EXTREMES OF THRESHOLD-DEPENDENT GAUSSIAN PROCESSES

LONG BAI, KRZYSZTOF DĘBICKI, ENKELEJD HASHORVA, AND LANPENG JI

Abstract: In this contribution we are concerned with the asymptotic behaviour, as $u \to \infty$, of $\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > u\right\}$, where $X_u(t), t \in [0,T], u > 0$ is a family of centered Gaussian processes with continuous trajectories. A key application of our findings concerns $\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t)) > u\right\}$, as $u \to \infty$, for X a centered Gaussian process and g some measurable trend function. Further applications include the approximation of both the ruin time and the ruin probability of the Brownian motion risk model with constant force of interest.

Key Words: Extremes; Gaussian processes; fractional Brownian motion; ruin probability; ruin time.

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1. Introduction

Let $X(t), t \ge 0$ be a centered Gaussian process with continuous trajectories. An important problem in applied and theoretical probability is the determination of the asymptotic behavior of

$$p(u) = \mathbb{P}\left\{\sup_{t\in[0,T]} (X(t)+g(t)) > u\right\}, \quad u \to \infty \tag{1}$$

for some $T > 0$ and $g(t), t \in [0,T]$ a bounded measurable function. For instance, if $g(t) = -ct$, then in the context of risk theory $p(u)$ has interpretation as the ruin probability over the finite-time horizon $[0,T]$. Dually, in the context of queueing theory, $p(u)$ is related to the buffer overload problem; see e.g., [1–5].

For the special case that $g(t) = 0, t \in [0,T]$ the exact asymptotics of (1) is well-known for both locally stationary and general non-stationary Gaussian processes, see e.g., [6–18]. Commonly, for X a centered non-stationary Gaussian process it is assumed that the standard deviation function σ is such that $t_0 = argmax_{t\in[0,T]}\sigma(t)$ is unique and $\sigma(t_0) = 1$. Additionally, if the correlation function r and the standard deviation function σ satisfy (hereafter $\sim$ means asymptotic equivalence)

$$1 - r(s,t) \sim a\,|t-s|^{\alpha}, \qquad 1 - \sigma(t_0+t) \sim b\,|t|^{\beta}, \qquad s,t \to t_0 \tag{2}$$

for some a, b, β positive and $\alpha \in (0,2]$, then we have (see [10][Theorem D.3])

$$p(u) \sim C_0 u^{(\frac{2}{\alpha}-\frac{2}{\beta})_+}\mathbb{P}\{X(t_0) > u\}, \quad u \to \infty, \tag{3}$$

where $(x)_+ = \max(0,x)$ and

$$C_0 = \begin{cases} a^{1/\alpha} b^{-1/\beta}\Gamma(1/\beta+1)\mathcal{H}_\alpha, & \text{if } \alpha < \beta, \\ \mathcal{P}_\alpha^{b/a}, & \text{if } \alpha = \beta, \\ 1, & \text{if } \alpha > \beta. \end{cases}$$

Here $\Gamma(\cdot)$ is the gamma function, and

$$\mathcal{H}_\alpha = \lim_{T\to\infty}\frac{1}{T}\mathbb{E}\left\{\sup_{t\in[0,T]} e^{W(t)}\right\}, \quad \mathcal{P}_\alpha^{b/a} = \mathbb{E}\left\{\sup_{t\in[0,\infty)} e^{W(t)-b/a|t|^{\alpha}}\right\}, \text{ with } W(t) = \sqrt{2}B_\alpha(t) - |t|^{\alpha},$$

are the Pickands and Piterbarg constants, respectively, where B_α is a standard fractional Brownian motion (fBm) with self-similarity index $\alpha/2 \in (0,1]$, see [19–25] for properties of both constants.

The more general case with non-zero g has also been considered in the literature; see, e.g., [1, 26–30]. However, most of the aforementioned contributions treat only restrictive trend functions g. For instance, in [26][Theorem 3] a Hölder-type condition for g is assumed, which excludes important cases of g that appear in applications. The restrictions are

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often so severe that simple cases such as the Brownian bridge with drift considered in Example 3.11 below cannot be covered.

A key difficulty when dealing with $p(u)$ is that $X+g$ is not a centered Gaussian process. It is however possible to get rid of the trend function g since for any bounded function g and all u large (1) can be re-written as

$$p_T(u)=\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t)>u\right\},\quad X_u(t)=\frac{X(t)}{1-g(t)/u},\quad t\in[0,T].$$

Here X_u is centered, however it depends on the threshold u, which complicates the analysis.

Extremes of threshold-dependent Gaussian processes $X_u(t), t\in\mathbb{R}$ have been already dealt with in several contributions, see e.g., [2, 3, 30–32]. Our principal result in Theorem 2.4 derives the asymptotics of $p_T(u)$ for quite general families of centered Gaussian processes X_u under tractable assumptions on the variance and correlation functions of X_u. To this end, in Theorem 2.2 we first derive the asymptotics of

$$p_\Delta(u)=\mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t)>u\right\},\quad u\to\infty$$

for some short compact intervals $\Delta(u)$.

Applications of our main results include derivation of Proposition 3.1 for a class of locally stationary Gaussian processes with trend and that of Proposition 3.6 for a class of non-stationary Gaussian processes with trend, as well as those of their corollaries. For instance, a direct application of Proposition 3.6 yields the asymptotics of (1) for a non-stationary X with standard deviation function σ and correlation function r satisfying (2) with $t_0 = argmax_{t\in[0,T]}\sigma(t)$. If further the trend function g is continuous in a neighborhood of t_0, $g(t_0)=\max_{t\in[0,T]} g(t)$ and

$$g(t)\sim g(t_0)-c|t-t_0|^\gamma,\quad t\to t_0 \tag{4}$$

for some positive constants c,γ, then (3) holds with C_0 specified in Proposition 3.9 and β,u being substituted by $\min(\beta,2\gamma)$ and $u-g(t_0)$ respectivelly.

Complementary, we investigate asymptotic properties of the first passage time (ruin time) of $X(t)+g(t)$ to u on the finite-time interval $[0,T]$, given the process has ever exceeded u during $[0,T]$. In particular, for

$$\tau_u=\inf\{t\ge 0 : X(t)>u-g(t)\}, \tag{5}$$

with $\inf\{\emptyset\}=\infty$, we are interested in the approximate distribution of $\tau_u|\tau_u\le T$, as $u\to\infty$. Normal and exponential approximations of various Gaussian models have been discussed in [30, 32–35]. In this paper, we derive general results for the approximations of the conditional passage time in Propositions 3.3, 3.10. The asymptotics of $p_\Delta(u)$ for a short compact intervals $\Delta(u)$ displayed in Theorem 2.2 plays a key role in the derivation of these results.

Organisation of the rest of the paper: In Section 2, the tail asymptotics of the supremum of a family of centered Gaussian processes indexed by u are given. Several applications and examples are displayed in Section 3. Finally, we present all the proofs in Section 4 and Section 5.

2. Main Results

Let $X_u(t), t\in\mathbb{R}, u>0$ be a family of threshold-dependent centered Gaussian processes with continuous trajectories, variance functions σ_u^2 and correlation functions r_u. Our main results concern the asymptotics of slight generalization of $p_\Delta(u)$ and $p_T(u)$ for families of centered Gaussian processes X_u satisfying some regularity conditions for variance and coavariance respectivelly.

Let $C_0^*(E)$ be the set of continuous real-valued functions defined on the interval E such that $f(0)=0$ and for some $\epsilon_2>\epsilon_1>0$

$$\lim_{|t|\to\infty, t\in E} f(t)/|t|^{\epsilon_1}=\infty,\qquad \lim_{|t|\to\infty, t\in E} f(t)/|t|^{\epsilon_2}=0, \tag{6}$$

if $\sup\{x : x\in E\}=\infty$ or $\inf\{x : x\in E\}=-\infty$.

In the following $\mathcal{R}_\alpha$ denotes the set of regularly varying functions at 0 with index $\alpha\in\mathbb{R}$, see [36–38] for details.

We shall impose the following assumptions where $\Delta(u)$ is a compact interval:

A1: For any large u, there exists a point $t_u \in \mathbb{R}$ such that $\sigma_u(t_u) = 1$.

A2: There exists some $\lambda > 0$ such that

$$\lim_{u\to\infty} \sup_{t\in\Delta(u)} \left| \frac{\left(\frac{1}{\sigma_u(t_u+t)} - 1\right) u^2 - f(u^\lambda t)}{f(u^\lambda t) + 1} \right| = 0 \tag{7}$$

holds for some non-negative continuous function f with $f(0) = 0$.

A3: There exists $\rho \in \mathcal{R}_{\alpha/2}, \alpha \in (0, 2]$ such that

$$\lim_{u\to\infty} \sup_{\substack{s,t\in\Delta(u)\\ t\neq s}} \left| \frac{1 - r_u(t_u + s, t_u + t)}{\rho^2(|t - s|)} - 1 \right| = 0$$

and $\eta := \lim_{s\to 0} \frac{\rho^2(s)}{s^{2/\lambda}} \in [0, \infty]$, with λ given in **A2**.

Remark 2.1. *If f satisfies $f(0) = 0$ and $f(t) > 0, t \neq 0$, then*

$$\lim_{u\to\infty} \sup_{t\in\Delta(u),t\neq 0} \left| \frac{\frac{1}{\sigma_u(t_u+t)} - 1}{u^{-2} f(u^\lambda t)} - 1 \right| = 0$$

for some $\lambda > 0$ implies that (7) is valid.

Next we introduce some further notation, starting with the Pickands-type constant defined by

$$\mathcal{H}_\alpha[0, T] = \mathbb{E} \left\{ \sup_{t\in[0,T]} e^{\sqrt{2}B_\alpha(t)-|t|^\alpha} \right\}, \quad T > 0,$$

where B_α is an fBm. Further, define for $f \in C_0^*([S, T])$ with $S, T \in \mathbb{R}, S < T$ and a positive constant a

$$\mathcal{P}^f_{\alpha,a}[S, T] = \mathbb{E} \left\{ \sup_{t\in[S,T]} e^{\sqrt{2a}B_\alpha(t)-a|t|^\alpha - f(t)} \right\},$$

and set

$$\mathcal{P}^f_{\alpha,a}[0, \infty) = \lim_{T\to\infty} \mathcal{P}^f_{\alpha,a}[0, T], \quad \mathcal{P}^f_{\alpha,a}(-\infty, \infty) = \lim_{S\to-\infty,T\to\infty} \mathcal{P}^f_{\alpha,a}[S, T].$$

The finiteness of $\mathcal{P}^f_{\alpha,a}[0, \infty)$ and $\mathcal{P}^f_{\alpha,a}(-\infty, \infty)$ is guaranteed under weak assumptions on f, which will be shown in the proof of Theorem 2.2, see [2, 3, 5, 7, 15, 25, 39–43] for various properties of $\mathcal{H}_\alpha$ and $\mathcal{P}^f_{\alpha,a}[0, \infty)$.

Denote by $\mathbb{I}_{\{\cdot\}}$ the indicator function. For the regularly varying function $\rho(\cdot)$, we denote by $\overleftarrow{\rho}(\cdot)$ its asymptotic inverse (which is asymptotically unique). Throughout this paper, we set $0 \cdot \infty = 0$ and $u^{-\infty} = 0$ if $u > 0$. Let $\Psi(u) := \mathbb{P} \{\mathcal{N} > u\}$, with $\mathcal{N}$ a standard normal random variable.

In the next theorem we shall consider two functions $x_1(u), x_2(u), u \in \mathbb{R}$ such that $x_1(\frac{1}{t}) \in \mathcal{R}_{\mu_1}, \ x_2(\frac{1}{t}) \in \mathcal{R}_{\mu_2}$ with $\mu_1, \mu_2 \geq \lambda$, and

$$\lim_{u\to\infty} u^\lambda x_i(u) = x_i \in [-\infty, \infty], i = 1, 2, \quad \text{with} \ \ x_1 < x_2. \tag{8}$$

Theorem 2.2. *Let $X_u(t), t \in \mathbb{R}$ be a family of centered Gaussian processes with variance functions σ_u^2 and correlation functions r_u. If **A1**-**A3** are satisfied with $\Delta(u) = [x_1(u), x_2(u)]$, and $f \in C_0^*([x_1, x_2])$, then for M_u satisfying $M_u \sim u, u \to \infty$, we have*

$$\mathbb{P} \left\{ \sup_{t\in\Delta(u)} X_u(t_u + t) > M_u \right\} \sim C \left(u^\lambda \overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}} \Psi(M_u), \quad u \to \infty, \tag{9}$$

where

$$C = \begin{cases} \mathcal{H}_\alpha \int_{x_1}^{x_2} e^{-f(t)}dt, & \text{if } \ \eta = \infty, \\ \mathcal{P}^f_{\alpha,\eta}[x_1, x_2], & \text{if } \ \eta \in (0, \infty), \\ \sup_{t\in[x_1,x_2]} e^{-f(t)}, & \text{if } \ \eta = 0, \end{cases} \tag{10}$$

and $\mathcal{P}^f_{\alpha,\eta}(-\infty, \infty) \in (0, \infty)$.

Remark 2.3. *Let $\alpha \in (0,2], a>0$ be given. If $f \in C_0^*([x_1,x_2])$ for $x_1,x_2,y \in \mathbb{R}, x_1<x_2$, as shown in Appendix, we have, with $f_y(t) := f(y+t), t\in\mathbb{R}$*

$$\mathcal{P}_{\alpha,a}^{f}[x_1,x_2] = \mathcal{P}_{\alpha,a}^{f_y}[x_1-y,x_2-y], \quad \mathcal{P}_{\alpha,a}^{f}[x_1,\infty) = \mathcal{P}_{\alpha,a}^{f_y}[x_1-y,\infty). \tag{11}$$

In particular, if $f(t)=ct, c>0$, then for any $x\in\mathbb{R}$

$$\mathcal{P}_{\alpha,a}^{ct}[x,\infty) = \mathcal{P}_{\alpha,a}^{cx+ct}[0,\infty) = e^{-cx}\mathcal{P}_{\alpha,a}^{ct}[0,\infty).$$

Next, for any fixed $T\in(0,\infty)$, in order to analyse $p_T(u)$ we shall suppose that:
A1': For all large u, $\sigma_u(t)$ attains its maximum over $[0,T]$ at a unique point t_u such that

$$\sigma_u(t_u)=1 \quad \text{and} \quad \lim_{u\to\infty} t_u = t_0 \in [0,T].$$

A4: For all u large enough

$$\inf_{t\in[0,T]\setminus(t_u+\Delta(u))} \frac{1}{\sigma_u(t)} \ge 1 + \frac{p(\ln u)^q}{u^2}$$

holds for some constants $p>0, q>1$.
A5: For some positive constants $G,\varsigma>0$

$$\mathbb{E}\left\{(\overline{X}_u(t)-\overline{X}_u(s))^2\right\} \le G|t-s|^{\varsigma}$$

holds for all $s,t\in\{x\in[0,T]:\sigma(x)\neq 0\}$ and $\overline{X}_u(t) = \frac{X_u(t)}{\sigma_u(t)}$.
Below we define for λ given in **A2** and ν, d positve

$$\Delta(u) = \begin{cases} [0,\delta_u] & \text{if } \ t_u \equiv 0, \\ [-t_u,\delta_u], & \text{if } \ t_u \sim du^{-\nu} \text{ and } \nu\ge\lambda, \\ [-\delta_u,\delta_u], & \text{if } \ t_u\sim du^{-\nu} \text{ or } T-t_u\sim du^{-\nu} \text{ when } \nu<\lambda, \text{ or } t_0\in(0,T), \\ [-\delta_u, T-t_u], & \text{if } \ T-t_u\sim du^{-\nu} \text{ and } \nu\ge\lambda, \\ [-\delta_u,0] & \text{if } \ t_u = T, \end{cases} \tag{12}$$

where $\delta_u = \left(\frac{(\ln u)^q}{u}\right)^{\lambda}$ with q given in **A4**.

Theorem 2.4. *Let $X_u(t), t\in[0,T]$ be a family of centered Gaussian processes with variance functions σ_u^2 and correlation functions r_u. Assume that **A1',A2-A5** are satisfied with $\Delta(u)=[c_1(u),c_2(u)]$ given in (12) and*

$$\lim_{u\to\infty} c_i(u)u^{\lambda} = x_i \in [-\infty,\infty], i=1,2, \quad x_1<x_2.$$

If $f\in C_0^*([x_1,x_2])$, then for M_u suc that $\lim_{u\to\infty} M_u/u=1$ we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > M_u\right\} \sim C\left(u^{\lambda}\overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}}\Psi(M_u), \quad u\to\infty, \tag{13}$$

where C is the same as in (10) if $\eta\in(0,\infty]$ and $C=1$ if $\eta=0$.

Remark 2.5. *Theorem 2.4 generalises both [26][Theorem 1] and [32][Theorem 4.1].*

3. Applications

3.1. **Locally stationary Gaussian processes with trend.** In this section we consider the asymptotics of (1) for $X(t), t\in[0,T]$ a centered locally stationary Gaussian process with unit variance and correlation function r satisfying

$$\lim_{h\to 0}\sup_{t\in[0,T]}\left|\frac{1-r(t,t+h)}{a(t)\,|h|^{\alpha}}\right| = 1 \tag{14}$$

with $\alpha\in(0,2]$, $a(\cdot)$ a positive continuous function on $[0,T]$ and further

$$r(s,t)<1, \ \forall s,t\in[0,T] \text{ and } s\neq t. \tag{15}$$

We refer to e.g., [9, 10, 44–46] for results on locally stationary Gaussian processes. Extensions of this class to $\alpha(t)$-locally stationary processes are discussed in [13, 47, 48].

Regarding the continuous trend function g, we define $g_m = \max_{t\in[0,T]} g(t)$ and set

$$H := \{s \in [0,T] : g(s) = g_m\}.$$

Set below, for any $t_0 \in [0,T]$

$$Q_{t_0} = 1 + \mathbb{I}_{\{t_0\in(0,T)\}}, \quad w_{t_0} = \begin{cases} -\infty, & \text{if } t_0 \in (0,T), \\ 0, & \text{if } t_0 = 0 \text{ or } t_0 = T. \end{cases} \tag{16}$$

Proposition 3.1. *Suppose that* (14) *and* (15) *hold for a centered locally stationary Gaussian process* $X(t), t \in [0,T]$ *and let* $g : [0,T] \to \mathbb{R}$ *be a continuous function.*
i) If $H = \{t_0\}$ *and* (4) *holds, then as* $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (X(t) + g(t)) > u\right\} \sim C_{t_0} u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+} \Psi(u - g_m), \tag{17}$$

where (set with $a = a(t_0)$*)*

$$C_{t_0} = \begin{cases} Q_{t_0} a^{1/\alpha} c^{-1/\gamma} \Gamma(1/\gamma + 1)\mathcal{H}_\alpha, & \text{if } \alpha < 2\gamma, \\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0}, \infty), & \text{if } \alpha = 2\gamma, \\ 1, & \text{if } \alpha > 2\gamma. \end{cases}$$

ii) If $H = [A, B] \subset [0,T]$ *with* $0 \le A < B \le T$*, then as* $u \to \infty$

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (X(t) + g(t)) > u\right\} \sim \mathcal{H}_\alpha \int_A^B (a(t))^{1/\alpha} dt u^{\frac{2}{\alpha}} \Psi(u - g_m).$$

Remarks 3.2. *i) If* $H = \{t_1, \ldots, t_n\}$*, then as mentioned in [10], the tail distribution of the corresponding supremum is easily obtained assuming that for each* t_i *the assumptions of Proposition 3.1 statement i) hold, implying that*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (X(t) + g(t)) > u\right\} \sim \Big(\sum_{j=1}^n C_{t_j}\Big) u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+} \Psi(u - g_m), \quad u \to \infty.$$

ii) The novelty of Proposition 3.1 statement i) is that for the trend function g *only a polynomial local behavior around* t_0 *is assumed. In the literature so far only the case that* (4) *holds with* $\gamma = 2$ *has been considered (see [28]).*
iii) By the proof of Proposition 3.1 statement i), if $g(t)$ *is a measurable function which is continuous in a neighborhood of* t_0 *and smaller than* $g_m - \varepsilon$ *for some* $\varepsilon > 0$ *in the rest part over* $[0,T]$*, then the results still hold.*

We present below the approximation of the conditional passage time $\tau_u | \tau_u \le T$ with τ_u defined in (5).

Proposition 3.3. *Suppose that* (14) *and* (15) *hold for a centered locally stationary Gaussian process* $X(t), t \in [0,T]$*. Let* $g : [0,T] \to \mathbb{R}$ *be a continuous function,* $H = \{t_0\}$ *and* (4) *holds.*
i) If $t_0 \in [0,T)$*, then for any* $x \in (w_{t_0}, \infty)$

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\gamma c^{1/\gamma} \int_{w_{t_0}}^x e^{-c|t|^\gamma} dt}{Q_{t_0} \Gamma(1/\gamma)}, & \text{if } \alpha < 2\gamma, \\ \frac{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0}, x]}{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0}, \infty)}, & \text{if } \alpha = 2\gamma, \\ \sup_{t\in[w_{t_0}, x]} e^{-c|t|^\gamma}, & \text{if } \alpha > 2\gamma, \end{cases}$$

ii) If $t_0 = T$*, then for any* $x \in (-\infty, 0)$

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\gamma c^{1/\gamma} \int_{-x}^\infty e^{-c|t|^\gamma} dt}{\Gamma(1/\gamma)}, & \text{if } \alpha < 2\gamma, \\ \frac{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[-x, \infty)}{\mathcal{P}_{\alpha,a}^{c|t|^\gamma}[0, \infty)}, & \text{if } \alpha = 2\gamma, \\ e^{-c|x|^\gamma}, & \text{if } \alpha > 2\gamma. \end{cases}$$

Example 3.4. *Let $X(t), t \in [0,T]$ be a centered stationary Gaussian process with unit variance and correlation function r that satisfies $r(t) = 1 - a|t|^\alpha(1+o(1)),\ t \to 0$ for some $a > 0,\ \alpha \in (0,2],$ and $r(t) < 1,$ for all $t \in (0,T]$. Let τ_u be defined as in (5) with $g(t) = -ct, c > 0$. Then we have*

$$\mathbb{P}\left\{\max_{t\in[0,T]} (X(t) - ct) > u\right\} \sim u^{(\frac{2}{\alpha}-1)_+}\Psi(u)\begin{cases} c^{-1}a^{1/\alpha}\mathcal{H}_\alpha, & \alpha \in (0,2),\\ \mathcal{P}^{ct}_{\alpha,a}[0,\infty), & \alpha = 2,\end{cases}$$

and for any x positive

$$\mathbb{P}\left\{u\tau_u \le x \middle| \tau_u \le T\right\} \sim \begin{cases} 1 - e^{-cx}, & \alpha \in (0,2),\\ \frac{\mathcal{P}^{ct}_{\alpha,a}[0,x]}{\mathcal{P}^{ct}_{\alpha,a}[0,\infty)}, & \alpha = 2.\end{cases}$$

Example 3.5. *Let $X(t), t > 0$ be a standardized fBm, i.e., $X(t) = B_\alpha(t)/t^{\alpha/2}$ with B_α an fBm. Let c, T be positive constants. Then for any $n \in \mathbb{N}$, we have*

$$\mathbb{P}\left\{\max_{t\in[T,(n+1)T]}\left(X(t) + c\sin\left(\frac{2\pi t}{T}\right)\right) > u\right\} \sim \left(\sum_{j=1}^n a_j^{\frac{1}{\alpha}}\right)\mathcal{H}_\alpha\frac{T}{\sqrt{2c\pi}}u^{\frac{2}{\alpha}-\frac{1}{2}}\Psi(u-c),$$

where $a_j = \frac{1}{2}\left(\frac{(4j+1)T}{4}\right)^{-\alpha}, j = 1, \ldots, n.$

3.2. **Non-stationary Gaussian processes with trend.** In this section we consider the asymptotics of (1) for $X(t), t \in [0,T]$ a centered Gaussian process with non-constant variance function σ^2. Define below whenever $\sigma(t) \neq 0$

$$\overline{X}(t) := \frac{X(t)}{\sigma(t)}, \quad t \in [0,T],$$

and set for a continuous function g

$$m_u(t) := \frac{\sigma(t)}{1 - g(t)/u}, \quad t \in [0,T], \quad u > 0. \tag{18}$$

Proposition 3.6. *Let X and g be as above. Assume that $t_u = argmax_{t\in[0,T]} m_u(t)$ is unique with $\lim_{u\to\infty} t_u = t_0$ and $\sigma(t_0) = 1$. Further, we suppose that **A2-A5** are satisfied with $\sigma_u(t) = \frac{m_u(t)}{m_u(t_u)}$, $r_u(s,t) = r(s,t)$, $\overline{X}_u(t) = \overline{X}(t)$ and $\Delta(u) = [c_1(u), c_2(u)]$ given in (12). If in **A2** $f \in C^*_0([x_1,x_2])$ and*

$$\lim_{u\to\infty} c_i(u)u^\lambda = x_i \in [-\infty,\infty], i = 1,2, \quad x_1 < x_2,$$

then we have

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t) + g(t)) > u\right\} \sim C\left(u^\lambda \overleftarrow{\rho}(u^{-1})\right)^{-\mathbb{I}_{\{\eta=\infty\}}}\Psi\left(\frac{u - g(t_u)}{\sigma(t_u)}\right), \quad u \to \infty, \tag{19}$$

where C is the same as in (10) when $\eta \in (0,\infty]$ and $C = 1$ when $\eta = 0$.

Remarks 3.7. *i) Proposition 3.6 extends [26][Theorem 3] and the results of [1] where (1) was analyzed for special X with stationary increments and special trend function g.*
ii) The assumption that $\sigma(t_0) = 1$ is not essential in the proof. In fact, for the general case where $\sigma(t_0) \neq 1$ we have that (19) holds with

$$C = \begin{cases} \sigma_0^{-\frac{2}{\alpha}}\mathcal{H}_\alpha\int_{x_1}^{x_2} e^{-\sigma_0^{-2}f(t)}dt, & \text{if } \ \eta = \infty,\\ \mathcal{P}^{\sigma_0^{-2}f}_{\alpha,\sigma_0^{-2}\eta}[x_1,x_2], & \text{if } \ \eta \in (0,\infty), \qquad \sigma_0 = \sigma(t_0).\\ 1, & \text{if } \ \eta = 0,\end{cases}$$

Proposition 3.8. *Under the notation and assumptions of Proposition 3.6 without assuming* **A3,A5**, *if X is differentiable in the mean square sense such that*

$$r(s,t) < 1, s \neq t, \quad \mathbb{E}\left\{X'^2(t_0)\right\} > \sigma'^2(t_0),$$

and $\mathbb{E}\left\{X'^2(t)\right\} - \sigma'^2(t)$ is continuous in a neighborhood of t_0, then (19) holds with

$$\alpha = 2, \quad \rho^2(t) = \frac{1}{2}\left(\mathbb{E}\left\{X'^2(t_0)\right\} - \sigma'^2(t_0)\right)t^2.$$

The next result is an extension of a classical theorem concerning the extremes of non-stationary Gaussian processes discussed in the Introduction, see [10][Theorem D.3].

Proposition 3.9. *Let $X(t), t \in [0,T]$ be a centered Gaussian process with correlation function r and variance function σ^2 such that $t_0 = argmax_{t\in[0,T]}\sigma(t)$ is unique with $\sigma(t_0) = \sigma > 0$. Suppose that g is a bounded measurable function being continuous in a neighborhood of t_0 such that (4) holds. If further (2) is satisfied, then*

$$\mathbb{P}\left\{\sup_{t\in[0,T]} (X(t) + g(t)) > u\right\} \sim C_0 u^{(\frac{2}{\alpha}-\frac{2}{\beta^*})_+}\Psi\left(\frac{u-g(t_0)}{\sigma}\right), \tag{20}$$

where $\beta^* = \min(\beta, 2\gamma)$,

$$C_0 = \begin{cases} \sigma^{-2/\alpha}a^{1/\alpha}\mathcal{H}_\alpha \int_{w_{t_0}}^{\infty} e^{-f(t)}dt, & \text{if } \alpha < \beta^*, \\ \mathcal{P}^f_{\alpha,\sigma^{-2}a}[w_{t_0},\infty), & \text{if } \alpha = \beta^*, \\ 1, & \text{if } \alpha > \beta^*, \end{cases}$$

with $f(t) = \frac{b}{\sigma^3}|t|^\beta \mathbb{I}_{\{\beta=\beta^*\}} + \frac{c}{\sigma^2}|t|^\gamma \mathbb{I}_{\{2\gamma=\beta^*\}}$ and w_{t_0} defined in (16).

Proposition 3.10. *i) Under the conditions and notation of Proposition 3.6, for any $x \in [x_1, x_2]$ we have*

$$\mathbb{P}\left\{u^\lambda(\tau_u - t_u) \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\int_{x_1}^{x} e^{-f(t)}dt}{\int_{x_1}^{x_2} e^{-f(t)}dt}, & \text{if } \eta = \infty, \\ \frac{\mathcal{P}^f_{\alpha,\eta}[x_1,x]}{\mathcal{P}^f_{\alpha,\eta}[x_1,x_2]}, & \text{if } \eta \in (0,\infty), \\ \sup_{t\in[x_1,x]} e^{-f(t)}, & \text{if } \eta = 0. \end{cases} \tag{21}$$

ii) Under the conditions and notation of Proposition 3.9, if $t_0 \in [0,T)$, then for $x \in (w_{t_0}, \infty)$

$$\mathbb{P}\left\{u^{2/\beta^*}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\int_{w_{t_0}}^{x} e^{-f(t)}dt}{\int_{w_{t_0}}^{\infty} e^{-f(t)}dt}, & \text{if } \alpha < \beta^*, \\ \frac{\mathcal{P}^f_{\alpha,a}[w_{t_0},x]}{\mathcal{P}^f_{\alpha,a}[w_{t_0},\infty)}, & \text{if } \alpha = \beta^*, \\ \sup_{t\in[w_{t_0},x]} e^{-f(t)}, & \text{if } \alpha > \beta^*, \end{cases}$$

and if $t_0 = T$, then for $x \in (-\infty, 0)$

$$\mathbb{P}\left\{u^{2/\beta^*}(\tau_u - t_0) \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\int_{-x}^{\infty} e^{-f(t)}dt}{\int_{0}^{\infty} e^{-f(t)}dt}, & \text{if } \alpha < \beta^*, \\ \frac{\mathcal{P}^f_{\alpha,a}[-x,\infty)}{\mathcal{P}^f_{\alpha,a}[0,\infty)}, & \text{if } \alpha = \beta^*, \\ e^{-f(x)}, & \text{if } \alpha > \beta^*. \end{cases}$$

Example 3.11. *Let $X(t) = B(t) - tB(1), t \in [0,1]$, where $B(t)$ is a standard Brownian motion and suppose that τ_u is defined by (5) with $g(t) = -ct$. Then*

$$\mathbb{P}\left\{\sup_{t\in[0,1]} (X(t) - ct) > u\right\} \sim e^{-2(u^2+cu)}, \tag{22}$$

$$\mathbb{P}\left\{u\left(\tau_u - \frac{u}{c+2u}\right) \le x \middle| \tau_u \le 1\right\} \sim \Phi(4x), \quad x \in (-\infty,\infty).$$

We note that according to [49][Lemma 2.7], the result in (22) is actually exact, i.e. for any $u > 0$, $\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t) - ct) > u\right\} = e^{-2(u^2+cu)}$.
Now, let $T = 1/2$. It appears that the asymptotics in this case is different, i.e.,

$$\mathbb{P}\left\{\sup_{t\in[0,1/2]} (X(t) - ct) > u\right\} \sim \Phi(c)e^{-2(u^2+cu)}, \tag{23}$$

and

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\le x\Big|\tau_u\le\frac{1}{2}\right\}\sim\frac{\Phi(4x)}{\Phi(c)},\quad x\in(-\infty,c/4].$$

Similarly, we have

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\left(X(t)+\frac{c}{2}-c\left|t-\frac{1}{2}\right|\right)>u\right\}\sim 2\Psi(c)e^{-2(u^2-cu)} \tag{24}$$

and

$$\mathbb{P}\left\{u\left(\tau_u-\frac{1}{2}\right)\le x\Big|\tau_u\le 1\right\}\sim\frac{\int_{-\infty}^{4x}e^{-\frac{(|t|+c)^2}{2}}dt}{2\sqrt{2\pi}\Psi(c)},\quad x\in(-\infty,\infty).$$

We conclude this section with an application of Proposition 3.6 to the calculation of the ruin probability of a Brownian motion risk model with constant force of interest over infinite-time horizon.

3.3. **Ruin probability in Gaussian risk model.** Consider risk reserve process $U(t)$, with interest rate δ, modeled by

$$U(t)=ue^{\delta t}+c\int_0^t e^{\delta(t-v)}dv-\sigma\int_0^t e^{\delta(t-v)}dB(v),\quad t\ge 0,$$

where c,δ,σ are some positive constants and B is a standard Brownian motion. The corresponding ruin probability over infinite-time horizon is defined as

$$p(u)=\mathbb{P}\left\{\inf_{t\in[0,\infty)}U(t)<0\right\}.$$

For this model we also define the ruin time $\tau_u=\inf\{t\ge 0: U(t)<0\}$. Set below

$$h(t)=\frac{\delta}{\sigma^2}\left(\sqrt{t+r^2}-r\right)^2,\quad t\in[0,\infty),\quad r=\frac{c}{\delta}.$$

We present next approximations of the ruin probability and the conditional ruin time $\tau_u|\tau_u<\infty$ as $u\to\infty$.

Proposition 3.12. *As* $u\to\infty$

$$p(u)\sim\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,\infty\right)\Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2+4cu}\right) \tag{25}$$

and for $x\in(-r^2,\infty)$

$$\mathbb{P}\left\{u^2\left(e^{-2\delta\tau_u}-\left(\frac{c}{\delta u+c}\right)^2\right)\le x\Big|\tau_u<\infty\right\}\sim\frac{\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,x\right]}{\mathcal{P}^h_{1,\delta/\sigma^2}\left[-r^2,\infty\right)}.$$

Remark 3.13. *According to [50] (see also [51]) we have*

$$\mathbb{P}\left\{\inf_{t\in[0,\infty)}U(t)<0\right\}=\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right)\Big/\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right). \tag{26}$$

By (25) *and* (11)

$$\begin{aligned}
\mathbb{P}\left\{\inf_{t\in[0,\infty]}U(t)<0\right\}&\sim\mathbb{E}\left\{\sup_{t\in[-r^2,\infty)}\exp\left(\sqrt{\frac{2\delta}{\sigma^2}}B(t)-\frac{\delta}{\sigma^2}\left(\sqrt{t+r^2}-r\right)^2-\frac{\delta}{\sigma^2}|t|\right)\right\}\Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2+4cu}\right)\\
&\sim\mathbb{E}\left\{\sup_{t\in[-\frac{c^2}{\sigma^2\delta},\infty)}\exp\left(\sqrt{2}B(t)-\left(t+\frac{c^2}{\sigma^2\delta}\right)+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t+\frac{c^2}{\sigma^2\delta}}-|t|\right)\right\}\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right)\\
&=\mathbb{E}\left\{\sup_{t\in[0,\infty)}\exp\left(\sqrt{2}B\left(t\right)-2t+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t}\right)\right\}\Psi\left(\frac{\sqrt{2\delta}}{\sigma}(u+r)\right),
\end{aligned}$$

which combined with (26) *implies that*

$$\mathbb{E}\left\{\sup_{t\in[0,\infty)}\exp\left(\sqrt{2}B\left(t\right)-2t+\frac{2c}{\sigma\sqrt{\delta}}\sqrt{t}\right)\right\}=\left(\Psi\left(\frac{\sqrt{2}c}{\sigma\sqrt{\delta}}\right)\right)^{-1}. \tag{27}$$

4. Proofs

In the proofs presented in this section $\mathbb{C}_i, i \in \mathbb{N}$ are some positive constants which may be different from line to line. We first give two preliminary lemmas, which play an important role in the proof of Theorem 2.2.

Lemma 4.1. *Let $\xi(t), t \in \mathbb{R}$ be a centered stationary Gaussian process with unit variance and correlation function r satisfying*

$$1 - r(t) \sim a\rho^2(|t|), \quad t \to 0, \tag{28}$$

with $a > 0$, and $\rho \in \mathcal{R}_{\alpha/2}$, $\alpha \in (0, 2]$. Let f be a continuous function, K_u be a family of index sets and

$$Z_u(t) := \frac{\xi(\overleftarrow{\rho}(u^{-1})t)}{1 + u^{-2} f(\overleftarrow{\rho}(u^{-1})u^{\lambda}t)}, \quad t \in [S_1, S_2],$$

where $\lambda > 0$ and $-\infty < S_1 < S_2 < \infty$. If $M_k(u), k \in K_u$ is such that

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \frac{M_k(u)}{u} - 1 \right| = 0, \tag{29}$$

then we have

$$\lim_{u\to\infty} \sup_{k\in K_u} \left| \frac{1}{\Psi(M_k(u))} \mathbb{P}\left\{ \sup_{t\in[S_1,S_2]} Z_u(t) > M_k(u) \right\} - \mathcal{R}_{\eta}^{f}[S_1, S_2] \right| = 0, \tag{30}$$

where

$$\mathcal{R}_{\eta}^{f}[S_1, S_2] := \mathbb{E}\left\{ \sup_{t\in[S_1,S_2]} e^{\sqrt{2a}B_\alpha(t) - a|t|^\alpha - f(\eta^{-1/\alpha}t)} \right\} = \begin{cases} \mathcal{H}_\alpha[a^{1/\alpha}S_1, a^{1/\alpha}S_2] & f(\cdot) \equiv 0, \\ \mathcal{P}_{\alpha,a}^{h}[S_1, S_2] & \text{otherwise}, \end{cases}$$

with $\eta := \lim_{t\downarrow 0} \frac{\rho^2(t)}{t^{2/\lambda}} \in (0, \infty]$ and $h(t) = f(\eta^{-1/\alpha}t)$ for $\eta \in (0, \infty)$, $h(t) = f(0)$ for $\eta = \infty$.

Proof of Lemma 4.1: We set $\eta^{-1/\alpha} = 0$ if $\eta = \infty$. The proof follows by checking the conditions of [52][Theorem 2.1] where the results still holds if we omit the requirements $f(0) = 0$ and $[S_1, S_2] \ni 0$. By (29)

$$\lim_{u\to\infty} \inf_{k\in K_u} M_k(u) = \infty.$$

By continuity of f we have

$$\lim_{u\to\infty} \sup_{k\in K_u, t\in[S_1,S_2]} \left| M_k^2(u)u^{-2} f(\overleftarrow{\rho}(u^{-1})u^{\lambda}t) - f(\eta^{-1/\alpha}t) \right| = 0. \tag{31}$$

Moreover, (28) implies

$$\mathrm{Var}(\xi(\overleftarrow{\rho}(u^{-1})t) - \xi(\overleftarrow{\rho}(u^{-1})t')) = 2 - 2r\left(\left|\overleftarrow{\rho}(u^{-1})(t - t')\right|\right) \sim 2a\rho^2\left(\left|\overleftarrow{\rho}(u^{-1})(t - t')\right|\right), \ u \to \infty,$$

holds for $t, t' \in [S_1, S_2]$. Thus

$$\lim_{u\to\infty} \sup_{k\in K_u} \sup_{t\neq t'\in[S_1,S_2]} \left| M_k^2(u) \frac{\mathrm{Var}(\xi(\overleftarrow{\rho}(u^{-1})t) - \xi(\overleftarrow{\rho}(u^{-1})t'))}{2au^2\rho^2\left(|\overleftarrow{\rho}(u^{-1})(t - t')|\right)} - 1 \right| = 0. \tag{32}$$

Since $\rho^2 \in \mathcal{R}_\alpha$ which satisfies the uniform convergence theorem (UCT) for regularly varying function, see, e.g., [53], i.e.,

$$\lim_{u\to\infty} \sup_{t,t'\in[S_1,S_2]} \left| u^2\rho^2\left(\left|\overleftarrow{\rho}(u^{-1})(t - t')\right|\right) - |t - t'|^\alpha \right| = 0, \tag{33}$$

and further by the Potter's bound for ρ^2, see [53] we have

$$\limsup_{u\to\infty} \sup_{\substack{t,t'\in[S_1,S_2] \\ t\neq t'}} \frac{u^2\rho^2\left(\left|\overleftarrow{\rho}(u^{-1})(t - t')\right|\right)}{|t - t'|^{\alpha-\varepsilon_1}} \le \mathbb{C}_1 \max\left(|S_1 - S_2|^{\alpha-\varepsilon_1}, |S_1 - S_2|^{\alpha+\varepsilon_1} \right) < \infty, \tag{34}$$

where $\varepsilon_1 \in (0, \min(1, \alpha))$. We know that for $\alpha \in (0, 2]$

$$\left| |t|^\alpha - |t'|^\alpha \right| \le \mathbb{C}_2 |t - t'|^{\alpha\wedge 1}, \ t, t' \in [S_1, S_2]. \tag{35}$$

By (28) for any small $\epsilon>0$, when u large enough

$$r(\overleftarrow{\rho}(u^{-1})t)\le 1-\rho^2(\overleftarrow{\rho}(u^{-1})\,|t|)(1-\epsilon),\quad r(\overleftarrow{\rho}(u^{-1})t)\ge 1-\rho^2(\overleftarrow{\rho}(u^{-1})\,|t|)(1+\epsilon) \tag{36}$$

hold for $t\in[S_1,S_2]$, then by (29) for u large enough

$$\begin{aligned}
&\sup_{k\in K_u}\sup_{|t-t'|<\varepsilon,t,t'\in[S_1,S_2]} M_k^2(u)\mathbb{E}\left\{[\xi(\overleftarrow{\rho}(u^{-1})t)-\xi(\overleftarrow{\rho}(u^{-1})t')]\xi(0)\right\}\\
&\le \mathbb{C}_3 u^2 \sup_{|t-t'|<\varepsilon,t,t'\in[S_1,S_2]}\left|r(\overleftarrow{\rho}(u^{-1})t)-r(\overleftarrow{\rho}(u^{-1})t')\right|\\
&\le \mathbb{C}_3 \sup_{|t-t'|<\varepsilon,t,t'\in[S_1,S_2]}\left(\left|u^2\rho^2(\overleftarrow{\rho}(u^{-1})\,|t|)-u^2\rho^2(\overleftarrow{\rho}(u^{-1})\,|t'|)\right|+\epsilon\left|u^2\rho^2(\overleftarrow{\rho}(u^{-1})\,|t|)\right|+\epsilon\left|u^2\rho^2(\overleftarrow{\rho}(u^{-1})\,|t'|)\right|\right)\\
&\le \mathbb{C}_3 \sup_{|t-t'|<\varepsilon,t,t'\in[S_1,S_2]}\left(\left|u^2\rho^2\left(\left|\overleftarrow{\rho}(u^{-1})(t)\right|\right)-|t|^{\alpha}\right|+\left|u^2\rho^2\left(\left|\overleftarrow{\rho}(u^{-1})(t')\right|\right)-|t'|^{\alpha}\right|+\left||t|^{\alpha}-|t'|^{\alpha}\right|\right.
\end{aligned}$$

$$\left.+\mathbb{C}_4\epsilon\left(|t|^{\alpha-\varepsilon_1}+|t'|^{\alpha-\varepsilon_1}\right)\right) \tag{37}$$

$$\le \mathbb{C}_5\varepsilon^{\alpha\wedge 1}+\mathbb{C}_6\epsilon,\ u\to\infty \tag{38}$$

$$\to 0,\varepsilon\to 0,\epsilon\to 0,$$

where in (37) we use (34) and (38) follows from (33) and (35).
Hence the proof follows from [52][Theorem 2.1]. □

Lemma 4.2. *Let $Z_u(s,t),(s,t)\in\mathbb{R}^2$ be a centered stationary Gaussian field with unit variance and correlation function $r_{Z_u}(\cdot,\cdot)$ satisfying*

$$1-r_{Z_u}(s,t)=au^{-2}\left(\left|\frac{s}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}+\left|\frac{t}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}\right),\quad (s,t)\in\mathbb{R}^2, \tag{39}$$

with $a>0$, $\rho^2\in\mathcal{R}_\alpha$ and $\alpha\in(0,2]$. Let K_u be some index sets. Then, for $M_k(u),k\in K_u$ satisfying (29) and for any $S_1,S_2,T_1,T_2\ge 0$ such that $\max(S_1,S_2)>0,\max(T_1,T_2)>0$, we have

$$\lim_{u\to\infty}\sup_{k\in K_u}\left|\frac{1}{\Psi(M_k(u))}\mathbb{P}\left\{\sup_{(s,t)\in D(u)}Z_u(s,t)>M_k(u)\right\}-\mathcal{F}(S_1,S_2,T_1,T_2)\right|=0,$$

where $D(u)=[-\overleftarrow{\rho}(u^{-1})S_1,\overleftarrow{\rho}(u^{-1})S_2]\times[-\overleftarrow{\rho}(u^{-1})T_1,\overleftarrow{\rho}(u^{-1})T_2]$ and

$$\mathcal{F}(S_1,S_2,T_1,T_2)=\mathcal{H}_{\alpha/2}[-a^{2/\alpha}S_1,a^{2/\alpha}S_2]\mathcal{H}_{\alpha/2}[-a^{2/\alpha}T_1,a^{2/\alpha}T_2].$$

Proof of Lemma 4.2: The proof follows by checking the conditions of [35][Lemma 5.3].
For $D=[-S_1,S_2]\times[-T_1,T_2]$ we have

$$\mathbb{P}\left\{\sup_{(s,t)\in D_u}Z_u(s,t)>M_k(u)\right\}=\mathbb{P}\left\{\sup_{(s,t)\in D}Z_u(\overleftarrow{\rho}(u^{-1})s,\overleftarrow{\rho}(u^{-1})t)>M_k(u)\right\}.$$

Since by (39)

$$\begin{aligned}
\mathrm{Var}(Z_u(\overleftarrow{\rho}(u^{-1})s,\overleftarrow{\rho}(u^{-1})t)-Z_u(\overleftarrow{\rho}(u^{-1})s',\overleftarrow{\rho}(u^{-1})t')) &= 2-2r_{Z_u}\left(\overleftarrow{\rho}(u^{-1})(s-s'),\overleftarrow{\rho}(u^{-1})(t-t')\right)\\
&= au^{-2}\left(|s-s'|^{\alpha/2}+|t-t'|^{\alpha/2}\right)
\end{aligned}$$

we obtain

$$\lim_{u\to\infty}\sup_{k\in K_u}\sup_{(s,t)\neq(s',t')\in D}\left|M_k^2(u)\frac{\mathrm{Var}(Z_u(\overleftarrow{\rho}(u^{-1})s,\overleftarrow{\rho}(u^{-1})t)-Z_u(\overleftarrow{\rho}(u^{-1})s',\overleftarrow{\rho}(u^{-1})t'))}{2a(|s-s'|^{\alpha/2}+|t-t'|^{\alpha/2})}-1\right|=0. \tag{40}$$

Further, since for $\alpha/2\in(0,1]$

$$\left||t|^{\alpha/2}-|t'|^{\alpha/2}\right|\le\mathbb{C}_1|t-t'|^{\alpha/2},\quad \left||s|^{\alpha/2}-|s'|^{\alpha/2}\right|\le\mathbb{C}_2|s-s'|^{\alpha/2}$$

holds for $t,t'\in[-T_1,T_2]$, $s,s'\in[-S_1,S_2]$, we have by (39)

$$\sup_{k\in K_u}\sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\(s,t),(s',t')\in D}}M_k^2(u)\mathbb{E}\left\{[Z_u(\overleftarrow{\rho}(u^{-1})s,\overleftarrow{\rho}(u^{-1})t)-Z_u(\overleftarrow{\rho}(u^{-1})s',\overleftarrow{\rho}(u^{-1})t')]Z_u(0,0)\right\}$$

$$
\begin{aligned}
&\le \mathbb{C}_3 u^2 \sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\ (s,t),(s',t')\in D}} \left| r_{Z_u}(\overleftarrow{\rho}(u^{-1})s, \overleftarrow{\rho}(u^{-1})t) - r_{Z_u}(\overleftarrow{\rho}(u^{-1})s', \overleftarrow{\rho}(u^{-1})t')\right|\\
&= \mathbb{C}_3 a \sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\ (s,t),(s',t')\in D}} \left| |s|^{\alpha/2} + |t|^{\alpha/2} - |s'|^{\alpha/2} - |t'|^{\alpha/2}\right|\\
&\le \mathbb{C}_3 a \sup_{\substack{|(s,t)-(s',t')|<\varepsilon\\ (s,t),(s',t')\in D}} \left(\left| |s|^{\alpha/2} - |s'|^{\alpha/2}\right| + \left| |t|^{\alpha/2} - |t'|^{\alpha/2}\right|\right)\\
&\le \mathbb{C}_4 \varepsilon^{\alpha/2} \to 0,\ \ u\to\infty, \varepsilon\to 0.
\end{aligned}
$$

Hence the claim follows from [35][Lemma 5.3]. □

Proof of Theorem 2.2: We have from **A3**

$$
\lim_{t\to 0}\frac{\rho^2(t)}{t^{2/\lambda}} = \eta \in [0,\infty], \quad \lim_{u\to\infty} u^{\lambda}\overleftarrow{\rho}(u^{-1}) = \eta^{-\lambda/2}.
$$

Without loss of generality, we consider only the case $t_u = 0$ for u large enough.

By **A2** for $t\in\Delta(u)$, for sufficiently large u,

$$
\frac{1}{\mathcal{F}_{u,+\varepsilon}(t)} \le \sigma_u(t) \le \frac{1}{\mathcal{F}_{u,-\varepsilon}(t)}, \quad \mathcal{F}_{u,\pm\varepsilon}(t) = 1 + u^{-2}\left[(1\pm\varepsilon) f(u^{\lambda}t) \pm \varepsilon\right] \tag{41}
$$

for small constant $\varepsilon\in(0,1)$. Since further

$$
\pi(u) := \mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t) > M_u\right\} = \mathbb{P}\left\{\sup_{t\in\Delta(u)} \overline{X_u}(t)\sigma_u(t) > M_u\right\} \tag{42}
$$

we have

$$
\pi(u) \le \mathbb{P}\left\{\sup_{t\in\Delta(u)} \frac{\overline{X_u}(t)}{\mathcal{F}_{u,-\varepsilon}(t)} > M_u\right\}, \quad \pi(u) \ge \mathbb{P}\left\{\sup_{t\in\Delta(u)} \frac{\overline{X_u}(t)}{\mathcal{F}_{u,+\varepsilon}(t)} > M_u\right\}.
$$

Set for some positive constant S

$$
I_k(u) = [k\overleftarrow{\rho}(u^{-1})S, (k+1)\overleftarrow{\rho}(u^{-1})S], \quad k\in\mathbb{Z}.
$$

Further, define

$$
\begin{aligned}
\mathcal{G}_{u,+\varepsilon}(k) &= M_u \sup_{s\in I_k(u)} \mathcal{F}_{u,+\varepsilon}(s),\ N_1(u) = \left\lfloor \frac{x_1(u)}{S\overleftarrow{\rho}(u^{-1})}\right\rfloor - \mathbb{I}_{\{x_1\le 0\}},\\
\mathcal{G}_{u,-\varepsilon}(k) &= M_u \inf_{s\in I_k(u)} \mathcal{F}_{u,-\varepsilon}(s),\ N_2(u) = \left\lfloor \frac{x_2(u)}{S\overleftarrow{\rho}(u^{-1})}\right\rfloor + \mathbb{I}_{\{x_2\le 0\}}.
\end{aligned}
$$

In view of [54], we can find centered stationary Gaussian processes $Y_{\pm\varepsilon}(t), t\in\mathbb{R}$ with continuous trajectories, unit variance and correlation function satisfying

$$
r_{\pm\varepsilon}(t) = 1 - (1\pm\varepsilon)\rho^2(|t|)(1+o(1)), \quad t\to 0.
$$

Case 1) $\eta=\infty$:

For any u positive

$$
\sum_{k=N_1(u)+1}^{N_2(u)-1} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\} - \sum_{i=1}^{2}\Lambda_i(u) \le \pi(u) \le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\}, \tag{43}
$$

where

$$
\Lambda_1(u) = \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u,\ \sup_{t\in I_{k+1}(u)} X_u(t) > M_u\right\},
$$

and

$$
\Lambda_2(u) = \sum_{N_1(u)\le k,l\le N_2(u), l\ge k+2} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u,\ \sup_{t\in I_l(u)} X_u(t) > M_u\right\}.
$$

Set below

$$
,\Theta(u) = \frac{\mathcal{H}_\alpha}{u^{\lambda}\overleftarrow{\rho}(u^{-1})}\int_{x_1}^{x_2} e^{-f(t)}dt\Psi(M_u).
$$

which is well-defined since $\int_{x_1}^{x_2} e^{-f(t)}dt < \infty$ follows by the assumption $f \in C_0^*([x_1, x_2])$. By Slepian inequality (see e.g., [55]), (42) and Lemma 4.1

$$
\begin{aligned}
\sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\} &\le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \overline{X_u}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&\le \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&= \sum_{k=N_1(u)}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \\
&\sim \sum_{k=N_1(u)}^{N_2(u)} \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(\mathcal{G}_{u,-\varepsilon}(k)) \\
&\sim \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(M_u) \sum_{k=N_1(u)}^{N_2(u)} e^{-M_u^2u^{-2}\inf_{s\in I_k(u)}\left[(1-\varepsilon)f(u^\lambda s)-\varepsilon\right]} \\
&\sim \frac{\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]}{Su^\lambda \overleftarrow{\rho}(u^{-1})}\int_{x_1}^{x_2} e^{-(1-\varepsilon)f(t)+\varepsilon}dt\Psi(M_u) \\
&\sim \Theta(u), \quad u\to\infty, S\to\infty, \varepsilon\to 0. \qquad (44)
\end{aligned}
$$

Similarly, we derive that

$$
\sum_{k=N_1(u)+1}^{N_2(u)-1} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > u\right\} \ge (1+o(1))\Theta(u), u\to\infty,\ S\to\infty,\ \varepsilon\to 0. \qquad (45)
$$

Moreover,

$$
\begin{aligned}
\Lambda_1(u) &\le \sum_{k=N_1(u)}^{N_2(u)} \Bigg(\mathbb{P}\left\{\sup_{t\in I_k(u)} Y_{+\varepsilon}(t) > \widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\} + \mathbb{P}\left\{\sup_{t\in I_{k+1}(u)} Y_{+\varepsilon}(t) > \widehat{\mathcal{G}}_{u,-\varepsilon}(k)\right\} \\
&\quad -\mathbb{P}\left\{\sup_{t\in I_k(u)\cup I_{k+1}(u)} Y_{-\varepsilon}(t) > \overline{\mathcal{G}}_{u,+\varepsilon}(k)\right\}\Bigg) \\
&\le \sum_{k=N_1(u)}^{N_2(u)} \left(2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S] - \mathcal{H}_\alpha[0,2(1-\varepsilon)^{1/\alpha}S]\right)\Psi(\widehat{\mathcal{G}}_{u,-\varepsilon}(k)) \\
&\le \left(2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S] - \mathcal{H}_\alpha[0,2(1-\varepsilon)^{1/\alpha}S]\right)\sum_{k=N_1(u)}^{N_2(u)} \Psi(\widehat{\mathcal{G}}_{u,-\varepsilon}(k)) \\
&= o(\Theta(u)),\ u\to\infty, S\to\infty, \varepsilon\to 0, \qquad (46)
\end{aligned}
$$

where

$$
\widehat{\mathcal{G}}_{u,-\varepsilon}(k) = \min(\mathcal{G}_{u,-\varepsilon}(k), \mathcal{G}_{u,-\varepsilon}(k+1)), \quad \overline{\mathcal{G}}_{u,+\varepsilon}(k) = \max(\mathcal{G}_{u,+\varepsilon}(k), \mathcal{G}_{u,+\varepsilon}(k+1)).
$$

By **A3** for any $(s,t) \in I_k(u)\times I_l(u)$ with $N_1(u) \le k, l \le N_2(u), l \ge k+2$ we have

$$
2 \le Var\left(\overline{X_u}(s) + \overline{X_u}(t)\right) = 4 - 2(1 - r_u(s,t)) \le 4 - \rho^2(|t-s|) \le 4 - \mathbb{C}_1 u^{-2}|(l-k-1)S|^{\alpha/2}
$$

and for $(s,t),(s',t') \in I_k(u)\times I_l(u)$ with $N_1(u) \le k, l \le N_2(u)$

$$
\begin{aligned}
&1 - Cov\left(\frac{\overline{X_u}(s)+\overline{X_u}(t)}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}}, \frac{\overline{X_u}(s')+\overline{X_u}(t')}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right) \\
&= \frac{1}{2}\mathbb{E}\left\{\left(\frac{\overline{X_u}(s)+\overline{X_u}(t)}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}} - \frac{\overline{X_u}(s')+\overline{X_u}(t')}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right)^2\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')+\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}\\
&\quad +Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)\left(\frac{1}{\sqrt{Var\left(\overline{X_u}(s)+\overline{X_u}(t)\right)}}-\frac{1}{\sqrt{Var\left(\overline{X_u}(s')+\overline{X_u}(t')\right)}}\right)^2\\
&\le 2\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')\right)^2\right\}+2\mathbb{E}\left\{\left(\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}+\mathbb{E}\left\{\left(\overline{X_u}(s)-\overline{X_u}(s')+\overline{X_u}(t)-\overline{X_u}(t')\right)^2\right\}\\
&\le 8(1-r_u(s,s')+1-r_u(t,t'))\\
&= 16u^{-2}\left(\left|\frac{s-s'}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}+\left|\frac{t-t'}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}\right).
\end{aligned}
$$

In view of our assumptions, we can find centered homogeneous Gaussian random fields $Z_u(s,t)$ with correlation

$$
r_{Z_u}(s,t)=\exp\left(-32u^{-2}\left(\left|\frac{s}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}+\left|\frac{t}{\overleftarrow{\rho}(u^{-1})}\right|^{\alpha/2}\right)\right).
$$

Slepian inequality, Lemma 4.2 and (44) imply

$$
\begin{aligned}
\Lambda_2(u) &\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2}\mathbb{P}\left\{\sup_{s\in I_k(u)}X_u(s)>M_u,\ \sup_{t\in I_l(u)}X_u(t)>M_u\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2}\mathbb{P}\left\{\sup_{(s,t)\in I_k(u)\times I_l(u)}(\overline{X_u}(s)+\overline{X_u}(t))>2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2}\mathbb{P}\left\{\sup_{(s,t)\in I_0(u)\times I_0(u)}Z_u(s,t)>\frac{2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)}{\sqrt{4-\mathbb{C}_1u^{-2}|(l-k-1)S|^{\alpha/2}}}\right\}\\
&\le \sum_{N_1(u)\le k,l\le N_2(u),l\ge k+2}\left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2\Psi\left(\frac{2\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)}{\sqrt{4-\mathbb{C}_1u^{-2}|(l-k-1)S|^{\alpha/2}}}\right)\\
&\le 2\sum_{k=N_1(u)}^{N_2(u)}\sum_{l=1}^{N_2(u)-N_1(u)}\left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2\Psi\left(\frac{2\mathcal{G}_{u,-\varepsilon}(k)}{\sqrt{4-\mathbb{C}_1u^{-2}(lS)^{\alpha/2}}}\right)\\
&\le 2\sum_{k=N_1(u)}^{N_2(u)}\left(\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\right)^2\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\sum_{l=1}^{\infty}e^{-\mathbb{C}_2(lS)^{\alpha/2}}\\
&\le 2\mathcal{H}_{\alpha/2}32^{2/\alpha}Se^{-\mathbb{C}_3S^{\alpha/2}}\sum_{k=N_1(u)}^{N_2(u)}\mathcal{H}_{\alpha/2}[0,32^{2/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\\
&= o(\Theta(u)),\quad u\to\infty,S\to\infty,\varepsilon\to 0,
\end{aligned}\tag{47}
$$

where $\widetilde{\mathcal{G}}_{u,-\varepsilon}(k,l)=\min(\mathcal{G}_{u,-\varepsilon}(k),\mathcal{G}_{u,-\varepsilon}(l))$. Combing (43)-(46) with (47), we obtain

$$
\pi(u)\sim\Theta(u),\quad u\to\infty.
$$

Case 2) $\eta\in(0,\infty)$: This implies $\lambda=2/\alpha$.

Set for any small constant $\theta\in(0,1)$ and any constant $S_1>0$

$$
S_1^*=\begin{cases}-S_1, & \text{if } x_1=-\infty;\\ (x_1+\theta)\eta^{1/\alpha}, & \text{if } x_1\in(-\infty,\infty),\end{cases}\qquad S_2^*=\begin{cases}(x_2-\theta)\eta^{1/\alpha}, & \text{if } x_2\in(-\infty,\infty);\\ S_1, & \text{if } x_2=\infty,\end{cases}\tag{48}
$$

$$
S_1^{**}=\begin{cases}-S, & \text{if } x_1=-\infty;\\ (x_1-\theta)\eta^{1/\alpha}, & \text{if } x_1\in(-\infty,\infty),\end{cases}\qquad S_2^{**}=\begin{cases}(x_2+\theta)\eta^{1/\alpha}, & \text{if } x_2\in(-\infty,\infty);\\ S, & \text{if } x_2=\infty.\end{cases}\tag{49}
$$

With $K^*=[\overleftarrow{\rho}(u^{-1})S_1^*,\overleftarrow{\rho}(u^{-1})S_2^*]$ and $K^{**}=[\overleftarrow{\rho}(u^{-1})S_1^{**},\overleftarrow{\rho}(u^{-1})S_2^{**}]$ we have for any $S_1>0$ and u large enough

$$
\pi(u)\ge\mathbb{P}\left\{\sup_{t\in K^*}X_u(t)>M_u\right\},\tag{50}
$$

$$\pi(u) \le \mathbb{P}\left\{\sup_{t\in K^{**}} X_u(t) > M_u\right\} + \sum_{\substack{k=N_1(u)\\ k\neq 0,-1}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\}. \tag{51}$$

Using Slepian inequality and Lemma 4.1, we have that

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in K^*} X_u(t) > M_u\right\} &\ge \mathbb{P}\left\{\sup_{t\in K^*} \frac{Y_{-\varepsilon}(t)}{\mathcal{F}_{u,+\varepsilon}(t)} > M_u\right\}\\ &\sim \mathcal{P}_{\alpha,1}^{h_{+\varepsilon}}[S_1^*, S_2^*]\Psi(M_u),\ u\to\infty,\end{aligned}$$

where $h_{\pm\varepsilon}(t) = (1\pm\varepsilon)f(\eta^{-1/\alpha}t)\pm\varepsilon$, and similarly

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in K^{**}} X_u(t) > M_u\right\} &\le \mathbb{P}\left\{\sup_{t\in K^{**}} \frac{Y_{+\varepsilon}(t)}{\mathcal{F}_{u,-\varepsilon}(t)} > M_u\right\}\\ &\sim \mathcal{P}_{\alpha,1}^{h_{-\varepsilon}}[S_1^{**}, S_2^{**}]\Psi(M_u),\ u\to\infty. \end{aligned}\tag{52}$$

Moreover, in light of (6), the Slepian inequality and Lemma 4.1

$$\begin{aligned}\sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} X_u(t) > M_u\right\} &\le \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \frac{Y_{+\varepsilon}(t)}{\mathcal{F}_{u,-\varepsilon}(t)} > M_u\right\}\\ &\le \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\}\\ &\sim \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right)\\ &\sim \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(M_u)\sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} e^{-\inf_{s\in[k,k+1]}\left((1-\varepsilon)f(s\eta^{-1/\alpha}S)-\varepsilon\right)}\\ &\sim \mathbb{C}_4\mathcal{H}_\alpha\Psi(M_u)Se^{-\mathbb{C}_5(\eta^{-1/\alpha}S)^{\epsilon_1/2}}e^{\varepsilon}\\ &= o\left(\Psi(M_u)\right),\ u\to\infty, S\to\infty,\varepsilon\to 0. \end{aligned}\tag{53}$$

Letting $\varepsilon\to 0,\ S_1\to\infty,\ S\to\infty$, and $\theta\to 0$ we obtain

$$\pi(u)\sim \mathcal{P}_{\alpha,\eta}^f[x_1,x_2]\Psi(M_u),\quad u\to\infty.$$

Next, if we set $x_1(u) = -\left(\frac{\ln u}{u}\right)^\lambda, x_2(u) = \left(\frac{\ln u}{u}\right)^\lambda$, then

$$x_1=-\infty,\quad x_2=\infty,\quad S_1^*=-S_1,\quad S_2^*=S_1,\quad S_1^{**}=-S,\quad S_2^{**}=S.$$

Inserting (52), (53) into (51) and letting $\varepsilon\to 0$ leads to

$$\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)} \le \mathcal{P}_{\alpha,\eta}^f[-S,S] + \mathbb{C}_4\mathcal{H}_\alpha Se^{-\mathbb{C}_5(\eta^{-1/\alpha}S)^{\epsilon_1/2}} < \infty.$$

By (50), we have

$$\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)} \ge \mathcal{P}_{\alpha,\eta}^f[-S_1,S_1] > 0.$$

Letting $S_1\to\infty, S\to\infty$ we obtain

$$\mathcal{P}_{\alpha,\eta}^f(-\infty,\infty)\in(0,\infty),\quad \pi(u)\sim\mathcal{P}_{\alpha,\eta}^f(-\infty,\infty)\Psi(M_u),\quad u\to\infty.$$

Case 3) $\eta=0$: Note that

$$\pi(u)\le \mathbb{P}\left\{\sup_{t\in((I_{-1}(u)\cup I_0(u))\cap\Delta(u))} \overline{X_u}(t)\sigma_u(t) > M_u\right\} + \sum_{\substack{k=N_1(u)\\ k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_k(u)} \overline{X_u}(t)\sigma_u(t) > M_u\right\} =: J_1(u)+J_2(u).$$

By (41)

$$\frac{1}{\mathcal{F}_{u,+\varepsilon}(t)} \le \sigma_u(t) \le \frac{1}{\mathcal{F}_{u,-\varepsilon}(t)} \le \frac{1}{1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^{\lambda}s)-\varepsilon]} \tag{54}$$

holds for all $t \in \Delta(u)$. Hence Lemma 4.1 implies

$$\begin{aligned} J_1(u) &\le \mathbb{P}\left\{\sup_{t\in[-\overleftarrow{\rho}(u^{-1})S,\overleftarrow{\rho}(u^{-1})S]} \overline{X_u}(t) > M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^{\lambda}s)-\varepsilon]\right)\right\} \\ &\le \mathbb{P}\left\{\sup_{t\in[-\overleftarrow{\rho}(u^{-1})S,\overleftarrow{\rho}(u^{-1})S]} Y_{+\varepsilon}(t) > M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^{\lambda}s)-\varepsilon]\right)\right\} \\ &\sim \mathcal{H}_\alpha[0,2(1+\varepsilon)^{1/\alpha}S]\Psi\left(M_u\left(1+u^{-2}\inf_{s\in\Delta(u)}[(1-\varepsilon)f(u^{\lambda}s)-\varepsilon]\right)\right) \\ &\sim \mathcal{H}_\alpha[0,2(1+\varepsilon)^{1/\alpha}S]\Psi\left(M_u\right)e^{-(1-\varepsilon)\omega^*+\varepsilon} \\ &\sim \Psi\left(M_u\right)e^{-\omega^*},\ u\to\infty,\ S\to 0,\ \varepsilon\to 0, \end{aligned}$$

where $\omega^* = \inf_{t\in[x_1,x_2]} f(t)$. Furthermore, by Lemma 4.1, for any $x>0$

$$\begin{aligned} J_2(u) &\le \sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathbb{P}\left\{\sup_{t\in I_0(u)} Y_{+\varepsilon}(t) > \mathcal{G}_{u,-\varepsilon}(k)\right\} \sim \sum_{\substack{k=N_1(u)\\k\neq -1,0}}^{N_2(u)} \mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi\left(\mathcal{G}_{u,-\varepsilon}(k)\right) \\ &\le 2\mathcal{H}_\alpha[0,(1+\varepsilon)^{1/\alpha}S]\Psi(M_u)\sum_{k=1}^{\infty} e^{-(1-2\varepsilon)(kxS)^{\epsilon_1/2}+2\varepsilon} \\ &\le \mathbb{C}_6\mathcal{H}_\alpha\Psi(M_u)Se^{-\mathbb{C}_7(xS)^{\epsilon_1/2}} = o\left(\Psi(M_u)\right),\ u\to\infty, x\to\infty,\ S\to 0, \end{aligned} \tag{55}$$

hence

$$\lim_{u\to\infty}\frac{\pi(u)}{\Psi(M_u)} \le e^{-\omega^*}, \quad u\to\infty.$$

Next, since $f \in C_0^*([x_1,x_2])$ there exists $y(u)\in\Delta(u)$ satisfying

$$\lim_{u\to\infty} y(u)u^{\lambda} = y \in \{z\in[x_1,x_2] : f(z)=\omega^*\}.$$

Consequently, in view of (54)

$$\begin{aligned} \pi(u) &\ge \mathbb{P}\left\{X_u(y(u)) > M_u\right\} \\ &\ge \mathbb{P}\left\{\overline{X}_u(y(u)) > M_u(1+[(1+\varepsilon)f(u^{\lambda}y(u))+\varepsilon]u^{-2})\right\} \\ &= \Psi\left(M_u(1+(1+\varepsilon)[f(u^{\lambda}y(u))+\varepsilon]u^{-2})\right) \\ &\sim \Psi\left(M_u\right)e^{-f(y)},\ u\to\infty,\ \varepsilon\to 0, \end{aligned}$$

which implies that

$$\pi(u) \sim \Psi\left(M_u\right)e^{-\omega^*}, \quad u\to\infty$$

establishing the proof. □

Proof of Theorem 2.4: Clearly, for any $u>0$

$$\pi(u) \le \mathbb{P}\left\{\sup_{t\in[0,T]} X_u(t) > M_u\right\} \le \pi(u)+\pi_1(u),$$

where with $D(u) := [0,T]\setminus(t_u+\Delta(u))$,

$$\pi(u) := \mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(t_u+t) > M_u\right\},\quad \pi_1(u) := \mathbb{P}\left\{\sup_{t\in D(u)} X_u(t) > M_u\right\}.$$

Next, we derive an upper bound for $\pi_1(u)$ which will finally imply that

$$\pi_1(u) = o(\pi(u)), \quad u\to\infty. \tag{56}$$

Thus by **A4**, **A5** and Piterbarg inequality (see e.g., [10][Theorem 8.1], [56][Theorem 3] and [35][Lemma 5.1])

$$\begin{aligned}\pi_1(u) &= \mathbb{P}\left\{\sup_{t\in D(u)} \overline{X}_u(t)\sigma_u(t) > M_u\right\}\\ &\le \mathbb{P}\left\{\sup_{t\in D(u)} \overline{X}_u(t) > M_u + \mathbb{C}_1\frac{p(\ln u)^q}{u}\right\}\\ &\le \mathbb{C}_2 T M_u^{2/\varsigma}\Psi\left(M_u + \mathbb{C}_1\frac{p(\ln u)^q}{u}\right)\\ &= o\left(\Psi\left(M_u\right)\right), \quad u\to\infty. \qquad (57)\end{aligned}$$

Since **A1'** implies **A1**, by Theorem 2.2 and **A2**, **A3**, we have

$$\pi(u) \sim \Psi\left(M_u\right)\begin{cases}\frac{\mathcal{H}_\alpha}{u^\lambda \overleftarrow{\rho}(u^{-1})}\int_{x_1}^{x_2} e^{-f(t)}dt, & \text{if } \eta=\infty,\\ \mathcal{P}_{\alpha,\eta}^f[x_1,x_2], & \text{if } \eta\in(0,\infty),\\ 1, & \text{if } \eta=0,\end{cases} \quad u\to\infty, \qquad (58)$$

where the result of case $\eta = 0$ comes from the fact that $f(t) \ge 0$ for $t \in [x_1, x_2]$, $f(0) = 0$ and $0 \in [x_1, x_2]$. Consequently, it follows from (57) and (58) that (56) holds, and thus the proof is complete. □

Proof of Proposition 3.1: Without loss of generality we assume that $g_m = g(t_0) = 0$.

i) We present first the proof for $t_0 \in (0,T)$. Let $\Delta(u) = [-\delta(u), \delta(u)]$, where $\delta(u) = \left(\frac{(\ln u)^q}{u}\right)^{1/\gamma}$ with some large $q > 1$. By (4) for u large enough and some small $\varepsilon \in (0,1)$

$$1+\frac{(1-\varepsilon)c\left|t\right|^\gamma}{u} \le \frac{1}{\sigma_u(t+t_0)} := \frac{u-g(t+t_0)}{u} = 1-\frac{g(t+t_0)}{u} \le 1+\frac{(1+\varepsilon)c\left|t\right|^\gamma}{u} \qquad (59)$$

holds for all $t \in [-\theta, \theta], \theta > 0$. It follows that

$$\Pi(u) \le \mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t)) > u\right\} \le \Pi(u) + \Pi_1(u),$$

with

$$\Pi_1(u) := \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta]}(X(t)+g(t)) > u\right\},$$

and

$$\Pi(u) := \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)+g(t)) > u\right\} = \mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]} X(t)\frac{u}{u-g(t)} > u\right\}.$$

By (59), we may further write

$$\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq0}\left|\frac{\frac{1}{\sigma_u(t_0+t)}-1}{cu^{-1}|t|^\gamma}-1\right| = \lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq0}\left|\frac{\frac{1}{\sigma_u(t_0+t)}-1}{cu^{-2}|u^{1/\gamma}t|^\gamma}-1\right| = 0, \qquad (60)$$

and

$$\inf_{t\in[-\theta,\theta]\setminus\Delta(u)}\frac{1}{\sigma_u(t+t_0)} \ge 1+\frac{(1-\varepsilon)c(\ln u)^q}{u^2}.$$

In addition, from (14) we have that

$$\lim_{u\to\infty}\sup_{\substack{s,t\in\Delta(u)\\ t\neq s}}\left|\frac{1-r(t_0+t,t_0+s)}{a|t-s|^\alpha}-1\right| = 0,$$

and

$$\sup_{s,t\in[t_0-\theta,t_0+\theta]}\mathbb{E}\left\{X(t)-X(s))^2\right\} \le \sup_{s,t\in[t_0-\theta,t_0+\theta]}(2-2r(s,t)) \le \mathbb{C}_1\left|t-s\right|^\alpha$$

hold when θ is small enough. Therefore, by Theorem 2.4

$$\Pi(u) \sim u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi(u)\begin{cases}\mathcal{H}_\alpha a^{\frac{1}{\alpha}}\int_{w_{t_0}}^\infty e^{-c|t|^\gamma}dt, & \text{if } \alpha<2\gamma,\\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[w_{t_0},\infty), & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma.\end{cases}$$

Moreover, since $g_\theta := \sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]} g(t)<0$ we have

$$\Pi_1(u)\le \mathbb{P}\left\{\sup_{t\in[0,T]\setminus[t_0-\theta,t_0+\theta]} X(t)>u-g_\theta\right\}\sim \mathcal{H}_\alpha\int_0^T\frac{1}{a(t)}dt\, u^{\frac{2}{\alpha}}\Psi(u-g_\theta)=o(\Pi(u)),\ u\to\infty,$$

hence the claims follow.

For $t_0=0$ and $t_0=T$, we just need to replace $\Delta(u)$ by $\Delta(u)=[0,\delta(u)]$ and $\Delta(u)=[-\delta(u),0]$, respectively.

ii) Applying [10][Theorem 7.1] we obtain

$$\mathbb{P}\left\{\sup_{t\in[A,B]}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in[A,B]}X(t)>u\right\}\sim\int_A^B(a(t))^{1/\alpha}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u).$$

Set $\Delta_\varepsilon=[A-\varepsilon,B+\varepsilon]\cap[0,T]$ for some $\varepsilon>0$, then we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}&\ge \mathbb{P}\left\{\sup_{t\in[A,B]}(X(t)+g(t))>u\right\},\\ \mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}&\le \mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}(X(t)+g(t))>u\right\}+\mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}(X(t)+g(t))>u\right\}.\end{aligned}$$

Since g is a continuous function and $g_\varepsilon:=\sup_{t\in[0,T]\setminus\Delta_\varepsilon}g(t)<0$

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}(X(t)+g(t))>u\right\}&\le \mathbb{P}\left\{\sup_{t\in[0,T]\setminus\Delta_\varepsilon}X(t)>u-g_\varepsilon\right\}\\ &\le \mathbb{C}_2u^{2/\alpha}\Psi(u-g_\varepsilon)=o\left(u^{2/\alpha}\Psi(u)\right),\quad u\to\infty,\varepsilon\to0.\end{aligned}$$

Further, we have

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}(X(t)+g(t))>u\right\}&\le \mathbb{P}\left\{\sup_{t\in\Delta_\varepsilon}X(t)>u\right\}\sim\int_{A-\varepsilon}^{B+\varepsilon}(a(t))^{\frac{1}{\alpha}}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u)\\ &\sim\int_A^B(a(t))^{\frac{1}{\alpha}}dt\mathcal{H}_\alpha u^{\frac{2}{\alpha}}\Psi(u),\quad u\to\infty,\varepsilon\to0.\end{aligned}$$

Hence the claims follow. □

Proof of Proposition 3.3: We give the proof only for $t_0=0$. In this case, $x\in(0,\infty)$. By definition

$$\mathbb{P}\left\{u^{1/\gamma}(\tau_u-t_0)\le x\middle|\tau_u\le T\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,u^{-1/\gamma}x]}(X(t)+g(t))>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}}.$$

Set $\Delta(u)=[0,u^{-1/\gamma}x]$. For all u large

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in\Delta(u)}X(t)\frac{u}{u-g(t)}>u\right\}.$$

Denote $X_u(t)=X(t)\frac{u}{u-g(t)}$ and $\sigma_u(t)=\frac{u}{u-g(t)}$. As in the proof of Proposition 3.1 i), by Theorem 2.2 we obtain

$$\mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(t)+g(t))>u\right\}\sim u^{(\frac{2}{\alpha}-\frac{1}{\gamma})_+}\Psi(u)\begin{cases}a^{\frac{1}{\alpha}}\mathcal{H}_\alpha\int_0^x e^{-c|t|^\gamma}dt, & \text{if } \alpha<2\gamma,\\ \mathcal{P}_{\alpha,a}^{c|t|^\gamma}[0,x], & \text{if } \alpha=2\gamma,\\ 1, & \text{if } \alpha>2\gamma.\end{cases}$$

Consequently, by Proposition 3.1 statement i), the results follow. □

Proof of Proposition 3.6: Clearly, for any $u>0$

$$\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}=\mathbb{P}\left\{\sup_{t\in[0,T]}\overline{X}(t)\frac{m_u(t)}{m_u(t_u)}>\frac{u-g(t_u)}{\sigma(t_u)}\right\},$$

and **A1'** is satisfied. By the continuity of $\sigma(t)$, $\lim_{u\to\infty} t_u = t_0$ and $\sigma(t_0)=1$, we have that for u large enough

$$\sigma(t_u)>0, \text{ and } \frac{u-g(t_u)}{\sigma(t_u)}\sim u,\ u\to\infty.$$

Set next

$$X_u(t) = \overline{X}(t)\frac{m_u(t)}{m_u(t_u)}, \quad t\in[0,T],$$

which has standard deviation function $\sigma_u(t)=\frac{m_u(t_u+t)}{m_u(t_u)}$ and correlation function $r_u(s,t)=r(s,t)$ satisfying assumptions **A2**–**A4**. Further, $\overline{X}_u(t)=\overline{X}(t)$ implies **A5**. Hence the claims follow from Theorem 2.4. □

Proof of Proposition 3.8: For all u large

$$1-r(t_u+t,t_u+s)=\frac{\mathbb{E}\left\{[X(t_u+t)-X(t_u+s)]^2\right\}-[\sigma(t_u+t)-\sigma(t_u+s)]^2}{2\sigma(t_u+t)\sigma(t_u+s)}. \tag{61}$$

Using that

$$\begin{aligned}\mathbb{E}\left\{[X(t_u+t)-X(t_u+s)]^2\right\} &= \mathbb{E}\left\{X'^2(t_u+s)\right\}(t-s)^2+o((t-s)^2),\\ [\sigma(t_u+t)-\sigma(t_u+s)]^2 &= \sigma'^2(t_u+t)(t-s)^2+o((t-s)^2),\end{aligned}$$

we have, as $u\to\infty$

$$1-r(t_u+t,t_u+s)=\frac{\mathbb{E}\left\{X'^2(t_u+t)\right\}-\sigma'^2(t_u+t)}{2\sigma(t_u+t)\sigma(t_u+s)}(t-s)^2+o((t-s)^2).$$

Since $D(s,t):=\frac{\mathbb{E}\left\{X'^2(t)\right\}-\sigma'^2(t)}{2\sigma(s)\sigma(t)}$ is continuous at (t_0,t_0), then setting $D=D(t_0,t_0)$ we obtain

$$\lim_{u\to\infty}\sup_{\substack{t\in\Delta(u),s\in\Delta(u)\\ t\neq s}}\left|\frac{1-r(t_u+t,t_u+s)}{D|t-s|^2}-1\right|=0,$$

which implies that **A3** is satisfied. Next we suppose that $\sigma(t)>\frac{1}{2}$ for any $t\in[0,T]$, since if we set $E_1=\{t\in[0,T]:\sigma(t)\le\frac{1}{2}\}$, by Borell-TIS inequality

$$\mathbb{P}\left\{\sup_{t\in E_1}(X(t)+g(t))>u\right\}\le\exp\left(-2\left(u-\sup_{t\in[0,T]}g(t)-\mathbb{C}_1\right)^2\right)=o\left(\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right)\right)$$

as $u\to\infty$, where $\mathbb{C}_1=\mathbb{E}\left\{\sup_{t\in[0,T]}X(t)\right\}<0$. Further by (61)

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(s))^2\right\}\le 2-2r(t,s)\le 4\left(\sup_{\theta\in[0,T]}\mathbb{E}\left\{X'^2(\theta)\right\}(t-s)^2-\inf_{\theta\in[0,T]}\sigma'^2(\theta)(t-s)^2\right),$$

then **A5** is satisfied. Consequently, the conditions of Proposition 3.6 are satisfied and hence the claim follows. □

Proof of Proposition 3.9: Without loss of generality we assume that $g(t)$ satisfies (4) with $g(t_0)=0$.

First we present the proof for $t_0\in(0,T)$. Clearly, m_u attains its maximum at the unique point t_0. Further, we have

$$\frac{m_u(t_0)}{m_u(t_0+t)}-1=\frac{1}{\sigma(t_0+t)}(1-\sigma(t_0+t))-\frac{g(t_0+t)}{u\sigma(t_0+t)}.$$

Consequently, by (2) and (4)

$$\frac{m_u(t_0)}{m_u(t_0+t)}=1+\left(b\,|t|^\beta+\frac{c}{u}\,|t|^\gamma\right)(1+o(1)),\quad t\to 0 \tag{62}$$

holds for all u large. Further, set $\Delta(u)=[-\delta(u),\delta(u)]$, where $\delta(u)=\left(\frac{(\ln u)^q}{u}\right)^{2/\beta^*}$ for some constant $q>1$ with $\beta^*=\min(\beta,2\gamma)$, and let $f(t)=b|t|^\beta\mathbb{I}_{\{\beta=\beta^*\}}+c|t|^\gamma\mathbb{I}_{\{2\gamma=\beta^*\}}$. We have

$$\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq0}\left|\frac{\left(\frac{m_u(t_0)}{m_u(t_0+t)}-1\right)u^2-f(u^{2/\beta^*}t)}{f(u^{2/\beta^*}t)+\mathbb{I}_{\{\beta\neq2\gamma\}}}\right|=0. \tag{63}$$

By (2)

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(s))^2\right\}=\mathbb{E}\left\{(\overline{X}(t))^2\right\}+\mathbb{E}\left\{(\overline{X}(s))^2\right\}-2\mathbb{E}\left\{\overline{X}(t)\overline{X}(s)\right\}=2-2r(s,t)\le\mathbb{C}_1|t-s|^\alpha \tag{64}$$

holds for $s, t \in [t_0 - \theta, t_0 + \theta]$, with $\theta > 0$ sufficiently small. By (62), for any $\varepsilon > 0$

$$\frac{m_u(t_0)}{m_u(t_0 + t)} \geq 1 + \mathbb{C}_2(1 - \varepsilon)\frac{(\ln u)^q}{u} \tag{65}$$

holds for all $t \in [-\theta, \theta] \setminus \Delta(u)$. Further

$$\Pi(u) := \mathbb{P}\left\{\sup_{t\in[t_0-\theta, t_0+\theta]} (X(t) + g(t)) > u\right\} \leq \mathbb{P}\left\{\sup_{t\in[0,T]} (X(t) + g(t)) > u\right\} \leq \Pi(u) + \Pi_1(u),$$

with

$$\Pi_1(u) := \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta, t_0+\theta])} (X(t) + g(t)) > u\right\}.$$

By(63), (2), (65), (64) which imply **A2–A5** and Proposition 3.6, we have

$$\Pi(u) \sim u^{(\frac{2}{\alpha} - \frac{2}{\beta^*})_+}\Psi(u) \begin{cases} \mathcal{H}_\alpha a^{1/\alpha} \int_{w_{t_0}}^{\infty} e^{-f(t)} dt, & \text{if } \alpha < \beta^*, \\ \mathcal{P}_{\alpha,a}^f[w_{t_0}, \infty), & \text{if } \alpha = \beta^*, \\ 1, & \text{if } \alpha > \beta^*. \end{cases} \tag{66}$$

In order to complete the proof it suffices to show that

$$\Pi_1(u) = o(\Pi(u)).$$

Since $\sigma_\theta := \max_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])} \sigma(t) < 1$, by the Borell-TIS inequality we have

$$\Pi_1(u) \leq \mathbb{P}\left\{\sup_{t\in([0,T]\setminus[t_0-\theta,t_0+\theta])} X(t) > u\right\} \leq \exp\left(-\frac{(u - \mathbb{C}_3)^2}{2\sigma_\theta^2}\right) = o(\Pi(u)),$$

where $\mathbb{C}_3 = \mathbb{E}\left\{\sup_{t\in[0,T]} X(t)\right\} < \infty$.
For the cases $t_0 = 0$ and $t_0 = T$, we just need to replace $\Delta(u)$ by $[0, \delta(u)]$ and $[-\delta(u), 0]$, respectively. Hence the proof is complete. □

Proof of Proposition 3.10: i) We shall present the proof only for the case $t_0 \in (0, T)$. In this case, $[x_1, x_2] = \mathbb{R}$. By definition, for any $x \in \mathbb{R}$

$$\mathbb{P}\left\{u^\lambda(\tau_u - t_u) \leq x \middle| \tau_u \leq T\right\} = \frac{\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}(X(t) + g(t)) > u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t) + g(t)) > u\right\}}.$$

For $u > 0$ define

$$X_u(t) = \overline{X}(t_u + t)\frac{m_u(t_u + t)}{m_u(t_u)}, \quad \sigma_u(t) = \frac{m_u(t_u + t)}{m_u(t_u)}.$$

As in the proof of Proposition 3.6, we obtain

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]} (X(t) + g(t)) > u\right\} = \mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]} X_u(t) > \frac{u - g(t_u)}{\sigma(t_u)}\right\},$$

and **A1'**, **A2–A5** are satisfied with $\Delta(u) = [-\delta_u, u^{-\lambda}x]$. Clearly, for any $u > 0$

$$\pi(u) \leq \mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]} X_u(t) > \frac{u - g(t_u)}{\sigma(t_u)}\right\} \leq \pi(u) + \pi_1(u),$$

where

$$\pi(u) = \mathbb{P}\left\{\sup_{t\in[t_u-\delta(u),t_u+u^{-\lambda}x]} X_u(t) > \frac{u - g(t_u)}{\sigma(t_u)}\right\}, \quad \pi_1(u) = \mathbb{P}\left\{\sup_{t\in[0,t_u-\delta(u)]} X_u(t) > \frac{u - g(t_u)}{\sigma(t_u)}\right\}.$$

Applying Theorem 2.2 we have

$$\pi(u) \sim \Psi\left(\frac{u - g(t_u)}{\sigma(t_u)}\right) \begin{cases} \frac{\mathcal{H}_\alpha}{u^\lambda \overleftarrow{\rho}(u^{-1})} \int_{-\infty}^{x} e^{-f(t)} dt, & \text{if } \eta = \infty, \\ \mathcal{P}_{\alpha,\eta}^f(-\infty, x], & \text{if } \eta \in (0, \infty), \\ \sup_{t\in(-\infty,x]} e^{-f(t)}, & \text{if } \eta = 0. \end{cases} \tag{67}$$

In view of (57)

$$\pi_1(u) = o\left(\Psi\left(\frac{u-g(t_u)}{\sigma(t_u)}\right)\right), \quad u \to \infty,$$

hence

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-\lambda}x]}(X(t)+g(t))>u\right\} \sim \pi(u), \quad u\to\infty$$

and thus the claim follows by (67) and Proposition 3.6.

ii) We give the proof of $t_0=T$. In this case $x\in(-\infty,0)$ implying

$$\mathbb{P}\left\{u^{2/\beta^*}(\tau_u-T)\le x \middle| \tau_u\le T\right\} = \frac{\mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t))>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t)+g(t))>u\right\}}.$$

Set $\delta_u = \left(\frac{(\ln u)^q}{u}\right)^{2/\beta^*}$ for some $q>1$ and let

$$\Delta(u) = [-\delta_u, u^{-2/\beta^*}x], \quad \sigma_u(t) = \frac{m_u(t)}{m_u(T)},$$

with

$$m_u(t) = \frac{\sigma(t)}{1-g(t)/u}, \qquad X_u(t) = \overline{X}(t)\frac{m_u(t)}{m_u(T)}.$$

For all u large, we have

$$\pi(u) \le \mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t))>u\right\} \le \pi(u) + \mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]}(X(t)+g(t))>u\right\},$$

where

$$\pi(u) := \mathbb{P}\left\{\sup_{t\in\Delta(u)}(X(T+t)+g(T+t))>u\right\} = \mathbb{P}\left\{\sup_{t\in\Delta(u)} X_u(T+t)>u\right\}.$$

As in the proof of Proposition 3.9 it follows that the Assumptions **A2**–**A5** hold with $\Delta(u) = [-\delta_u, u^{-2/\beta^*}x]$. Hence an application of Theorem 2.2 yields

$$\pi(u) \sim u^{(\frac{2}{\alpha}-\frac{2}{\beta^*})_+}\Psi(u)\begin{cases} a^{1/\alpha}\mathcal{H}_\alpha\int_{-x}^\infty e^{-f(t)}dt, & \text{if } \alpha<\beta^*,\\ \mathcal{P}_{\alpha,a}^f[-x,\infty), & \text{if } \alpha=\beta^*,\\ e^{-f(x)}, & \text{if } \alpha>\beta^*.\end{cases} \tag{68}$$

In view of (57)

$$\mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]}(X(t)+g(t))>u\right\} = \mathbb{P}\left\{\sup_{t\in[0,T-\delta_u]} X_u(t)>u\right\} = o\left(\Psi(u)\right), \quad u\to\infty$$

implying

$$\mathbb{P}\left\{\sup_{t\in[0,T+u^{-2/\beta^*}x]}(X(t)+g(t))>u\right\} \sim \pi(u), \quad u\to\infty.$$

Consequently, the proof follows by (68) and Proposition 3.9. □

Proof of Proposition 3.12: Set next $A(t)=\int_0^t e^{-\delta v}dB(v)$ and define

$$\widetilde{U}(t) = u + c\int_0^t e^{-\delta v}dv - \sigma A(t), \quad t\ge 0.$$

Since

$$\sup_{t\in[0,\infty)}\mathbb{E}\left\{[A(t)]^2\right\} = \frac{1}{2\delta}$$

implying $\sup_{t\in[0,\infty)}\mathbb{E}\left\{|A(t)|\right\}<\infty$, then by the martingale convergence theorem in [57] we have that $\widetilde{U}(\infty) := \lim_{t\to\infty}\widetilde{U}(t)$ exists and is finite almost surely. Clearly, for any $u>0$

$$p(u) \quad = \quad \mathbb{P}\left\{\inf_{t\in[0,\infty)}\widetilde{U}(t)<0\right\}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{\sup_{t\in[0,\infty]}\left(\sigma A(t)-c\int_0^t e^{-\delta v}dv\right)>u\right\}\\
&= \mathbb{P}\left\{\sup_{t\in[0,1]}\left(\sigma A(-\frac{1}{2\delta}\ln t)-\frac{c}{\delta}(1-t^{\frac{1}{2}})\right)>u\right\}.
\end{aligned}
$$

The proof will follow by applying Proposition 3.6, hence we check next the assumptions therein for this specific model. Below, we set $Z(t)=\sigma A(-\frac{1}{2\delta}\ln t)$ with variance function given by

$$
V_Z^2(t)=Var\left(\sigma\int_0^{-\frac{1}{2\delta}\ln t}e^{-\delta v}dB(v)\right)=\frac{\sigma^2}{2\delta}(1-t),\quad t\in[0,1].
$$

We show next that for u sufficiently large, the function

$$
M_u(t):=\frac{uV_Z(t)}{G_u(t)}=\frac{\frac{\sigma}{\sqrt{2\delta}}\sqrt{1-t}}{1+\frac{c}{\delta u}(1-t^{1/2})},\quad 0\le t\le 1,
$$

with $G_u(t):=u+\frac{c}{\delta}(1-t^{\frac{1}{2}})$ attains its maximum at the unique point $t_u=\left(\frac{c}{\delta u+c}\right)^2$. In fact, we have

$$
\begin{aligned}
[M_u(t)]_t:=\frac{dM_u(t)}{dt} &= \frac{dV_Z(t)}{dt}\cdot\frac{u}{G_u(t)}-\frac{V_Z(t)}{G_u^2(t)}\left(-\frac{cu}{2\delta}t^{-\frac{1}{2}}\right)=\frac{u}{2G_u^2(t)V_z(t)}\left[\frac{dV_Z^2(t)}{dt}G_u(t)+V_Z^2(t)\frac{ct^{-\frac{1}{2}}}{\delta}\right]\\
&= \frac{u\sigma^2t^{-1/2}}{4\delta G_u^2(t)V_Z(t)}\left[\frac{c}{\delta}-\left(u+\frac{c}{\delta}\right)t^{\frac{1}{2}}\right]. \qquad (69)
\end{aligned}
$$

Letting $[M_u(t)]_t=0$, we get $t_u=\left(\frac{c}{\delta u+c}\right)^2$. By (69), $[M_u(t)]_t>0$ for $t\in(0,t_u)$ and $[M_u(t)]_t<0$ for $t\in(t_u,1]$, so t_u is the unique maximum point of $M_u(t)$ over $[0,1]$. Further

$$
M_u:=M_u(t_u)=\frac{\sigma u}{\sqrt{2\delta u^2+4cu}}=\frac{\sigma}{\sqrt{2\delta}}(1+o(1)),\ u\to\infty.
$$

We set $\delta(u)=\left(\frac{(\ln u)^q}{u}\right)^2$ for some $q>1$, and $\Delta(u)=[-t_u,\delta(u)]$. Next we check the assumption **A2**. It follows that

$$
\frac{M_u}{M_u(t_u+t)}-1=\frac{[G_u(t_u+t)V_Z(t_u)]^2-[G_u(t_u)V_Z(t_u+t)]^2}{V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u)+V_Z(t_u+t)G_u(t_u)]}.
$$

We further write

$$
\begin{aligned}
&[G_u(t_u+t)V_Z(t_u)]^2-[G_u(t_u)V_Z(t_u+t)]^2\\
&\quad=\left[\left(u+\frac{c}{\delta}\right)-\frac{c}{\delta}\sqrt{t_u+t}\right]^2\frac{\sigma^2}{2\delta}(1-t_u)-\left[\left(u+\frac{c}{\delta}\right)-\frac{c}{\delta}\sqrt{t_u}\right]^2\frac{\sigma^2}{2\delta}(1-t_u-t)\\
&\quad=\left(u+\frac{c}{\delta}\right)^2\frac{\sigma^2}{2\delta}t-2\left(u+\frac{c}{\delta}\right)\frac{c\sigma^2}{2\delta^2}(\sqrt{t_u+t}-\sqrt{t_u})(1-t_u)-\frac{c^2\sigma^2}{2\delta^3}t\\
&\quad=\left(u+\frac{c}{\delta}\right)^2\frac{\sigma^2}{2\delta}t(1-t_u)-2\left(u+\frac{c}{\delta}\right)^2\frac{\sigma^2}{2\delta}(1-t_u)\sqrt{t_u}(\sqrt{t_u+t}-\sqrt{t_u})\\
&\quad=\frac{\sigma^2}{2\delta}\left[\left(u+\frac{c}{\delta}\right)^2-\left(\frac{c}{\delta}\right)^2\right](\sqrt{t+t_u}-\sqrt{t_u})^2\\
&\quad=\frac{\sigma^2}{2\delta}\left(u^2+\frac{2c}{\delta}u\right)(\sqrt{t+t_u}-\sqrt{t_u})^2.
\end{aligned}
$$

Since for any $t\in\Delta(u)$

$$
\sqrt{\frac{\sigma^2}{2\delta}(1-t_u-\delta(u))}\le V_Z(t_u+t)\le\sqrt{\frac{\sigma^2}{2\delta}},\ u+\frac{c}{\delta}-\frac{c}{\delta}\sqrt{t_u+\delta(u)}\le G_u(t_u+t)\le u+\frac{c}{\delta},
$$

we have for all large u

$$
V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u)+V_Z(t_u+t)G_u(t_u)]\le\frac{\sigma^2}{\delta}\left(u+\frac{c}{\delta}\right)^2
$$

and

$$
V_Z(t_u+t)G_u(t_u)[G_u(t_u+t)V_Z(t_u)+V_Z(t_u+t)G_u(t_u)]\quad\ge\quad\frac{\sigma^2}{\delta}(1-t_u-\delta(u))\left(u+\frac{c}{\delta}-\frac{c}{\delta}\sqrt{t_u+\delta(u)}\right)^2
$$

$$\geq \frac{\sigma^2}{\delta}\left[\left(u+\frac{c}{\delta}\right)^2-u\right].$$

Thus as $u\to\infty$

$$\inf_{t\in\Delta(u),t\neq 0}\frac{M_u/M_u(t_u+t)-1}{\frac{1}{2}\left(\sqrt{u^2t+\frac{c^2}{\delta^2}}-\frac{c}{\delta}\right)^2u^{-2}}-1\geq\frac{\frac{1}{2}\frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2}(\sqrt{t+t_u}-\sqrt{t_u})^2}{\frac{1}{2}\left(\sqrt{t+\frac{c^2}{(\delta u)^2}}-\frac{c}{\delta u}\right)^2}-1\geq\frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2}-1\to 0, \tag{70}$$

where we used the fact that for $t\in\Delta(u)$

$$(\sqrt{t+t_u}-\sqrt{t_u})^2\geq\left(\sqrt{t+\frac{c^2}{(\delta u)^2}}-\frac{c}{\delta u}\right)^2.$$

Furthermore, since

$$\begin{aligned}0&\leq\frac{\sqrt{t+t_u}-\sqrt{t_u}}{\sqrt{t+\frac{c^2}{(\delta u)^2}}-\frac{c}{\delta u}}-1=\frac{\sqrt{t+\frac{c^2}{(\delta u)^2}}+\frac{c}{\delta u}}{\sqrt{t+t_u}+\sqrt{t_u}}-1\leq\frac{\sqrt{t+\frac{c^2}{(\delta u)^2}}-\sqrt{t+t_u}}{\sqrt{t+t_u}+\sqrt{t_u}}\\&=\frac{\frac{c^2}{(\delta u)^2}-t_u}{(\sqrt{t+t_u}+\sqrt{t_u})(\sqrt{t+\frac{c^2}{(\delta u)^2}}+\sqrt{t+t_u})}\leq\frac{\sqrt{\frac{c^2}{(\delta u)^2}-t_u}}{\sqrt{t_u}}=\sqrt{\left(1+\frac{c}{\delta u}\right)^2-1},\end{aligned}$$

we have as $u\to\infty$

$$\begin{aligned}\sup_{t\in\Delta(u),t\neq 0}\frac{M_u/M_u(t_u+t)-1}{\frac{1}{2}\left(\sqrt{u^2t+\frac{c^2}{\delta^2}}-\frac{c}{\delta}\right)^2u^{-2}}-1&\leq\frac{\frac{1}{2}\frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2-u}(\sqrt{t+t_u}-\sqrt{t_u})^2}{\frac{1}{2}\left(\sqrt{t+\frac{c^2}{(\delta u)^2}}-\frac{c}{\delta u}\right)^2}-1\\&\leq\frac{u^2+\frac{2c}{\delta}u}{(u+\frac{c}{\delta})^2-u}\left(1+\sqrt{\left(1+\frac{c}{\delta u}\right)^2-1}\right)^2-1\to 0.\end{aligned} \tag{71}$$

Consequently, (70) and (71) imply

$$\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq 0}\left|\frac{M_u/M_u(t_u+t)-1}{\frac{1}{2}\left(\sqrt{u^2t+\frac{c^2}{\delta^2}}-\frac{c}{\delta}\right)^2u^{-2}}-1\right|=0. \tag{72}$$

Since for $0\leq t'\leq t<1$, the correlation function of $Z(t)$ equals

$$r(t,t')=\frac{\mathbb{E}\left\{(\sigma\int_0^{-\frac{1}{2\delta}\ln t}e^{-\delta v}dB(v))(\sigma\int_0^{-\frac{1}{2\delta}\ln t'}e^{-\delta v}dB(v))\right\}}{\sqrt{\frac{\sigma^2}{2\delta}(1-t)}\sqrt{\frac{\sigma^2}{2\delta}(1-t')}}=\frac{\sqrt{1-t}}{\sqrt{1-t'}}=1-\frac{t-t'}{\sqrt{1-t'}(\sqrt{1-t'}+\sqrt{1-t})},$$

we have

$$\begin{aligned}\sup_{t,t'\in\Delta(u),t'\neq t}\left|\frac{1-r(t_u+t,t_u+t')}{\frac{1}{2}|t-t'|}-1\right|&=\sup_{t,t'\in\Delta(u),t'\neq t}\left|\frac{2}{\sqrt{1-t-t_u}(\sqrt{1-t'-t_u}+\sqrt{1-t-t_u})}-1\right|\\&\leq\frac{1}{1-(\frac{c}{c+\delta u})^2-(\frac{(\ln u)^q}{u})^2}-1\to 0,\quad u\to\infty.\end{aligned} \tag{73}$$

Further, for some small $\theta\in(0,1)$, we obtain (set below $\overline{Z}(t)=\frac{Z(t)}{V_Z(t)}$)

$$\mathbb{E}\left(\overline{Z}(t)-\overline{Z}(t')\right)^2=2-2r(t,t')\leq\mathbb{C}_1|t-t'| \tag{74}$$

for $t,t'\in[0,\theta]$. For all u large

$$\Pi(u):=\mathbb{P}\left\{\sup_{t\in[0,\theta]}\left(Z(t)-\frac{c}{\delta}(1-t^{\frac{1}{2}})\right)>u\right\}\leq p(u)\leq\Pi(u)+\widetilde{\Pi}(u),$$

where

$$\widetilde{\Pi}(u):=\mathbb{P}\left\{\sup_{t\in[\theta,1]}\left(Z(t)-\frac{c}{\delta}(1-t^{\frac{1}{2}})\right)>u\right\}\leq\mathbb{P}\left\{\sup_{t\in[\theta,1]}Z(t)>u\right\}.$$

Moreover, for all u large

$$\begin{aligned}\frac{1}{M_u(t)}-\frac{1}{M_u} &\geq \frac{[G_u(t)V_Z(t_u)]^2-[G_u(t_u)V_Z(t)]^2}{2uV_Z^3(t_u)G_u(t_u)}=\frac{\frac{\sigma^2}{2\delta}(u^2+\frac{2c}{\delta}u)(\sqrt{t}-\sqrt{t_u})^2}{2u[\frac{\sigma^2}{2\delta}(1-t_u)]^{3/2}[u+\frac{c}{\delta}(1-\sqrt{t_u})]}\\ &\geq \mathbb{C}_2(\sqrt{t}-\sqrt{t_u})^2\geq \frac{\mathbb{C}_2\delta^2(u)}{\left(\sqrt{\delta(u)+t_u}+\sqrt{t_u}\right)^2}\geq \mathbb{C}_3\frac{(\ln u)^{2q}}{u^2} \qquad (75)\end{aligned}$$

holds for any $t\in[t_u+\delta(u),\theta]$, therefore

$$\inf_{t\in[t_u+\delta(u),\theta]}\frac{M_u}{M_u(t)}\geq 1+\mathbb{C}_3\frac{(\ln u)^q}{u^2}.$$

The above inequality combined with (72), (73), (74) and Proposition 3.6 yields

$$\Pi(u)\sim \mathcal{P}^h_{1,\delta/\sigma^2}\left[-\frac{c^2}{\delta^2},\infty\right)\Psi\left(\frac{1}{\sigma}\sqrt{2\delta u^2+4cu}\right),\ u\to\infty.$$

Finally, since

$$\sup_{t\in[\theta,1]}V_Z^2(t)\leq\frac{\sigma^2}{2\delta}(1-\theta),\quad\text{and}\quad \mathbb{E}\left\{\sup_{t\in[\theta,1]}Z(t)\right\}\leq\mathbb{C}_4<\infty,$$

by Borell-TIS inequality

$$\widetilde{\Pi}(u)\leq\mathbb{P}\left\{\sup_{t\in[\theta,1]}Z(t)>u\right\}\leq\exp\left(-\frac{\delta(u-\mathbb{C}_4)^2}{\sigma^2(1-\theta)}\right)=o(\Pi(u)),\quad u\to\infty,$$

which establishes the proof. Next, we consider that

$$\begin{aligned}\mathbb{P}\left\{u^2\left(e^{-2\delta\tau_u}-\left(\frac{c}{\delta u+c}\right)^2\right)\leq x\Big|\tau_u<\infty\right\}&=\frac{\mathbb{P}\left\{\inf_{t\in[-\frac{1}{2\delta}\ln(t_u+u^{-2}x),\infty)}\widetilde{U}(t)<0\right\}}{\mathbb{P}\left\{\inf_{t\in[0,\infty)}\widetilde{U}(t)<0\right\}}\\ &=\frac{\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-2}x]}\left(\sigma A(-\frac{1}{2\delta}\ln t)-\frac{c}{\delta}(1-t^{\frac12})\right)>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,1]}\left(\sigma A(-\frac{1}{2\delta}\ln t)-\frac{c}{\delta}(1-t^{\frac12})\right)>u\right\}}\\ &=\mathbb{P}\left\{u^2\left(\tau_u^*-t_u\right)\leq x\big|\tau_u^*<1\right\},\end{aligned}$$

where

$$\tau_u^*=\{t\in[0,1]:\sigma A(-\frac{1}{2\delta}\ln t)-\frac{c}{\delta}(1-t^{\frac12})>u\}.$$

The proof follows by Proposition 3.10 i). □

5. Appendix

Proof of (11): Let $\xi(t),t\in\mathbb{R}$ be a centered stationary Gaussian process with unit variance and correlation function r satisfying

$$1-r(t)\sim a|t|^\alpha,\ t\to 0,\quad a>0,\ \alpha\in(0,2].$$

In view of by Theorem 2.2, for $-\infty<x_1<x_2<\infty$ and $f\in C_0^*([x_1,x_2])$ we have

$$\mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}x_1,u^{-2/\alpha}x_2]}\frac{\xi(t)}{1+u^{-2}f(u^{2/\alpha}t)}>u\right\}\sim\Psi(u)\mathcal{P}^f_{\alpha,a}[x_1,x_2],\quad u\to\infty$$

and for any $y\in\mathbb{R}$

$$\begin{aligned}&\mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}x_1,u^{-2/\alpha}x_2]}\frac{\xi(t)}{1+u^{-2}f(u^{2/\alpha}t)}>u\right\}\\ &=\mathbb{P}\left\{\sup_{t\in[u^{-2/\alpha}(x_1-y),u^{-2/\alpha}(x_2-y)]}\frac{\xi(t+yu^{-2/\alpha})(1+u^{-2}f(y))}{1+u^{-2}f(y+u^{2/\alpha}t)}>u(1+u^{-2}f(y))\right\}\\ &\sim\Psi(u(1+u^{-2}f(y)))\mathcal{P}^{f_y(t)-f(y)}_{\alpha,a}[x_1-y,x_2-y]\\ &\sim\Psi(u)\mathcal{P}^{f_y(t)}_{\alpha,a}[x_1-y,x_2-y].\end{aligned}$$

Let

$$Z_u(t) = \frac{\xi(t + yu^{-2/\alpha})(1 + u^{-2}f(y))}{1 + u^{-2}f(y + u^{2/\alpha}t)}, \quad t \in [u^{-2/\alpha}(x_1 - y), u^{-2/\alpha}(x_2 - y)]$$

and denote its variance function by $\sigma^2_{Z_u}(t)$. Then

$$\left(\frac{1}{\sigma_{Z_u}(t)} - 1\right)u^2 = \left(\frac{1 + u^{-2}f(y + u^{2/\alpha}t)}{1 + u^{-2}f(y)} - 1\right)u^2 = \frac{f(y + u^{2/\alpha}t) - f(y)}{1 + u^{-2}f(y)},$$

i.e.,

$$\lim_{u\to\infty} \sup_{t\in[u^{-2/\alpha}(x_1-y),u^{-2/\alpha}(x_2-y)]} \left| \frac{\left(\frac{1}{\sigma_{Z_u}(t)} - 1\right)u^2}{f(y + u^{2/\alpha}t) - f(y)} - 1 \right| = 0.$$

Consequently, we have

$$\mathcal{P}^f_{\alpha,a}[x_1, x_2] = \mathcal{P}^{f_y}_{\alpha,a}[x_1 - y, x_2 - y].$$

Further, letting $x_2 \to \infty$ yields $\mathcal{P}^f_{\alpha,a}[x_1, \infty) = \mathcal{P}^{f_y}_{\alpha,a}[x_1 - y, \infty)$. This completes the proof. □

Proof of Example 3.4: We have $t_0 = 0, \gamma = 1, g_m = 0$. Then by Proposition 3.1 statement i)

$$\mathbb{P}\left\{\max_{t\in[0,T]} (X(t) - ct) > u\right\} \sim \Psi(u) \begin{cases} c^{-1}a^{1/\alpha}u^{2/\alpha-1}\mathcal{H}_\alpha, & \alpha \in (0, 2), \\ \mathcal{P}^{ct}_{\alpha,a}[0, \infty), & \alpha = 2. \end{cases}$$

Since for all u large

$$\mathbb{P}\left\{u\tau_u \le x \middle| \tau_u \le T\right\} = \frac{\mathbb{P}\left\{\sup_{t\in[0,u^{-1}x]}(X(t) - g(t)) > u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,T]}(X(t) - g(t)) > u\right\}},$$

then using Proposition 3.3, we obtain for $x \in (0, \infty)$

$$\mathbb{P}\left\{u\tau_u \le x \middle| \tau_u \le T\right\} \sim \begin{cases} \frac{\int_0^x e^{-ct}dt}{\int_0^\infty e^{-ct}dt}, & \alpha \in (0, 2), \\ \frac{\mathcal{P}^{ct}_{\alpha,a}[0,x]}{\mathcal{P}^{ct}_{\alpha,a}[0,\infty)}, & \alpha = 2. \end{cases}$$

Proof of Example 3.5: We have that $X(t) = \frac{B_\alpha(t)}{\sqrt{Var(B_\alpha(t))}}$ is locally stationary with correlation function

$$r_X(t, t + h) = \frac{|t|^\alpha + |t + h|^\alpha - |h|^\alpha}{2\,|t(t + h)|^{\alpha/2}} = 1 - \frac{1}{2t^\alpha}|h|^\alpha + o(|h|^\alpha), \quad h \to 0$$

for any $t > 0$. Since $g(t) = c\sin\left(\frac{2\pi t}{T}\right), t \in [T, (n + 1)T]$ attains its maximum at $t_j = \frac{(4j+1)T}{4}, j \le n$ and

$$g(t) = c - 2c\left(\frac{\pi}{T}\right)^2 |t - t_j|^2(1 + o(1)),\ t \to t_j,\ j \le n$$

the claim follows by applying Remarks 3.2 statement i). □

Proof of Example 3.11: First note that the variance function of $X(t)$ is given by $\sigma^2(t) = t(1 - t)$ and correlation function is given by $r(t, s) = \frac{\sqrt{s(1-t)}}{\sqrt{t(1-s)}}, 0 \le s < t \le 1$.

Case 1) The proof of (22): Clearly, $m_u(t) := \frac{\sqrt{t(1-t)}}{1+ct/u}$ attains its maximum over $[0, 1]$ at the unique point $t_u = \frac{u}{c+2u} \in (0, 1)$ which converges to $t_0 = \frac{1}{2}$ as $u \to \infty$, and $m^*_u := m_u(t_u) = \frac{1}{2\sqrt{1+c/u}}$. Furthermore, we have

$$\begin{aligned} \frac{m^*_u}{m_u(t)} - 1 &= \frac{u + ct}{\sqrt{t(1 - t)}} \frac{\sqrt{t_u(1 - t_u)}}{u + ct_u} - 1 = \frac{(u + ct)\sqrt{t_u(1 - t_u)} - (u + ct_u)\sqrt{t(1 - t)}}{\sqrt{t(1 - t)}(u + ct_u)} \\ &= \frac{(u + ct)^2t_u(1 - t_u) - (u + ct_u)^2t(1 - t)}{\sqrt{t(1 - t)}(u + ct_u)[(u + ct)\sqrt{t_u(1 - t_u)} + (u + ct_u)\sqrt{t(1 - t)}]}. \end{aligned} \tag{76}$$

Setting $\Delta(u) = \left[-\frac{(\ln u)^q}{u}, \frac{(\ln u)^q}{u}\right]$, and $(t_u + \Delta(u)) \subset [0, \frac{1}{2}]$ for all u large, we have

$$\begin{aligned} (u + ct)^2t_u(1 - t_u) - (u + ct_u)^2t(1 - t) &= u^2[(t_u - t_u^2) - (t - t^2)] + 2cutt_u(t - t_u) + c^2tt_u(t - t_u) \\ &= (t - t_u)^2u(u + c) \end{aligned} \tag{77}$$

and

$$\frac{u^4}{2\left(u+\frac{c}{2}\right)^2}-u^{-1/2}\le 2(u+ct)^2[t(1-t)]\le \frac{1}{2}\left(u+\frac{c}{2}\right)^2$$

for all $t\in(t_u+\Delta(u))$. Then

$$\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq 0}\left|\frac{m_u^*/m_u(t_u+t)-1}{2t^2}-1\right|=\lim_{u\to\infty}\sup_{t\in\Delta(u),t\neq 0}\left|\frac{m_u^*/m_u(t_u+t)-1}{2(ut)^2u^{-2}}-1\right|=0. \tag{78}$$

Furthermore, since

$$r(t,s)=\frac{\sqrt{s(1-t)}}{\sqrt{t(1-s)}}=1+\frac{\sqrt{s(1-t)}-\sqrt{t(1-s)}}{\sqrt{t(1-s)}}=1-\frac{t-s}{\sqrt{t(1-s)}(\sqrt{s(1-t)}+\sqrt{t(1-s)})},$$

and

$$\frac{1}{2}-\frac{1}{u}\le\sqrt{t(1-s)}(\sqrt{s(1-t)}+\sqrt{t(1-s)})\le\frac{1}{2}+\frac{1}{u}$$

for all $s<t,\ s,t\in(t_u+\Delta(u))$, we have

$$\lim_{u\to\infty}\sup_{\substack{t,s\in\Delta(u)\\ t\neq s}}\left|\frac{1-r(t_u+t,t_u+s)}{2|t-s|}-1\right|=0.$$

Next for some small $\theta\in(0,\frac{1}{2})$, we have

$$\mathbb{E}\left\{(\overline{X}(t)-\overline{X}(s))^2\right\}=2(1-r(t,s))\le\frac{|t-s|}{(\frac{1}{2}-\theta)^2}$$

holds for all $s,t\in[\frac{1}{2}-\theta,\frac{1}{2}+\theta]$. Moreover, by (76), (77) and

$$2(u+ct)^2[t(1-t)]\le 2\left[u+c\left(\frac{1}{2}+\theta\right)\right]^2\left(\frac{1}{2}+\theta\right)^2$$

for all $t\in[\frac{1}{2}-\theta,\frac{1}{2}+\theta]$, we have that for any $t\in[\frac{1}{2}-\theta,\frac{1}{2}+\theta]\setminus(t_u+\Delta(u))$

$$\frac{m_u^*}{m_u(t)}-1\ge\frac{(\ln u)^{2q}}{2[u+c(\frac{1}{2}+\theta)]^2(\frac{1}{2}+\theta)^2},$$

and further

$$\frac{m_u^*}{m_u(t)}\ge 1+\mathbb{C}_1\frac{(\ln u)^q}{u^2},\quad t\in[\frac{1}{2}-\theta,\frac{1}{2}+\theta]\setminus(t_u+\Delta(u)). \tag{79}$$

Consequently, by Proposition 3.6

$$\mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1u\int_{-\infty}^{\infty}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right)\sim e^{-2(u^2+cu)}.$$

In addition, since $\sigma_\theta:=\max_{t\in[0,1]/[t_0-\theta,t_0+\theta]}\sigma(t)<\sigma(t_0)=\frac{1}{2}$, by Borell-TIS inequality

$$\begin{aligned}\mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}&\le\mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]}X(t)>u\right\}\le\exp\left(-\frac{\left(u-\mathbb{E}\left\{\sup_{t\in[0,1]}X(t)\right\}\right)^2}{2\sigma_\theta^2}\right)\\ &=o(e^{-2(u^2+cu)}).\end{aligned} \tag{80}$$

Thus, by the fact that

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\ge\mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}$$

and

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\le\mathbb{P}\left\{\sup_{t\in[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\}+\mathbb{P}\left\{\sup_{t\in[0,1]\setminus[t_0-\theta,t_0+\theta]}(X(t)-ct)>u\right\},$$

we conclude that

$$\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}\sim e^{-2(u^2+cu)}.$$

For any $u>0$

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\le x\Big|\tau_u\le 1\right\}=\frac{\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-1}x]}(X(t)-ct)>u\right\}}{\mathbb{P}\left\{\sup_{t\in[0,1]}(X(t)-ct)>u\right\}}$$

and by Theorem 2.2

$$\mathbb{P}\left\{\sup_{t\in\left[t_u-\frac{(\ln u)^q}{u},t_u+u^{-1}x\right]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{x}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right).$$

The above combined with (79) and (80) implies that as $u\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,t_u+u^{-1}x]}(X(t)-ct)>u\right\}\sim\mathbb{P}\left\{\sup_{t\in\left[t_u-\frac{(\ln u)^q}{u},t_u+u^{-1}x\right]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{x}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right).$$

Consequently,

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\le x\Big|\tau_u\le 1\right\}\sim\frac{\int_{-\infty}^{x}e^{-8t^2}dt}{\int_{-\infty}^{\infty}e^{-8t^2}dt}=\Phi(4x),\quad x\in(-\infty,\infty).$$

Case 2) The proof of (23): We have $t_u=\frac{u}{c+2u}\in(0,\frac{1}{2})$ which converge to $t_0=\frac{1}{2}$ as $u\to\infty$. Since

$$\frac{1}{2}-t_u\sim\frac{c}{4u},\quad u\to\infty,$$

by Proposition 3.6

$$\mathbb{P}\left\{\sup_{t\in[0,1/2]}(X(t)-ct)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{c/4}e^{-8t^2}dt\Psi\left(2\sqrt{cu+u^2}\right)\sim\Phi(c)e^{-2(u^2+cu)}.$$

As for the proof of Case 1) we obtain further

$$\mathbb{P}\left\{u\left(\tau_u-\frac{u}{c+2u}\right)\le x\Big|\tau_u\le\frac{1}{2}\right\}\sim\frac{\int_{-\infty}^{x}e^{-8t^2}dt}{\int_{-\infty}^{c/4}e^{-8t^2}dt}\sim\Phi(4x)/\Phi(c),\quad x\in(-\infty,c/4].$$

Case 3) The proof of (24): We have that $\sigma(t)$ attains its maximum over $[0,1]$ at the unique point $t_0=\frac{1}{2}$, which is also the unique maximum point of $\frac{c}{2}-c\left|t-\frac{1}{2}\right|,t\in[0,1]$. Furthermore,

$$\sigma(t)=\sqrt{t(1-t)}\sim\frac{1}{2}-\left(t-\frac{1}{2}\right)^2,\quad t\to\frac{1}{2}$$

and

$$r(t,s)\sim 1-2|t-s|,\quad s,t\to\frac{1}{2}.$$

By Proposition 3.9 as $u\to\infty$

$$\mathbb{P}\left\{\sup_{t\in[0,1]}\left(X(t)+\frac{c}{2}-c\left|t-\frac{1}{2}\right|\right)>u\right\}\sim 8\mathcal{H}_1 u\int_{-\infty}^{\infty}e^{-\left(8|t|^2+4c|t|\right)}dt\Psi\left(2u-c\right)\sim 2\Psi(c)e^{-2(u^2-cu)}$$

and in view of Proposition 3.10 ii)

$$\mathbb{P}\left\{u\left(\tau_u-\frac{1}{2}\right)\le x\Big|\tau_u\le 1\right\}\sim\frac{\int_{-\infty}^{x}e^{-\left(8|t|^2+4c|t|\right)}dt}{\int_{-\infty}^{\infty}e^{-\left(8|t|^2+4c|t|\right)}dt},\ u\to\infty.$$

□

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Long Bai, Department of Actuarial Science, University of Lausanne UNIL-Dorigny, 1015 Lausanne, Switzerland
E-mail address: Long.Bai@unil.ch

Krzysztof Dębicki, Mathematical Institute, University of Wrocław, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
E-mail address: Krzysztof.Debicki@math.uni.wroc.pl

Enkelejd Hashorva, Department of Actuarial Science, University of Lausanne,, UNIL-Dorigny, 1015 Lausanne, Switzerland
E-mail address: Enkelejd.Hashorva@unil.ch

Lanpeng Ji, Department of Actuarial Science, University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland
E-mail address: jilanpeng@126.com