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Cosmological models of dimensional segregation

Jon Yearsley and John D Barrow
Astronomy Centre, University of Sussex, Brighton BN1 9QH, UK

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Abstract. We study a range of cosmological models permitting dimensional reductions in which subsets of the dimensions of space can expand at different rates. A specific inflationary model of this sort proposed by Linde and Zelnikov is studied in detail and shown to be destabilized by the addition of an isotropic radiation field to the magnetic and scalar stresses which permit anisotropic inflation.

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1. Introduction

The idea that space contains more than three dimensions, with all but three residing now in some compactified form, is a persistent one. It played a leading role in the modern formulation of Kaluza–Klein (Freund 1982, Cremmer *et al* 1983) and supergravity theories (Witten 1985, Derendinger 1986), and persists in the context of contemporary superstring theories (Dine *et al* 1985, Green *et al* 1987, Green and Schwarz 1984), even though specific mechanisms of dimensional compactification remain undiscovered. Extra spatial dimensions can only exhibit temporal changes in the very early universe, otherwise they create conflicts with our observations of the constancy in time of those 'constants of nature' that define the bare strengths of the fundamental forces (Freund 1982, Marciano 1984, Barrow 1987b). It is expected, but not demanded, that any such changes would have been completed, along with the process of compactification, very soon after the Planck epoch at an energy close to 10^{19} GeV.

A simple description of cosmological models with separate internal and external spaces can be achieved by studying anisotropic cosmologies with an imposed product space structure. Although this has the disadvantage of requiring all the cosmological matter fields and associated interactions to be put in by hand, it allows some aspects of the stability of particular geometrical configurations to be studied. Some simple examples have been studied by Chodos and Detweiler (1980), Barrow and Stein Schabes (1985, 1986), and Barrow and Burd (1988), but we are motivated primarily by the study of Linde and Zelnikov (1988) who present a dimensionally reduced solution to a $(5+1)$-dimensional Kantowski–Sachs spacetime containing a single scalar field, a magnetic field and a non-zero cosmological constant. Inflation occurs in only three dimensions of this solution because of the slow rolling of the scalar field and creates a cosmology in which three dimensions of space become large whilst the rest remain small, with constant radius, as a result of the stresses imposed by the other matter fields. In the context of chaotic or stochastic inflation, this general pattern could provide a basis for a cosmological evolution in which different regions of the Universe display different numbers of large and compactified dimensions.

However, the example given by Linde and Zelnikov relies upon geometric idiosyncrasies of the Kantowski–Sachs $S^2 \times S^1$ metric which are known from the study of no-hair theorems for inflation in anisotropic cosmologies with three space dimensions (Moniz 1993, Jensen and Stein Schabes 1986, Weber 1984). Therefore, it is important to investigate the condition under which such solutions are stable.

In order to do this we shall consider first the asymptotic stability of $(N+1)$-dimensional spacetimes containing a scalar field with an exponential self-interaction potential and then investigate the stability of the Linde–Zelnikov solution in the presence of other realistic matter fields.

2. Anisotropic models with a scalar field and an exponential potential

Before we study the examples of Linde and Zelnikov (1988) in which dimensional segregation occurs, we will consider the fate of universes evolving under the influence of an exponential scalar-field potential. Since this potential has no minimum there is simple asymptotic behaviour which is characteristic of the behaviour of potentials with minima which are exponentially steep over part of their ranges. We have chosen the metric to illustrate how it is possible for there to exist both solutions in which all directions inflate towards an isotropic state in accord with the expectations of the no-hair theorems and solutions in which there occurs the dimensional segregation envisaged by Linde and Zelnikov. The specific choice of an exponential potential allows us to capture the behaviour more precisely with exact solutions. Other potentials exhibit the same general behaviour but we would need to use approximations to describe them.

We consider homogeneous anisotropic spacetimes with N space dimensions, of which d form a sub-space with metric g_{ij}, $1 \leqslant i, j \leqslant d$, and $D = N - d$ form a sub-space with metric g_{IJ}, $d+1 \leqslant I, J \leqslant N$. The metric for the entire spacetime is taken to have the simple form

$$ds^2 = -dt^2 + a^2(t) g_{ij}\, dx^i\, dx^j + b^2(t) g_{IJ}\, dx^I\, dx^J \tag{1}$$

where $a(t)$ and $b(t)$ are the expansion scale factors associated with the internal and external spaces respectively. We take g_{ij} to be a Euclidean metric and g_{IJ} to be a D-dimensional homogeneous space of constant curvature characterized by a constant k which can be scaled to take the value 0, $+1$, or -1 if the space has zero, positive, or negative curvature respectively. If $k = 0$ then the spacetime is the N-dimensional version of a Bianchi type I spacetime; $k = -1$ similarly corresponds to a Bianchi type III spacetime and $k = 1$ corresponds to a Kantowski–Sachs spacetime. The only matter content is assumed to be a scalar field, φ, which has an exponential potential, $V(\varphi)$, of the form

$$V(\varphi) = V_0 \exp(-\lambda\varphi) \tag{2}$$

where λ and $V_0 \geqslant 0$ are constants. The dynamical Einstein equations are (with units such that $8\pi G = c = 1$)

$$\ddot{a} + \left[(d-1)\frac{\dot{a}}{a} + D\frac{\dot{b}}{b}\right]\dot{a} + \frac{\partial}{\partial a} W_a(a, \varphi) = 0 \tag{3}$$

$$\ddot{b} + \left[d\frac{\dot{a}}{a} + (D-1)\frac{\dot{b}}{b}\right]\dot{b} + \frac{\partial}{\partial b} W_b(b, \varphi) = 0. \tag{4}$$

Here $\dot{} = d/dt$ and the potentials W_a and W_b take the form,

$$W_a(a, \varphi) = -\frac{1}{N-1}\left(V(\varphi) + \Lambda\right) a^2 \tag{5}$$

and

$$W_b(b, \varphi) = (D-1)k\ln(b) - \frac{1}{N-1}(V(\varphi)+\Lambda)b^2 \tag{6}$$

where Λ is a cosmological constant. Use can be made of the first integral

$$d(d-1)\frac{\dot{a}^2}{a^2} + D(D-1)\frac{\dot{b}^2}{b^2} + 2dD\frac{\dot{a}}{a}\frac{\dot{b}}{b} + D(D-1)\frac{k}{b^2} - 2\Lambda = 2\left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right) \tag{7}$$

to eliminate k from the above equations. The first integral can then be replaced by the conservation equation for the scalar field:

$$\ddot{\varphi} + \left(d\frac{\dot{a}}{a} + D\frac{\dot{b}}{b}\right)\dot{\varphi} = -\frac{\partial}{\partial\varphi}V(\varphi). \tag{8}$$

Using the exponential potential (2), equations (3), (4), and (8) form a closed three-dimensional system of autonomous equations for $\dot{a}$, $\dot{b}$, and $\dot{\varphi}$. Equations (3) and (4) can be viewed as a classical dynamical equation with a damping term $\gamma_a\dot{a}$ or $\gamma_b\dot{b}$ (where $\gamma_a = (d-1)\dot{a}/a + D\dot{b}/b$ and $\gamma_b = d\dot{a}/a + (D-1)\dot{b}/b$) and a potential W_a or W_b. For a slowly rolling field, $|\dot{\varphi}/\varphi| \ll \dot{a}/a$ and $|\dot{\varphi}/\varphi| \ll \dot{b}/b$, and the solutions to these equations can be viewed as a point 'rolling' from the top of the potentials W_a and W_b. The general form of the potential W_b is sketched in figure 1. For $k = 0$ and -1 the solution takes a single form and evolves down the right-hand side of the potential W_b. For the closed case, $k = 1$, the potential possesses a maximum and two types of solution are possible. One corresponds to the left-hand side of the maximum, and the other to the right-hand side (see curve (a) in figure 1). Solutions rolling down the right-hand side of the potential converge on the $k = 0$ solution, and so tend asymptotically to 'flatness'.

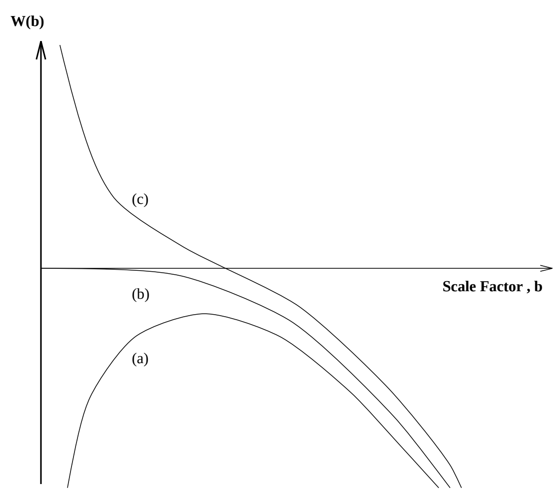

Figure 1. The potential W_b, defined by equation (6), for the three possible values of k and fixed ϕ: (a) positive curvature, $k = 1$, note that a maximum occurs at $b^2 = (D-1)(N-1)/2V(\varphi)$; (b) zero curvature, $k = 0$; (c) negative curvature, $k = -1$.

By rewriting equations (3) and (4) in terms of the volume expansion ($\theta \equiv d\dot{a}/a + D\dot{b}/b$) and the shear ($\sigma \equiv \sqrt{dD/2N}\left(\dot{a}/a - \dot{b}/b\right)$), and defining a new time variable τ, via

$$\sqrt{V(\varphi)}\mathrm{d}\tau = \mathrm{d}t, \tag{9}$$

the Einstein and scalar-field equations become

$$\theta' + \frac{\theta^2}{N} + \frac{1}{2}\theta\varphi' \frac{\partial}{\partial\varphi} \ln(V(\varphi)) = \frac{2}{N-1}\left(1 + \frac{\Lambda}{V(\varphi)}\right) - \varphi'^2 \tag{10}$$

$$\sigma' + \theta\sigma + \frac{1}{2}\sigma\varphi' \frac{\partial}{\partial\varphi} \ln(V(\varphi)) = \sqrt{\frac{d}{DN}}\left[\left(1 + \frac{\Lambda}{V(\varphi)}\right) + \frac{\varphi'^2}{2} + \sigma^2 - \frac{N-1}{2N}\theta^2\right] \tag{11}$$

$$\varphi'' + \theta\varphi' = -\left(\frac{1}{2}\varphi'^2 + 1\right)\frac{\partial}{\partial\varphi} \ln(V(\varphi)) \tag{12}$$

from which the critical points of the corresponding phase plane can be found when $\theta' = \sigma' = \varphi'' = 0$, $(\,' = \mathrm{d}/\mathrm{d}\tau)$.

Besides the case when $\lambda = 0$, there are two other cases which need to be investigated according to whether the cosmological constant, Λ, is zero or non-zero (solutions are valid whilst $(1 + V(\varphi)/\Lambda)^{-1} \ll \theta\varphi' + 1$). For positive semi-definite values of the scalar field each case exhibits two solutions given by

$\Lambda = 0$; $\lambda = 0$

$$\begin{aligned} a(t) &= a(t_0)\exp\left(\frac{t}{t_0}\right) \\ b(t) &= b(t_0)\exp\left(\frac{t}{t_0}\right) \\ \varphi - \varphi(t_0) &= 0 \end{aligned} \tag{13a}$$

$\Lambda = 0$; $\lambda \neq 0$

$$\begin{aligned} a(t) &= a(t_0)\left(\frac{t}{t_0}\right)^p \\ b(t) &= b(t_0)\left(\frac{t}{t_0}\right)^q \\ \varphi - \varphi(t_0) &= \frac{2}{\lambda}\ln\frac{t}{t_0} \end{aligned}$$

$\Lambda \neq 0$; $\lambda \neq 0$ (for $\Lambda/(\Lambda + V(\varphi)) \ll \theta\varphi' + 1$)

$$\begin{aligned} a(t) &= a(t_0)\left[\cosh\frac{C\lambda}{2\sqrt{\Lambda}}(t - t_0)\right]^p \\ b(t) &= b(t_0)\left[\cosh\frac{C\lambda}{2\sqrt{\Lambda}}(t - t_0)\right]^q \\ \varphi - \varphi(t_0) &= \frac{2}{\lambda}\ln\frac{t}{t_0} \end{aligned}$$

where

$$\begin{aligned} p = q &= \frac{4}{\lambda^2(N-1)} \\ C^2 &= \frac{2\lambda^2(N-1)}{4N - \lambda(N-1)} \end{aligned} \tag{13b}$$

or

$$\begin{aligned} p &= \frac{4}{\lambda^2(N-1)} \\ q &= 1 \\ C^2 &= \frac{2\lambda^2(N-1)}{\lambda^2(N-1)(D-1) - 4d} \end{aligned} \tag{13c}$$

and where t_0 is an integration constant. The critical points given by equations (13a)–(13c) are the stationary solutions of equations (10)–(12). Two types of solution behaviour can occur around each critical point. Either all solutions as they evolve forward in time approach the point, in which case the critical point is stable, or some solutions diverge away from the point, in which case the critical point is unstable. If a critical point solution is stable then it is an attractor in the space of solutions. Thus a stable critical point is to some extent (depending upon the attractors domain of attraction) a generic behaviour for the system's asymptotic time development.

The solution given by equation (13a) displays the usual $(N + 1)$ version of de Sitter inflation, in accordance with no-hair theorems. Since $\sigma = 0$ the solution is isotropic and thus corresponds to a metric with $k = 0$.

The solution given by equations (13b) also has $\sigma = 0$, so the solution is isotropic and corresponds to a metric with $k = 0$. For $\Lambda = 0$, and $\lambda^2 \leqslant 4/(N - 1)$, this solution is the $(N + 1)$-dimensional generalization of the usual isotropic power-law inflationary solution studied by Barrow (1987a), Halliwell (1987), and Barrow and Burd (1988). This is as expected from no-hair theorems. Non-zero Λ values create a solution which asymptotes towards $(N + 1)$-dimensional de Sitter expansion at large t.

The solutions given by equations (13c) are anisotropic and correspond to a metric whose curvature depends upon the value of λ. If the cosmological constant is zero ($\Lambda = 0$) both scale factors, $a(t)$ and $b(t)$, exhibit power-law expansion. If $a(t)$ is inflating ($\lambda^2 = 4/(N - 1)$) the curvature is positive ($k = +1$) whilst if $a(t)$ is non-inflating the curvature is negative ($k = -1$). However $b(t)$ in general does not have the same behaviour as $a(t)$. The exponent for $a(t)$ depends upon λ, whereas the exponent for $b(t)$ is always 1. Therefore $b(t)$ decouples from the $a(t)$ evolution and always exhibits critical power-law inflation, contrary to the expectations of the no-hair theorem. This is not a contravention of the theorem though, since the positive curvature case is always excluded from such theorems (Wald 1983, Jensen and Stein Schabes 1986, Barrow and Götz 1989). The exception is when $\lambda^2 = 4/(N - 1)$, then $a(t) \propto b(t)$ and this 'anisotropic' solution is isotropic. In this special case the curvature of the space is zero. This anisotropic inflationary behaviour is characteristic of the geometrical structure of Kantowski–Sachs universes. It will not occur in Bianchi type IX closed universes because of the SO(N) symmetry.

The scale factor $b(t)$ is required to be proportional to t to compensate the accelerating affect of the curvature. Thus the behaviour of $b(t)$ is 'driven' by the curvature of the space and is independent of λ and the scalar-field potential. Since the behaviour of $b(t)$ is fixed, the shear can be used to determine the behaviour of $a(t)$. Positive curvature increases the shear, and thus requires $a(t)$ to be inflating (negative curvature decreases the shear and requires $a(t)$ to be non-inflating).

For a non-zero cosmological constant the solutions give anisotropic de Sitter expansion at late times.

Now by linearizing about these stationary solutions the eigenvalues for the particular solutions can be found and their stability properties established. Their character depends upon the value of the parameter λ defining the slope of the scalar-field potential in equation (2) and by the total space dimension, N, as follows:

(i) $\lambda = 0$. The two time variables t and τ are proportional and the solution given by equation (13b) reduces to $a(t) \propto b(t) \propto \exp(t\sqrt{2V_0/[N(N-1)]})$, which is de Sitter expansion ($\Lambda = 0$). The potential in equation (2) is constant, $V(\varphi) = V_0$, and plays the role of a cosmological constant (if $\Lambda \neq 0$ then V_0 in the above equation should be replaced by $V_0 + \Lambda$). All the eigenvalues are real and negative definite so the critical point is a stable node and the isotropic solution is asymptotically stable as $t \to \infty$ in accordance with the expectation of the cosmic no-hair conjecture.

For the solution given by equation (13*c*) the eigenvalues have different signs and thus the critical point is a saddle point. The solution, which is anisotropic with a non-zero cosmological constant and positive curvature, is unstable. The shear, σ, has a value $\sqrt{DV_0/[N(N-1)]}$; the scale factors evolve such that $a(t) \propto \exp(t\sqrt{2V_0/[d(N-1)]})$ and $b(t) = \text{constant}$.

(ii) $0 < \lambda < 2/\sqrt{N-1}$. For this range of λ the eigenvalues for the isotropic solution given by equation (13*b*) are once again real and negative, so the critical point remains a stable node and the model approaches isotropy as $t \to \infty$. For $\Lambda = 0$ the scale factors have a power-law evolution where the exponent is greater than 1. This specifies power-law inflation, and therefore this model solves the horizon and flatness problems. The curvature of the sub-space with scale factor b is zero.

The anisotropic solution given by equation (13*c*) still has eigenvalues of different signs and is a saddle point in the phase plane. The curvature is positive and decelerates the scale factor b.

(iii) $\lambda = 2/\sqrt{N-1}$. In this case the two solutions coalesce to one isotropic solution and the scale factors evolve with $a(t) \propto b(t) \propto t$ if $\Lambda = 0$. The eigenvalues are all real: two are negative definite, whilst one is zero, and hence the stability is determined at second order. Using a standard Lyapunov procedure outlined in Barrow and Sonoda (1986), the solution is found to be unstable. The zero eigenvalue, which is associated with the shear, is due to the curvature and the energy density of the scalar field evolving at the same rate. The scalar field tries to isotropize the space, whilst the curvature tries to produce an anisotropy. The combination of the two gives a logarithmic decay of the shear and hence a zero eigenvalue in the linear perturbation about the critical point. The metric has zero curvature.

(iv) $2/\sqrt{N-1} < \lambda < \sqrt{4N/(N-1)}$. In this range the isotropic solution given by equation (13*b*) is no longer stable since the eigenvalues are complex and have different signs. The critical point is therefore an unstable spiral. The metric is still flat.

The anisotropic solution given by equation (13*c*) is stable since its complex eigenvalues have positive definite real parts. The critical point is therefore a stable spiral. The curvature of the space for the scale factor b is now negative and accelerates the scale factor b.

(v) $\lambda = \sqrt{4N/(N-1)}$. At this value of λ the solution given by equation (13*b*) becomes singular, and the critical point goes off to infinity. The remaining anisotropic solution again is stable and the critical point is a stable spiral. The curvature of the model is negative.

These stability results are summarized in table 1. These examples display the possibility of obtaining different amounts of inflation in different sub-dimensions of space. In models where the isotropic (de Sitter or power-law) inflationary behaviour is a stable attractor, all dimensions inflate at a similar rate towards isotropy, and there is no possibility of leaving some dimensions small whilst others enlarge by inflation. In contrast, where the anisotropic expansion remains a stable attractor, it is possible for a subset of the dimensions to inflate to large size whilst others remain relatively small. Since the model we have considered is very simple we shall now consider some further generalizations obtained by adding further matter fields.

3. Anisotropic solutions with a cosmological magnetic field

We have seen how the particular geometrical structure of the closed anisotropic universes of Kantowski–Sachs form permits a violation of the expectations of the cosmic no-hair theorem. Different groups of dimensions can expand at different rates, so creating a segregation of dimensions. Having seen this explicitly in the simplest soluble case we now consider the

Table 1. A summary of the stability results discussed in the text. The model contains only a scalar field and a cosmological constant, Λ. The scalar field has an exponential potential of the form $V(\varphi) = V_0 \exp(-\lambda\varphi)$. Scale factors take the form $a(t) \propto t^p$ and $b(t) \propto t^q$. Inflation of $a(t)$ occurs when $\ddot{a} > 0$.

p	q	λ	Isotropy and curvature	Stability	Inflation by $a(t)$	Inflation by $b(t)$
$\frac{4}{\lambda^2(N-1)}$	$\frac{4}{\lambda^2(N-1)}$	$0 \leqslant \lambda < \sqrt{\frac{4}{N-1}}$	Isotropic, $k = 0$	Stable	Yes	Yes
$\frac{4}{\lambda^2(N-1)}$	1	$0 \leqslant \lambda < \sqrt{\frac{4}{N-1}}$	Anisotropic, $k = +1$	Unstable	Yes	No
1	1	$\lambda = \sqrt{\frac{4}{N-1}}$	Isotropic, $k = 0$	Unstable	No	No
$\frac{4}{\lambda^2(N-1)}$	$\frac{4}{\lambda^2(N-1)}$	$\sqrt{\frac{4}{N-1}} < \lambda < \sqrt{\frac{4N}{N-1}}$	Isotropic, $k = 0$	Unstable	No	No
$\frac{4}{\lambda^2(N-1)}$	1	$\sqrt{\frac{4}{N-1}} < \lambda < \sqrt{\frac{4N}{N-1}}$	Anisotropic, $k = -1$	Stable	No	No
$\frac{1}{N}$	1	$\lambda = \sqrt{\frac{4N}{N-1}}$	Anisotropic, $k = -1$	Stable	No	No

more complicated scenario created by Linde and Zelnikov, in which a magnetic field is included to contribute an anisotropic pressure and the potential of the scalar field is left unspecified (assuming only that it is shallow enough to create slow roll inflation). We shall extend this model by adding an isotropic radiation field to show that the dimensional segregation found by Linde and Zelnikov is destabilized by its presence. The phase plane analysis presented in the last section can be extended to models containing both a source-free magnetic field and an isotropic perfect fluid. In terms of the expansion and shear, the Einstein and conservation equations for this case are:

$$\theta' + \frac{\theta^2}{N} + \frac{1}{2}\theta\varphi'\frac{\partial}{\partial\varphi}\ln(V(\varphi)) = \frac{2}{N-1}\left(1 + \frac{\Lambda}{V(\varphi)}\right) - \varphi'^2 - \frac{N\gamma - 2}{N-1}\frac{\rho}{V(\varphi)} - \frac{2}{N-1}\frac{\rho_{\mathrm{m}}}{V(\varphi)} \tag{14}$$

$$\sigma' + \theta\sigma + \frac{1}{2}\sigma\varphi'\frac{\partial}{\partial\varphi}\ln(V(\varphi)) = \frac{1}{D}\sqrt{\frac{dD}{2N}}\left[2\left(1 + \frac{\Lambda}{V(\varphi)}\right) + \varphi'^2 + 2\sigma^2 - \frac{N-1}{N}\theta^2 + \frac{2}{V(\varphi)}(\rho + \rho_{\mathrm{m}} + p_d - p_D)\right] \tag{15}$$

$$\varphi'' + \theta\varphi' = -\left(\frac{1}{2}\varphi'^2 + 1\right)\frac{\partial}{\partial\varphi}\ln(V(\varphi)) \tag{16}$$

$$\rho' + \frac{2-\gamma}{V(\varphi)}\theta\rho = 0 \tag{17}$$

$$\rho_{\mathrm{m}}' + \frac{N-1}{V(\varphi)}\frac{\theta}{N}\rho_{\mathrm{m}} = \sqrt{\frac{2dD}{N}}(p_D - p_d)\sigma \tag{18}$$

where $' = \mathrm{d}/\mathrm{d}\tau = V(\varphi)^{-\frac{1}{2}}\mathrm{d}/\mathrm{d}t$, ρ is a perfect fluid density with pressure p which obeys an equation of state $p = (\gamma - 1)\rho$; ρ_{m} is the energy density of the magnetic field, p_d is its pressure in the d-dimensional sub-space and p_{D} is its pressure in the D-dimensional sub-space. The critical points of the five-dimensional system (14)–(18) can be calculated for $\Lambda = 0$ and for $\Lambda \neq 0$ if the assumption that $V(\varphi)/\Lambda \approx$ constant holds, as it does once de Sitter inflation begins. The general results are lengthy and will not be presented here. Instead we confine our attention to a (5 + 1)-dimensional Kantowski–Sachs model containing only a scalar and magnetic field. The results can be compared with the work of Linde and Zelnikov (1988). They considered a spacetime with space-like hypersurfaces $M^3 \times S^2$, where the magnetic field has a monopole configuration (i.e. its vector potential has only one component, which is purely azimuthal in the S^2 space, $A_\phi = f(1 - \cos\theta)$, where f is a constant and θ is the poloidal coordinate in the S^2 space). This gives an electromagnetic field tensor

$$F_{ij} = 0 \tag{19}$$

$$F_{IJ} = f\sqrt{|\det g_{IJ}|}\epsilon_{IJ} \tag{20}$$

where $\epsilon_{IJ} = -\epsilon_{JI}$. Such a configuration gives a homogeneous source-free magnetic flux only in the M^3 space. The topology of the S^2 space forbids a source-free magnetic field from having non-zero components there. If $V(\varphi) = 0$ at its minimum then $\Lambda = 8/f^2$ and a 'radius of compactification' b_0 can be defined for the compact space as

$$b_0 = \frac{f}{2} = \sqrt{\frac{2}{\Lambda}}. \tag{21}$$

This value for the scale factor, $b(t)$, is the critical stability case identified by Moniz (1993) for the Kantowski–Sachs model containing a scalar field with an exponential potential of the form (2) and a positive cosmological constant.

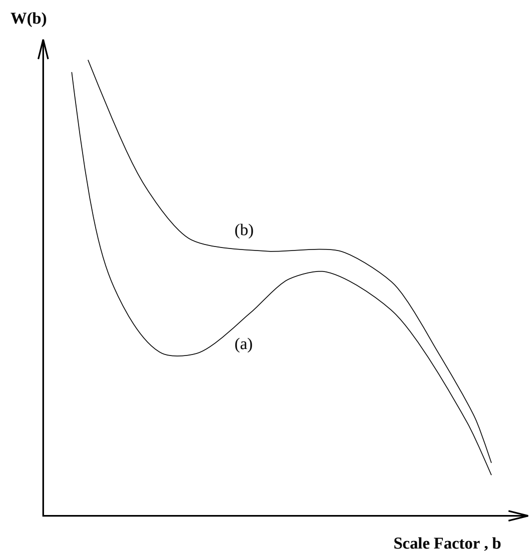

Figure 2. The general form of the effective potential $W(b)$ at constant φ, when (a) $V(\varphi) = 0$, and (b) $V(\varphi) > \Lambda/3$, as found by Linde and Zelnikov. The depth of the minimum has been exaggerated in case (a).

As before, the Einstein equations can be written in the form of classical dynamical equations for the scale factors. The potential W_b becomes,

$$W(b, \varphi) = \frac{3}{8}\left(\frac{b_0}{b}\right)^2 + \ln\frac{b}{b_0} - \frac{1}{8}\left(1 + \frac{V(\varphi)}{\Lambda}\right)\frac{b^2}{b_0^2}. \tag{22}$$

This potential is shown in figure 2. For a slow-rolling field, as defined in the previous section, Linde and Zelnikov found two solutions, one isotropic and the other anisotropic. The anisotropic solution, which exists when $V(\varphi)/\Lambda < 3$, lets the scale factor $b(t)$ 'sit' in the minimum of the potential $W(b, \varphi)$, so $b(t)$ asymptotically approaches b_0 as $t \to \infty$. Perturbing the scale factors by $\alpha(t)$ and $\beta(t)$ such that

$$a(t) \longrightarrow a(t)(1 + \alpha(t)) \tag{23}$$

$$b(t) \longrightarrow b(t)(1 + \beta(t)) \tag{24}$$

and then linearizing the equations gives the stability criterion

$$\left(\frac{b}{b_0}\right)^2 \leqslant \frac{3}{2} \tag{25}$$

for this anisotropic solution. This inequality is satisfied for the anisotropic solution, provided that $V(\varphi) < \Lambda/3$. Thus the anisotropic solution is stable to metric perturbations of the form (23)–(24) whenever the solution exists.

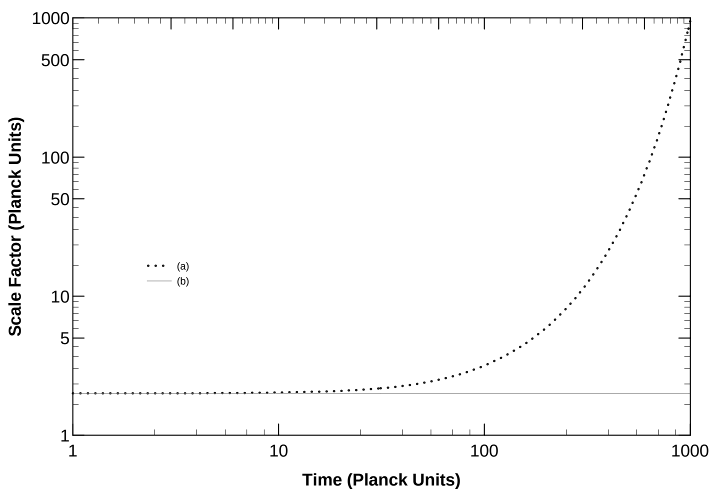

Figure 3. The anisotropic evolution of the Linde–Zelnikov solution. Line (a) is the scale factor of the three flat dimensions, whilst line (b) is the scale factor for the remaining two-dimensional compact space. $\Lambda = 0.005$, $\dot{\varphi}(t = 1) = 0.06$, $\rho_{\rm m}(t = 1) = 0.005$, $\dot{b}(t = 1) = 0$ (all units are appropriate Planck units).

The stability can be studied more generally using the phase plane analysis. Although this requires an exponential scalar field potential whilst the work of Linde and Zelnikov assumed no particular form for the potential, it does not rely upon a slow-roll condition, and can be used for any value of λ. For small λ when $V(\varphi)/\Lambda < 3$ the eigenvalues of the critical point, corresponding to the anisotropic solution, are complex with negative definite real parts. Thus the solution for a slowly rolling field is asymptotically stable. For large values of λ ($\lambda \gtrsim 1$) one eigenvalue has a positive definite real part, and thus the anisotropic solution becomes unstable. Within the constraints imposed by Linde and Zelnikov their anisotropic solution is asymptotically stable. A numerical evolution of the model is shown in figure 3 for the scale factors a and b, and in figure 4 for the shear and energy densities. Figure 4 shows the shear oscillating about the minimum of the potential $W(b, \varphi)$ before finally settling down such that the compact space has a compactification radius of several Planck lengths. Recall that there is no perfect fluid present in this case ($\rho = p = 0$).

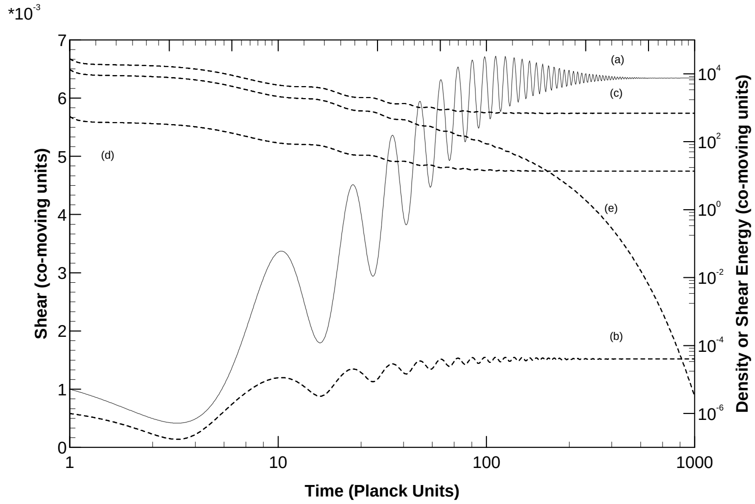

Figure 4. The anisotropic solution of Linde and Zelnikov for a $(5+1)$-dimensional Kantowski–Sachs spacetime. Curve (a) shows the co-moving shear (on the left-hand scale). Curves (b)–(e) correspond to the right-hand axis. Curve (b) shows $(\sigma/\theta)^2$. Curve (c) shows the curvature. Curve (d) shows the magnetic density and curve (e) shows the scalar field energy density. There is no perfect fluid present. $\Lambda = 0.005$, $\dot{\varphi}(t=1) = 0.06$, $\rho_{\rm m}(t=1) = 0.005$, $\dot{b}(t=1) = 0$ (all units are appropriate Planck units).

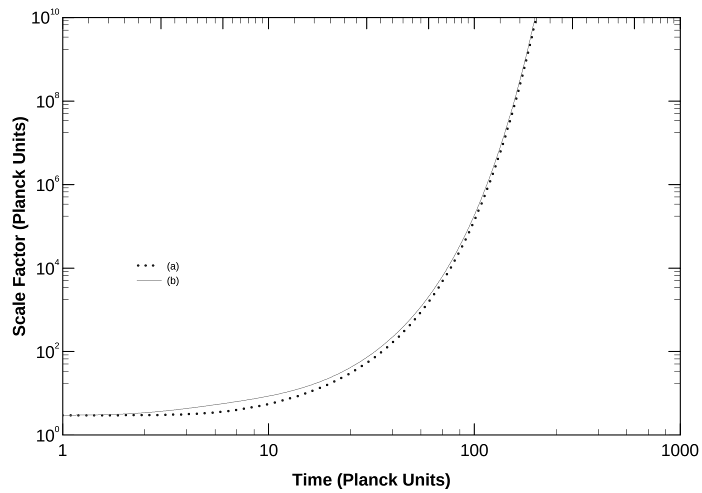

Figure 5. The isotropic evolution of the Linde–Zelnikov solution. Line (a) is the scale factor of the three flat dimensions, whilst line (b) is the scale factor for the remaining two-dimensional compact space. Initial scale factors give Kasner behaviour. $\Lambda = 0.005$, $\dot{\varphi}(t=1) = 0.06$, $\rho_m(t=1) = 0.005$ (all units are appropriate Planck units).

Although the solution is stable, it is very dependent upon the initial conditions chosen. The local minimum of the potential $W(b,\varphi)$, shown in figure 2 is extremely shallow, and even very small perturbations can push the model into the isotropic solution, shown in figure 5. The analysis can be extended to a model containing an isotropic perfect fluid obeying the equation of state $p = (\gamma - 1)\rho$. The effect of this extra matter field adds another term to the potential $W(b, \varphi)$, which becomes for $\gamma \neq 1$,

$$W(b, \varphi) = \frac{3}{8}\left(\frac{b_0}{b}\right)^2 + \ln\frac{b}{b_0} - \frac{1}{8}\left(1 + \frac{V(\varphi)}{\Lambda}\right)\frac{b^2}{b_0^2} + \frac{1}{8a^{3\gamma}}\frac{2-\gamma}{\gamma-1}b^{2(1-\gamma)} \tag{26}$$

(if $\gamma = 1$ (dust) then the last term on the right is $a^{-3}/8\ln(b)$). This extra term adds a 'tilt' to the potential $W(b, \varphi)$. If the perfect fluid is stiff ($\gamma = 2$), then the extra term is zero, and it has no effect upon results. This is to be expected since a $\gamma = 2$ fluid mimics a free-scalar field with $V(\varphi) = 0$. For all other possible values of γ ($1 \leqslant \gamma < 2$) the perfect fluid contributes an expanding force to the scale factor $b(t)$. The most interesting case is $\gamma = \frac{4}{3}$, where it describes the ambient medium of radiation fields expected from asymptotically free relativistic particles in thermal equilibrium. In this case the extra term evolves proportional to $b^{-\frac{2}{3}}a^{-4}$. In general, the effect of the extra isotropic stress term is twofold. First, it mixes the scale factor $a(t)$ into the dynamics for $b(t)$, so that any stationary solution for $b(t)$ cannot exist whilst $a(t)$ is evolving (it is a special property of the magnetic field chosen by Linde and Zelnikov that its density is only a function of the scale factor $b(t)$; a more general matter configuration will have a density which is a function of both scale factors and therefore incorporates a mixing of the scale factors in the matter sector of the model). Second, it reduces or eliminates the minimum in $W(b, \varphi)$ even at fixed $a(t)$. Since the original minimum is very weak, the amount of isotropic fluid required before the minimum no longer exists need not be appreciable. For initial conditions where the starting energy densities of the perfect isotropic fluid and the magnetic field are of the same order, as would be required by an equipartition principle near the Planck time (Barrow 1994), the effect of the additional matter is to remove the minimum in the potential $W(b, \varphi)$. Looking at the eigenvalues (see appendix A for more details), one is now positive definite for all values of λ and thus the solution is no longer asymptotically stable. The destabilizing effect of the radiation can be viewed as an additional stochastic feature adding to the likelihood that this type of dimensional segregation would occur in different ways in different places in a chaotic or self-reproducing inflationary universe as envisaged by Linde and Zelnikov. We also note that any collisionless radiation present in all dimensions, for example gravitons, would have a strong isotropizing effect on the N-dimensional dynamics and would oppose dimensional segregation at very early times. These effects will be examined in more detail elsewhere.

4. Discussion

We have considered a number of anisotropic general-relativistic cosmologies with N space dimensions that allow simple models to be constructed for a cosmological evolution that allows a subset of these dimensions (which can have any number, including 3) to inflate to large size whilst the remainder expand more slowly or remain static. In order for this to occur during an inflationary phase the N-dimensional de Sitter or power-law inflationary asymptotes, which occur when inflation is driven by a scalar field with an exponential potential, must not be global attractors. This is possible in metrics with the Kantowski–Sachs form, for which exceptional counter-examples to the cosmic no-hair conjecture are known in $(3 + 1)$-dimensional spacetimes. However, such behaviour is special. In order to test the stability of the simplest cosmological models which exhibit this form of dimensional segregation we first examined models containing only a scalar field with exponential potential and found (see table 1) that if the scalar field gives rise to inflation then the asymptote exhibiting anisotropic dimensional segregation is unstable. Only the isotropic solution is stable. The situation is reversed if the solution is non-inflationary. We then considered more general models containing a scalar field stress and a magnetic field in order to encompass the models considered earlier by Linde and Zelnikov. These solutions exhibit dimensional segregation in the presence of slow-roll inflation. The compact sub-space is kept static whilst a flat three-dimensional sub-space is allowed to expand. So long as the field always obeys such a slow-roll approximation and the scalar field potential and the cosmological constant always obey the inequality $3V(\varphi) < \Lambda$ this solution is asymptotically stable. For the exponential potential studied, the solution becomes unstable once the gradient of the potential becomes too great. The addition of an isotropic perfect fluid with initial density comparable to that of the magnetic field also made the solution unstable. The addition of a perfect fluid with a changing equation of state, as would be more realistic for such an anisotropic model, would be expected to have an even stronger isotropizing effect than the perfect fluid (Lukash and Starobinskiĭ 1974).

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Appendix. Stability of the critical solutions

To study the stability of the critical solutions discussed in the main text we examine the behaviour of the system in the local domain of each critical solution. This is a standard technique in studies of dynamical systems, and here we present some more details of the steps we have used to arrive at the results discussed in sections 2 and 3.

From equations (14)–(18) critical solutions are found by requiring $\theta' = \sigma' = \varphi' = \rho' = \rho'_{\mathrm{m}} = 0$. Since the explicit solutions are often long, we will write them as $\theta = \theta_0$, $\sigma = \sigma_0$, $\varphi' = \varphi'_0$, $\rho = \rho_0$ and $\rho_{\mathrm{m}} = \rho_{\mathrm{m}0}$. When *only* a scalar field is present, as discussed in section 2, the solutions are more manageable, and can be written:

Isotropic solutions:

$$\theta_0 = \sqrt{\frac{2}{N-1}}\frac{2N}{\sqrt{4N-\lambda^2(N-1)}} \tag{A1}$$

$$\sigma_0 = 0 \tag{A2}$$

$$(\varphi'_0)^2 = \frac{2\lambda^2(N-1)}{4N-\lambda^2(N-1)}. \tag{A3}$$

Anisotropic solutions:

$$\theta_0 = \sqrt{\frac{1}{2(N-1)}}\frac{4d+\lambda^2 D(N-1)}{\sqrt{4d+\lambda^2(N-1)(D-1)}} \tag{A4}$$

$$\sigma_0 = \sqrt{\frac{dD}{4N(N-1)}}\frac{4-\lambda^2(N-1)}{\sqrt{4d+\lambda^2(N-1)(D-1)}} \tag{A5}$$

$$(\varphi'_0)^2 = \frac{2\lambda^2(N-1)}{4d-\lambda^2(N-1)(D-1)}. \tag{A6}$$

To study the stability of a critical solution the standard method is to define new variables such that the solution of interest is moved to the origin, and then to look at the linearized dynamical equations in these new variables. We have chosen the following new variables,

$$\tilde{\theta} = \theta - \theta_0 \tag{A7}$$

$$\tilde{\sigma} = \sigma - \sigma_0 \tag{A8}$$

$$\tilde{y} = \varphi' - \varphi_0' \tag{A9}$$

$$\tilde{u} = \frac{\rho}{V(\varphi)} - \frac{\rho_0}{V(\varphi_0)} \tag{A10}$$

$$\tilde{v} = \frac{\rho_{\rm m}}{V(\varphi)} - \frac{\rho_{\rm m0}}{V(\varphi_0)}. \tag{A11}$$

Looking at the linear terms of equations (14)–(18) in these new variables, and taking $\Lambda/V(\varphi) \approx$ constant, we obtain

$$\frac{\rm d}{{\rm d}\tau}\tilde{\theta} = \left(\frac{\lambda}{2}\varphi_0' - \frac{2\theta_0}{N}\right)\tilde{\theta} + \left(\frac{\lambda}{2}\theta_0 - 2\varphi_0'\right)\tilde{y} - \frac{N\gamma - 2}{N-1}\tilde{u} - \frac{2}{N-1}\tilde{v} \tag{A12}$$

$$\frac{\rm d}{{\rm d}\tau}\tilde{\sigma} = -\left[\sqrt{\frac{2d}{DN}}\left(\frac{N-1}{N}\right)\theta_0 + \sigma_0\right]\tilde{\theta} + \left(2\sigma_0\sqrt{\frac{2d}{DN}}\tilde{\sigma} + \frac{\lambda}{2}\varphi_0' - \theta_0\right)\tilde{\sigma} + \left[\frac{\lambda}{2}\sigma_0 + \sqrt{\frac{2d}{DN}}\varphi_0'\right]\tilde{y} + \sqrt{\frac{2d}{DN}}[\tilde{u} + (1 + \mu_d - \mu_{\rm D})\tilde{v}] \tag{A13}$$

$$\frac{\rm d}{{\rm d}\tau}\tilde{y} = -\varphi_0\tilde{\theta} + (\lambda\varphi_0' - \theta_0)\tilde{y} \tag{A14}$$

$$\frac{\rm d}{{\rm d}\tau}\tilde{u} = -(2-\gamma)\frac{\rho_0}{V(\varphi_0)}\tilde{\theta} - (2-\gamma)\,\theta_0\tilde{u} \tag{A15}$$

$$\frac{\rm d}{{\rm d}\tau}\tilde{v} = -\frac{N-1}{N}\frac{\rho_{\rm m}}{V(\varphi_0)}\tilde{\theta} + \sqrt{\frac{2dD}{N}}(\mu_D - \mu_d)\rho_{\rm m0}\tilde{\sigma} + \left[\sqrt{\frac{2dD}{N}}(\mu_D - \mu_d)\sigma_0 - \frac{N-1}{N}\theta_0\right]\tilde{v} \tag{A16}$$

where $\mu_d = p_d/\rho_{\rm m}$ and $\mu_D = p_D/\rho_{\rm m}$ and all other variables are defined previously. Eigenvalues of the linear equations (A12)–(A16) have been found using the algebraic mathematical package Maple. For a critical solution to be time asymptotically stable at first order, and hence for it to be of interest to cosmology, all the eigenvalues are required to be negative (the stability when one or more eigenvalues are zero cannot be decided at first order).

General expressions for these eigenvalues are cumbersome, but specific examples can be given for the situations looked at in section 3, where the solution of Linde and Zelnikov was considered with the addition of radiation. Taking $N = 5$, $d = 3$, $D = 2$, $\mu_d - \mu_D = -2$, $\gamma = 4/5$, $\lambda = 0.5$, and $\Lambda = 0.005$ (Planck units), the eigenvalues are calculated to be,

$$E_1 = 0.45 \tag{A17}$$

$$E_2 = -1.59 \tag{A18}$$

$$E_3 = -1.42 \tag{A19}$$

$$E_4 = -1.70 \tag{A20}$$

$$E_5 = -0.41. \tag{A21}$$

The appearance of the positive eigenvalue shows that in this case the anisotropic solution where $\rho_{\rm m0} \neq 0$ is unstable. Thus such a solution is not going to be a generic solution for a cosmological model.

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