{2,0}^{Rt^\lambda}$ for $1<\lambda<2,\ R>0$.
Set $f(t)=\sqrt{2}\zeta t-t^\lambda-Rt^\lambda, g(t)=\sqrt{2}\zeta t-t^2-Rt^2$. Since

$$f'(t)=\sqrt{2}\zeta-\lambda(1+R)t^{\lambda-1}=0,\quad g'(t)=\sqrt{2}\zeta-2(1+R)t=0,$$

then $f'(t_3)=g'(t_4)=0$ for $t_3=\left[\frac{\sqrt{2}\zeta}{(1+R)\lambda}\right]^{\frac{1}{\lambda-1}}$ and $t_4=\frac{\sqrt{2}\zeta}{2(1+R)}$, respectively. Moreover, $f(t)$ increases in $(-\infty,t_3]$ and decreases in (t_3,∞), implying that t_3 is the unique maximizer of $f(\cdot)$ over $\mathbb{R}$. Similarly, t_4 is the unique maximizer of $g(\cdot)$ over $\mathbb{R}$. Hence

$$\sup_{t\in[0,1]}f(t)=\begin{cases}0, & \text{if } \zeta\in(-\infty,0],\\ D(\zeta), & \text{if } \zeta\in(0,\frac{(1+R)\lambda}{\sqrt{2}}],\\ \sqrt{2}\zeta-(1+R), & \text{if } \zeta\in(\frac{(1+R)\lambda}{\sqrt{2}},\infty),\end{cases}$$

where $D(\zeta):=[(1+R)\lambda]^{\frac{1}{1-\lambda}}\left(1-\frac{1}{\lambda}\right)[\sqrt{2}\zeta]^{\frac{\lambda}{\lambda-1}}$ and

$$\sup_{t\in[1,T]}g(t)=\begin{cases}\sqrt{2}\zeta-(1+R), & \text{if } \zeta\in(-\infty,\sqrt{2}(1+R)],\\ \frac{\zeta^2}{2(1+R)}, & \text{if } \zeta\in(\sqrt{2}(1+R),\sqrt{2}(1+R)T],\\ \sqrt{2}\zeta T-(1+R)T^2, & \text{if } \zeta\in(\sqrt{2}(1+R)T,\infty).\end{cases}$$

Further, for $T>1$

$$\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right)$$
$$=\begin{cases}0, & \text{if } \zeta\in(-\infty,0],\\ D(\zeta), & \text{if } \zeta\in(0,\frac{(1+R)\lambda}{\sqrt{2}}],\\ \sqrt{2}\zeta-(1+R), & \text{if } \zeta\in(\frac{(1+R)\lambda}{\sqrt{2}},\sqrt{2}(1+R)],\\ \frac{\zeta^2}{2(1+R)}, & \text{if } \zeta\in(\sqrt{2}(1+R),\sqrt{2}(1+R)T],\\ \sqrt{2}\zeta T-T^2-RT^2, & \text{if } \zeta\in(\sqrt{2}(1+R)T,\infty).\end{cases}$$

Consequently,

$$\begin{aligned}\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) &\ge \mathbb{E}\left\{\exp\left(\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right)\right)\right\}\\ &=: I_{21}+I_{22}+I_{23}+I_{24}+I_{25},\end{aligned}$$

where

$$\begin{aligned}I_{21} &= \mathbb{E}\left\{\exp(0);\zeta\le 0\right\}=\frac{1}{2},\\ I_{22} &= \mathbb{E}\left\{e^{D(\zeta)};0<\zeta\le\frac{(1+R)\lambda}{\sqrt{2}}\right\}=\frac{1}{\sqrt{2\pi}}\int_0^{\frac{(1+R)\lambda}{\sqrt{2}}}e^{A(x)}dx,\\ I_{23} &= \mathbb{E}\left\{\exp(\sqrt{2}\zeta-(1+R));\frac{(1+R)\lambda}{\sqrt{2}}<\zeta\le\sqrt{2}(1+R)\right\}\\ &= \frac{1}{\sqrt{2\pi}}e^{-R}\int_{\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}}^{\sqrt{2}R}e^{-\frac{x^2}{2}}dx=e^{-R}\left[\Phi\left(\sqrt{2}R\right)-\Phi\left(\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}\right)\right],\\ I_{24} &= \mathbb{E}\left\{\exp\left(\frac{\zeta^2}{2(1+R)}\right);\sqrt{2}(1+R)<\zeta\le\sqrt{2}(1+R)T\right\}\\ &= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}(1+R)}^{\sqrt{2}(1+R)T}\exp\left(-\frac{R}{2(1+R)}x^2\right)dx\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\frac{R+1}{R}}\left[\Phi\left(\sqrt{2R(1+R)T}\right) - \Phi\left(\sqrt{2R(1+R)}\right)\right],\\
I_{25} &= \mathbb{E}\left\{\exp(\sqrt{2}\zeta T - T^2 - RT^2); \zeta > \sqrt{2}(1+R)T\right\}\\
&= \frac{1}{\sqrt{2\pi}} e^{-RT^2}\int_{\sqrt{2}(1+R)T}^{\infty} \exp\left(-\frac{1}{2}(x-\sqrt{2}T)^2\right) dx = e^{-RT^2}\left[1-\Phi\left(\sqrt{2}RT\right)\right].
\end{aligned}
$$

It follows that for $T > 1$

$$
\begin{aligned}
\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \ \geq\ & \frac{1}{\sqrt{2\pi}}\int_0^{\frac{(1+R)\lambda}{\sqrt{2}}} e^{A(x)}dx + \frac{1}{2} + e^{-R}\left[\Phi\left(\sqrt{2}R\right) - \Phi\left(\frac{(1+R)\lambda}{\sqrt{2}} - \sqrt{2}\right)\right]\\
&+\sqrt{\frac{R+1}{R}}\left[\Phi\left(\sqrt{2R(1+R)T}\right) - \Phi\left(\sqrt{2R(1+R)}\right)\right] + e^{-RT^2}\left[1-\Phi\left(\sqrt{2}RT\right)\right].
\end{aligned}
$$

Hence the lower bound is obtained by letting $T \to \infty$.
Similarly, we obtain for any $T > 1$

$$
\begin{aligned}
&\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t - t^2 - Rt^2\right), \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t - t^2 - Rt\right)\right)\\
&= \begin{cases}
0, & \text{if } \zeta \in (-\infty, 0],\\
\frac{\zeta^2}{2(1+R)}, & \text{if } \zeta \in (0, \frac{1+R+\sqrt{1+R}}{\sqrt{2}}],\\
\frac{(\sqrt{2}\zeta - R)^2}{4}, & \text{if } \zeta \in (\frac{1+R+\sqrt{1+R}}{\sqrt{2}}, \sqrt{2}(T+\frac{R}{2})],\\
\sqrt{2}\zeta T - T^2 - RT, & \text{if } \zeta \in (\sqrt{2}(T+\frac{R}{2}), \infty),
\end{cases}
\end{aligned}
$$

implying that

$$
\begin{aligned}
\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \ &\leq\ \mathbb{E}\left\{e^{\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t - t^2 - Rt^2\right),\ \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t - t^2 - Rt\right)\right)}\right\}\\
&=: I_{31} + I_{32} + I_{33} + I_{34},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{31} &= \mathbb{E}\left\{\exp(0); \zeta \leq 0\right\},\\
I_{32} &= \mathbb{E}\left\{\exp\left(\frac{\zeta^2}{2(1+R)}\right); 0 < \zeta \leq \frac{1+R+\sqrt{1+R}}{\sqrt{2}}\right\}\\
&= \frac{1}{\sqrt{2\pi}}\sqrt{\frac{1+R}{R}}\int_0^{\frac{\sqrt{R(1+R)}+\sqrt{R}}{\sqrt{2}}} \exp\left(-\frac{x^2}{2}\right)dx = \sqrt{\frac{1+R}{R}}\left[\Phi\left(\frac{\sqrt{R(1+R)}+\sqrt{R}}{\sqrt{2}}\right) - \frac{1}{2}\right],\\
I_{33} &= \mathbb{E}\left\{\exp\left(\frac{(\sqrt{2}\zeta - R)^2}{4}\right); \frac{1+R+\sqrt{1+R}}{\sqrt{2}} < \zeta \leq \sqrt{2}(T+\frac{R}{2})\right\}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\frac{1+R+\sqrt{1+R}}{\sqrt{2}}}^{\sqrt{2}(T+\frac{R}{2})} \exp\left(-\frac{\sqrt{2}}{2}Rx + \frac{R^2}{4}\right)dx = \frac{e^{\frac{R^2}{4}}}{R\sqrt{\pi}}\left[e^{-\frac{R}{2}(1+R+\sqrt{1+R})} - e^{-R(T+\frac{R}{2})}\right],\\
I_{34} &= \mathbb{E}\left\{\exp(\sqrt{2}\zeta T - T^2 - RT); \sqrt{2}(T+\frac{R}{2}) < \zeta\right\}\\
&= \frac{1}{\sqrt{2\pi}}\int_{\sqrt{2}(T+\frac{R}{2})}^{\infty} \exp\left(\sqrt{2}xT - T^2 - RT - \frac{x^2}{2}\right)dx\\
&= \frac{e^{-RT}}{\sqrt{2\pi}}\int_{\frac{R\sqrt{2}}{2}}^{\infty} \exp\left(-\frac{x^2}{2}\right)dx = e^{-RT}\left[1-\Phi\left(\frac{R\sqrt{2}}{2}\right)\right].
\end{aligned}
$$

Hence, we get

$$
\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \ \leq\ \frac{1}{2} + \sqrt{\frac{1+R}{R}}\left[\Phi\left(\frac{\sqrt{R(1+R)}+\sqrt{R}}{\sqrt{2}}\right) - \frac{1}{2}\right]
$$

$$+\frac{1}{R\sqrt{\pi}}\left[e^{-\frac{R}{2}(1+\frac{1}{2}R+\sqrt{1+R})}-e^{-R(T+\frac{R}{4})}\right]+e^{-RT}\left[1-\Phi\left(\frac{R\sqrt{2}}{2}\right)\right].$$

The proof is established by letting $T\to\infty$.

iii) If $\lambda>2$ and $T>1$, then we shall use the following bounds

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right) \geq \max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right)\right),$$

$$\sup_{t\in[0,T]}\left(\sqrt{2}\zeta t-t^2-Rt^\lambda\right) \leq \max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right).$$

It is straightforward to check that

$$\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right)\right)$$
$$=\begin{cases}0, & \text{if } \zeta\in(-\infty,0],\\ \frac{\zeta^2}{2(1+R)}, & \text{if } \zeta\in(0,\sqrt{2}(1+R)],\\ \sqrt{2}\zeta-(1+R), & \text{if } \zeta\in(\sqrt{2}(1+R),\frac{(1+R)\lambda}{\sqrt{2}}],\\ D(\zeta), & \text{if } \zeta\in(\frac{(1+R)\lambda}{\sqrt{2}},\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}],\\ \sqrt{2}\zeta T-T^\lambda-RT^\lambda, & \text{if } \zeta\in(\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}},\infty),\end{cases}$$

where we put $D(\zeta):=[(1+R)\lambda]^{\frac{1}{1-\lambda}}\left(1-\frac{1}{\lambda}\right)[\sqrt{2}\zeta]^{\frac{\lambda}{\lambda-1}}$. Consequently,

$$\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \geq \mathbb{E}\left\{e^{\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right),\sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right)\right)}\right\}$$
$$=: I_{41}+I_{42}+I_{43}+I_{44}+I_{45},$$

where

$$I_{41} = \mathbb{E}\{\exp(0);\zeta\leq 0\}=\frac{1}{2},$$
$$I_{42} = \mathbb{E}\left\{\exp\left(\frac{\zeta^2}{2(1+R)}\right);0<\zeta\leq\sqrt{2}(1+R)\right\}$$
$$= \frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}(1+R)}\exp\left(-\frac{Rx^2}{2(1+R)}\right)dx=\sqrt{\frac{1+R}{R}}\left(\Phi\left(\sqrt{2R(1+R)}\right)-\frac{1}{2}\right),$$
$$I_{43} = \mathbb{E}\left\{\exp(\sqrt{2}\zeta-(1+R));\sqrt{2}(1+R)<\zeta\leq\frac{(1+R)\lambda}{\sqrt{2}}\right\}$$
$$= \frac{1}{\sqrt{2\pi}}e^{-R}\int_{\sqrt{2}(1+R)}^{\frac{(1+R)\lambda}{\sqrt{2}}}\exp\left(-\frac{1}{2}(x-\sqrt{2})^2\right)dx$$
$$= \frac{1}{\sqrt{2\pi}}e^{-R}\int_{\sqrt{2}R}^{\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}}e^{-\frac{x^2}{2}}dx=e^{-R}\left[\Phi\left(\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}\right)-\Phi\left(\sqrt{2}R\right)\right],$$
$$I_{44} = \mathbb{E}\left\{e^{D(\zeta)};\frac{(1+R)\lambda}{\sqrt{2}}<\zeta\leq\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}\right\},$$
$$I_{45} = \mathbb{E}\left\{\exp(\sqrt{2}\zeta T-T^\lambda-RT^\lambda);\zeta>\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}\right\}$$
$$= \frac{1}{\sqrt{2\pi}}e^{T^2-T^\lambda-RT^\lambda}\int_{\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}}^{\infty}\exp\left(-\frac{1}{2}(x-\sqrt{2}T)^2\right)dx$$
$$= e^{T^2-T^\lambda-RT^\lambda}\left[1-\Phi\left(\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}-\sqrt{2}T\right)\right].$$

It follows that

$$\begin{aligned}\mathcal{P}_{2,0}^{Rt^\lambda}([0,T]) \;\geq\;& \frac{1}{2}+\sqrt{\frac{1+R}{R}}\left(\Phi\left(\sqrt{2R(1+R)}\right)-\frac{1}{2}\right)+e^{-R}\left[\Phi\left(\frac{(1+R)\lambda}{\sqrt{2}}-\sqrt{2}\right)-\Phi\left(\sqrt{2}R\right)\right]\\ &+\frac{1}{\sqrt{2\pi}}\int_{\frac{(1+R)\lambda}{\sqrt{2}}}^{\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}} e^{A(x)}dx+e^{T^2-T^\lambda-RT^\lambda}\left[1-\Phi\left(\frac{(1+R)\lambda T^{\lambda-1}}{\sqrt{2}}-\sqrt{2}T\right)\right].\end{aligned}$$

Hence the lower bound is derived by letting $T\to\infty$.
Next, we get that

$$\begin{aligned}&\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right),\ \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right)\\ &=\begin{cases}0, & \text{if } \zeta\in(-\infty,0],\\ D(\zeta), & \text{if } \zeta\in(0,B],\\ \frac{\zeta^2}{2(1+R)}, & \text{if } \zeta\in(B,\sqrt{2}(1+R)T],\\ \sqrt{2}\zeta T-T^2-RT^2, & \text{if } \zeta\in(\sqrt{2}(1+R)T,\infty),\end{cases}\end{aligned}$$

$$D(\zeta):=((1+R)\lambda)^{\frac{1}{1-\lambda}}\left(1-\frac{1}{\lambda}\right)(\sqrt{2}\zeta)^{\frac{\lambda}{\lambda-1}},\quad B:=(1+R)2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}.$$

Consequently,

$$\mathcal{P}_{2,0}^{Rt^\lambda}([0,T])\leq\mathbb{E}\left\{e^{\max\left(\sup_{t\in[0,1]}\left(\sqrt{2}\zeta t-t^\lambda-Rt^\lambda\right),\ \sup_{t\in[1,T]}\left(\sqrt{2}\zeta t-t^2-Rt^2\right)\right)}\right\}=:I_{51}+I_{52}+I_{53}+I_{54},$$

where

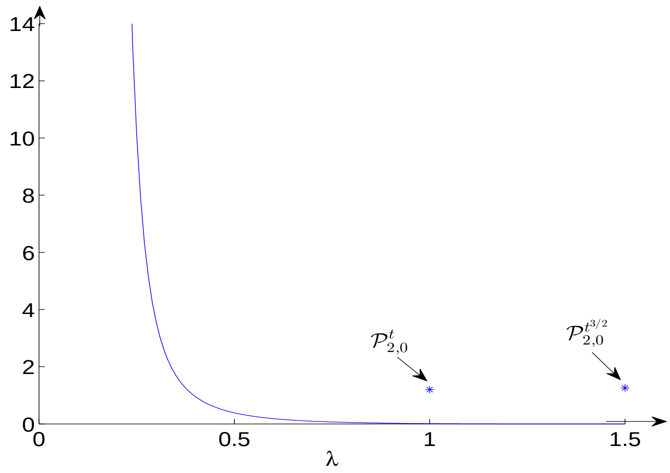
$$\begin{aligned}I_{51}&=\mathbb{E}\{\exp(0);\zeta\leq 0\}=\frac{1}{2},\\ I_{52}&=\mathbb{E}\left\{e^{D(\zeta)};0<\zeta\leq B\right\},\\ I_{53}&=\mathbb{E}\left\{\exp\left(\frac{\zeta^2}{2(1+R)}\right);B<\zeta\leq\sqrt{2}(1+R)T\right\}\\ &=\frac{1}{\sqrt{2\pi}}\sqrt{\frac{1+R}{R}}\int_{2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}\sqrt{R(1+R)}}^{\sqrt{2R(1+R)}T}\exp\left(-\frac{x^2}{2}\right)dx\\ &=\sqrt{\frac{1+R}{R}}\left[\Phi\left(\sqrt{2R(1+R)}T\right)-\Phi\left(2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}\sqrt{R(1+R)}\right)\right],\\ I_{54}&=\mathbb{E}\left\{\exp(\sqrt{2}\zeta T-T^2-RT^2);\zeta>\sqrt{2}(1+R)T\right\}\\ &=\frac{1}{\sqrt{2\pi}}e^{-RT^2}\int_{\sqrt{2}RT}^{\infty}\exp\left(-\frac{x^2}{2}\right)dx=e^{-RT^2}\left[1-\Phi\left(\sqrt{2}RT\right)\right].\end{aligned}$$

Hence

$$\begin{aligned}\mathcal{P}_{2,0}^{Rt^\lambda}([0,T])\;\leq\;&\frac{1}{2}+\sqrt{\frac{1+R}{R}}\left[\Phi\left(\sqrt{2R(1+R)}T\right)-\Phi\left(2^{\frac{3\lambda-2}{2(\lambda-2)}}\lambda^{\frac{\lambda}{2-\lambda}}(\lambda-1)^{\frac{\lambda-1}{\lambda-2}}\sqrt{R(1+R)}\right)\right]\\ &+\frac{1}{\sqrt{2\pi}}\int_0^B e^{A(x)}dx+e^{-RT^2}\left[1-\Phi\left(\sqrt{2}RT\right)\right]\end{aligned}$$

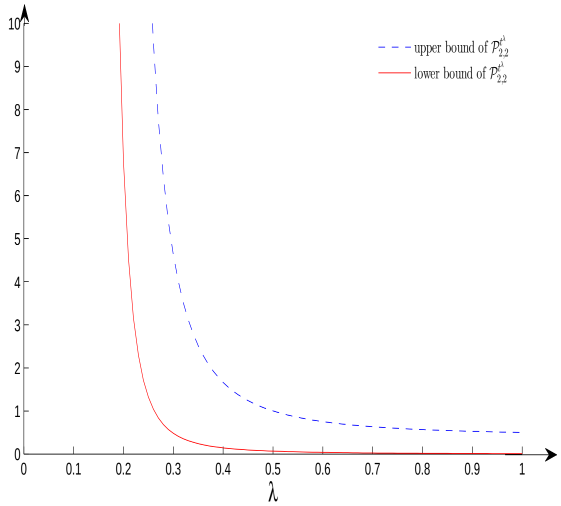
and thus the proof follows by letting $T\to\infty$. □

We conclude this section with some graphical illustrations of bounds obtained in Proposition 9.2.2 and Proposition 9.2.3.

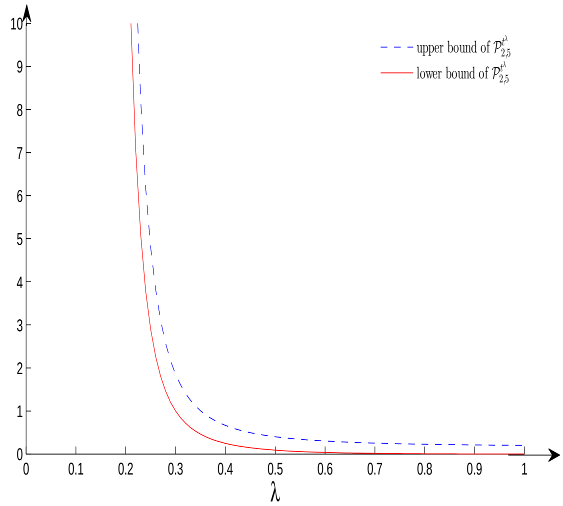


(a)

In Fig.(a) we plot a lower bound of $\mathcal{P}_{2,0}^{t^\lambda}$ for $\lambda \in (0, 1]$ according to the case $\delta = 0$, Proposition 9.2.2 ii). The exact values of $\mathcal{P}_{2,0}^{t}$, $\mathcal{P}_{2,0}^{t^{3/2}}$ are taken from Proposition 9.1.1 i), ii). It follows that $\mathcal{P}_{2,0}^{t^\lambda}$ tends to infinity when λ tends to zero.

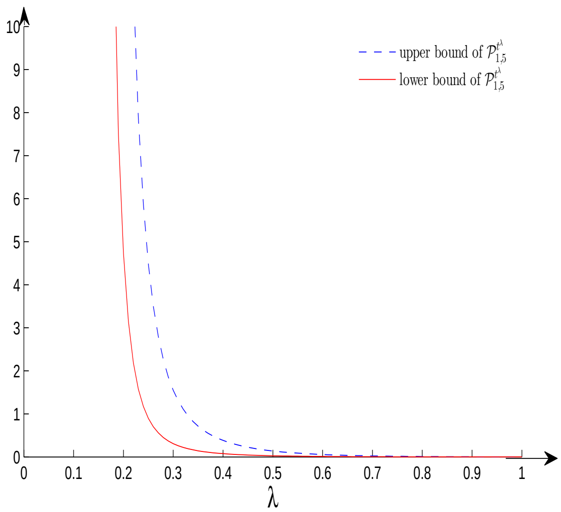


(b)

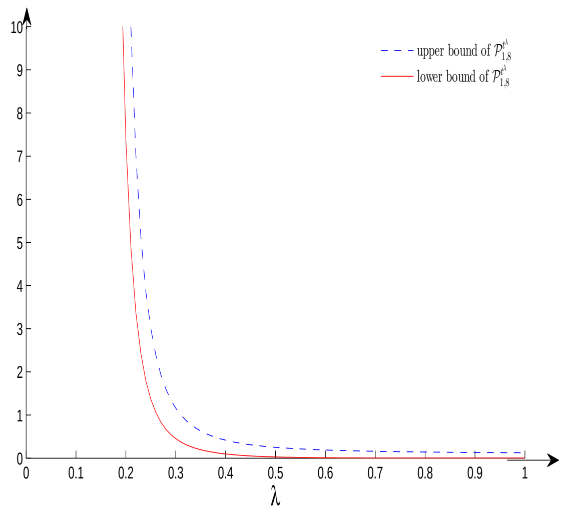


(c)

In Fig.(b) and Fig.(c), following Proposition 9.2.2, we give the upper and lower bounds of $\mathcal{P}_{2,2}^{t^\lambda}$ and $\mathcal{P}_{2,5}^{t^\lambda}$ respectively.

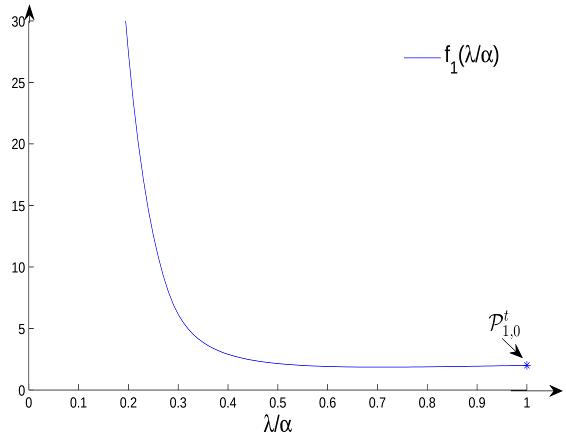


(d)

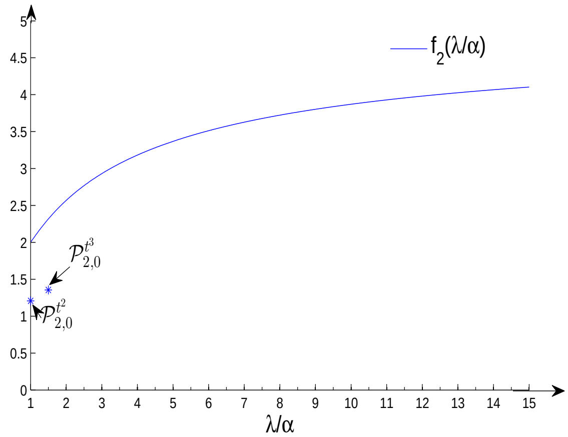


(e)

In Fig.(d) and Fig.(e) we give the upper and lower bounds of $\mathcal{P}_{1,5}^{t^\lambda}$ and $\mathcal{P}_{1,8}^{t^\lambda}$, respectively.



(f)



(g)

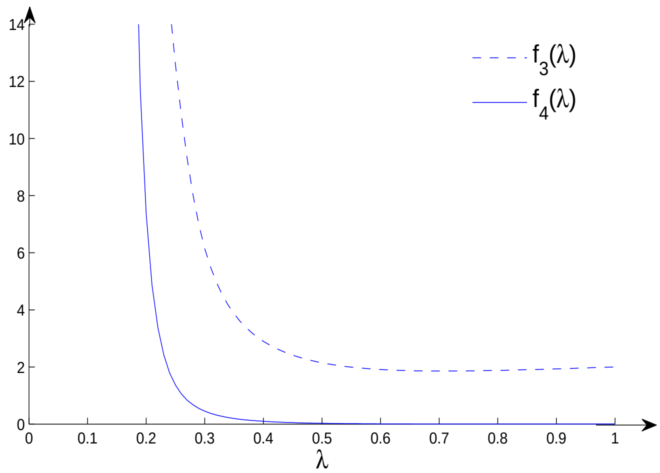
Let

$$f_1(\lambda/\alpha) = 1 + \max_{y\geq 0}\left(\frac{\alpha}{\lambda}y^{1-\lambda/\alpha}e^{(\lambda/\alpha-1)y^{\lambda/\alpha}(1+\lambda/\alpha y^{\lambda/\alpha-1})}\right)$$

and

$$f_2(\lambda/\alpha) = \min_{y\geq 0}\left(\left(1+\frac{\alpha}{\lambda}y^{1-\lambda/\alpha}\right)e^{(\lambda/\alpha-1)y^{\lambda/\alpha}}\right).$$

In Fig.(f) we give a lower bound of $\mathcal{P}^{t^\lambda}_{\alpha,0}$ according to Proposition 9.2.3 i) and exact value of $\mathcal{P}^{t}_{1,0}$ for $0<\lambda\leq\alpha\leq 1$. The lower bound of $\mathcal{P}^{t^\lambda}_{\alpha,0}$ tends to infinity when λ/α goes to 0 and is decreasing when λ/α goes to 1. In Fig.(g) we give an upper bound of $\mathcal{P}^{t^\lambda}_{\alpha,0}$ according to Proposition 9.2.3 ii) and exact value of $\mathcal{P}^{t^2}_{2,0}$, $\mathcal{P}^{t^3}_{2,0}$ for $\lambda\geq\alpha\geq 1$. The upper bound of $\mathcal{P}^{t^\lambda}_{\alpha,0}$ is increasing when λ/α becomes large.



(h)

In Fig.(h) we compare $f_3(\lambda) = 1+\max_{y\geq 0}\left(\frac{1}{\lambda}y^{1-\lambda}e^{(\lambda-1)y^\lambda(1+\lambda y^{\lambda-1})}\right)$ which is the lower bound of $\mathcal{P}^{t^\lambda}_{1,0}$ from Proposition 9.2.3 i) and $f_4(\lambda) = \frac{1}{16}\int_8^\infty e^{-x^\lambda}dx$ which is the lower bound of $\mathcal{P}^{t^\lambda}_{1,0}$ from Proposition 9.2.2 iii). The lower bound given by Proposition 9.2.3 i) is more precise, while Proposition 9.2.2 iii) holds for general h. Both lower bounds go to infinity when λ goes to 0 and are decreasing when λ goes to 1.

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