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MODELLING DEPENDENT INSURANCE RISKS

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Lausanne, October 2016 *Gildas Ratovomirija*

Dedicated to my mother and father.

*"Learn from yesterday, live for today, hope for tomorrow.
The important thing is not to stop questioning."* Albert Einstein

"Ataovy tsara ny soratra." My mother

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Chapter 1

Introduction

Recent insurance risk management is encountered with complex dependent risk factors. Accordingly, one of the main tasks of actuaries is the modelling of the dependence structure between insurance risks in numerous applications such as premiums calculation, reserving, risk capital quantification and capital allocation. An extensive literature has been developed on multivariate distributions, we refer to Joe [54], Balakrishnan et al. [10, 59] for a general overview of the existing methods and their applications. In particular, multivariate parametric distributions are a dominant choice in insurance applications. According to Harry Joe in [54] p.84, an ideal parametric family of multivariate distributions satisfies the following four desirable properties:

" A. interpretability, which could mean something like a mixture, stochastic or latent variable representation;
B. the closure property under the taking of margins, in particular the bivariate margins belonging to the same parametric family (this is especially important if, in statistical modelling, one thinks first about appropriate univariate margins and sequentially to higher -order margins);
C. a flexible and wide range of dependence (with type of dependence structure depending on applications);
D. a closed representation of the cdf and density (a closed-form cdf is useful if the data are discrete and a continuous latent random vector is used), and if not closed-form, then a cdf and density that are computationally feasible to work with.

This thesis is concerned with the modelling of dependence among insurance risks using two families of parametric multivariate distributions, namely the Sarmanov distribution and the concept of copula. On the one hand, the Sarmanov distribution has been introduced by Sarmanov [84] in the bivariate case and extended by Lee [62] for the multivariate framework. One key advantage of the Sarmanov distribution is its flexibility to join different types of marginal; it also allows to model highly dependent risks. In insurance applications numerous contributions have been flourished in using the Sarmanov distribution as a dependence model for multivariate insurance risks, see e.g., Yang and Hashorva [97], Abdallah et al. [1], Vernic [92], Bahraoui et al. [8], Hashorva and Ratovomirija [49], Ratovomirija [80] among others.

On the other hand, copula models were introduced by Sklar [86] in order to specify the joint distribution function (df) of a random vector as a function of the df of the marginals. We refer to Nelsen [74] for a general introduction to copula theory. Due to its allowance to separate marginals and the dependence structure, copula models are practically attractive with regard to the modelling of dependence between risk factors in finance and insurance, see for instance Frees and Valdes [38], Denuit et al. [25], Tang et Valdes [87], Constantinescu et al. [18] and the special issue of *Insurance Mathematics and Economics* [42] and references therein.

This thesis includes results in the area of dependent insurance risks, which are structured in two parts. The first part (Chapter 2 and Chapter 3) is devoted to the analysis of multivariate insurance risks using the Sarmanov distribution. In this regards, with mixed Erlang distribution as a marginal distribution, we address risk aggregation, capital allocation and diversification effects. The second part (Chapter 4 and Chapter 5) is concerned with multivariate insurance risks using the copula approach. Specifically, we investigate in Chapter 4 the effect of age difference on the level of dependence between husband and wife lifetimes. In Chapter 5, we introduce a new family of multivariate distribution derived from the collective risk model. The results presented in Chapter 2, Chapter 3 and Chapter 5 are published in actuarial journals while the content of Chapter 4 is submitted for journal publication.

Next, we discuss a brief insight of the contributions of each chapter.

In the risk management framework, such as Solvency II in the European market, reinsurance and insurance companies are required to hold a certain amount of capital, namely the risk capital, in order to be covered from unexpected losses. The quantification of risk capital involves the aggregation of dependent risks and thus the choice of an appropriate multivariate model for the dependence structure between risks. Significant developments have been devoted to the aggregation of dependent risks, specifically in insurance and finance, see e.g., Alink et al. [2], McNeil et al. [68], Embrechts et al. [33] and references therein. Motivated by Cossette et al. [20], in Chapter 2 we introduce the Sarmanov distribution as a model of the dependence structure between mixed Erlang insurance risks. In particular, the aggregate risk derived from the convolution of Sarmanov dependent mixed Erlang risks belongs again to the class of Erlang mixtures. In this respect, we obtain analytical expressions for the risk capital needed for the whole portfolio as well as the contribution of each individual risk to the risk capital. To demonstrate the flexibility and the wideness of the dependence range of Sarmanov mixed Erlang distributions, we also calculate commonly used dependence measures, namely Pearson's correlation coefficient, Spearman's rho and Kendall's tau, .

Chapter 3 is focused on the aggregation of reinsurance risks in which the dependence among the ceding insurer(s) is governed by the Sarmanov distribution and each individual risk belongs to the class of Erlang mixtures. The main contribution of this chapter is the investigation of the effects of the ceding insurer(s) risks dependency on the reinsurer risk profile which has only stop loss reinsurance portfolios. We derive analytical expression for the df of the aggregate stop loss reinsurance risk. This allows us to determine the risk capital required for the entire portfolio of the reinsurer. Diversification effects stemmed from aggregating reinsurance risks are also examined and a closed expression for the allocated risk capital to each business unit of the reinsurer are gathered. Furthermore, we illustrate the results with numerical applications for different Sarmanov distribution families.

The aim of Chapter 4 is to analyse the dependence structure between lifetimes of a married couple. Life insurance and annuity products insuring numerous lives require the modelling of the joint distribution of future lifetimes. Commonly in actuarial practice, the future lifetimes between a group of people are assumed to be independent. However, this simplifying hypothesis is not supported by real insurance data. In this chapter, we model the joint distribution of the future lifetimes of husband and wife utilizing the concept of copula. Based on a data from a large Canadian insurance company we demonstrate that the age difference and the gender of the elder partner have an impact on the level of dependence. In particular, maximum likelihood approach is implemented for parameters estimation. Not only do the results show that the correlation decreases with age difference, but also the dependence between the lifetimes is higher when husband is older than wife. Goodness-of-fit procedures are performed in order to assess the validity of our model. Additionally, we consider several products available on the life insurance market to conclude the chapter with practical illustrations.

In the literature a broad range of methods has been developed with regard to the construction of multivariate distributions , see e.g., Joe [54], Johnson et al. [56] and Denuit et al. [25] . In general, multivariate distributions family is elaborated through methods like stochastic representations and mixtures. With motivation from Zhang and Lin [100], in Chapter 5 we propose a flexible family of copulas derived from the joint distribution of the largest claim sizes of two insurance portfolios. In this respect, we consider the classical collective model over a fixed time period of two insurance portfolios with (X_i, Y_i) modelling the i-th claim sizes of both portfolios and N the total number of claims. The df F of the maximum claim amount, denoted by $(X_{N:N}, Y_{N:N})$, in both portfolios is given by

$$F(x, y) = L_\Lambda(-\ln G(x, y)), \quad x, y \geqslant 0,$$

where $\Lambda = N|N > 1$, L_Λ its Laplace transform and G is the df of (X_i, Y_i). We demonstrate that the distributions F and their copulas have some interesting distributional and extremal properties. Interestingly, the extremal properties of F are similar to those of G. We discuss parameter estimation with three applications to concrete insurance data set and Monte Carlo simulations for parametric families of bivariate df's induced by F. We also present two applications of the results, on the one hand, we quantify stop loss and excess of loss reinsurance premiums of a Swiss insurance Loss and ALAE data set. On the other hand, we examine by simulation the influence of the sum of largest claims observed in two insurance portfolios $X_{N:N} + Y_{N:N}$ on the distribution of $S_N = \sum_{i=1}^{N}(X_i + Y_i)$. Furthermore, using the covariance capital allocation principle we quantify the impact of $X_{N:N}$ and $Y_{N:N}$ on the total loss S_N. The latter application is of importance when designing risk management and reinsurance strategies especially in proportional reinsurance.

Chapter 2

Sarmanov Mixed Erlang Risks in Insurance Applications

This chapter is based on E. Hashorva, G. Ratovomirija: On Sarmanov Mixed Erlang risks in Insurance applications, published in the *Astin Bulletin*, **45**:175–205, 2015.

2.1 Introduction

Analysis of aggregated risk is important for insurance business, it allows the insurers to assess and to monitor their risks through the risk management framework. In the classical framework of independent and identically distributed risks, explicit analytical formulas for quantities of interest including Value-at-Risk (VaR), Tail Value-at-Risk (TVaR) or Stop-loss premium formula for the aggregated risk can be derived explicitly for few tractable cases. For instance Willmot and Lin [94], Lee and Lin [63, 64] and Cossette et al. [20] have shown that this is the case if we choose the mixed Erlang distribution as a model for claim sizes. One reason for the tractability of the mixed Erlang distribution is the fact that the convolution of such risks is again mixed Erlang, see Klugman et al. [58].

Since insurance data clearly shows that insurance risks are commonly dependent, in order to be able to get closed-form formulas for quantities of interest, an important task is the adequate choice of the dependence structure between the risks. Even for the simple case of the dependence specified by a log-normal framework with stochastic volatility, as shown in the recent contributions of Embrechts et al. [35] , Hashorva and Li [48] and Hashorva and Kortschak [47] only asymptotic results can be derived.

With motivation from Cossette et al. [20] where the aggregation of FGM mixed

Erlang risks is considered, in this contribution we shall investigate the Sarmanov mixed Erlang risks. The Sarmanov distribution includes the FGM distribution as a special case. One key advantage of the Sarmanov distribution is its flexibility; it also allows to model highly dependent risks, see e.g., Lee [62], Bairamov et al. [9]. The aim of this chapter is to provide analytical results and properties of the aggregated dependent risks with mixed Erlang marginals by using the Sarmanov distribution as a model for the dependence structure. This model is promising in risk aggregation practice as it satisfies the four desirable properties of a multivariate parametric model mentioned in Joe [54] p.84, namely the interpretability property, the closure property, the flexibility and the wideness of the range of dependence, and the representation of the distribution function (df) and the probability density function (pdf) in analytical form.
The chapter is organised as follows. In Section 2.2, we describe the background of the Sarmanov mixed Erlang distribution by exploring some definitions and properties of the Sarmanov distribution as a model for the dependence structure and the mixed Erlang distribution with a common scale parameter as a model for claim size distribution in insurance. In Section 2.3, we demonstrate that the distribution of the aggregated risk belongs to the class of Erlang mixtures; numerical illustrations and simulation studies are performed to validate the results. In Section 2.4, we derive explicit expressions for the allocated capital to each individual risk $X_i, i = 1, 2$ under the TVaR and the covariance capital allocation rules. We present some useful results and properties of the mixed Erlang distribution in Section 2.2.2. In Section 2.6, an extension of the results in the bivariate case to the multivariate framework is presented with numerical examples. All the proofs are relegated to Section 2.7. In the Appendix, the flexibility and the wideness of the dependence range of Sarmanov mixed Erlang distributions are discussed by calculating commonly used dependence measures, namely Pearson's correlation coefficient, Sperman's rho and Kendall's tau.

2.2 Preliminaries

2.2.1 Sarmanov Distribution

The Sarmanov distribution introduced in Sarmanov [84] has proved valuable in numerous insurance applications. For instance Hernandez et al. [51] used the multivariate Sarmanov distribution to model the dependence structure between risk profiles for the calculation of Bayes premiums in the collective risk model. Sarabia and Gómez-Déniz [83] fitted multivariate insurance count data using the Sarmanov distribution with Poisson-Beta marginals. As shown in Yang and Wang [98] and Yang and Hashorva [97] the Sarmanov distribution allows for tractable asymptotic formulas in the context of ruin probabilities. Referring to Sarmanov [84] a bivariate risk (X_1, X_2) has the Sarmanov distribution with joint pdf h given by

$$h(x_1, x_2) = f_1(x_1) f_2(x_2)\Big(1 + \alpha_{12}\phi_1(x_1)\phi_2(x_2)\Big), \quad \alpha_{12} \in \mathbb{R}, \tag{2.1}$$

where f_i is the pdf of $X_i, i = 1, 2$, and ϕ_1, ϕ_2 are two kernel functions, which are assumed to be bounded and non-constant such that

$$\mathbb{E}\{\phi_1(X_1)\} = \mathbb{E}\{\phi_2(X_2)\} = 0, \quad 1 + \alpha_{12}\phi_1(x_1)\phi_2(x_2) \geqslant 0, \tag{2.2}$$

is valid. If $\phi_i(x_i) = 2F_i(x_i) - 1$ with F_i the df of X_i, then h is the joint pdf of the FGM distribution introduced by Morgenstern [71] for Cauchy marginals and developed by Gumbel [45] for exponential margins and generalized by Farlie [37]. Lee [62] proposed some general methods for finding the kernel function $\phi_i(x_i)$ with different types of marginals. Yang and Hashorva [97] considered $\phi_i(x_i) = g_i(x_i) - \mathbb{E}(g_i(X_i))$. When $g_i(x_i) = e^{-x_i}$ the corresponding kernel function coincides with the one explored by Lee [62] for marginal distributions with support in $[0, \infty)$. We have

$$\phi_i(x_i) = e^{-x_i} - \mathbb{E}\left\{e^{-X_i}\right\} = e^{-x_i} - \mathcal{L}_i(1), \tag{2.3}$$

where $\mathcal{L}_i(t) = \mathbb{E}\left\{e^{-tX_i}\right\}, t > 0$ is the Laplace transform of X_i. In the rest of the chapter, we set

$$\mathcal{L}_i := \mathcal{L}_i(1), \quad \mathcal{L}_i' := \mathcal{L}_i'(1).$$

The joint pdf h is thus given by

$$h(x_1, x_2) \;=\; f_1(x_1) f_2(x_2)\left((1+\gamma) + \alpha_{12}(e^{-x_1-x_2} - e^{-x_1}\mathcal{L}_2 - e^{-x_2}\mathcal{L}_1)\right), \tag{2.4}$$

where $\gamma = \alpha_{12}\mathcal{L}_1\mathcal{L}_2$.

Remarks 2.2.1. If (X_1, X_2) has a Sarmanov distribution with kernel functions given in (2.3), additionally if $X_i, i = 1, 2$ follows a mixture of Gamma distributions where the mixture components share the same scale parameter $\beta_i \in (0, \infty)$, then the joint df of (X_1, X_2) follows easily from integrating the pdf in (2.4). Specifically, we have for H the joint df of (X_1, X_2)

$$H(x_1, x_2) \;=\; (1+\gamma)F_1(x_1, \beta_1)F_2(x_2, \beta_2) + \gamma F_1(x_1, \beta_1 + 1)F_2(x_2, \beta_2 + 1)$$

$$-\gamma F_1(x_1, \beta_1 + 1)F_2(x_2, \beta_2) - \gamma F_1(x_1, \beta_1)F_2(x_2, \beta_2 + 1),$$

where $F_i(x_i, \beta_i) = \sum_{k=1}^{\infty} q_k W_k(x_i, \beta_i), i = 1, 2$ with $W_k(x_i, \beta_i)$ is the df of the Gamma distribution with scale parameter β_i and shape parameter $k \in (0, \infty)$ and q_k is the mixing weight such that $\sum_{k=1}^{\infty} q_k = 1$.

Compared to the FGM distribution which has $[-1/3, 1/3]$ as the range of Pearson's correlation coefficient ρ_{12} the Sarmanov distribution has a wider range of ρ_{12}, which is useful in the aggregation of strongly dependent insurance risks. For the Sarmanov case we have the explicit formula for ρ_{12}, namely

$$\rho_{12}(X_1, X_2) = \frac{\alpha_{12}\nu_1\nu_2}{\sigma_1\sigma_2}, \quad \nu_i = \mathbb{E}\{X_i\phi_i(X_i)\}, \quad \sigma_i = \sqrt{Var(X_i)}, \quad i = 1, 2. \tag{2.5}$$

In the particular case that the kernels are given by (2.3), for two positive Sarmanov risks with finite variances the range of α_{12} is (see Lee [62])

$$\frac{-1}{\max\{\mathcal{L}_1\mathcal{L}_2, (1-\mathcal{L}_1)(1-\mathcal{L}_2)\}} \leqslant \alpha_{12} \leqslant \frac{1}{\max\{\mathcal{L}_1(1-\mathcal{L}_2), (1-\mathcal{L}_1)\mathcal{L}_2\}}, \tag{2.6}$$

where $\nu_i = -\mathcal{L}_i' - \mathcal{L}_i\mu_i$ and $\mu_i = \mathbb{E}\{X_i\}, i = 1, 2$. Lee [62] extended the Sarmanov distribution to the multivariate case by defining the joint pdf h of $(X_1, \ldots, X_n)$ as

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)(1 + R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x})), \quad \boldsymbol{x} := (x_1, \ldots, x_n), \tag{2.7}$$

where

$$\begin{aligned} R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x}) &= 1 + \sum_{j_1<}^{n-1}\sum_{j_2}^{n} \alpha_{j_1,j_2}\phi_{j_1}(x_{j_1})\phi_{j_2}(x_{j_2}) \\ &\quad + \sum_{j_1<}^{n-2}\sum_{j_2<}^{n-1}\sum_{j_3}^{n} \alpha_{j_1,j_2,j_3}\phi_{j_1}(x_{j_1})\phi_{j_2}(x_{j_2})\phi_{j_3}(x_{j_3}) \\ &\quad + \ldots + \alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\phi_i(x_i), \end{aligned}$$

such that

$$1 + R_{\phi_1,\ldots,\phi_n,\Omega_n}(\boldsymbol{x}) \geqslant 0 \tag{2.8}$$

is fulfilled for all $x_i \in \mathbb{R}$ with $\Omega_n = \{\alpha_{j_1,j_2}, \alpha_{j_1,j_2,j_3}, \ldots, \alpha_{1,2,\ldots,n}\} \in \mathbb{R}$. If the kernel functions are specified by (2.3), then h is given by (set $\Delta(x_i) := e^{-x_i} - \mathcal{L}_i$)

$$\begin{aligned} h(\boldsymbol{x}) &= \prod_{i=1}^{n} f_i(x_i)\left(1 + \sum_{j_1<}^{n-1}\sum_{j_2}^{n} \alpha_{j_1,j_2}\Delta(x_{j_1})\Delta(x_{j_2})\right. \\ &\quad + \sum_{j_1<}^{n-2}\sum_{j_2<}^{n-1}\sum_{j_3}^{n} \alpha_{j_1,j_2,j_3}\Delta(x_{j_1})\Delta(x_{j_2})\Delta(x_{j_3}) \\ &\quad \left. + \ldots + \alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\Delta(x_i)\right). \end{aligned} \tag{2.9}$$

2.2.2 Mixed Erlang Claim Sizes

These last decades, modelling claim size in insurance with the mixed Erlang distribution with a common scale parameter has been well developed. In risk theory, Dickson and Willmot [31] and Dickson [30] have explored an analytical form of the finite time ruin probability, using the mixed Erlang distribution as a claim size model. Recently, using the EM algorithm, mixed Erlang distribution has been fitted to catastrophic loss data in the United States by Lee and Lin [63] and also to censored and truncated data by Verbelen et al. [90] .

Moreover, Lee and Lin [64], Willmot and Woo [96] have developed the multivariate mixed Erlang distribution to overcome some drawbacks of the copula approach while Badescu et al. [7] have used the multivariate mixed Poisson distribution with mixed Erlang claim sizes to model operational risks. Furthermore, Cossette et al. [20] have introduced a risk aggregation in the multivariate set-up with mixed Erlang marginals and the FGM copula to capture the dependence structure.

In the sequel, we denote respectively

$$\begin{aligned} w_k(x,\beta) &= \frac{\beta^k x^{k-1} e^{-\beta x}}{(k-1)!}, \\ W_k(x,\beta) &= \sum_{j=k}^{\infty} \frac{(\beta x)^j e^{-\beta x}}{j!}, \\ \overline{W}_k(x,\beta) &= \sum_{j=0}^{k-1} \frac{(\beta x)^j e^{-\beta x}}{j!}, x > 0, \end{aligned} \tag{2.10}$$

the pdf, the df and the survival function of an Erlang distribution, where $k \in \mathbb{N}^*$ is the shape parameter and $\beta > 0$ is the scale parameter. As its name indicates, the mixed Erlang distribution is elaborated from the Erlang distribution, its pdf and df are respectively defined by

$$f_X(x, \beta, \underset{\sim}{V}) = \sum_{k=1}^{\infty} q_k w_k(x, \beta), \quad F_X(x, \beta, \underset{\sim}{V}) = \sum_{k=1}^{\infty} q_k W_k(x, \beta), \tag{2.11}$$

where $\underset{\sim}{Q} = (q_1, q_2, \ldots)$ is a vector of non-negative weights satisfying $\sum_{k=1}^{\infty} q_k = 1$. In the following we write $X \sim ME(\beta, \underset{\sim}{Q})$ if X has pdf given by (2.11).

As discussed in Willmot and Lin [94], Lee and Lin [63, 64] and Cossette et al. [20], one of the important advantages of employing the mixed Erlang distribution in insurance loss modelling is the fact that many useful risk related quantities, such as moments and mean excess function can be calculated explicitly by simple formulas. For instance, the quantile function (or VaR) of the mixed Erlang distribution can be easily obtained given the tractable form of the df. From the df (2.11), at a confidence level $p \in (0, 1)$, the VaR of X, denoted by x_p, is the solution of

$$e^{-\beta x_p} \sum_{k=1}^{\infty} q_k \sum_{j=0}^{k-1} \frac{(\beta x_p)^j}{j!} = 1 - p,$$

which can be solved numerically. Further, since for the mean excess function of X, we have (see Willmot and Lin [94], p.7)

$$\mathbb{E}((X - d)|X > d) = \frac{\sum_{k=0}^{\infty} Q_k^* \frac{(\beta d)^k}{k!}}{\beta \sum_{j=1}^{\infty} Q_j \frac{(\beta d)^{j-1}}{(j-1)!}}, \quad d > 0, \tag{2.12}$$

where $Q_k^* = \sum_{j=k+1}^{\infty} Q_j$ with $Q_j = \frac{\sum_{k=j}^{\infty} q_k}{\sum_{k=1}^{\infty} k q_k}$, then the TVaR of X at a confidence level $p \in (0, 1)$ is given by the following explicit formula

$$TVaR_X(p) = \frac{\sum_{k=0}^{\infty} Q_k^* \frac{(\beta x_p)^k}{k!}}{\beta \sum_{j=1}^{\infty} Q_j \frac{(\beta x_p)^{j-1}}{(j-1)!}} + x_p. \tag{2.13}$$

Remark that above we assume that $\mathbb{E}(X) = \sum_{k=1}^{\infty} k q_k$ is finite. Additionally, the mixed Erlang distribution is a tractable marginal distribution for the Sarmanov distribution. Next we present a result for the 2-dimensional set-up, see Section 2.6 for the same results in higher dimensions.

2.3 Aggregation of SmE Risks

Let (X_1, X_2) have a bivariate Sarmanov risk with kernel functions $\phi_i(x) = e^{-x_i} - \mathcal{L}_i$ for $i = 1, 2$. We shall assume that both X_1 and X_2 follow a mixed Erlang distribution, i.e.,

$$X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), \quad i = 1, 2,$$

where β_i is the scale parameter, $\underset{\sim}{Q}_i = (q_{i,1}, q_{i,2}, \ldots)$ denotes the mixing probabilities. The joint distribution of the random vector (X_1, X_2) will be referred to as a bivariate Sarmanov mixed Erlang (SmE) distribution and we shall abbreviate this as

$$(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2),$$

where $\boldsymbol{\beta} = (\beta_1, \beta_2)$. The dependence structure of the bivariate random vector (X_1, X_2) can be analysed by calculating commonly used dependence measures such as Pearson's correlation coefficient or Kendall's tau, see Appendix 2.8.1. For given vectors of the mixing probabilities $\underset{\sim}{V}_i = (v_{i1}, v_{i2}, \ldots), i = 1, 2$ we define in the following $\pi_1\{\underset{\sim}{V}_1, \underset{\sim}{V}_2\} = 0$ and for $k > 1$

$$\pi_k\{\underset{\sim}{V}_1, \underset{\sim}{V}_2\} = \sum_{j=1}^{k-1} v_{1,j} v_{2,k-j}.$$

The main result in this section is the derivation of the distribution of the aggregated risk $S_2 = X_1 + X_2$.

Proposition 2.3.1. If $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$ with $\beta_1 \leqslant \beta_2$, then $S_2 \sim ME(\beta_2 + 1, \underset{\sim}{P})$ where the mixing weights p_k are given by (set $\gamma := \alpha_{12}\mathcal{L}_1\mathcal{L}_2$, $\overline{\beta_i} := \beta_i/(\beta_i + 1)$)

$$\begin{aligned} p_k \;=\; & (1+\gamma)\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1), \underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2)\} + \gamma\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1), \underset{\sim}{\Psi}_2(\underset{\sim}{\Theta}_2)\} \\ & -\gamma\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1), \underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2)\} - \gamma\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1), \underset{\sim}{\Psi}_2(\underset{\sim}{\Theta}_2)\}, \end{aligned} \tag{2.14}$$

where for $i = 1, 2$ the components of $\underset{\sim}{\Theta}_i = (\theta_{i,1}, \theta_{i,2}, \ldots)$ are defined by

$$\theta_{i,k} = \frac{q_{i,k}\overline{\beta_i}^k}{\sum_{j=1}^{\infty} q_{i,j}\overline{\beta_i}^j},$$

whereas the components of $\underset{\sim}{\Psi}_i(\underset{\sim}{Q}_i) = (\psi_{i,1}, \psi_{i,2}, \ldots)$ are

$$\psi_{i,k} = \sum_{j=1}^{k} q_{i,j} \binom{k-1}{j-1} \left(\frac{\beta_i}{\beta_2 + 1}\right)^j \left(1 - \frac{\beta_i}{\beta_2 + 1}\right)^{k-j}.$$

Example 2.3.1. As an illustration, let

$$(X_1, X_2) \sim SME_2\left(\boldsymbol{\beta} = \begin{pmatrix} 0.9 \\ 0.95 \end{pmatrix}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2\right),$$

with $\underset{\sim}{Q}_1 = (0.4, 0.2, 0.3, 0.1), \underset{\sim}{Q}_2 = (0.3, 0.5, 0.1, 0.1)$ and $\alpha_{12} = 2.87$. According to (2.11), one can write the pdf of X_1 and X_2 as follows

$$\begin{aligned} f_1(x_1) &= 0.4w_1(x_1, 0.9) + 0.2w_2(x_1, 0.9) + 0.3w_3(x_1, 0.9) + 0.1w_4(x_1, 0.9), \\ f_2(x_2) &= 0.3w_1(x_2, 0.95) + 0.5w_2(x_2, 0.95) + 0.1w_3(x_2, 0.95) + 0.1w_4(x_2, 0.95). \end{aligned}$$

Following (2.4), the joint density of (X_1, X_2) is given by

$$h(x_1, x_2) = f_1(x_1)f_2(x_2)(1.22 + 2.87e^{-x_1-x_2} - 0.81e^{-x_1} - 0.78e^{-x_2}).$$

Table 2.1 below presents the central moments of the marginals.

	Mean	Variance	Skewness	Kurtosis
X_1	2.33	4.44	1.38	5.49
X_2	2.11	3.10	1.49	6.12

Table 2.1: Central moments of X_1 and X_2.

It follows that the distribution of S_2 is a mixed Erlang distribution with scale parameter $\beta_{S_2} = 1.95$ and mixing probabilities partially shown in 2.2. We notice that the higher the value of k is, the smaller the value of p_k.

k	p_k	k	p_k	k	p_k	k	p_k	k	p_k
1	0.0000	11	0.0664	21	0.0046	31	8.963E-05	41	9.294E-07
2	0.0675	12	0.0564	22	0.0033	32	5.803E-05	42	5.751E-07
3	0.0839	13	0.0465	23	0.0023	33	3.737E-05	43	3.547E-07
4	0.0645	14	0.0373	24	0.0016	34	2.393E-05	44	2.180E-07
5	0.0700	15	0.0292	25	0.0011	35	1.525E-05	45	1.336E-07
6	0.0740	16	0.0223	26	0.0007	36	9.668E-06	46	8.159E-08
7	0.0811	17	0.0168	27	0.0005	37	6.103E-06	47	4.970E-08
8	0.0840	18	0.0125	28	0.0003	38	3.835E-06	48	3.02E-08
9	0.0816	19	0.0091	29	0.0002	39	2.400E-06	49	1.828E-08
10	0.0753	20	0.0065	30	0.0001	40	1.496E-06	50	1.105E-08

Table 2.2: Mixing probabilities of the distribution of $S_2 = X_1 + X_2$, with scale parameter $\beta_{S_2} = 1.95$.

In order to validate our results, SmE risks have been simulated (see in Appendix 2.8.2 the details about the simulation algorithm). In this respect, analytical and simulated results on the aggregated risk $S_2 = X_1+X_2$ are presented and analysed. As displayed in Table 2.3, based on the VaR and TVaR risk measures the comparison of the exact and the simulated values shows that our results are consistent for different values of the tolerance level p. Furthermore, it can be seen that VaR is more sensitive to the change of the tolerance level than TVaR.

	Analytical formula		Simulated		Percentage difference (%)	
p (%)	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$	$VaR_{S_2}(p)$	$TVaR_{S_2}(p)$
90.00	8.26	10.24	8.22	10.21	0.49	0.29
92.50	8.88	10.80	8.86	10.77	0.23	0.28
95.00	9.71	11.56	9.66	11.53	0.52	0.26
97.50	11.05	12.82	10.98	12.82	0.64	0.00
99.00	12.71	14.41	12.79	14.46	-0.63	-0.35
99.50	13.92	15.56	13.87	15.43	0.36	0.84
99.90	16.57	18.13	16.61	17.86	-0.24	1.51
99.99	20.15	21.62	19.42	20.79	3.62	3.84

Table 2.3: Exact and simulated values of VaR and TVaR of $S_2 = X_1 + X_2$.

Similarly, by changing the level of the dependence between marginals which is described by α_{12} and for a tolerance level of 99%, the comparison of the exact and the simulated values of VaR and TVaR is displayed in Table 2.4. Note in passing that the maximum attainable value of α_{12}, in our example, is 4.87 while the minimum is -1.91.

	Analytical formula		Simulated		Percentage difference (%)	
α_{12}	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$	$VaR_{S_2}(0.99)$	$TVaR_{S_2}(0.99)$
-1.91	12.24	13.92	12.26	13.91	-0.16	0.10
-0.87	12.35	14.04	12.38	14.03	-0.25	0.06
0	12.44	14.13	12.48	14.13	-0.31	0.03
0.87	12.53	14.22	12.57	14.22	-0.29	0.01
1.87	12.62	14.31	12.66	14.32	-0.33	-0.02
2.87	12.71	14.41	12.74	14.41	-0.24	-0.05
3.87	12.80	14.49	12.82	14.50	-0.14	-0.08
4.87	12.88	14.57	12.90	14.59	-0.14	-0.10

Table 2.4: Dependence level and sensitiveness of risk measures.

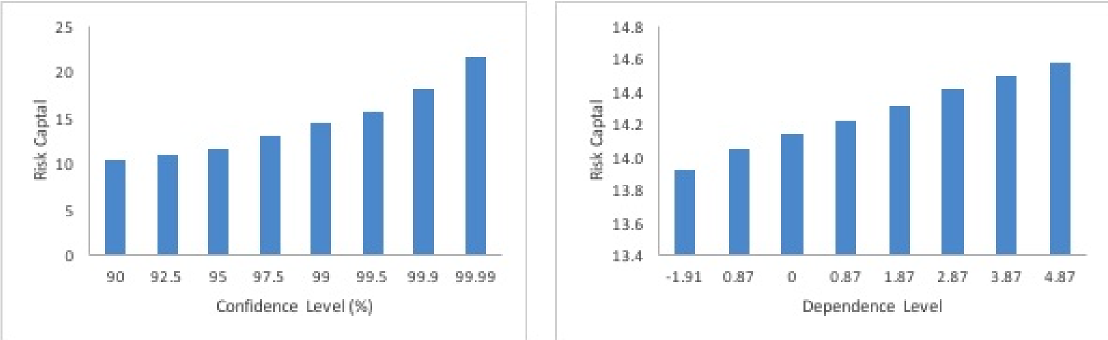


Figure 2.1: Risk Capital as a function the confidence level and the dependence parameters.

It can be seen from Figure 2.1 that not only does the risk capital, measured as the TVaR of the aggregated risk, increase with the confidence level but also with the level of dependence between individual risks.

2.4 Capital Allocation

In this section, we derive analytical expressions for the amount of capital allocated to each individual risk under the TVaR and the covariance principles. Evaluating the economic capital for the entire portfolio that an insurance company needs to absorb large unexpected losses is of importance in enterprise risk management. In this respect, the so-called capital allocation consists in determining the contribution of each individual risk to the aggregate economic capital. This allows the insurance company to identify and to monitor efficiently their risks. In the literature, many capital allocation techniques have been developed, see Cummins [21], Tasche [88, 89], Dhaene et al. [29], McNeil et al. [68] and references therein. In practice, the TVaR and the covariance allocation principle are commonly used, since they take into account the dependence structure between risks. More precisely, if $S_n = \sum_{i=1}^{n} X_i$ is the aggregate risk where X_i is a continuous random variable (rv) with finite mean that represents the individual risk, the amount of capital T_i allocated to each risk X_i, for $i = 1, \ldots, n$, is defined as (for a tolerance level $p \in (0, 1)$, denote $T_i = T_p(X_i, S_n)$ under the TVaR allocation principle, $T_i = K_p(X_i, S_n)$ under the covariance allocation principle)

$$T_p(X_i, S_n) = \frac{\mathbb{E}(X_i \mathbb{1}_{\{S_n > VaR_{S_n}(p)\}})}{1-p}, \tag{2.15}$$

$$K_p(X_i, S_n) = \mathbb{E}(X_i) + \frac{Cov(X_i, S_n)}{Var(S_n)}(TVaR_{S_n}(p) - \mathbb{E}(S_n)), \tag{2.16}$$

where we assume that S_n has finite and positive variance. We have

$$\sum_{i=1}^{n} T_i = \sum_{i=1}^{n} T_p(X_i, S_n) = \sum_{i=1}^{n} K_p(X_i, S_n) = TVaR_{S_n}(p),$$

which means that for both allocation principle, based on TVaR as a risk measure, the capital required for the entire portfolio is equal to the sum of the allocated capital of each risk within the portfolio. Given some vector $\underset{\sim}{V} = (v_1, v_2, \ldots)$ with non-negative components such that $\sum_{j=1}^{\infty} jv_j < \infty$ we define the new vector $\underset{\sim}{G}(\underset{\sim}{V}) = (g_1, g_2, \ldots)$ where

$$g_k = \begin{cases} 0 & \text{for} \quad k = 1, \\ \frac{(k-1)v_{k-1}}{\sum_{j=1}^{\infty} jv_j} & \text{for} \quad k > 1. \end{cases}$$

For notational simplicity we shall also write in the following $\overline{\beta_i}$ instead of $\beta_i/(\beta_i+1)$. We derive next an explicit form of $T_p(X_i, S_2)$ and $K_p(X_i, S_2), i = 1, 2$, in the case of SmE type risks.

Proposition 2.4.1. Let $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$ with $\beta_1 \leqslant \beta_2$, further let $\underset{\sim}{\Theta}_i$ and $\underset{\sim}{\Psi}_i$ be defined as in Theorem 2.3.1. If for $i = 1, 2$ both $\mu_i := \frac{1}{\beta_i}\sum_{k=1}^{\infty} kq_{i,k}$ and $\widetilde{\mu}_i := \frac{1}{\beta_i+1}\sum_{k=1}^{\infty} k\theta_{i,k}$ are finite, then for any $p \in (0, 1)$ the amount of capital allocated to each risk $X_i, i = 1, 2$, under the TVaR principle is

$$T_p(X_i, S_2) = \frac{1}{1-p}\sum_{k=1}^{\infty} z_{ik}\overline{W}_k(VaR_{S_2}(p), \beta_2 + 1), \tag{2.17}$$

where $\gamma = \alpha_{12}\mathcal{L}_1\mathcal{L}_2$,

$$\begin{aligned} z_{i,k} \quad = \quad & (1+\gamma)\mu_i\pi_k\{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{Q}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{Q}_j)\} + \gamma\widetilde{\mu}_i\pi_k\{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{\Theta}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{\Theta}_j)\} \\ & -\gamma\widetilde{\mu}_i\pi_k\{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{\Theta}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{Q}_j)\} - \gamma\mu_i\pi_k\{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{Q}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{\Theta}_j)\}, \quad i \neq j, \end{aligned}$$

and the contribution of each risk $X_i, i = 1, 2$ to the economic capital of the entire portfolio, under the covariance principle, is given by

$$K_p(X_i, S_2) = \sum_{k=1}^{\infty} \frac{L_{i,k}}{\beta_2 + 1},$$

where

$$L_{i,k} = k\psi_{i,k} + \varepsilon_{i,j}\left(\frac{P_k^*((\beta_2+1)VaR_{S_2}(p))^k}{\varphi k!} + (\beta_2+1)VaR_{S_2}(p) - kp_k\right), i \neq j,$$

with

$$\begin{aligned}
\varepsilon_{i,j} &= \frac{\sum_{m=1}^{\infty}(m^2+m)\psi_{im} - (\sum_{m=1}^{\infty} m\psi_{im})^2}{\sum_{m=1}^{\infty}(m^2+m)p_m - (\sum_{m=1}^{\infty} mp_m)^2} \\
&+\left(\frac{\alpha_{12}(\beta_2+1)^2}{\sum_{m=1}^{\infty}(m^2+m)p_m - (\sum_{m=1}^{\infty} mp_m)^2}\right) \\
&\times\left(\frac{1}{\beta_i+1}\sum_{m=1}^{\infty} mq_{i,m}\overline{\beta_i}^m - \frac{1}{\beta_i}\sum_{m=1}^{\infty} q_{i,m}\overline{\beta_i}^m\sum_{m=1}^{\infty} mq_{i,m}\right) \\
&\times\left(\frac{1}{\beta_j+1}\sum_{m=1}^{\infty} mq_{j,m}\overline{\beta_j}^m - \frac{1}{\beta_j}\sum_{m=1}^{\infty} q_{j,m}\overline{\beta_j}^m\sum_{m=1}^{\infty} mq_{j,m}\right),
\end{aligned} \tag{2.18}$$

$$\varphi = \sum_{j=1}^{\infty}\frac{P_j((\beta_2+1)VaR_{S_2}(p))^{j-1}}{(j-1)!}, \quad P_k^* = \sum_{j=k}^{\infty} P_j, \quad P_j = \frac{\sum_{k=j}^{\infty} p_k}{\sum_{k=1}^{\infty} kp_k}$$

and p_k is given in (2.14).

Example 2.4.1. In this example, we consider the same marginals and dependence parameters as in Example 2.3.1. For different level of the dependence between X_1 and X_2, which is described by α_{12}, TVaRs have been calculated on the aggregated risk $S_2 = X_1 + X_2$ at a tolerance level $p = 99\%$. Furthermore, the allocated capital to each risk $X_i, i = 1, 2$, under the TVaR and the covariance capital allocation principle are also evaluated. Table 2.5 demonstrates that risk measures on the aggregated risk are sensitive to the level of dependence between individual risks. Actually, due to the relationship between dependence level and the diversification effect, the more X_1 and X_2 are dependent, the more the portfolio is risky, hence more capital is needed to cover the risks. In this respect, more capital is allocated to risk X_1 compared to the amount allocated to risk X_2 under the TVaR and the covariance principle.

α_{12}	$TVaR_{S_2}(0.99)$	$T_{0.99}(X_1, S_2)$	$T_{0.99}(X_2, S_2)$	$K_{0.99}(X_1, S_2)$	$K_{0.99}(X_2, S_2)$
-1.91	13.92	7.70	6.22	7.69	6.23
-0.87	14.04	7.74	6.30	7.73	6.31
0	14.13	7.77	6.36	7.75	6.38
0.87	14.22	7.80	6.42	7.78	6.44
1.87	14.31	7.84	6.47	7.81	6.50
2.87	14.41	7.87	6.54	7.84	6.57
3.87	14.49	7.90	6.59	7.87	6.62
4.87	14.57	7.93	6.64	7.89	6.68

Table 2.5: Analytical formula: dependence level, TVaR and allocated capital to each risk $X_i, i = 1, 2$, under the TVaR and the covariance capital allocation principle.

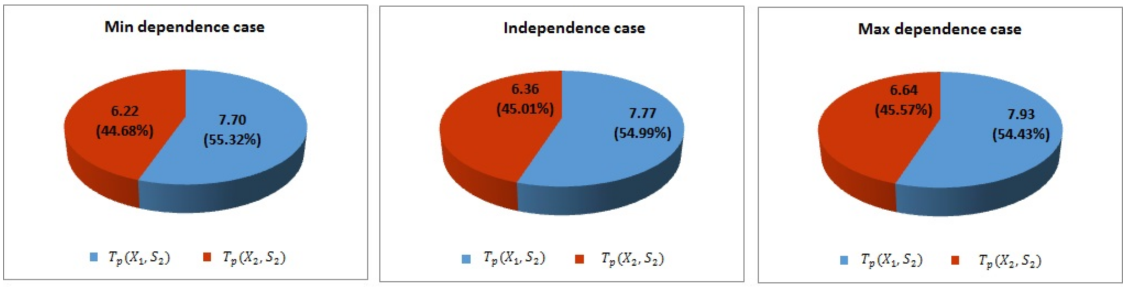


Figure 2.2: Contribution of X_1 and X_2 to the risk capital under the TVaR capital allocation principle with respect to the dependence level.

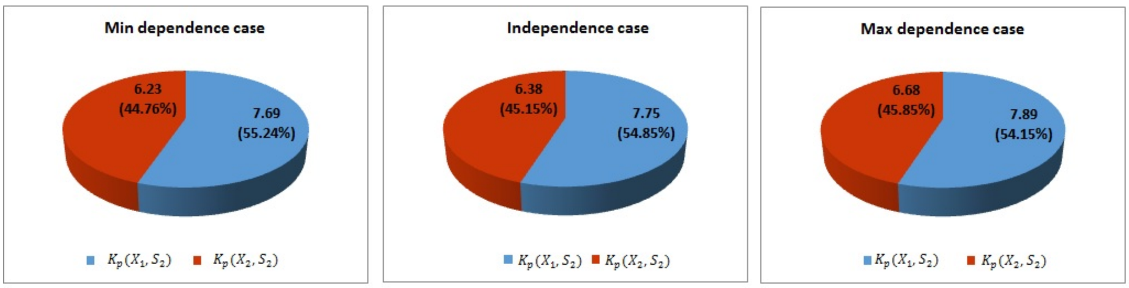


Figure 2.3: Contribution of X_1 and X_2 to the risk capital under the covariance capital allocation principle with respect to the dependence level.

Values from Table 2.5 are depicted in Figure 2.2 and Figure 2.3. The minimum dependence case, the independence case and the maximum dependence represent the allocated capital to each individual risk when the dependence parameter $\alpha_{1,2} = -1.91$, $\alpha_{1,2} = -1.91$, $\alpha_{1,2} = 4.87$, respectively. Under both TVaR and covariance capital allocation principle the relative contribution of X_2 to the risk capital increases with the dependence level between X_1 and X_2. This is due to the fact that the distribution of X_2 is more skewed to the right than the distribution of X_1.

2.5 Auxiliary Results

One of the main features of the mixed Erlang distribution is that its pdf can be used to derive some results in an analytical way. In this respect, this section presents some useful properties of the mixed Erlang distribution.

Lemma 2.5.1. If X is a rv from the mixed Erlang distribution with pdf $g(x, \beta, \underset{\sim}{Q})$, then

$$g^{\theta}(x, \beta+1, \underset{\sim}{\Theta}) = \frac{e^{-x} g(x, \beta, \underset{\sim}{Q})}{\mathcal{L}},$$

with $\mathcal{L} = \mathbb{E}\left\{e^{-X}\right\}$, is again a pdf of the mixed Erlang distribution with mixing probabilities $\underset{\sim}{\Theta} = (\theta_1, \theta_2, \ldots)$ and scale parameter $\beta + 1$ and we have

$$g^{\theta}(x, \beta+1, \underset{\sim}{\Theta}) = \sum_{k=1}^{\infty} \theta_k w_k(x, \beta+1),$$

where $\theta_k = \frac{q_k \overline{\beta}^k}{\sum_{j=1}^{\infty} q_j \overline{\beta}^j}$ with $\overline{\beta} = \frac{\beta}{\beta+1}$.

The results presented in the next two lemmas can be found in Section 2.2 of Willmot and Woo [95], and Section 7.2 of Lee and Lin [63], respectively.

Lemma 2.5.2. If $X \sim ME(\beta_1,\ \underset{\sim}{Q})$, then for any positive constant $\beta_2 \geq \beta_1$ we have

$$X \sim ME(\beta_2,\ \underset{\sim}{\Psi}(\underset{\sim}{Q})),$$

where the mixing probabilities $\underset{\sim}{\Psi}(\underset{\sim}{Q}) = (\psi_1, \psi_2, \ldots)$ and its individual components are given by

$$\psi_k = \sum_{i=1}^{k} q_i \binom{k-1}{i-1} \left(\frac{\beta_1}{\beta_2}\right)^i \left(1 - \frac{\beta_1}{\beta_2}\right)^{k-i}, \quad k \geq 1.$$

Lemma 2.5.3. Let X_1, X_2 be two independent rv. If $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), i = 1, 2$, then $S_2 = X_1 + X_2 \sim ME(\beta, \underset{\sim}{\Pi}\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2\})$, provided that $\beta_1 = \beta_2 = \beta$ with

$$\pi_l\{\underset{\sim}{Q}_1, \underset{\sim}{Q}_2\} = \begin{cases} 0 & \text{for} \quad l = 1, \\ \sum_{j=1}^{l-1} q_{1,j}\ q_{2,l-j} & \text{for} \quad l > 1. \end{cases}$$

Remarks 2.5.4. According to Cossette et al. [19] (Remark 2.1), the results in Lemma 2.5.3 can be extended to $S_n = \sum_{i=1}^{n} X_i$, as long as $X_i, \ldots, X_n$ are independent, $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i)$ and $\beta_i = \beta$ for $i = 1, \ldots, n$. Specifically, $S_n \sim ME(\beta, \underset{\sim}{\Pi}\{\underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n\})$

where the individual mixing probabilities can be evaluated iteratively as follows

$$\pi_l\{\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_{n+1}\} = \begin{cases} 0 & \text{for} \quad l=1,\ldots,n, \\ \sum_{j=n}^{l-1} \pi_j\{\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_n\}\ q_{n+1,l-j} & \text{for} \quad l=n+1,n+2,\ldots. \end{cases}$$

2.6 Multivariate SmE Risks

In this section, we assume that the joint distribution of the random vector $(X_1,\ldots,X_n)$ will be referred to as a multivariate SmE distribution and we shall abbreviate this as $(X_1,\ldots,X_n) \sim SME_n(\boldsymbol{\beta},\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_n)$ where $\boldsymbol{\beta} = (\beta_1,\ldots,\beta_n)$ with $X_i \sim ME(\beta_i,\underset{\sim}{Q}_i), i=1,\ldots,n$. Furthermore, we shall set

$$\widetilde{f}_i(x_i) := e^{-x_i} f_i(x_i).$$

2.6.1 Distribution of S_n

By decomposing the joint pdf of $(X_1,\ldots,X_n)$ in (2.9) and using some rules of integration, we show in the next proposition that the distribution of $S_n = \sum_{i=1}^n X_i$ belongs to the class of Erlang mixtures.

Proposition 2.6.1. If $(X_1,\ldots,X_n) \sim SME_n(\boldsymbol{\beta},\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_n)$ with $\beta_i \leqslant \beta_n$, for $i = 1,\ldots,n-1$, then $S_n \sim ME(\beta_n+1,\underset{\sim}{P})$. The components of $\underset{\sim}{P} = (p_1,p_2,\ldots)$ are given by

$$\begin{aligned} p_k \quad = \quad & \Big(1+\sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ & + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^n\mathcal{L}_i\Big)\overline{\pi}^{(k)} \\ & + \sum_{j_1}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ & + \ldots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1\}}\mathcal{L}_i\Bigg)\overline{\pi}^{(k)}_{j_1} \\ & + \sum_{j_1}\sum_{j_2}\Bigg(\alpha_{j_1,j_2} - \sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\ & + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Bigg)\overline{\pi}^{(k)}_{j_1,j_2} \\ & + \sum_{j_1}\sum_{j_2}\sum_{j_3}\Bigg(\alpha_{j_1,j_2,j_3} - \sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4} + \sum_{j_4}\sum_{j_5}\alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5} \end{aligned}$$

$$+\ldots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{j_1,j_2\}}\mathcal{L}_i\Bigg)\overline{\pi}^{(k)}_{j_1,j_2,j_3}$$
$$+\ldots+\sum_{j_1}\sum_{j_2}\cdots\sum_{j_{n-1}}\left(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n}\right)\overline{\pi}^{(k)}_{j_1,\ldots,j_{n-1}}+\alpha_{1,2,\ldots,n}\overline{\pi}^{(k)}_{1,\ldots,n}, \tag{2.19}$$

where
$\overline{\pi}^{(k)}=\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1}=\mathcal{L}_{j_1}\pi_k\{\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{Q}_{j_2}),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1,j_2}=\mathcal{L}_{j_1}\mathcal{L}_{j_2}\pi_k\{\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2}),\underset{\sim}{\Psi}_{j_3}(\underset{\sim}{Q}_{j_3})\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1,j_2,j_3}=\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3}\pi_k\{\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2}),\underset{\sim}{\Psi}_{j_3}(\underset{\sim}{\Theta}_{j_3})\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\overline{\pi}^{(k)}_{j_1,\ldots,j_{n-1}}=\mathcal{L}_{j_1}\cdots\mathcal{L}_{j_{n-1}}\pi_k\{\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\ldots,\underset{\sim}{\Psi}_{j_{n-1}}(\underset{\sim}{\Theta}_{j_{n-1}}),\underset{\sim}{\Psi}_{j_n}(\underset{\sim}{Q}_{j_n})\}$,
$\overline{\pi}^{(k)}_{1,\ldots,n}=\mathcal{L}_1\cdots\mathcal{L}_n\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{\Theta}_n)\}$,
with $C=\{1,\ldots,n\}$, $j_1\in C, j_2\in C\backslash\{j_1\}, j_3\in C\backslash\{j_1,j_2\},\ldots,j_n\in C\backslash\{j_1,\ldots,j_{n-1}\}$.

Example 2.6.1. Let $(X_1,X_2,X_3)\sim SME_3(\boldsymbol{\beta},\underset{\sim}{Q}_1,\underset{\sim}{Q}_2,\underset{\sim}{Q}_3)$ with $\beta_i\leqslant\beta_3, i=1,2$ then $S_3\sim ME(\beta_3+1,\underset{\sim}{P})$ where the components of $\underset{\sim}{P}=(p_1,p_2,\ldots)$ are given by (with $C=\{1,2,3\}$)

$$\begin{aligned}
p_k &= (1+\sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2}-\alpha_{1,2,3}\prod_{i=1}^{3}\mathcal{L}_i)\overline{\pi}^{(k)}\\
&\quad+\sum_{j_1}\left(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2}+\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3}+\alpha_{1,2,3}\prod_{i\in C\backslash\{j_1\}}\mathcal{L}_i\right)\overline{\pi}^{(k)}_{j_1}\\
&\quad+\sum_{j_1}\sum_{j_2}\left(\alpha_{j_1,j_2}-\alpha_{1,2,3}\mathcal{L}_{j_3}\right)\overline{\pi}^{(k)}_{j_1,j_2}+\alpha_{1,2,3}\overline{\pi}^{(k)}_{1,2,3}\\
&= (1+\alpha_{1,2}\mathcal{L}_1\mathcal{L}_2+\alpha_{1,3}\mathcal{L}_1\mathcal{L}_3+\alpha_{2,3}\mathcal{L}_2\mathcal{L}_3-\alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3)\\
&\quad\times\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{Q}_3)\}\\
&\quad+(-\alpha_{1,2}\mathcal{L}_2-\alpha_{1,3}\mathcal{L}_3+\alpha_{1,2,3}\mathcal{L}_2\mathcal{L}_3)\mathcal{L}_1\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{Q}_3)\}\\
&\quad+(-\alpha_{1,2}\mathcal{L}_1-\alpha_{2,3}\mathcal{L}_3+\alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_3)\mathcal{L}_2\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{\Theta}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{Q}_3)\}\\
&\quad+(-\alpha_{1,3}\mathcal{L}_1-\alpha_{2,3}\mathcal{L}_2+\alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2)\mathcal{L}_3\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{\Theta}_3)\}\\
&\quad+(\alpha_{1,3}-\alpha_{1,2,3}\mathcal{L}_2)\mathcal{L}_1\mathcal{L}_3\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{Q}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{\Theta}_3)\}\\
&\quad+(\alpha_{2,3}-\alpha_{1,2,3}\mathcal{L}_1)\mathcal{L}_2\mathcal{L}_3\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\underset{\sim}{\Psi}_3(\underset{\sim}{\Theta}_3)\}\\
&\quad+(\alpha_{1,2}-\alpha_{1,2,3}\mathcal{L}_3)\mathcal{L}_1\mathcal{L}_2\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{\Theta}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{Q}_3)\}\\
&\quad+\alpha_{1,2,3}\mathcal{L}_1\mathcal{L}_2\mathcal{L}_3\pi_k\{\underset{\sim}{\Psi}_1(\underset{\sim}{\Theta}_1),\underset{\sim}{\Psi}_2(\underset{\sim}{\Theta}_2),\underset{\sim}{\Psi}_3(\underset{\sim}{\Theta}_3)\}.
\end{aligned}$$

2.6.2 Capital Allocation

The following propositions provide analytical formulas for the allocated capital to each individual risk X_m, $m = 1, \ldots, n$, under the TVaR and the covariance rules.

Proposition 2.6.2. Let $(X_1, \ldots, X_n) \sim SME_n(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_n)$ with $\beta_m \leqslant \beta_n$, for $m = 1, \ldots, n-1$. Provided that both $\mu_m = \frac{1}{\beta_m}\sum_{k=1}^{\infty} k q_{m,k}$ and $\widetilde{\mu}_m = \frac{1}{\beta_m+1}\sum_{k=1}^{\infty} k\theta_{mk}$, $m = 1, \ldots, n$ are finite , then for $m = 1, \ldots, n$ and $p \in (0,1)$ the amount of capital allocated to each risk X_m under the TVaR principle is given by (set $C := \{1, \ldots, n\}$)

$$T_p(X_m, S_n) = \frac{1}{1-p}\sum_{k=1}^{\infty} z_{m,k}\overline{W}_k(VaR_{S_n}(p), \beta_n + 1),$$

where

$$\begin{aligned}
z_{m,k} \;=\; & \Bigg(1 + \sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\
& + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\mathcal{L}_i\Bigg)\mu_m\widetilde{\pi}^{(k)} \\
& + \sum_{j_1\neq m}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\
& + \ldots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{j_1\}}\mathcal{L}_i\Bigg)\mu_m\widetilde{\pi}^{(k)}_{j_1} \\
& + \Bigg(-\sum_{j_2\neq m}\alpha_{m,j_2}\mathcal{L}_{j_2} + \sum_{j_2\neq m}\sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\
& + \ldots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{m\}}\mathcal{L}_i\Bigg)\widetilde{\mu}_m\widetilde{\pi}^{(k)}_m \\
& + \sum_{j_1\neq m}\sum_{j_2}\Bigg(\alpha_{j_1,j_2} - \sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\
& + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{j_1,j_2\}}\mathcal{L}_i\Bigg)\mu_m\widetilde{\pi}^{(k)}_{j_1,j_2} \\
& + \sum_{j_2\neq m}\Bigg(\alpha_{m,j_2} - \sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3\neq m}\sum_{j_4\neq m}\alpha_{m,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\
& + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\backslash\{m,j_2\}}\mathcal{L}_i\Bigg)\widetilde{\mu}_m\widetilde{\pi}^{(k)}_{m,j_2}
\end{aligned}$$

$$+\ldots+\sum_{j_1\neq m}\sum_{j_2}\cdots\sum_{j_{n-1}}\left(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_m\right)\mu_m\widetilde{\pi}^{(k)}_{j_1,\ldots,j_{n-1}}$$
$$+\sum_{j_2\neq m}\cdots\sum_{j_{n-1}\neq m}\left(\alpha_{m,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n\neq m}\right)\widetilde{\mu}_m\widetilde{\pi}^{(k)}_{m,j_2,\ldots,j_{n-1}}+\alpha_{1,2,\ldots,n}\widetilde{\pi}^{(k)}_{1,\ldots,n},\tag{2.20}$$

where
$\widetilde{\pi}^{(k)}=\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{Q}_m),\underset{\sim}{\Psi}_1(\underset{\sim}{Q}_1),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\widetilde{\pi}^{(k)}_{j_1}=\mathcal{L}_{j_1}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{Q}_m),\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\widetilde{\pi}^{(k)}_{m}=\mathcal{L}_{m}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{\Theta}_m),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\widetilde{\pi}^{(k)}_{j_1,j_2}=\mathcal{L}_{j_1}\mathcal{L}_{j_2}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{Q}_m),\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1}),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2}),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\widetilde{\pi}^{(k)}_{m,j_2}=\mathcal{L}_{m}\mathcal{L}_{j_2}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{\Theta}_m),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2}),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{Q}_n)\}$,
$\widetilde{\pi}^{(k)}_{j_1,\ldots,j_{n-1}}=\mathcal{L}_{j_1}\cdots\mathcal{L}_{j_{n-1}}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{Q}_m),\underset{\sim}{\Psi}_{j_1}(\underset{\sim}{\Theta}_{j_1})\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{\Theta}_{j_{n-1}})\}$,
$\widetilde{\pi}^{(k)}_{m,j_2,\ldots,j_{n-1}}=\mathcal{L}_{m}\mathcal{L}_{j_2}\cdots\mathcal{L}_{j_{n-1}}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{\Theta}_m),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2})\ldots,\underset{\sim}{\Psi}_{j_{n-1}}(\underset{\sim}{\Theta}_{j_{n-1}}),\underset{\sim}{\Psi}_{j_n}(\underset{\sim}{Q}_{j_n})\}$,
$\widetilde{\pi}^{(k)}_{1,\ldots,n}=\mathcal{L}_{m}\mathcal{L}_{1}\cdots\mathcal{L}_{n}\pi_k\{\underset{\sim}{\Psi}_m(\underset{\sim}{G}_m(\underset{\sim}{\Theta}_m),\underset{\sim}{\Psi}_{j_2}(\underset{\sim}{\Theta}_{j_2}),\ldots,\underset{\sim}{\Psi}_n(\underset{\sim}{\Theta}_{j_n}\}$.

Proposition 2.6.3. Let $\beta_m\leqslant\beta_n, m\leq n-1$, and consider

$$(X_1,\ldots,X_n)\sim SME_n(\boldsymbol{\beta},\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_n).$$

If S_n has a finite and positive variance, then for any index $m\leq n$ and $p\in(0,1)$ we have

$$K_p(X_m,S_n)=\sum_{k=1}^{\infty}\frac{L_{m,k}}{\beta_n+1},$$

where K_p is defined in (2.16),

$$L_{m,k}=k\psi_{m,k}+\sum_{j\neq m}\varepsilon_{m,j}\left(\frac{P^*_k((\beta_n+1)VaR_{S_n}(p))^k}{\varphi k!}+(\beta_n+1)VaR_{S_n}(p)-kp_k\right),$$

with for $m\neq j$

$$\begin{aligned}\varepsilon_{m,j} &= \frac{\sum_{s=1}^{\infty}(s^2+s)\psi_{m,s}-(\sum_{s=1}^{\infty}s\psi^2_{m,s})}{\sum_{s=1}^{\infty}(s^2+s)p_s-(\sum_{s=1}^{\infty}sp_s)^2}\\ &+\sum_{j=1}^{n}\left(\frac{\alpha_{mj}(\beta_n+1)^2}{\sum_{s=1}^{\infty}(s^2+s)p_s-(\sum_{s=1}^{\infty}sp_s)^2}\right)\end{aligned}$$

$$\times\left(\frac{1}{\beta_m+1}\sum_{s=1}^{\infty} s q_{m,s}\overline{\beta}_m^s - \sum_{s=1}^{\infty} q_{m,s}\overline{\beta}_m^s \frac{1}{\beta_m}\sum_{s=1}^{\infty} s q_{m,s}\right)$$
$$\times\left(\frac{1}{\beta_j+1}\sum_{s=1}^{\infty} s q_{j,s}\overline{\beta}_j^s - \sum_{s=1}^{\infty} q_{j,s}\overline{\beta}_j^s \frac{1}{\beta_j}\sum_{s=1}^{\infty} s q_{j,s}\right),$$

$$\varphi = \sum_{s=1}^{\infty} \frac{P_s((\beta_n+1)VaR_{S_n}(p))^{s-1}}{(s-1)!}, \quad P_k^* = \sum_{s=k}^{\infty} P_s, \quad P_s = \frac{\sum_{k=s}^{\infty} p_s}{\sum_{s=1}^{\infty} s p_s},$$

and p_s is given in (2.19).

Proof. The proof is similar to the bivariate case and is therefore omitted.

2.6.3 Trivariate SmE Risks: Numerical Illustrations

Let (X_1, X_2, X_3) have a trivariate SmE risk, with $\alpha_{12} = 2.03$, $\alpha_{13} = 3.62$, $\alpha_{23} = -1.54$ and $\alpha_{123} = -1.03$ the dependence parameters. The parameters have been chosen so that the condition in (2.8) is fullfilled. Assume $\boldsymbol{\beta} = (0.75, 0.9, 0.95)$, $\underset{\sim}{Q}_1 = (0.2, 0.6, 0.2)$, $\underset{\sim}{Q}_2 = (0.4, 0.3, 0.1, 0.2)$ and $\underset{\sim}{Q}_3 = (0.6, 0.1, 0.2, 0.1)$. In view of (2.9) the joint pdf of (X_1, X_2, X_3) are given by

$$\begin{aligned} h(\boldsymbol{x}) \;\; = \;\; & \prod_{i=1}^{3} f_i(x_i)\Big(2.03(e^{-x_1}-0.21)(e^{-x_2}-0.28) + 3.62(e^{-x_1}-0.21)(e^{-x_3}-0.34) \\ & -1.54(e^{-x_2}-0.28)(e^{-x_3}-0.34) \\ & -1.03(e^{-x_1}-0.21)(e^{-x_2}-0.28)(e^{-x_3}-0.34)\Big). \end{aligned}$$

In light of Proposition 2.6.1, $S_3 = X_1+X_2+X_3$ follows the mixed Erlang distribution with scale parameter $\beta_{S_3} = 1.95$ and mixing probabilities $\underset{\sim}{P} = (p_1, p_2, \ldots)$, the first 50 values of $\underset{\sim}{P}$ are given in Table 2.6.

k	p_k	k	p_k	k	p_k	k	p_k	k	p_k
1	0.0000	11	0.0670	21	0.0256	31	0.0022	41	8.729E-05
2	0.0000	12	0.0676	22	0.0211	32	0.0017	42	5.751E-05
3	0.0121	13	0.0662	23	0.0172	33	0.0012	43	4.289E-05
4	0.0295	14	0.0631	24	0.0138	34	0.0009	44	2.988E-05
5	0.0366	15	0.0588	25	0.0109	35	0.0006	45	2.019E-05
6	0.0409	16	0.0536	26	0.0086	36	0.0005	46	9.869E-06
7	0.0466	17	0.0478	27	0.0067	37	0.0003	47	9.612E-06
8	0.0533	18	0.0419	28	0.0051	38	0.0002	48	4.635E-06
9	0.0596	19	0.0361	29	0.0039	39	0.0002	49	4.513E-06
10	0.0643	20	0.0307	30	0.0030	40	0.0001	50	3.161E-06

Table 2.6: Mixing probabilities of the distribution of $S_3 = X_1 + X_2 + X_3$, with scale parameter $\beta_{S_3} = 1.95$.

For different tolerance level p, Table 2.7 and Table 2.8 show the TVaR of $S_3 = X_1 + X_2 + X_3$ and the allocated capital to each risk under the TVaR and the covariance capital allocation rules, respectively. Furthermore, the corresponding relative contribution of X_1, X_2 and X_3 are presented in Figure 2.4 and Figure 2.5.

p	$TVaR_{S_3}(p)$	$T_p(X_1, S_3)$	$T_p(X_2, S_3)$	$T_p(X_3, S_3)$
90.0 %	14.16	5.53	4.73	3.90
92.5 %	14.84	5.79	4.96	4.09
95.0 %	15.77	6.13	5.29	4.35
97.5 %	17.29	6.70	5.82	4.77
99.0 %	19.20	7.45	6.47	5.28
99.5 %	20.58	8.01	6.94	5.63

Table 2.7: Exact values: TVaR of $S_3 = X_1 + X_2 + X_3$ and allocated capital to each risk $X_i, i = 1, 2, 3$, under the TVaR capital allocation principle.

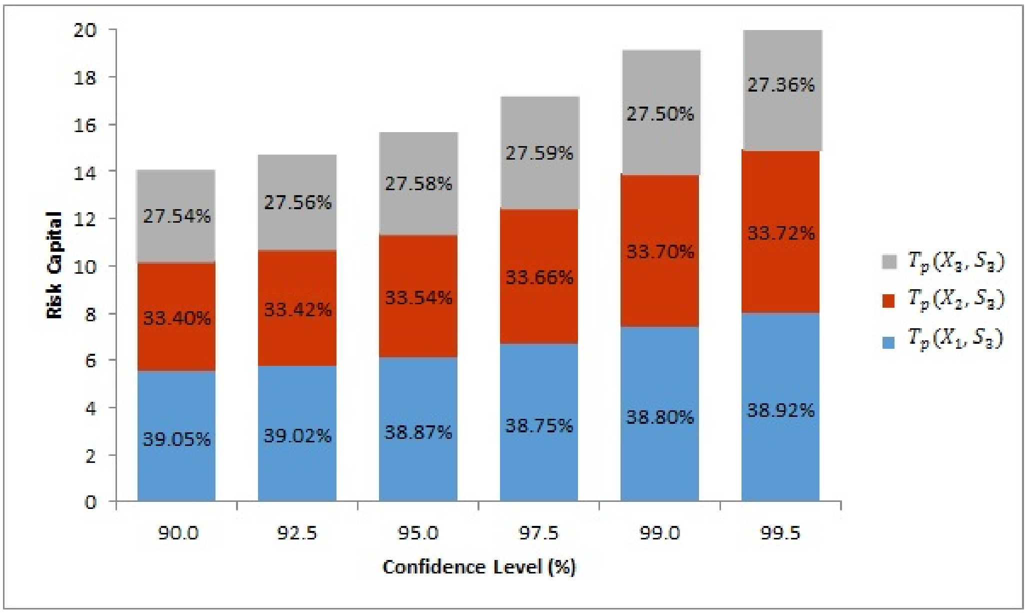


Figure 2.4: Contribution of X_1, X_2 and X_3 to the risk capital under the TVaR capital allocation principle with respect to the confidence level.

p	$TVaR_{S_3}(p)$	$K_p(X_1, S_3)$	$K_p(X_2, S_3)$	$K_p(X_1, S_3)$
90.0 %	14.16	5.56	4.70	3.90
92.5 %	14.84	5.84	4.93	4.07
95.0 %	15.77	6.20	5.23	4.34
97.5 %	17.29	6.82	5.72	4.75
99.0 %	19.20	7.58	6.35	5.27
99.5 %	20.58	8.13	6.80	5.65

Table 2.8: Exact values: TVaR of $S_3 = X_1 + X_2 + X_3$ and allocated capital to each risk $X_i, i = 1, 2, 3$, under the covariance capital allocation principle.

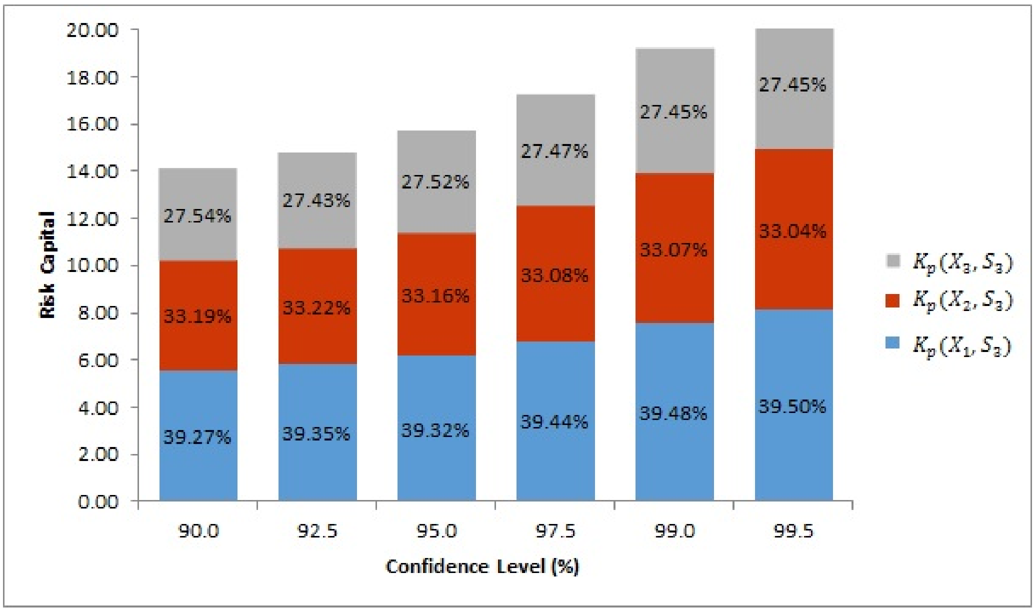


Figure 2.5: Contribution of X_1, X_2 and X_3 to the risk capital under the covariance capital allocation principle with respect to the confidence level.

2.7 Proofs

Proof of Proposition 2.3.1 The pdf f of S_2 is given in terms of the joint pdf of (X_1, X_2) as follows

$$f_{S_2}(s) = \int_0^s h(y, s-y)dy.$$

Taking (2.1) into account the pdf of S_2 becomes

$$\begin{aligned} f_{S_2}(s) &= (1+\alpha_{12}\mathcal{L}_1\mathcal{L}_2)\int_0^s f_1(y)f_2(s-y)dy + \alpha_{12}\int_0^s e^{-y}f_1(y)e^{-(s-y)}f_2(s-y)dy \\ &\quad -\alpha_{12}\mathcal{L}_2\int_0^s e^{-y}f_1(y)f_2(s-y)dy - \alpha_{12}\mathcal{L}_1\int_0^s e^{-(s-y)}f_2(s-y)f_1(y)dy. \end{aligned}$$

Let $A(s), B(s), C(s), D(s)$ be the four terms of the expression of $f_{S_2}(s)$, respectively. According to Lemma 2.5.2,

$$A(s) = (1+\alpha_{12}\mathcal{L}_1\mathcal{L}_2)\int_0^s \; f_1^{\psi}(s, \beta_2+1, \Psi_1(\underset{\sim}{Q_1}))f_2^{\psi}(s-y, \beta_2+1, \Psi_2(\underset{\sim}{Q_2}))dy,$$

and from Lemma 2.5.3, $A(s)$ can be expressed as a pdf of the mixed Erlang distribution as follows

$$A(s) = (1 + \alpha_{12}\mathcal{L}_1\mathcal{L}_2)\sum_{k=1}^{\infty}\pi_k(\utilde{\Psi}_1(\utilde{Q}_1), \utilde{\Psi}_2(\utilde{Q}_2))w_k(s, \beta_2 + 1).$$

In view of Lemma 2.5.1 and Lemma 2.5.2, the expression of $B(s)$ becomes

$$\begin{aligned} B(s) &= \alpha_{12}\int_0^s \mathcal{L}_1 f_1^{\theta}(s, \beta_1 + 1, \utilde{\Theta}_1)\mathcal{L}_2 f_2^{\theta}(s - y, \beta_2 + 1, \utilde{\Theta}_2)dy \\ &= \alpha_{12}\mathcal{L}_1\mathcal{L}_2\int_0^s f_1^{\psi}(s, \beta_2 + 1, \utilde{\Psi}_1(\utilde{\Theta}_1))f_2^{\psi}(s - y, \beta_2 + 1, \utilde{\Psi}_2(\utilde{\Theta}_2))dy. \end{aligned}$$

From Lemma 2.5.3 one can write $B(s)$ as

$$B(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\utilde{\Psi}_1(\utilde{\Theta}_1), \utilde{\Psi}_2(\utilde{\Theta}_2))w_k(s, \beta_2 + 1),$$

which is again a pdf of some mixed Erlang distribution. Similarly to $B(s)$, using Lemma 2.5.1, 2.5.2 and Lemma 2.5.3 one can express $C(s)$ and $D(s)$ as pdfs of mixed Erlang distribution as follows

$$C(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\utilde{\Psi}_1(\utilde{\Theta}_1), \utilde{\Psi}_2(\utilde{Q}_2))w_k(s, \beta_2 + 1),$$

$$D(s) = \alpha_{12}\mathcal{L}_1\mathcal{L}_2\sum_{k=1}^{\infty}\pi_k(\utilde{\Psi}_1(\utilde{Q}_1), \utilde{\Psi}_2(\utilde{\Theta}_2)w_k(s, \beta_2 + 1),$$

hence the claim follows. □

Proof of Proposition 2.4.1 For $j \neq i$, we have

$$\begin{aligned} \mathbb{E}(X_i\mathbb{1}_{\{S_2=s\}}) &= \int_0^s yh(y, s - y)dy \\ &= (1 + \alpha_{12}\mathcal{L}_i\mathcal{L}_j)\int_0^s yf_i(y)f_j(s - y)dy \\ &\quad +\alpha_{12}\int_0^s ye^{-y}f_i(y)e^{-(s-y)}f_j(s - y)dy \\ &\quad -\alpha_{12}\mathcal{L}_j\int_0^s ye^{-y}f_i(y)f_j(s - y)dy \\ &\quad -\alpha_{12}\mathcal{L}_i\int_0^s yf_i(y)e^{-(s-y)}f_j(s - y)dy. \end{aligned}$$

Let $A(s), B(s), C(s), D(s)$ be the four terms of the expression of $\mathbb{E}(X_i\mathbb{1}_{\{S_2=s\}})$, respectively. In light of Cossette et al. [20] Lemma 2.5, if $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i)$ then $\frac{x_i f_i(x_i, \beta_i, \underset{\sim}{Q}_i)}{\mathbb{E}(X_i)}$ can be expressed as a pdf of mixed Erlang distribution with mixing probabilities $\underset{\sim}{G}_i(\underset{\sim}{Q}_i) = (g_1, g_2, \ldots)$ where the k-th individual mixing probability is given by

$$g_k = \begin{cases} 0 & \text{for} \quad k = 1, \\ \frac{(k-1)q_{i,k-1}}{\sum_{j=1}^{k-1} j q_{i,j}} & \text{for} \quad k > 1. \end{cases} \tag{2.21}$$

If we set $\mu_i := \mathbb{E}(X_i) = \frac{1}{\beta_i} \sum_{k=1}^{\infty} k q_{ik}$, $\gamma := \alpha_{12}\mathcal{L}_1\mathcal{L}_2$, then using (2.21), Lemma 2.5.1, 2.5.2 and 2.5.3, one can write $A(s)$ as

$$A(s) = (1+\gamma)\mu_i \sum_{k=1}^{\infty} \pi_k \{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{Q}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{Q}_j)\} w_k(s, \beta_2 + 1).$$

Setting $\widetilde{\mu}_i := \frac{1}{\beta_i + 1} \sum_{k=1}^{\infty} k\theta_{ik}$, in light of (2.21), Lemma 2.5.1, 2.5.2 and 2.5.3, similarly to $A(s)$, we get the expression of the last three terms of $\mathbb{E}(X_i \mathbb{1}_{\{S_2 = s\}})$ as follows

$$\begin{aligned} B(s) &= \gamma\widetilde{\mu}_i \textstyle\sum_{k=1}^{\infty} \pi_k \{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{\Theta}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{\Theta}_j)\} w_k(s, \beta_2 + 1), \\ C(s) &= -\gamma\widetilde{\mu}_i \textstyle\sum_{k=1}^{\infty} \pi_k \{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{\Theta}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{Q}_j)\} w_k(s, \beta_2 + 1), \\ D(s) &= -\gamma\mu_i \textstyle\sum_{k=1}^{\infty} \pi_k \{\underset{\sim}{\Psi}_i(\underset{\sim}{G}_i(\underset{\sim}{Q}_i)), \underset{\sim}{\Psi}_j(\underset{\sim}{\Theta}_j)\} w_k(s, \beta_2 + 1). \end{aligned}$$

Hence, in view of (2.15)

$$T_p(X_i, S_2) \;=\; \frac{1}{1-p} \sum_{k=1}^{\infty} z_{i,k} \overline{W}_k(VaR_{S_2}(p), \beta_2 + 1),$$

where z_{ik} is given in (2.17). Next, by Lemma 2.5.2, since $\beta_1 \leqslant \beta_2$ we obtain

$$\mathbb{E}(X_i) = \frac{1}{\beta_2 + 1} \sum_{k=1}^{\infty} k\psi_{i,k},$$

$$Var(X_i) = \frac{1}{(\beta_2+1)^2} \left(\sum_{m=1}^{\infty} (m^2 + m)\psi_{i,m} - \left(\sum_{m=1}^{\infty} m\psi_{i,m} \right)^2 \right).$$

In light of (2.26), we know that for $i \neq j$

$$Cov(X_i, X_j) \;=\; \alpha_{12} \left(\frac{1}{\beta_i + 1} \sum_{m=1}^{\infty} m q_{i,m} \overline{\beta_i}^m - \sum_{m=1}^{\infty} q_{i,m} \overline{\beta_i}^m \frac{1}{\beta_i} \sum_{m=1}^{\infty} m q_{i,m} \right)$$

$$\times\left(\frac{1}{\beta_j+1}\sum_{s=1}^{\infty} sq_{j,s}\overline{\beta_j}^s - \sum_{s=1}^{\infty} q_{j,s}\overline{\beta_j}^s\frac{1}{\beta_j}\sum_{s=1}^{\infty} sq_{j,s}\right),$$

Furthermore, Proposition 2.3.1 and (2.13) yield

$$\mathbb{E}(S_2) = \frac{1}{\beta_2+1}\sum_{k=1}^{\infty} kp_k,$$

$$Var(S_2) = \frac{1}{(\beta_2+1)^2}\left(\sum_{m=1}^{\infty}(m^2+m)p_m - \Big(\sum_{m=1}^{\infty} mp_m\Big)^2\right),$$

$$TVaR_{S_2}(p) = \frac{1}{(\beta_2+1)\varphi}\sum_{k=0}^{\infty}\frac{P_k^*((\beta_2+1)VaR_{S_2}(p))^k}{k!} + VaR_{S_2}(p).$$

where

$$\varphi = \sum_{j=1}^{\infty}\frac{P_j((\beta_2+1)VaR_{S_2}(l))^{j-1}}{(j-1)!}, \quad P_k^* = \sum_{j=k}^{\infty} P_j, \quad P_j = \frac{\sum_{k=j}^{\infty} p_k}{\sum_{k=1}^{\infty} kp_k},$$

and p_k is given in (2.14). Setting

$$L_{i,k} := k\psi_{i,k} + \varepsilon_{i,j}\left(\frac{P_k^*((\beta_2+1)VaR_{S_2}(p))^k}{\varphi k!} + (\beta_2+1)VaR_{S_2}(p) - kp_k\right),$$

and plugging the value of $\mathbb{E}(X_i)$, $Var(X_i)$, $Cov(X_i,X_j)$, $Var(S_2)$, $TVaR_{S_2}(p)$ and $\mathbb{E}(S_2)$ in (2.16), we obtain the desired result for $K_p(X_i,S_2)$ where $\varepsilon_{i,j}$ is given in (2.18). $\square$

Proof of Lemma 2.5.1 We have

$$\begin{aligned} g^{\theta}(x,\beta+1,\underset{\sim}{\Theta}) &= \frac{e^{-x}g(x,\beta,\underset{\sim}{Q})}{\mathcal{L}} \\ &= \sum_{k=1}^{\infty} q_k\frac{\beta^k x^{k-1}e^{-\beta x}}{(k-1)!}\frac{e^{-x}}{\mathcal{L}} \\ &= \sum_{k=1}^{\infty}\frac{q_k\left(\frac{\beta}{\beta+1}\right)^k}{\sum_{j=1}^{\infty} q_j\left(\frac{\beta}{\beta+1}\right)^j} w_k(x,\beta+1) \\ &= \sum_{k=1}^{\infty}\theta_k w_k(x,\beta+1). \end{aligned}$$

□

Proof of Proposition 2.6.1 By definition

$$f_{S_n}(s) = \int_0^s \int_0^{s-x_1} \dots \int_0^{s-x_1-\dots-x_{n-2}} h(x_1, x_2, \dots, s - x_1 - \dots - x_{n-1}) \, dx_{n-1} \dots dx_2 dx_1. \tag{2.22}$$

For $C = \{1, \dots, n\}$, if we decompose the pdf h in (2.9), we obtain

$$\begin{aligned}
h(\boldsymbol{x}) = {} & \Bigg(1 + \sum_{j_1}\sum_{j_2} \alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3} \alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\
& + \dots + (-1)^n \alpha_{1,2,\dots,n} \prod_{i=1}^n \mathcal{L}_i\Bigg) \prod_{i=1}^n f_i(x_i) \\
& + \sum_{j_1}\Bigg(- \sum_{j_2} \alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3} \alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\
& + \dots + (-1)^{n+1} \alpha_{1,2,\dots,n} \prod_{i \in C\backslash\{j_1\}} \mathcal{L}_i\Bigg) \widetilde{f}_{j_1}(x_{j_1}) \prod_{i \in C\backslash\{j_1\}} f_i(x_i) \\
& + \sum_{j_1}\sum_{j_2}\Bigg(\alpha_{j_1,j_2} - \sum_{j_3} \alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4} \alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\
& + \dots + (-1)^n \alpha_{1,2,\dots,n} \prod_{i \in C\backslash\{j_1,j_2\}} \mathcal{L}_i\Bigg) \widetilde{f}_{j_1}(x_{j_1})\widetilde{f}_{j_2}(x_{j_2}) \prod_{i \in C\backslash\{j_1,j_2\}} f_i(x_i) \\
& + \sum_{j_1}\sum_{j_2}\sum_{j_3}\Bigg(\alpha_{j_1,j_2,j_3} - \sum_{j_4} \alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4} + \sum_{j_4}\sum_{j_5} \alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5} \\
& + \dots + (-1)^{n+1} \alpha_{1,2,\dots,n} \prod_{i \in C\backslash\{j_1,j_2\}} \mathcal{L}_i\Bigg) \\
& \times \widetilde{f}_{j_1}(x_{j_1})\widetilde{f}_{j_2}(x_{j_2})\widetilde{f}_{j_3}(x_{j_3}) \prod_{i \in C\backslash\{j_1,j_2,j_3\}} f_i(x_i) \\
& + \dots + \sum_{j_1}\sum_{j_2}\dots\sum_{j_{n-1}}\Bigg(\alpha_{j_1,j_2,\dots,j_{n-1}} - \alpha_{1,2,\dots,n}\mathcal{L}_{j_n}\Bigg) \\
& \times \widetilde{f}_{j_1}(x_{j_1}) \times \dots \times \widetilde{f}_{j_{n-1}}(x_{j_{n-1}}) f_{j_n}(x_{j_n}) + \alpha_{1,2,\dots,n} \prod_{i=1}^n \widetilde{f}_i(x_i),
\end{aligned} \tag{2.23}$$

where $j_1 \in C, j_2 \in C\backslash\{j_1\}, j_3 \in C\backslash\{j_1, j_2\}, \dots, j_n \in C\backslash\{j_1, \dots, j_{n-1}\}$. Hence, using

(2.23), one can express (2.22) as follows

$$
\begin{aligned}
f_{S_n}(s) &= \Bigg(1+\sum_{j_1}\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2}-\sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3}\\
&\quad+\ldots+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i=1}^{n}\mathcal{L}_i\Bigg)\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}\\
&\quad\prod_{i=1}^{n-1}f_i(x_i)f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1\\
&\quad+\sum_{j_1}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2}+\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3}\\
&\quad+\ldots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1\}}\mathcal{L}_i\Bigg)\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}\\
&\quad\widetilde{f}_{j_1}(x_{j_1})\prod_{i\in C\setminus\{j_1\}}f_i(x_i)f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1\\
&\quad+\sum_{j_1}\sum_{j_2}\Bigg(\alpha_{j_1,j_2}-\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3}+\sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4}\\
&\quad+\ldots+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Bigg)\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}\\
&\quad\widetilde{f}_{j_1}(x_{j_1})\widetilde{f}_{j_2}(x_{j_2})\prod_{i\in C\setminus\{j_1,j_2\}}f_i(x_i)f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1\\
&\quad+\sum_{j_1}\sum_{j_2}\sum_{j_3}\Bigg(\alpha_{j_1,j_2,j_3}-\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_4}+\sum_{j_4}\sum_{j_5}\alpha_{j_1,j_2,j_3,j_4,j_5}\mathcal{L}_{j_4}\mathcal{L}_{j_5}\\
&\quad+\ldots+(-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Bigg)\\
&\quad\times\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}\widetilde{f}_{j_1}(x_{j_1})\widetilde{f}_{j_2}(x_{j_2})\widetilde{f}_{j_3}(x_{j_3})\\
&\quad\prod_{i\in C\setminus\{j_1,j_2,j_3\}}f_i(x_i)f_n(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1\\
&\quad+\ldots+\sum_{j_1}\sum_{j_2}\ldots\sum_{j_{n-1}}\Big(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n}\Big)\\
&\quad\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}\\
&\quad\widetilde{f}_1(x_1)\times\ldots\times\widetilde{f}_{j_{n-1}}(x_{j_{n-1}})f_{j_n}(s-x_1-\ldots-x_{n-1})dx_{n-1}\ldots dx_2dx_1\\
&\quad+\alpha_{1,2,\ldots,n}\int_0^s\int_0^{s-x_1}\ldots\int_0^{s-x_1-\ldots-x_{n-2}}
\end{aligned}
$$

$$\prod_{i=1}^{n-1} \widetilde{f}_i(x_i)\widetilde{f}_n(s - x_1 - \ldots - x_{n-1})dx_{n-1}\ldots dx_2 dx_1.$$

It can be seen that the pdf of S_n is a sum of convolutions of mixed Erlang distributions. Thus, as in the case of S_2, S_n follows a mixed Erlang distribution with scale parameter $\beta_n + 1$ and mixing probabilities $\underset{\sim}{P} = (p_1, p_2, \ldots)$, we write $S_n \sim ME(\beta_n + 1, \underset{\sim}{P})$. For $k \in \mathbb{N}^*$, the k-th component p_k of $\underset{\sim}{P}$ is given in (2.19). □

Proof of Proposition 2.6.2 In view of (2.15) we need to evaluate

$$\begin{aligned}\mathbb{E}(X_m \mathbb{1}_{\{S_n=s\}}) &= \int_0^s \int_0^{s-x_1} \ldots \int_0^{s-x_1-\ldots-x_{n-2}} x_m h(x_1, x_2, \ldots, s - x_1 - \ldots - x_{n-1}) \\ &\quad dx_{n-1}\ldots dx_2 dx_1.\end{aligned} \tag{2.24}$$

If we decompose $x_m h(\boldsymbol{x})$, we have

$$\begin{aligned}x_m h(\boldsymbol{x}) &= \Bigg(1 + \sum_{j_1}\sum_{j_2} \alpha_{j_1,j_2}\mathcal{L}_{j_1}\mathcal{L}_{j_2} - \sum_{j_1}\sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_1}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ &\quad + \ldots + (-1)^n \alpha_{1,2,\ldots,n}\prod_{i=1}^n \mathcal{L}_i\Bigg)\Bigg(x_m f_m(x_m)\prod_{i\neq m} f_i(x_i)\Bigg) \\ &\quad + \sum_{j_1\neq m}\Bigg(-\sum_{j_2}\alpha_{j_1,j_2}\mathcal{L}_{j_2} + \sum_{j_2}\sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ &\quad + \ldots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1\}}\mathcal{L}_i\Bigg)\Bigg(x_m f_m(x_m)\widetilde{f}_{j_1}(x_{j_1})\prod_{i\in C\setminus\{m,j_1\}} f_i(x_i)\Bigg) \\ &\quad + \Bigg(-\sum_{j_2\neq m}\alpha_{m,j_2}\mathcal{L}_{j_2} + \sum_{j_2\neq m}\sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_2}\mathcal{L}_{j_3} \\ &\quad + \ldots + (-1)^{n+1}\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{m\}}\mathcal{L}_i\Bigg)\Bigg(x_m\widetilde{f}_m(x_m)\prod_{i\in C\setminus\{m\}} f_i(x_i)\Bigg) \\ &\quad + \sum_{j_1\neq m}\sum_{j_2}\Bigg(\alpha_{j_1,j_2} - \sum_{j_3}\alpha_{j_1,j_2,j_3}\mathcal{L}_{j_3} + \sum_{j_3}\sum_{j_4}\alpha_{j_1,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4} \\ &\quad + \ldots + (-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{j_1,j_2\}}\mathcal{L}_i\Bigg) \\ &\quad \times\Bigg(x_m f_m(x_m)\widetilde{f}_{j_1}(x_{j_1})\widetilde{f}_{j_2}(x_{j_2})\prod_{i\in C\setminus\{j_1,j_2,m\}} f_i(x_i)\Bigg)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j_2\neq m}\left(\alpha_{m,j_2}-\sum_{j_3\neq m}\alpha_{m,j_2,j_3}\mathcal{L}_{j_3}+\sum_{j_3\neq m}\sum_{j_4\neq m}\alpha_{m,j_2,j_3,j_4}\mathcal{L}_{j_3}\mathcal{L}_{j_4}\right.\\
&\left.+\ldots+(-1)^n\alpha_{1,2,\ldots,n}\prod_{i\in C\setminus\{m,j_2\}}\mathcal{L}_i\right)\left(x_m\widetilde{f}_m(x_m)\widetilde{f}_{j_2}(x_{j_2})\prod_{i\in C\setminus\{m,j_2\}}f_i(x_i)\right)\\
&+\ldots+\sum_{j_1\neq m}\sum_{j_2}\cdots\sum_{j_{n-1}}\left(\alpha_{j_1,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_m\right)\\
&\times\left(x_m f_m(x_m)\prod_{k=1,j_k\neq m}^{n}\widetilde{f}_{j_k}(x_{j_k})\right)\\
&+\sum_{j_2\neq m}\cdots\sum_{j_{n-1}\neq m}\left(\alpha_{m,j_2,\ldots,j_{n-1}}-\alpha_{1,2,\ldots,n}\mathcal{L}_{j_n\neq m}\right)\\
&\times\left(x_m\widetilde{f}_m(x_m)f_{j_n}(x_{j_n})\prod_{k=1,j_k\neq m}^{n-1}\widetilde{f}_{j_k}(x_{j_k})\right)\\
&+\alpha_{1,2,\ldots,n}x_m\widetilde{f}_m(x_m)\prod_{i\neq m}\widetilde{f}_i(x_i).
\end{aligned}
\tag{2.25}
$$

Plugging (2.25) in (2.24) and using (2.21), Lemma 2.5.1, 2.5.2, and 2.5.3, similarly to the bivariate case one may express (2.24) as follows

$$
\mathbb{E}(X_m\mathbb{1}_{\{S_n=s\}})=\sum_{k=1}^{\infty}z_{m,k}\overline{W}_k(VaR_{S_n}(p),\beta_n+1),
$$

where $z_{m,k}$ is given in (2.20). Hence, the proof follows easily. □

2.8 Appendix

2.8.1 Dependence Measures

Pearson's correlation coefficient has been widely used as a measure of the dependence between two rv X_1 and X_2. In this respect, the concept of dependence is assumed to be the linear relationship between the two rv. However, in practice the dependence structure is not always linear hence is why the concept of concordance has been introduced, see e.g., Nelsen [73], McNeil et al. [68] and Denuit et al. [25]. By definition, a rv X_1 is concordant with a rv X_2 if they tend to vary together. The two measures of association of X_1 and X_2, namely Spearman's rho and Kendall's tau are based on this concept. Probabilistically speaking, if (Y_1, Y_2) and (Z_1, Z_2) are independent copies of the pair of continuous rv (X_1, X_2), then Kendall's tau is defined as

$$\tau(X_1, X_2) = \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) > 0\} - \mathbb{P}\{(X_1 - Y_1)(X_2 - Y_2) < 0\},$$

and Spearman's rho is defined as

$$\rho_S(X_1, X_2) = 3\{\mathbb{P}((X_1 - Y_1)(X_2 - Z_2) > 0) - \mathbb{P}((X_1 - Y_1)(X_2 - Z_2) < 0)\},$$

where Y_1 and Z_2 are independent. If $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \underset{\sim}{Q}_2)$ and further $X_i, i = 1, 2$ has finite mean, then we have:

1. **Pearson's correlation coefficient**:
   If we set $\eta_{ik} := \frac{1}{\beta_i+1} \sum_{k=1}^{\infty} k q_{i,k} \overline{\beta}_i^k$ and $\Gamma_{ik} := \sum_{k=1}^{\infty} q_{i,k} \overline{\beta}_i^k \mu_i$ for $i = 1, 2$, then by (2.5) Pearson's correlation coefficient of the bivariate SmE risks has an explicit form as

   $$\rho_{12}(X_1, X_2) = \frac{\alpha_{12}(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\sigma_1 \sigma_2},$$

   where μ_i is the expected value of $X_i, i = 1, 2$ and σ_i is its standard deviation.

   Remarks 2.8.1. According to (2.6), the maximal value of Pearson's correlation coefficient of the bivariate SmE risks can be written as follows

   $$\rho_{12}^{max}(X_1, X_2) = \frac{(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\max\{\mathcal{L}_1(1 - \mathcal{L}_2), (1 - \mathcal{L}_1)\mathcal{L}_2\}\sigma_1 \sigma_2},$$

   and its minimal value can be expressed as

   $$\rho_{12}^{min}(X_1, X_2) = \frac{-(\eta_{1,k} - \Gamma_{1,k})(\eta_{2,k} - \Gamma_{2,k})}{\max\{\mathcal{L}_1\mathcal{L}_2, (1 - \mathcal{L}_1)(1 - \mathcal{L}_2)\}\sigma_1 \sigma_2}.$$

   In the following example, we show that the SmE distribution is flexible as a model for dependent risks.

   Example 2.8.1. **Extremal dependence**
   In this example, we analyse the bounds of Pearson's correlation coefficient of a bivariate mixed Erlang distribution with marginals which share the same scale parameter and consist of 9 Erlang components. The mixture parameters are summarized in Table 2.9. Figure 2.6 presents the lower and the upper bound of Pearson's correlation coefficient as a function of the common scale parameter β. We can see that ρ_{12}^{max} and ρ_{12}^{min} tend to reach the extremal dependence case which correspond to values of 1 and -1, respectively. The strongest negative

correlation $\rho_{12}^{min} = -0.87545$ is attained for $\beta = 21.5723$ while the value of $\beta = 153.0315$ yields the maximal positive correlation $\rho_{12}^{max} = 0.96871$. Hence, not only is the range of the dependence flexible but also wide. Moreover, the simulated values of ρ_{12}^{max} and ρ_{12}^{min}, presented in dotted red lines in Figure 2.6, correspond well with the exact values, this demonstrates again the robustness of our results.

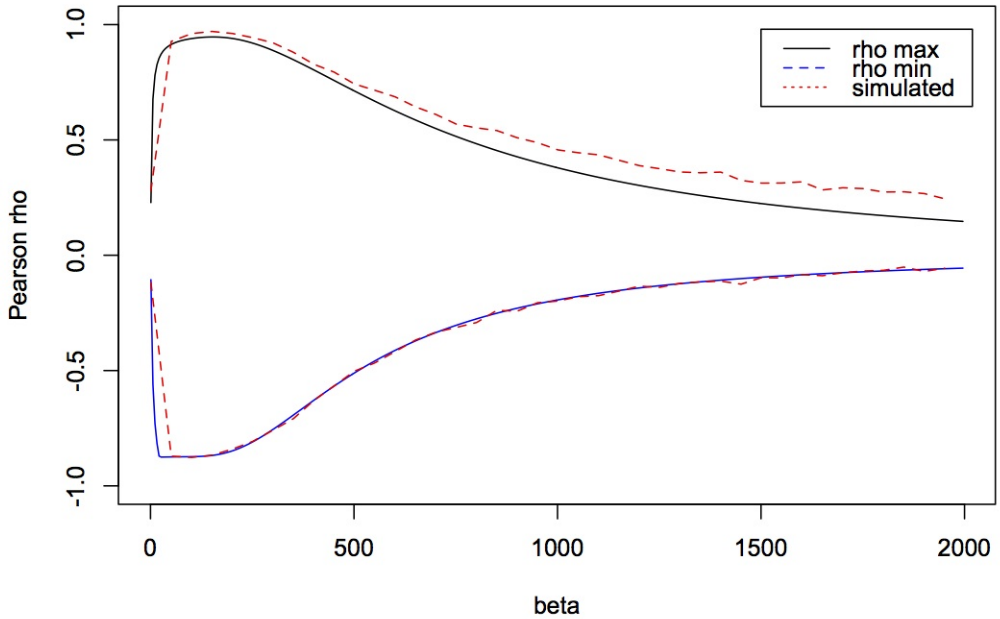


Figure 2.6: ρ_{12}^{max} and ρ_{12}^{min} as a function the common scale parameter β.

X_1		X_2	
k	$q_{1,k}$	k	$q_{2,k}$
1	0.5270	1	0.5050
40	0.0005	8	0.0150
50	0.0020	30	0.0105
75	0.0010	50	0.0020
150	0.0015	70	0.0015
345	0.0005	95	0.0010
902	0.0050	850	0.0055
970	0.4375	995	0.1050
993	0.0250	1000	0.3545

Table 2.9: Mixture parameters of marginals.

2. **Spearman's Rho**: Spearman's rho of the bivariate SmE risks can be expressed explicitly as follows

$$\rho_S(X_1,X_2)=3(1+\gamma)+6\alpha_{12}[2\zeta_1\zeta_2-\mathcal{L}_1\zeta_2-\mathcal{L}_2\zeta_1]-3,$$

where $\zeta_i=\sum_{k=1}^{\infty}q_{i,k}\overline{\beta_i}^k\sum_{m=1}^{\infty}\sum_{j=0}^{k-1}q_{i,m}\begin{pmatrix}j+m-1\\ m-1\end{pmatrix}\frac{\beta_i^m(\beta_i+1)^j}{(2\beta+1)^{m+j}}$, for $i=1,2$.

3. **Kendall's Tau**: Kendall's tau of the bivariate SmE is given by the following closed formula

$$\tau(X_1,X_2)=4\left[(1+\gamma)12(\rho_S(X_1,X_2)+3)+\alpha_{12}\tau_1-\alpha_{12}\mathcal{L}_2\tau_2-\alpha_{12}\mathcal{L}_1\tau_3\right]-1,$$

where $\rho_S(X_1,X_2)$ is Spearman's rho,

$$\begin{aligned}\tau_1&=(1+\gamma)Z_1Z_2+\alpha_{12}T_1T_2-\alpha_{12}\mathcal{L}_1Z_1T_2-\alpha_{12}\mathcal{L}_2Z_2T_1,\\ \tau_2&=\frac{1}{2}(1+\gamma)Z_1+\alpha_{12}T_1\zeta_2-\alpha_{12}\mathcal{L}_1Z_1\zeta_2-\frac{1}{2}\alpha_{12}\mathcal{L}_2T_1,\\ \tau_3&=\frac{1}{2}(1+\gamma)Z_2+\alpha_{12}\zeta_1T_2-\alpha_{12}\mathcal{L}_2\zeta_1Z_2-\frac{1}{2}\alpha_{12}\mathcal{L}_1T_2,\end{aligned}$$

with

$Z_i=\sum_{k=1}^{\infty}q_{i,k}\sum_{m=1}^{\infty}\sum_{j=0}^{k-1}q_{i,m}\begin{pmatrix}j+m-1\\ m-1\end{pmatrix}\left(\frac{\beta_i}{2\beta_i+1}\right)^{m+j}$, for $i=1,2$,

$T_i=\sum_{k=1}^{\infty}q_{i,k}\overline{\beta_i}^k\sum_{m=1}^{\infty}\sum_{j=0}^{k-1}q_{i,m}\begin{pmatrix}j+m-1\\ m-1\end{pmatrix}\frac{\beta_i^m(\beta_i+2)^j}{(2\beta_i+2)^{m+j}}$, for $i=1,2$.

2.8.2 Simulation of SmE Risks

In simulation, in order to remove the dependence between two risks X_1 and X_2, the Rosenblatt transform introduced by Rosenblatt [82] is widely used. In fact, to simulate X_2 this approach consists in using the conditional quantile function of X_2 given the value of X_1. Hence, the conditional df of X_2 is found accordingly. The following lemma yields how this can be done for the case of the bivariate SmE distribution.

Lemma 2.8.2. Let $(X_1,X_2)\sim SME_2(\boldsymbol{\beta},\underset{\sim}{Q}_1,\underset{\sim}{Q}_2)$, for a given value of X_1 the conditional df of X_2 is described as follows

$$F_{2|1}(x_2|x_1)=\lambda F_2(x_2,\beta_2,\underset{\sim}{Q}_2)+\alpha_{12}\Delta_1\sum_{k=1}^{\infty}q_{2,k}\overline{\beta}_2^kW_k(x_2,\beta_2+1),\tag{2.26}$$

where

$$\lambda = 1 + \alpha_{12}\mathcal{L}_2(\mathcal{L}_1 - e^{-x1}), \quad \Delta_1 = (e^{-x1} - \mathcal{L}_1).$$

Proof. For a given value of X_1, one can define the conditional distribution function of X_2 as

$$F_{2|1}(x_2|x_1) = \frac{\int_0^{x_2} h(x_1, s)ds}{f_1(x_1)}.$$

According to (2.1)

$$\begin{aligned} h(x_1, s) &= (1 + \alpha_{12}\mathcal{L}_1\mathcal{L}_2)f_1(x_1)f_2(s) + \alpha_{12}e^{-x_1}f_1(x_1)e^{-s}f_2(s) \\ &\quad -\alpha_{12}\mathcal{L}_2e^{-x_1}f_1(x_1)f_2(s) - \alpha_{12}\mathcal{L}_1e^{-s}f_2(s)f_1(x_1) \\ &= (1 + \alpha_{12}\mathcal{L}_1\mathcal{L}_2 - \alpha_{12}\mathcal{L}_2e^{-x_1})f_1(x_1)f_2(s) \\ &\quad +\alpha_{12}(e^{-x_1} - \mathcal{L}_1)f_1(x_1)e^{-s}f_2(s). \end{aligned}$$

Setting

$$\lambda := 1 + \alpha_{12}\mathcal{L}_2(\mathcal{L}_1 - e^{-x1}) \textit{ and } \Delta_1 := e^{-x1} - \mathcal{L}_1,$$

the expression of $h(x_1, s)$ becomes

$$h(x_1, s) = \lambda f_1(x_1)f_2(s) + \alpha_{12}\Delta_1 f_1(x_1)e^{-s}f_2(s).$$

Hence

$$\begin{aligned} F_{2|1}(x_2|x_1) &= \frac{\int_0^{x_2} \lambda f_1(x_1)f_2(s) + \alpha_{12}\Delta_1 f_1(x_1)e^{-s}f_2(s)ds}{f_1(x_1)} \\ &= \lambda \int_0^{x_2} f_2(s)ds + \alpha_{12}\Delta_1 \int_0^{x_2} e^{-s}f_2(s)ds \\ &= \lambda F_2(x_2, \beta_2, \underset{\sim}{Q}_2) + \alpha_{12}\Delta_1 \int_0^{x_2} e^{-s} \sum_{k=1}^{\infty} q_{2k}\frac{\beta_2^k}{(k-1)!}s^{k-1}e^{-\beta_2 s}ds \\ &= \lambda F_2(x_2, \beta_2, \underset{\sim}{Q}_2) + \alpha_{12}\Delta_1 \sum_{k=1}^{\infty} q_{2k}\left(\frac{\beta_2}{\beta_2 + 1}\right)^k W_k(x_2, \beta_2 + 1). \end{aligned}$$

The inverse of $F_{2|1}$ can be computed numerically and as a result the Rosenblatt transform can be implemented efficiently. The simulation algorithm can be summarised as follows:

- Simulate two independent rv u_1 and u_2 uniformly distributed.

- Simulate X_1 using the inverse transform: $x_1 = F_1^{-1}(u_1)$.
- Simulate X_2 using the Rosenblatt transform: $x_2 = F_{2|1}^{-1}(u_2|x_1)$.
- Simulate the aggregate rv $S_2 = X_1 + X_2$.

Remarks 2.8.3. The result in Lemma 2.8.2 can be generalized for the multivariate case. Specifically, if $(X_1, \ldots, X_n)$ has a multivariate SmE distribution with $X_i \sim ME(\beta_i, \underset{\sim}{Q}_i), i = 1, \ldots, n$, for given values of $X_1, \ldots, X_{n-1}$ one can express the conditional distribution of X_n as follows (set $C := \{1, \ldots, n\}$)

$$F_{n|1,\ldots,n-1}(x_n|x_1, \ldots, x_{n-1}) = \lambda F_n(x_n, \beta_n, \underset{\sim}{Q}_n) + \Delta \sum_{k=1}^{\infty} q_{n,k} \overline{\beta}_n^k W_k(x_n, \beta_n + 1),$$

where

$$\begin{aligned}
\lambda = & \frac{1}{D(x_1, \ldots, x_{n-1})} \Bigg\{ (1+\gamma) + \sum_{j_1 \neq n} \Bigg(- \sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_2} + \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_2} \mathcal{L}_{j_3} \\
& + \ldots + (-1)^{n+1} \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1\}} \mathcal{L}_i \Bigg) e^{-x_{j_1}} \\
& + \sum_{j_1 \neq n} \sum_{j_2 \neq n} \Bigg(\alpha_{j_1,j_2} - \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_3} + \sum_{j_3} \sum_{j_4} \alpha_{j_1,j_2,j_3,j_4} \mathcal{L}_{j_3} \mathcal{L}_{j_4} \\
& + \ldots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1, j_2\}} \mathcal{L}_i \Bigg) e^{-x_{j_1} - x_{j_2}} \\
& + \ldots + (\alpha_{1,2,\ldots,n-1} - \alpha_{1,2,\ldots,n} \mathcal{L}_n) e^{-x_1 - \ldots - x_{n-1}} \Bigg\},
\end{aligned}$$

$$\begin{aligned}
\Delta = & \frac{1}{D(x_1, \ldots, x_{n-1})} \Bigg\{ \Bigg(- \sum_{j_2 \neq n} \alpha_{j_2,n} \mathcal{L}_{j_2} + \sum_{j_2 \neq n} \sum_{j_3 \neq n} \alpha_{j_2,j_3,n} \mathcal{L}_{j_2} \mathcal{L}_{j_3} \\
& + \ldots + (-1)^{n+1} \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{n\}} \mathcal{L}_i \Bigg) \\
& + \sum_{j_2 \neq n} \Bigg(\alpha_{j_2,n} - \sum_{j_3 \neq n} \alpha_{j_2,j_3,n} \mathcal{L}_{j_3} + \sum_{j_3 \neq n} \sum_{j_4 \neq n} \alpha_{j_2,j_3,j_4,n} \mathcal{L}_{j_3} \mathcal{L}_{j_4} \\
& + \ldots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i \in C \setminus \{j_1, n\}} \mathcal{L}_i \Bigg) e^{-x_{j_2}} \\
& + \ldots + \sum_{j_1 \neq n} \sum_{j_2 \neq n} \cdots \sum_{j_{n-1} \neq n} \Bigg(\alpha_{j_1,\ldots,j_{n-1}} - \alpha_{1,\ldots,n} \mathcal{L}_{l, l \in C \setminus \{j_1 \ldots, j_{n-1}\}} \Bigg)
\end{aligned}$$

$$e^{-x_{j_1}-\ldots-x_{j_{n-2}}} + \alpha_{1,2,\ldots,n} e^{-x_1} - \ldots - x_{n-1} \Bigg\},$$

with

$$\begin{aligned} D(x_1, \ldots, x_{n-1}) &= \Bigg(1 + \sum_{j_1 \neq n} \sum_{j_2 \neq n} \alpha_{j_1,j_2} (e^{-x_{j_1}} - \mathcal{L}_{j_1})(e^{-x_{j_2}} - \mathcal{L}_{j_2}) \\ &\quad + \sum_{j_1 \neq n} \sum_{j_2 \neq n} \sum_{j_3 \neq n} \alpha_{j_1,j_2,j_3} (e^{-x_{j_1}} - \mathcal{L}_{j_1})(e^{-x_{j_2}} - \mathcal{L}_{j_2})(e^{-x_{j_3}} - \mathcal{L}_{j_3}) \\ &\quad + \ldots + \alpha_{1,2,\ldots,n-1} \prod_{i=1}^{n-1} (e^{-x_i} - \mathcal{L}_i) \Bigg), \end{aligned}$$

$$\gamma = \sum_{j_1} \sum_{j_2} \alpha_{j_1,j_2} \mathcal{L}_{j_1} \mathcal{L}_{j_2} - \sum_{j_1} \sum_{j_2} \sum_{j_3} \alpha_{j_1,j_2,j_3} \mathcal{L}_{j_1} \mathcal{L}_{j_2} \mathcal{L}_{j_3} + \ldots + (-1)^n \alpha_{1,2,\ldots,n} \prod_{i=1}^{n} \mathcal{L}_i,$$

$j_1 \in C, j_2 \in C \backslash \{j_1\}, j_3 \in C \backslash \{j_1, j_2\}, \ldots, j_n \in C \backslash \{j_1, \ldots, j_{n-1}\}$.
Similarily to the simulation of two dependent SmE risks, one can simulate n dependent SmE risks iteratively.

Chapter 3

Mixed Erlang Reinsurance Risk: Aggregation, Capital Allocation and Default Risk

This chapter is based on G. Ratovomirija: Mixed Erlang reinsurance risk: aggregation, capital allocation and default risk, published in the *European Actuarial Journal*, **6(1)**:149–175, 2016.

3.1 Introduction

Reinsurance companies operate in many regions in the world and insure various insurance business lines. In this respect, it is well recognised that the ceding insurer(s) losses are dependent. This risk dependency can be seen between individual risks within each insurance portfolio and also across business lines. Furthermore, the phenomena of dependence also occurs from global risk factors which generate claims simultaneously to each business line, for instance an hurricane damages buildings or cars which affect property lines, at the same time, causes people injuries which influence accident lines. In the risk management framework, for instance the Swiss Solvency Test (SST), similarly to insurance companies, reinsurance companies are obliged to hold a certain level of risk capital in order to be protected from unexpected large losses. The determination of this capital requires the aggregation of the losses generated from each reinsurance portfolio whose distribution depends on the loss distribution of the ceding insurer(s). Meyers et al. [69] is one of the first contribution which has addressed the aggregation of dependent reinsurance risks to evaluate risk capital. In this regard, in order to derive explicit formula for the

measure of risk capital including VaR, TVaR for the aggregated risk, an important task is the appropriate choice of the marginals and the dependence structure between risks. For our framework, mixed Erlang distribution has been chosen as a claim size model for the individual risk of the ceding insurer(s). One of the reason of the tractability of this distribution is the fact that the convolution of such risks belongs again to the class of Erlang mixtures, see Klugman et al. [58]. Thus stop loss and excess of loss premiums have a closed expression which are very usefull in reinsurance risk modelling, see Lee and Lin [63, 64]. In this contribution, we address the dependence structure between risks by the Sarmanov distribution. The main features of this distribution are its flexibility to model the dependence among risks and its tractable form which allows exact results in calculus.
The aim of this chapter is to analyse the effects of the ceding insurer(s) risk dependencies on the reinsurer risk profile which has only stop loss reinsurance portfolios. Diversification effects from aggregating reinsurance risks are examined by deriving a closed expression for the risk capital needed for the whole portfolio and also the allocated risk capital for each business unit. The effects of the reinsurer default are also analysed. The chapter is organised as follows: in Section 3.2 we describe the background of the Sarmanov distribution as a model for the dependence structure between insurance risks. The risk model of the ceding insurer is explored in Section 3.3, with numerical examples, by deriving the joint tail probability of the aggregated risk of n portfolios. In Section 3.4, the aggregation of stop loss mixed Erlang risks of a reinsurer is addressed by determining a closed form for the df of the aggregated risk. Capital allocation and diversification effects are also presented with numerical studies. All the proofs are relegated to Section 3.5. Some properties of the mixed Erlang distribution and closed expressions for the Pearson's correlation coefficient are presented in the Appendix.

3.2 Preliminaries

3.2.1 Multivariate Sarmanov Distribution

Due to its flexibility to model the dependence structure between rv, the Sarmanov distribution, introduced by Sarmanov [84], have been widely used in many fields. Concerning insurance applications, Abdallah et al. [1] have used the Sarmanov distribution to deal with the calendar, the accident and the development year effects in loss reserving framework . Vernic [91] has derived some formulas for the density of the sum of several rv associated by the Sarmanov distribution with exponential marginals. Based on Vernic [91], Vernic [92] has explored analytical formulas in the context of capital allocation problem. Furthermore, to evaluate the total loss of a motor insurance line, Zuhair et al. [8] have addressed the dependence between the cost of property damage and the cost of medical expense using bivariate Sarmanov distribution with truncated extreme value marginals. In addition, Hashorva and Ratovomirija [49] have addressed risk aggregation and capital allocation with mixed Erlang marginals and the Sarmanov distribution.

For the purpose of presentation, it is common to assume $\alpha_{j_1,\ldots,j_h} = 0$ for all $h \geqslant 3$ in the joint density of $(X_1, \ldots, X_n)$ described in (2.7), see e.g., Mari and Kotz [67]. In that particular case the corresponding joint density can be expressed as follows

$$h(\boldsymbol{x}) = \prod_{i=1}^{n} f_i(x_i)\left(1 + \sum_{j<}^{n-1}\sum_{h}^{n} \alpha_{j,h}\phi_j(x_j)\phi_h(x_h)\right), \tag{3.1}$$

such that

$$\mathbb{E}\left\{\phi_i(X_i)\right\} = 0,$$

$$1 + \sum_{j<}^{n-1}\sum_{h}^{n} \alpha_{j,h}\phi_j(x_j)\phi_h(x_h) \geqslant 0, \quad \forall x_i \in \mathbb{R} \tag{3.2}$$

are fulfilled. Some general methods for finding the kernel function ϕ_i were specified by Lee [62] for different types of marginals. In particular, it is common to choose $\phi_i(x_i) = g_i(x_i) - \mathbb{E}\left\{g_i(X_i)\right\}$ for marginal distributions with support in $\mathbb{R}_+$ (see e.g., Yang and Hashorva [97]). The following three cases are the tractable specifications of $g_i(x_i)$:

(i) $g_i(x_i) = 2\overline{F}_i(x_i)$ which corresponds to the Farlie-Gumbel-Morgenstern (FGM) distribution, where $\overline{F}_i$ is the survival function of X_i,

(ii) $g_i(x_i) = x_i^t$ such that the t-th moment $\mathbb{E}\left\{X_i^t\right\}$ of X_i is finite,

(iii) $g_i(x_i) = e^{-tx_i}$ where $\mathbb{E}\left\{e^{-tX_i}\right\} < \infty$ is the Laplace transform of X_i at t.

Referring to the bivariate risk, provided that $\mathbb{E}\left\{g_i(X_i)\right\}$ is finite it follows from (2.3) that the range of $\alpha_{1,2}$ is (set $\gamma_i := \mathbb{E}\left\{g_i(X_i)\right\}$)

$$\frac{-1}{\max\{\gamma_1\gamma_2, (1-\gamma_1)(1-\gamma_2)\}} \leqslant \alpha_{12} \leqslant \frac{1}{\max\{\gamma_1(1-\gamma_2), (1-\gamma_1)\gamma_2\}}. \tag{3.3}$$

Next we present results for the correlated insurance portfolios where the dependence structure between individual risk is governed by the Sarmanov distribution.

3.3 Ceding Insurance Risk Model

3.3.1 Joint density of n Aggregate Insurance Risks

One of the main features of the Sarmanov distribution is that its pdf can be used to derive some results in analytical way. For instance Vernic [91] have derived general formula for the density of the sum of several rv joined by the Sarmanov distribution. In this section, we consider n insurance portfolios, each of which consists of k risks and we denote $S_{i,k} = \sum_{j=(i-1)k+1}^{ik} X_j, i = 1, \ldots, n$ the aggregated risk of each portfolio where $X_i, i = 1, \ldots, nk$ is a positive continuous rv with finite mean. Below we derive the joint density of $(S_{1,k}, \ldots, S_{n,k})$. Furthermore, we assume that the joint distribution of the multivariate risks $(X_1, \ldots, X_{nk})$ has the Sarmanov distribution where the joint density is defined in (2.7) with any kernel function satisfying (2.8).

Theorem 3.3.1. The joint density of $(S_{1,k}, \ldots, S_{n,k})$ is given by

$$\zeta(u_1, \ldots, u_n) = \prod_{i=1}^{n} f_{S_{i,k}}(u_i) + \sum_{h=2}^{nk} \sum_{1 \leqslant j_1 < j_2 < \ldots < j_h \leqslant nk} \alpha_{j_1,\ldots,j_h} \prod_{i=1}^{n} \widetilde{f}_{S_{i,k},J_h}(u_i),$$

where

$$f_{S_{i,k}}(u_i) = (f_{(i-1)k+1} * \ldots * f_{ik})(u_i),$$
$$\widetilde{f}_{S_{i,k},J_h}(u_i) = (\widetilde{f}_{(i-1)k+1,J_h} * \ldots * \widetilde{f}_{ik,J_h})(u_i),$$

with

$$J_h = \{j_1, j_2, \ldots, j_h\},$$

for $m = 1, \ldots, nk$

$$\widetilde{f}_{m,J_h}(x_m) = \begin{cases} f_m(x_m) & \text{if} \quad m \notin J_h, \\ \phi_m(x_m) f_m(x_m) & \text{if} \quad m \in J_h. \end{cases}$$

3.3.2 Joint density of aggregate mixed Erlang risks

Hereafter, we derive a special case of Theorem 3.3.1 where we assume

$$X_i \sim ME(\beta_i, \underset{\sim}{Q}_i),$$

with $\underset{\sim}{Q}_i = (q_{i,1}, q_{i,2}, \ldots), i = 1, \ldots, nk$. Furthermore, the dependence structure between individual risks within and across the portfolio is assumed to be governed by the Sarmanov distribution where the joint density is specified in (3.1) with kernel function

$$\phi_i(x_i) = g_i(x_i) - \mathbb{E}(g_i(X_i)),$$

which shall be abbreviated as

$$(X_1, \ldots, X_{nk}) \sim SME_{nk}(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_{nk}),$$

where $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_{nk})$. In the rest of the chapter we consider for g_i one of the three cases described in $(i), (ii)$ and (iii).

We define two vectors of mixing weights $\underset{\sim}{\Theta}(\underset{\sim}{Q}_i)$ and $\underset{\sim}{\Psi}(\underset{\sim}{Q}_i)$ where their components depend on the kernel function ϕ_i. In particular, the components of $\underset{\sim}{\Theta}(\underset{\sim}{Q}_i) = (\theta_{i,1}, \theta_{i,2}, \ldots)$ are given by:

- for $g_i(x_i) = 2\overline{F}(x_i)$,

$$\theta_{i,s} = \frac{1}{2^{s-1}} \sum_{j=1}^{s} \binom{s-1}{j-1} q_{i,j} \sum_{l=s-j+1}^{\infty} q_{i,l}, \, s = 1, 2, \ldots, \tag{3.4}$$

- for $g_i(x_i) = e^{-tx_i}$,

$$\theta_{i,s} = \frac{q_{i,s}\overline{\beta}^s}{\sum_{j=1}^{\infty} q_{i,j}\overline{\beta}^j}, \tag{3.5}$$

  with $\overline{\beta} = \frac{\beta}{\beta+t}, s = 1, 2, \ldots,$

- for $g_i(x_i) = x_i^t$,

$$\theta_{i,s} = \begin{cases} 0 & \text{for} \quad s \leqslant t, \\ \frac{q_{i,s-t}\frac{\Gamma(s)}{\Gamma(s-t)}}{\sum_{j=1}^{\infty} q_{i,j}\frac{\Gamma(j+t)}{\Gamma(j)}} & \text{for} \quad s > t, \end{cases} \tag{3.6}$$

  where $\Gamma(.)$ is the Gamma function,

whilst the components of $\underset{\sim}{\Psi}(\underset{\sim}{Q}_i) = (\psi_{i,1}, \psi_{i,2}, \ldots)$ are given by

$$\psi_{i,s} = \sum_{j=1}^{s} q_{i,j} \binom{s-1}{j-1} \left(\frac{\beta_i}{Z(\beta_{nk})}\right)^j \left(1 - \frac{\beta_i}{Z(\beta_{nk})}\right)^{s-j}, \tag{3.7}$$

where

$$Z(\beta_{nk}) = \begin{cases} 2\beta_{nk} & \text{for} \quad g_i(x_i) = 2\overline{F}(x_i), \\ \beta_{nk} & \text{for} \quad g_i(x_i) = x_i^t, \\ \beta_{nk} + t & \text{for} \quad g_i(x_i) = e^{-tx_i}. \end{cases}$$

Moreover, for given mixing weights $\underset{\sim}{V}_i = (v_{i,1}, v_{i,2}, \ldots), i = 1, \ldots, k$ we define the elements of the vector of mixing probabilities $\Pi(\underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_k)$ as follows

$$\pi_l\{\underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_k\} = \begin{cases} 0 & \text{for} \quad l = 1, \ldots, k-1, \\ \sum_{j=k-1}^{l-1} \pi_j\{\underset{\sim}{V}_1, \ldots, \underset{\sim}{V}_{k-1}\} v_{k,l-j} & \text{for} \quad l = k, k+1, \ldots. \end{cases} \tag{3.8}$$

We present next the main result of this section.

Proposition 3.3.2. If $(X_1, \ldots, X_{nk}) \sim SME_{nk}(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_{nk})$ with $\gamma_i < \infty$ and $\beta_{nk} \geq \beta_i, i = 1, \ldots, nk$, then the joint tail probability of $(S_{1,k}, S_{2,k}, \ldots, S_{n,k})$ is given by

$$\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2, \ldots, S_{n,k} > u_n) \quad = \quad \sum_{l=1}^{4} \xi_l \overline{F}_{S_{1,k}^{(l)}}(u_1) \overline{F}_{S_{2,k}^{(l)}}(u_2) \times \ldots \times \overline{F}_{S_{n,k}^{(l)}}(u_n),$$

where

$$\xi_1 = 1 + \sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h}\gamma_j\gamma_h, \quad \xi_2 = \xi_3 = -\xi_4 = -\sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h}\gamma_j\gamma_h,$$

$$\begin{aligned}
S_{1,k}^{(l)} &\sim ME(Z(\beta_{nk}), \Pi\{\underset{\sim}{\Psi}(\underset{\sim}{Q}_{1,j,h}^{(l)}), \ldots, \underset{\sim}{\Psi}(\underset{\sim}{Q}_{k,j,h}^{(l)})\}), \\
S_{2,k}^{(l)} &\sim ME(Z(\beta_{nk}), \Pi\{\underset{\sim}{\Psi}(\underset{\sim}{Q}_{k+1,j,h}^{(l)}), \ldots, \underset{\sim}{\Psi}(\underset{\sim}{Q}_{2k,j,h}^{(l)})\}), \\
S_{n,k}^{(l)} &\sim ME(Z(\beta_{nk}), \Pi\{\underset{\sim}{\Psi}(\underset{\sim}{Q}_{(n-1)k+1,j,h}^{(l)}), \ldots, \underset{\sim}{\Psi}(\underset{\sim}{Q}_{nk,j,h}^{(l)})\}), \quad l = 1, 2, 3, 4,
\end{aligned}$$

and for $i = 1, \ldots, 2k$

$$\begin{aligned}
\underset{\sim}{Q}_{i,j,h}^{(1)} &= \underset{\sim}{Q}_i, \\
\underset{\sim}{Q}_{i,j,h}^{(2)} &= \begin{cases} \underset{\sim}{Q}_i & \text{if} \quad i \neq j, \\ \Theta(\underset{\sim}{Q}_i) & \text{if} \quad i = j, \end{cases} \\
\underset{\sim}{Q}_{i,j,h}^{(3)} &= \begin{cases} \underset{\sim}{Q}_i & \text{if} \quad i \neq h, \\ \Theta(\underset{\sim}{Q}_i) & \text{if} \quad i = h, \end{cases} \\
\underset{\sim}{Q}_{i,j,h}^{(4)} &= \begin{cases} \underset{\sim}{Q}_i & \text{if} \quad i \notin \{j, h\}, \\ \Theta(\underset{\sim}{Q}_i) & \text{if} \quad i \in \{j, h\}, \end{cases}
\end{aligned}$$

where the components of $\underset{\sim}{\Theta}()$ are defined in (3.4), (3.5) and (3.6), respectively for $g_i(x_i) = 2\overline{F}(x_i)$, $g_i(x_i) = e^{-tx_i}$ and $g_i(x_i) = x_i^t$ and the elements of $\underset{\sim}{\Psi}()$ and $\underset{\sim}{\Pi}()$ are respectively described in (3.7) and (3.8).

Remarks 3.3.3. A special case of the aggregate mixed Erlang risk $S_{i,k}^{(l)}, i = 1, \ldots, n, l =$

$1, 2, 3, 4$ can be found in Cossette et al. [20] and Hashorva and Ratovomirija [49].

Example 3.3.4. **Joint survival probability for** $n = k = 2$
To illustrate the result presented in Proposition 3.3.2, we assume that the ceding insurer has two portfolios ($n = 2$) each of which is made up of two risks ($k = 2$). Provided that $\beta_4 > \beta_i, i = 1, 2, 3$, the probability that $S_{1,2}$ and $S_{2,2}$ exceed simultaneously some threshold u_1 and u_2 is given by (set $p(u_1, u_2) := \mathbb{P}(S_{1,2} > u_1, S_{2,2} > u_2)$)

$$p(u_1, u_2) = \sum_{l=1}^{4} \xi_l \overline{F}_{S_{1,2}^{(l)}}(u_1) \overline{F}_{S_{2,2}^{(l)}}(u_2),$$

where

$$\xi_1 = 1 + \sum_{j<}^{3} \sum_{h}^{4} \alpha_{j,h} \gamma_j \gamma_h, \quad \xi_2 = \xi_3 = -\xi_4 = -\sum_{j<}^{3} \sum_{h}^{4} \alpha_{j,h} \gamma_j \gamma_h,$$

$$S_{1,2}^{(l)} \sim ME(Z(\beta_4), \Pi\{\underset{\sim}{\Psi}(\underset{\sim}{Q}_{1,j,h}^{(l)}), \underset{\sim}{\Psi}(\underset{\sim}{Q}_{2,j,h}^{(l)})\}),$$
$$S_{2,2}^{(l)} \sim ME(Z(\beta_4), \Pi\{\underset{\sim}{\Psi}(\underset{\sim}{Q}_{3,j,h}^{(l)}), \underset{\sim}{\Psi}(\underset{\sim}{Q}_{4,j,h}^{(l)})\}), \quad l = 1, 2, 3, 4.$$

Example 3.3.5. **Numerical illustrations**
Assume that the ceding insurer has two portfolios say Portfolio A and Portfolio B. Concerning the dependence structure between risks, three cases of kernel function are considered, namely:

- $g_i(x_i) = 2\overline{F}_i(x_i)$ which defines the FGM distribution as explored in Cossette et al. [20],
- $g_i(x_i) = e^{-x_i}$ introduced by Hashorva and Ratovomirija [49] for mixed Erlang marginals; in the rest of the chapter we refer to the latter as the Laplace case,
- $g_i(x_i) = x_i$ explored in Lee [62] and Vernic [91], hereafter this will be referred to as the Moment case.

Table 2.1 presents the parameters of each individual risk $X_i, i = 1, \ldots, 4$ and their central moments.

	X_i	β_i	$\underset{\sim}{Q}_i$	Mean	Variance	Skewness	Kurtosis
Portfolio A	X_1	0.12	(0.4,0.6)	13.33	127.78	1.55	6.50
	X_2	0.14	(0.3,0.7)	12.14	97.45	1.49	4.33
Portfolio B	X_3	0.15	(0.5,0.5)	10.00	77.78	1.62	6.80
	X_4	0.16	(0.8,0.2)	7.50	53.13	1.88	8.16

Table 3.1: Parameters and central moments of $X_i, i = 1, 2, 3, 4$.

Referring to (2.11), one can express the pdf of X_1, X_2, X_3 and X_4 respectively as follows

$$\begin{aligned} f_1(x_1) &= 0.4w_1(x_1, 0.12) + 0.6w_2(x_1, 0.12), \\ f_2(x_2) &= 0.3w_1(x_2, 0.14) + 0.7w_2(x_2, 0.14), \\ f_3(x_3) &= 0.5w_1(x_3, 0.15) + 0.5w_2(x_3, 0.15), \\ f_4(x_4) &= 0.8w_1(x_4, 0.16) + 0.2w_2(x_4, 0.16). \end{aligned}$$

As demonstrated in Appendix 3.6.2 and the numerical illustrations in Table 3.2, not only does the dependence level between two risks X_i and $X_j, i \neq j$ depends on the dependence parameter $\alpha_{i,j}$ but also on the marginals.

	$\alpha_{i,j}$	$\rho_{1,2}(X_1, X_2)$	$\rho_{3,4}(X_3, X_4)$
Moment	9E-5	0.0100	0.0057
	-5E-5	-0.0056	-0.0032
Laplace	14	0.0403	0.1094
	-3	-0.0086	-0.0234
FGM	0.6	0.1653	0.1315
	-0.2	-0.0551	-0.0526

Table 3.2: Pearson's correlation with different $\alpha_{i,j}, i \neq j$ and different marginals.

Hereafter, we consider the dependence parameters between X_1, X_2, X_3 and X_4 displayed in Table 3.3. We note that these dependence parameters have been chosen so that (3.2) holds. Note in passing that since the joint density in (3.1) has a tractable form, the dependence parameters $\alpha_{i,j}, i \neq j$ can be estimated easily with the maximum likelihood approach for a given dataset, see e.g., Abdallah et al. [1].

	$\alpha_{1,2}$	$\alpha_{1,3}$	$\alpha_{1,4}$	$\alpha_{2,3}$	$\alpha_{2,4}$	$\alpha_{3,4}$
FGM	0.6	0.1	0.1	0.1	0.04	0.5
Laplace	16	5	3	5	3	8
Moment	9E-5	9E-5	3E-5	3E-5	1E-3	1.7E-3

Table 3.3: Dependence parameters of (X_1, X_2, X_3, X_4).

Considering the dependence parameters presented in Table 3.3, the Pearson's correlation between each individual risk of the ceding insurer are gathered in Table 3.4. In this respect, all the individual risks are positively correlated, this means that the aggregate portfolio is riskier when taking the dependency into account than with the independence assumption.

	Moment	Laplace	FGM
$\rho_{1,2}(X_1, X_2)$	0.0100	0.0461	0.1653
$\rho_{1,3}(X_1, X_3)$	0.0089	0.0205	0.0272
$\rho_{1,4}(X_1, X_4)$	0.0024	0.0168	0.0264
$\rho_{2,3}(X_2, X_3)$	0.0026	0.0200	0.0274
$\rho_{2,4}(X_2, X_4)$	0.0071	0.0165	0.0107
$\rho_{3,4}(X_3, X_4)$	0.0109	0.0625	0.1315

Table 3.4: Pearson's correlation of X_i and $X_j, i \neq j$.

According to (3.1), considering the FGM case the joint density of (X_1, X_2, X_3, X_4) is given by

$$\begin{aligned} h(x_1, x_2, x_3, x_4) &= \prod_{i=1}^{4} f_i(x_i)(1 + 0.6\delta(x_1, x_2) + 0.1\delta(x_1, x_3) + 0.1\delta(x_1, x_4) \\ &\quad +0.1\delta(x_2, x_3) + 0.04\delta(x_2, x_4) + 0.5\delta(x_3, x_4)), \end{aligned}$$

where $\delta(x_j, x_h) = (2\overline{F}_j(x_j) - 1)(2\overline{F}_h(x_h) - 1), j \neq h$. Similarly one can express the joint density of (X_1, X_2, X_3, X_4) for the Laplace case and for the Moment case. It can be seen from Table 3.5 that the interdependence between the two insurance portfolios yields high probability for the aggregated risk of each portfolio to exceed simultaneously some threshold u_1 and u_2.

Thresholds	Independence	Moment	Laplace	FGM
(u_1, u_2)	$p(u_1, u_2)$	$p(u_1, u_2)$	$p(u_1, u_2)$	$p(u_1, u_2)$
(20,15)	0.1494	0.1495	0.1601	0.1573
(25,20)	0.0697	0.0698	0.0759	0.0806
(30,25)	0.0304	0.0305	0.0335	0.0386
(35,30)	0.0125	0.0126	0.0140	0.0173

Table 3.5: Joint tail probability of $S_{1,2} = X_1 + X_2$ and $S_{2,2} = X_3 + X_4$.

3.4 Reinsurance Risk Model

In this section, we denote $R_{n,k} = \sum_{i=1}^{n} T_{i,k}$ the aggregate reinsurance stop loss risk, where for $i = 1, \ldots, n$, $T_{i,k} = (S_{i,k} - d_i)_+$ represent the stop loss reinsurance portfolios of the reinsurer with $S_{i,k} = \sum_{j=(i-1)k+1}^{ik} X_i$ the ceding insurer aggregated risk and d_i some positive deductible.

In the enterprise risk management framework, reinsurers are obliged to hold a certain amount of capital $K > 0$, known as the risk capital, in order to be covered from unexpected large losses. The risk capital is determined so that the reinsurer will be able to honor its liabilities even in the worst case with high probability. For instance, in the SST, K is quantified as the TVaR at a tolerance level of 99% of the aggregated risk $R_{n,k}$. This means that for 99% probability the reinsurer has enough buffer to pay its obligations. However, in case $R_{n,k} > K$ the reinsurer is in default and thus the ceding insurers are not protected from losses exceeding K i.e. $R_{n,k} - K$. By analogy to the case between the insurer and the policyholders, see Myers and Read [72], the quantity $(R_{n,k} - K)_+$ is called the *default option of the reinsurer* or in other words the *ceding insurers deficit* with $U(K) := \mathbb{E}\left\{(R_{n,k} - K)_+\right\}$ the value of the default option.

Without loss of generality, we present next the results for $R_{2,k}$. Additionally, for a given risk $X \sim ME(\beta, \underset{\sim}{V})$ with df F and for a deductible $d > 0$ we denote in the rest of the chapter

$$F_X(d+y) = \sum_{k=0}^{\infty} \Delta_k(d, \beta, \underset{\sim}{V}) W_{k+1}(y, \beta), y > 0,$$

$$\overline{U}_X(y, d, \beta) = \int_y^{\infty} u f_X(d+u) du = \frac{1}{\beta} \sum_{k=0}^{\infty} (k+1) \Delta_k(d, \beta, \underset{\sim}{V}) \overline{W}_{k+2}(y, \beta), y > 0,$$

with

$$\Delta_k(d, \beta, \underset{\sim}{V}) = \frac{1}{\beta} \sum_{j=0}^{\infty} q_{j+k+1} w_{j+1}(d, \beta).$$

Furthermore, for $X_i \sim ME(\beta, \underset{\sim}{Q}_i)$, with $d_i > 0, i = 1, 2$ we define

$$F_{X_1+X_2}(d_1, d_2, y) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \Delta_k(d_1, \beta, \underset{\sim}{Q}_1) \Delta_j(d_2, \beta, \underset{\sim}{Q}_2) W_{k+j+2}(y, \beta),$$

$$\overline{U}_{X_1}(y,d_1,d_2,\beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(k+1)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2)\overline{W}_{k+j+3}(y,\beta),$$

$$\overline{U}_{X_2}(y,d_1,d_2,\beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(j+1)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2)\overline{W}_{k+j+3}(y,\beta),$$

$$\overline{U}_{X_1+X_2}(y,d_1,d_2,\beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(k+j+2)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2)\overline{W}_{k+j+3}(y,\beta).$$

3.4.1 Aggregation of Reinsurance Stop Loss Risks

In the next results we show that the df of the aggregated stop loss risk $R_{2,k}$ has a closed form which allows us to derive analytical formula for its mean excess function and for $U(K)$.

Proposition 3.4.1. If $(X_1,\ldots,X_{2k}) \sim SME_{2k}(\boldsymbol{\beta},\underset{\sim}{Q}_1,\ldots,\underset{\sim}{Q}_{2k})$ with $\gamma_i < \infty$, $\beta_{2k} \geq \beta_i, i=1,\ldots,2k$ and $d_s > 0, s=1,2$, then the df of the aggregated stop loss risk $R_{2,k}$ is given by

$$F_{R_{2,k}}(y) = \begin{cases} F_{S_{1,k},S_{2,k}}(d_1,d_2) & \text{for} \quad y=0, \\ F_{S_{1,k},S_{2,k}}(d_1+y,d_2+y) & \text{for} \quad y>0, \end{cases} \tag{3.9}$$

where

$$\begin{aligned} F_{S_{1,k},S_{2,k}}(d_1,d_2) &= \sum_{l=1}^{4}\xi_l F_{S_{1,k}^{(l)}}(d_1)F_{S_{2,k}^{(l)}}(d_2), \\ F_{S_{1,k},S_{2,k}}(d_1+y,d_2+y) &= \sum_{l=1}^{4}\xi_l\Bigg(F_{S_{1,k}^{(l)}}(d_1)F_{S_{2,k}^{(l)}}(d_2+y)+F_{S_{1,k}^{(l)}}(d_1+y)F_{S_{2,k}^{(l)}}(d_2) \\ &\quad +F_{S_{1,k}^{(l)}+S_{2,k}^{(l)}}(d_1,d_2,y)\Bigg), \end{aligned}$$

with $\xi_l, S_{1,k}^{(l)}, S_{2,k}^{(l)}, l=1,2,3,4$ are defined in Proposition 3.3.2.
It follows that the default probability of the reinsurer has a closed form. Specifically, we have

$$\mathbb{P}(R_{2,k} > K) = 1 - F_{S_{1,k},S_{2,k}}(d_1+K,d_2+K),$$

where $K > 0$ is the risk capital.

Remarks 3.4.2. Given the tractable form of the df in (3.9), many risk related quantities for $R_{2,k}$ have an explicit form, for instance, for $c > 0$ the mean excess function of $R_{2,k}$ is given by (set $e(c) := \mathbb{E}\{R_{2,k} - c | R_{2,k} > c\}$)

$$\begin{aligned} e(c) &= \frac{1}{\overline{F}_{R_{2,k}}(c)} \sum_{l=1}^{4} \xi_l \Bigg(F_{S_{1,k}^{(l)}}(d_1)\overline{U}_{S_{2,k}^{(l)}}(c, d_2, Z(\beta_{2k})) + F_{S_{2,k}^{(l)}}(d_2)\overline{U}_{S_{1,k}^{(l)}}(c, d_1, Z(\beta_{2k})) \\ &\quad + \overline{U}_{S_{1,k}^{(l)}+S_{2,k}^{(l)}}(c, d_1, d_2, Z(\beta_{2k})) \Bigg) - c. \end{aligned} \tag{3.10}$$

In light of (3.10) and Proposition 3.4.1 one can express $U(K)$ analytically as follows

$$\begin{aligned} U(K) &= \mathbb{E}\{(R_{2,k} - K)_+\} = \overline{F}_{R_{2,k}}(K)e(K) \\ &= \sum_{l=1}^{4} \xi_l \Bigg(F_{S_{1,k}^{(l)}}(d_1)\overline{U}_{S_{2,k}^{(l)}}(K, d_2, Z(\beta_{2k})) + F_{S_{2,k}^{(l)}}(d_2)\overline{U}_{S_{1,k}^{(l)}}(K, d_1, Z(\beta_{2k})) \\ &\quad + \overline{U}_{S_{1,k}^{(l)}+S_{2,k}^{(l)}}(K, d_1, d_2, Z(\beta_{2k})) \Bigg) - K\overline{F}_{R_{2,k}}(K), \end{aligned}$$

where $\overline{F}_{R_{2,k}} = 1 - F_{R_{2,k}}$ and $F_{R_{2,k}}$ is defined in (3.9) with $K > 0$.

Example 3.4.3. In this illustration, we consider the same parameters of each individual risk of the ceding insurer portfolios as in Table 3.1. Furthermore, we assume that the ceding insurer re-insures its two portfolios to a reinsurer with stop loss programs where the deductibles are $d_1 = 40$ and $d_2 = 30$ for Portfolio A and for Portfolio B, respectively. In practice, it is recognised that risk measures on the aggregated risk are sensitive to the strength of the dependence between individual risks. Actually, by taking into account the dependence within and across the ceding insurer portfolios which is determined by the parameters in Table 3.3, the aggregated risk $R_{2,2}$ of the reinsurer is riskier than in the independence case. Therefore, based on VaR and TVaR as risk measures, the reinsurer needs much more risk capital in the dependence case. Furthermore, for a different confidence level p, it can be seen that the deviation from the independence assumption is greater for VaR than for TVaR.

	p (%)	$VaR_{R_{2,2}}(p)$	$TVaR_{R_{2,2}}(p)$
Independence	95.00	19.47	30.10
	97.50	26.97	37.40
	99.00	36.64	46.85
	99.90	60.08	69.92
Moment	95.00	19.49	30.15
	97.50	27.01	37.46
	99.00	36.69	46.93
	99.90	60.19	70.06
Laplace	95.00	19.91	30.60
	97.50	27.45	37.93
	99.00	37.17	47.43
	99.90	60.71	70.58
FGM	95.00	22.14	33.25
	97.50	30.03	40.85
	99.00	40.10	50.63
	99.90	64.23	74.27

Table 3.6: Deviation of VaR and TVaR from the independence case.

It is well known that risk diversification across portfolios arises from aggregating their individual risks, see e.g., Tasche [89], Tang and Valdes [87]. In this respect, by considering the TVaR as a measure for the risk capital, the diversification benefits D_p are quantified as the relative reduction of the risk capital required for the whole portfolio of the reinsurer from aggregating the stop loss risk $T_{1,k}$ and $T_{2,k}$ as follows

$$D_p = 1 - \frac{TVaR_{R_{2,k}}(p)}{TVaR_{T_{1,k}}(p) + TVaR_{T_{2,k}}(p)}.$$

As presented in Table 3.7 and Figure 3.1, diversification benefits increase with the confidence level. Conversely, the deviation from the independence case yields a reduction of the diversification benefits which is obvious since the full diversification effects are attained when risks are independent.

	p (%)	$TVaR_{R_{2,2}}(p)$	$TVaR_{T_{1,2}}(p)$	$TVaR_{T_{2,2}}(p)$	D_p (%)
Independence	95.00	30.10	24.87	18.26	30.19
	97.50	37.40	32.34	24.26	33.92
	99.00	46.85	41.89	31.97	36.56
	99.90	69.92	64.84	50.55	39.40
Moment	95.00	30.15	24.89	18.29	30.18
	97.50	37.46	32.36	24.31	33.91
	99.00	46.93	41.93	32.04	36.54
	99.90	70.06	64.90	50.68	39.38
Laplace	95.00	30.60	25.19	18.67	30.13
	97.50	37.93	32.66	24.69	33.86
	99.00	47.43	42.22	32.40	36.43
	99.90	70.58	65.15	50.99	39.22
FGM	95.00	33.25	27.71	18.64	28.27
	97.50	40.85	35.40	24.69	32.01
	99.00	50.63	45.14	32.44	34.73
	99.90	74.27	68.32	51.08	37.80

Table 3.7: Diversification benefits based on TVaR of the aggregate risk $R_{2,2}$ and the individual risk $T_{i,2}, i = 1, 2$.

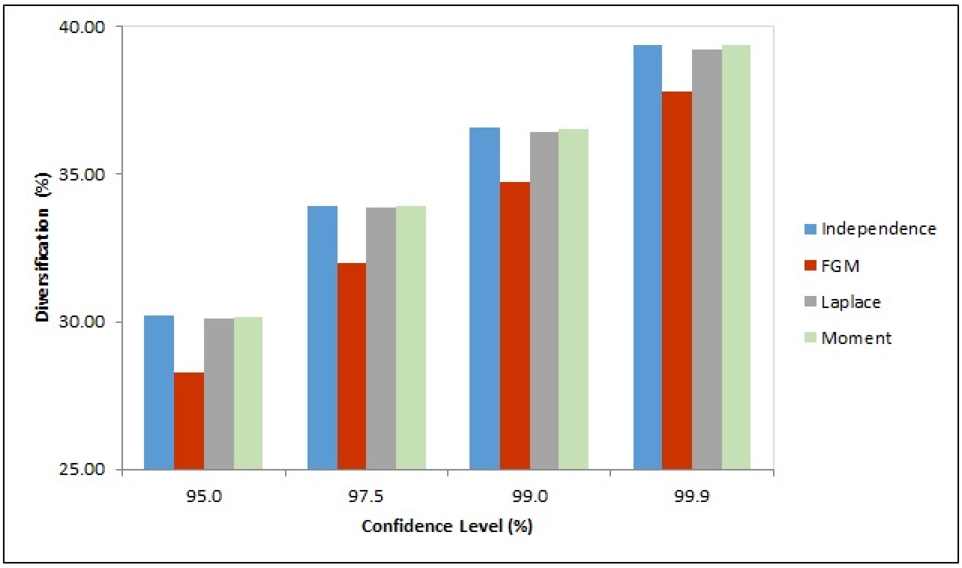


Figure 3.1: Diversification benefits as a function of the confidence level.

3.4.2 TVaR Capital Allocation

In this section, we derive an analytical expression for the amount of capital allocated to each individual risk of the reinsurer under the TVaR principle. In the risk management framework, the so-called capital allocation consists in attributing the risk capital to each individual line. This allows the reinsurance company to identify and to manage conveniently their risks. In practice, it is well known that the TVaR principle takes into account the dependence structure between risks and satisfies the full allocation principle. More precisely, if $R_{n,k} = \sum_{i=1}^{n} T_{i,k}$ is the aggregate risk where $T_{i,k}$ is a rv with finite mean that represents the individual risk of the reinsurer, the amount of capital $T_p(T_{i,k}, R_{n,k})$ required for each risk $T_{i,k}$, for $i = 1, \ldots, n$, is defined as

$$T_p(T_{i,k}, R_{n,k}) = \frac{\mathbb{E}(T_{i,k}\mathbb{1}_{\{R_{n,k} > VaR_{R_{n,k},k}(p)\}})}{1-p}, \tag{3.11}$$

where $p \in (0,1)$ is the tolerance level. The full allocation principle implies

$$TVaR_{R_{n,k}}(p) = \sum_{i=1}^{n} T_p(T_{i,k}, R_{n,k}),$$

which means that, based on TVaR as a risk measure for the risk capital, the capital required for the entire portfolio is equal to the sum of the required capital of each risk within the portfolio. The following proposition develops an explicit form for $T_p(T_{i,k}, R_{2,k}), i = 1, 2$, in the case of stop loss mixed Erlang type risks. In addition, we define below $\xi_l, S_{1,k}^{(l)}, S_{2,k}^{(l)}, l = 1, 2, 3, 4$ as in Proposition 3.3.2 and we denote $x_p := VaR_{R_{2,k}}(p)$.

Proposition 3.4.4. Let $(X_1, \ldots, X_{2k}) \sim SME_{2k}(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_{2k})$ with $\gamma_i < \infty$, $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k$ and $d_i > 0, i = 1, 2$. If further $T_{i,k}, i = 1, 2$ has finite mean then

$$\begin{aligned} T_p(T_{1,k}, R_{2,k}) \quad &= \quad \frac{1}{1-p}\sum_{l=1}^{4} \xi_l \Bigg(F_{S_{2,k}^{(l)}}(d_2)\overline{U}_{S_{1,k}^{(l)}}(x_p, d_1, Z(\beta_{2k})) \\ &\quad + \overline{U}_{S_{1,k}^{(l)}}(x_p, d_1, d_2, Z(\beta_{2k}))\Bigg). \end{aligned}$$

Example 3.4.5. In this example, we consider the same individual risks and dependence parameters as in Example 3.3.5 and the reinsurance programs as in Example 3.4.3. Based on $TVaR$ as a risk measure for quantifying the risk capital required for the whole portfolio, the required capital of each stop loss risk $T_{i,2}, i = 1, 2$ are evaluated for different confidence level p. Since $T_{1,2}$ is riskier than $T_{2,2}$, as shown in

Table 3.8 and the relative contribution of each individual risk in Figure 3.2, more capital is required for $T_{1,2}$ compared to the amount needed for $T_{2,2}$.

	p (%)	$TVaR_{R_{2,2}}(p)$	$T_p(T_{1,2}, R_{2,2})$	$T_p(T_{2,2}, R_{2,2})$
Independence	95.00	30.10	19.69	10.41
	97.50	37.40	25.47	11.93
	99.00	46.85	33.35	13.50
	99.90	69.92	53.59	16.33
Moment	95.00	30.15	19.70	10.45
	97.50	37.46	25.48	11.98
	99.00	46.93	33.36	13.57
	99.90	70.06	53.56	16.51
Laplace	95.00	30.60	19.91	10.69
	97.50	37.93	25.68	12.25
	99.00	47.43	33.54	13.89
	99.90	70.58	53.63	16.95
FGM	95.00	33.25	22.15	11.10
	97.50	40.85	28.25	12.60
	99.00	50.63	36.41	14.22
	99.90	74.27	56.63	17.64

Table 3.8: TVaR and allocated capital to each stop loss risk $T_{i,2}, i = 1, 2$, under the TVaR capital allocation principle.

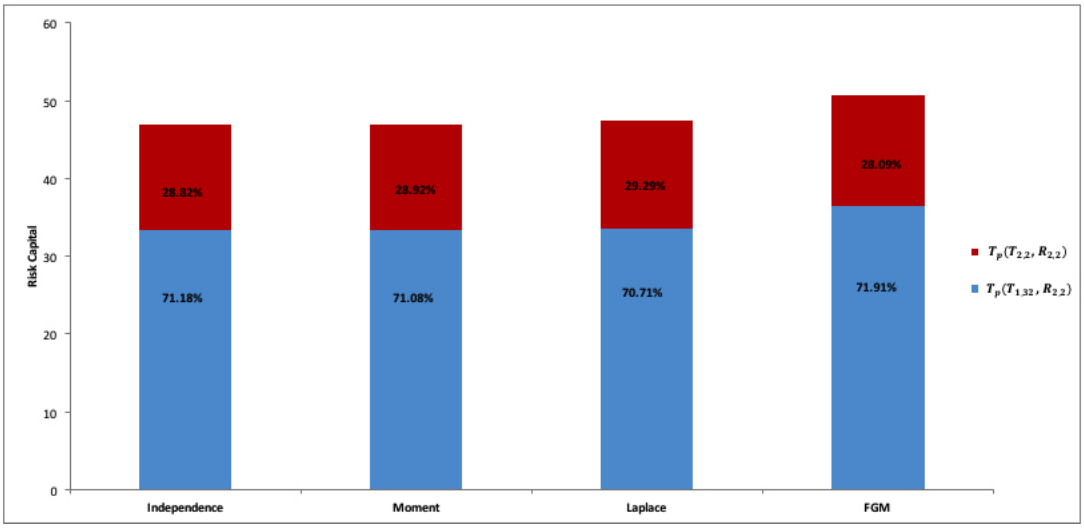


Figure 3.2: Contribution of $T_{i,2}, i = 1, 2$, to the risk capital under the TVaR capital allocation principle at a confidence level of 99%.

In view of the full capital allocation principle, for a given risk capital K required for the entire portfolio of the reinsurer, if $K_i, i = 1, \ldots, n$ is the risk capital needed for each individual risk then $K = \sum_{i=1}^{n} K_i$. Therefore, the value of the default option $U(K)$ is also defined as the sum of the value of the unpaid losses $U(K_i, K) := \mathbb{E}\left\{(T_{i,k} - K_i)\mathbb{1}_{\{R_{n,k}>K\}}\right\}$ of each ceding insurer(s) reinsured lines of business, specifically (see e.g., Dhaene et al. [29])

$$U(K) = \sum_{i=1}^{n} U(K_i, K).$$

Corollary 3.4.1. *Let $K_i, i = 1, 2$ be the capital required for each stop loss reinsurance portfolio of the reinsurer such that $K = K_1 + K_2$. Given that the reinsurer is in default, if $(X_1, \ldots, X_{2k}) \sim SME_{2k}(\boldsymbol{\beta}, \underset{\sim}{Q}_1, \ldots, \underset{\sim}{Q}_{2k})$ with $\gamma_i < \infty$, $\beta_{2k} \geq \beta_i, i = 1, \ldots, 2k$, $d_i > 0$ and $T_{i,k}, i = 1, 2$ has finite mean then*

$$\begin{aligned} U(K_1, K) &= \sum_{l=1}^{4} \xi_l \bigg(F_{S^{(l)}_{2,k}}(d_2)\overline{U}_{S^{(l)}_{1,k}}(K, d_1, Z(\beta_{2k})) + \overline{U}_{S^{(l)}_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \bigg) \\ &\quad - K_1 \overline{F}_{R_{2,k}}(K). \end{aligned}$$

3.5 Proofs

Proof of Theorem 3.3.1 The joint density of $(S_{1,k}, \ldots, S_{n,k})$ is determined in term of the joint density of $(X_1, \ldots, X_{nk})$ as follows

$$\zeta(u_1, \ldots, u_n) = \int \cdots \int_{s_{1,k}=u_1, s_{2,k}=u_2, \ldots, s_{n,k}=u_n} h(\boldsymbol{x}) dx_1 \ldots dx_{nk-1}, \qquad (3.12)$$

with

$$\boldsymbol{x} = (x_1, \ldots, x_{nk}),$$

$$s_{1,k} = x_1 + \ldots + x_k,$$

$$s_{2,k} = x_{k+1} + \ldots + x_{2k},$$

$$s_{n,k} = x_{(n-1)k+1} + \ldots + x_{nk}.$$

Referring to (2.7),

$$h(\boldsymbol{x}) = \prod_{i=1}^{nk} f_i(x_i) \bigg(1 + \sum_{h=2}^{nk} \sum_{1 \leqslant j_1 < j_2 < \ldots < j_h \leqslant nk} \alpha_{j_1, \ldots, j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k}) \bigg)$$

$$
\begin{aligned}
&= \prod_{i=1}^{nk} f_i(x_i) + \sum_{h=2}^{nk} \sum_{1\leqslant j_1<j_2<\ldots<j_h\leqslant nk} \alpha_{j_1,\ldots,j_h} \prod_{k=1}^{h} \phi_{j_k}(x_{j_k}) f_{j_k}(x_{j_k}) \prod_{m\notin J_h} f_m(x_m) \\
&= \prod_{i=1}^{nk} f_m(x_m) + \sum_{h=2}^{nk} \sum_{1\leqslant j_1<j_2<\ldots<j_h\leqslant nk} \alpha_{j_1,\ldots,j_h} \prod_{m=1}^{nk} \widetilde{f}_{m,J_h}(x_m),
\end{aligned}
$$

where

$$
J_h = \{j_1, j_2, \ldots, j_h\},
$$

for $m = 1, \ldots, nk$

$$
\widetilde{f}_{m,,J_h}(x_m) = \begin{cases} f_m(x_m) & \text{if} \quad m \notin J_h, \\ \phi_m(x_m) f_m(x_m) & \text{if} \quad m \in J_h. \end{cases}
$$

Therefore, one can express (3.12) as a sum of convolutions as follows (set $d\boldsymbol{x}_{ik} := dx_{(i-1)k+1} \ldots, dx_{ik-1}, i = 1, \ldots, n$)

$$
\begin{aligned}
\zeta(u_1, \ldots, u_n) &= \prod_{i=1}^{n} \int_{\mathbb{R}^{k-1}} \int \prod_{m=(i-1)k+1}^{ik-1} f_m(x_m) f_{ik}(u_i - \sum_{m=(i-1)k+1}^{ik-1} x_m) d\boldsymbol{x}_{ik} \\
&\quad + \sum_{h=2}^{nk} \sum_{1\leqslant j_1<j_2<\ldots<j_h\leqslant nk} \alpha_{j_1,\ldots,j_h} \\
&\quad \times \prod_{i=1}^{n} \int_{\mathbb{R}^{k-1}} \int \prod_{m=(i-1)k+1}^{ik-1} \widetilde{f}_{m,J_h}(x_m) \widetilde{f}_{ik,J_h}(u_j - \sum_{m=(i-1)k+1}^{ik-1} x_m) d\boldsymbol{x}_{ik} \\
&= \prod_{i=1}^{n} f_{S_i}(u_i) + \sum_{h=2}^{nk} \sum_{1\leqslant j_1<j_2<\ldots<j_h\leqslant nk} \alpha_{j_1,\ldots,j_h} \prod_{i=1}^{n} \widetilde{f}_{S_i,J_h}(u_i),
\end{aligned}
$$

establishing the proof. □

Proof of Proposition 3.3.2 The joint tail probability of $(S_{1,k}, S_{2,k}, \ldots, S_{n,k})$ is determined in terms of the joint density of $(X_1, \ldots, X_{nk})$ as follows

$$
\mathbb{P}(S_{1,k} > u_1, \ldots, S_{n,k} > u_n) = \int \ldots \int_{s_{1,k}>u_1,\ldots,s_{n,k}>u_n} h(\boldsymbol{x}) dx_1 \ldots dx_{nk}. \tag{3.13}
$$

Refering to (3.1), the joint density of $(X_1, \ldots, X_{nk})$ is given by

$$
\begin{aligned}
h(\boldsymbol{x}) &= \prod_{i=1}^{nk} f_i(x_i) \Big(1 + \sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h}(g_j(x_j) - \gamma_j)(g_h(x_h) - \gamma_h)\Big) \\
&= \prod_{i=1}^{nk} f_i(x_i) \Big(1 + \sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h} \gamma_j \gamma_h -
\end{aligned}
$$

$$\begin{aligned}
&\alpha_{j,h}g_j(x_j)\gamma_h - \alpha_{j,h}\gamma_j g_h(x_h) + \alpha_{j,h}g_j(x_j)g_h(x_h)\Big) \\
= \; & (1+\sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h)\prod_{i=1}^{nk}f_i(x_i) - \sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h\prod_{i=1}^{nk}f_{i,j}(x_i) \\
& - \sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h\prod_{i=1}^{nk}f_{i,h}(x_i) + \sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h\prod_{i=1}^{nk}f_{i,j,h}(x_i),
\end{aligned}$$

where for $i = 1, \ldots, nk$ and a=j,h,

$$\begin{aligned}
f_{i,a}(x_i) &= \begin{cases} f_i(x_i) & \text{if} \quad i \neq a, \\ \frac{g(x_i)f_i(x_i)}{\gamma_i} & \text{if} \quad i = a, \end{cases} \\
f_{i,h,j}(x_i) &= \begin{cases} f_i(x_i) & \text{if} \quad i \notin \{j,h\}, \\ \frac{g(x_i)f_i(x_i)}{\gamma_i} & \text{if} \quad i \in \{j,h\}. \end{cases}
\end{aligned}$$

By Lemma 3.6.4, $\frac{g(x_i)f_i(x_i)}{\gamma_i}, i = 1, \ldots, nk$ is a pdf of a mixed Erlang distribution, therefore one can write (3.13) as a sum product of convolutions of mixed Erlang risks as follows

$$\begin{aligned}
&\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2, \ldots, S_{n,k} > u_n) \\
&= (1+\sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h) \\
&\times \int_{u_1}^{\infty}\int_{u_1-x_1}^{\infty}\cdots\int_{u_1-x_1-\ldots-x_{k-2}}^{\infty}\prod_{i=1}^{k-1}f_i(x_i)\overline{F}_k(u_1-x_1-\ldots-x_{k-1})dx_{k-1}\ldots dx_1 \\
&\times \int_{u_2}^{\infty}\int_{u_2-x_{k+1}}^{\infty}\cdots\int_{u_2-x_{k+1}-\ldots-x_{2k-2}}^{\infty}\prod_{i=k+1}^{2k-1}f_i(x_i) \\
&\overline{F}_{2k}(u_2-x_{k+1}-\ldots-x_{2k-1})dx_{2k-1}\ldots dx_{k+1} \\
&\times\ldots\times\int_{u_n}^{\infty}\int_{u_n-x_{(n-1)k+1}}^{\infty}\cdots\int_{u_n-x_{(n-1)k+1}-\ldots-x_{nk-2}}^{\infty}\prod_{i=(n-1)k+1}^{nk-1}f_i(x_i) \\
&\overline{F}_{nk}(u_n-x_{(n-1)k+1}-\ldots-x_{nk-1})dx_{nk-1}\ldots dx_{(n-1)k+1} \\
&-\sum_{j<}^{nk-1}\sum_{h}^{nk}\alpha_{j,h}\gamma_j\gamma_h\int_{u_1}^{\infty}\int_{u_1-x_1}^{\infty}\cdots\int_{u_1-x_1-\ldots-x_{k-2}}^{\infty}\prod_{i=1}^{k-1}f_{i,j}(x_i) \\
&\overline{F}_{k,j}(u_1-x_1-\ldots-x_{k-1})dx_{k-1}\ldots dx_1 \\
&\times \int_{u_2}^{\infty}\int_{u_2-x_{k+1}}^{\infty}\cdots\int_{u_2-x_{k+1}-\ldots-x_{2k-2}}^{\infty}\prod_{i=k+1}^{2k-1}f_{i,j}(x_i)
\end{aligned}$$

$$
\begin{aligned}
&\overline{F}_{2k,j}(u_2 - x_{k+1} - \ldots - x_{2k-1}) dx_{2k-1} \ldots dx_{k+1} \\
&\times \ldots \times \int_{u_n}^{\infty} \int_{u_n - x_{(n-1)k+1}}^{\infty} \cdots \int_{u_n - x_{(n-1)k+1} - \ldots - x_{nk-2}}^{\infty} \prod_{i=(n-1)k+1}^{nk-1} f_i(x_i) \\
&\overline{F}_{nk}(u_n - x_{(n-1)k+1} - \ldots - x_{nk-1}) dx_{nk-1} \ldots dx_{(n-1)k+1} \\
&- \sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h} \gamma_j \gamma_h \int_{u_1}^{\infty} \int_{u_1 - x_1}^{\infty} \cdots \int_{u_1 - x_1 - \ldots - x_{k-2}}^{\infty} \prod_{i=1}^{k-1} f_{i,h}(x_i) \\
&\overline{F}_{k,h}(u_1 - x_1 - \ldots - x_{k-1}) dx_{k-1} \ldots dx_1 \\
&\times \int_{u_2}^{\infty} \int_{u_2 - x_{k+1}}^{\infty} \cdots \int_{u_2 - x_{k+1} - \ldots - x_{2k-2}}^{\infty} \prod_{i=k+1}^{2k-1} f_{i,h}(x_i) \\
&\overline{F}_{2k,h}(u_2 - x_{k+1} - \ldots - x_{2k-1}) dx_{2k-1} \ldots dx_{k+1} \\
&\times \ldots \times \int_{u_n}^{\infty} \int_{u_n - x_{(n-1)k+1}}^{\infty} \cdots \int_{u_n - x_{(n-1)k+1} - \ldots - x_{nk-2}}^{\infty} \prod_{i=(n-1)k+1}^{nk-1} f_i(x_i) \\
&\overline{F}_{nk}(u_n - x_{(n-1)k+1} - \ldots - x_{nk-1}) dx_{nk-1} \ldots dx_{(n-1)k+1} \\
&+ \sum_{j<}^{nk-1} \sum_{h}^{nk} \alpha_{j,h} \gamma_j \gamma_h \int_{u_1}^{\infty} \int_{u_1 - x_1}^{\infty} \cdots \int_{u_1 - x_1 - \ldots - x_{k-2}}^{\infty} \prod_{i=1}^{k-1} f_{i,j,h}(x_i) \\
&\overline{F}_{k,j,h}(u_1 - x_1 - \ldots - x_{k-1}) dx_{k-1} \ldots dx_1 \\
&\times \int_{u_2}^{\infty} \int_{u_2 - x_{k+1}}^{\infty} \cdots \int_{u_2 - x_{k+1} - \ldots - x_{2k-2}}^{\infty} \prod_{i=k+1}^{2k-1} f_{i,j,h}(x_i) \\
&\overline{F}_{2k,j,h}(u_2 - x_{k+1} - \ldots - x_{2k-1}) dx_{2k-1} \ldots dx_{k+1} \\
&\times \ldots \times \int_{u_n}^{\infty} \int_{u_n - x_{(n-1)k+1}}^{\infty} \cdots \int_{u_n - x_{(n-1)k+1} - \ldots - x_{nk-2}}^{\infty} \prod_{i=(n-1)k+1}^{nk-1} f_i(x_i) \\
&\overline{F}_{nk}(u_n - x_{(n-1)k+1} - \ldots - x_{nk-1}) dx_{nk-1} \ldots dx_{(n-1)k+1}.
\end{aligned}
\tag{3.14}
$$

Provided that $\beta_{nk} \geq \beta_i, i = 1, \ldots, nk$, by Lemma 2.5.2 each $i-$th mixed Erlang component of (3.14) can be transformed into a new mixed Erlang distribution with a common scale parameter $Z(\beta_{nk})$.

In addition, according to Remark 2.5.4 the convolution of mixed Erlang distributions belongs to the class of Erlang mixtures distribution. Therefore (3.14) can be expressed as a sum product of n mixed Erlang survival functions as follows

$$
\begin{aligned}
&\mathbb{P}(S_{1,k} > u_1, S_{2,k} > u_2, \ldots, S_{n,k} > u_n) \\
&= (1 + \sum_{h<}^{nk-1} \sum_{j}^{nk} \alpha_{j,h}\gamma_j\gamma_h)\overline{F}_{S^{(1)}_{1,k}}(u_1)\overline{F}_{S^{(1)}_{2,k}}(u_2)\ldots\overline{F}_{S^{(1)}_{n,k}}(u_n) \\
&- \sum_{h<}^{nk-1} \sum_{j}^{nk} \alpha_{j,h}\gamma_j\gamma_h\overline{F}_{S^{(2)}_{1,k}}(u_1)\overline{F}_{S^{(2)}_{2,k}}(u_2)\ldots\overline{F}_{S^{(2)}_{n,k}}(u_n) \\
&- \sum_{h<}^{nk-1} \sum_{j}^{nk} \alpha_{j,h}\gamma_j\gamma_h\overline{F}_{S^{(3)}_{1,k}}(u_1)\overline{F}_{S^{(3)}_{2,k}}(u_2)\ldots\overline{F}_{S^{(3)}_{n,k}}(u_n) \\
&+ \sum_{h<}^{nk-1} \sum_{j}^{nk} \alpha_{j,h}\gamma_j\gamma_h\overline{F}_{S^{(4)}_{1,k}}(u_1)\overline{F}_{S^{(4)}_{2,k}}(u_2)\ldots\overline{F}_{S^{(4)}_{n,k}}(u_n),
\end{aligned}
$$

where

$$
\begin{aligned}
&S^{(l)}_{1,k} \sim ME(Z(\beta_{nk}), \Pi\{\Psi(\underset{\sim}{Q}^{(l)}_{1,j,h}), \ldots, \Psi(\underset{\sim}{Q}^{(l)}_{k,j,h})\}), \\
&S^{(l)}_{2,k} \sim ME(Z(\beta_{nk}), \Pi\{\Psi(\underset{\sim}{Q}^{(l)}_{k+1,j,h}), \ldots, \Psi(\underset{\sim}{Q}^{(l)}_{2k,j,h})\}), \\
&S^{(l)}_{n,k} \sim ME(Z(\beta_{nk}), \Pi\{\Psi(\underset{\sim}{Q}^{(l)}_{(n-1)k+1,j,h}), \ldots, \Psi(\underset{\sim}{Q}^{(l)}_{nk,j,h})\}), \quad l = 1, 2, 3, 4,
\end{aligned}
$$

with $\underset{\sim}{Q}^{(l)}_{i,j,h}$, $i = 1, \ldots, nk$, $l = 1, 2, 3, 4$ is defined in (3.9). Thus the proof is complete.
□

Proof of Proposition 3.4.1 Similarly to the independence case described in Lemma 3.6.2, the df of $R_{2,k}$ is of mixed distribution and can be expressed in terms of the joint df of $(T_{1,k}, T_{2,k})$ as follows

$$
\begin{aligned}
F_{R_{2,k}}(y) &= \begin{cases} \mathbb{P}(T_{1,k} = 0, T_{2,k} = 0) & \text{for} \quad y = 0 \\ \mathbb{P}(T_{1,k} = 0, 0 < T_{2,k} \leqslant y) + \mathbb{P}(0 < T_{1,k} \leqslant y, T_{2,k} = 0) & \\ +\mathbb{P}(T_{1,k} + T_{2,k} \leqslant y, 0 < T_{1,k} \leqslant y, 0 < T_{2,k} \leqslant y) & \text{for} \quad y > 0 \end{cases} \\
&=: \begin{cases} F_{S_{1,k},S_{2,k}}(d_1, d_2) & \text{for} \quad y = 0, \\ F_{S_{1,k},S_{2,k}}(d_1 + y, d_2 + y) & \text{for} \quad y > 0. \end{cases}
\end{aligned}
$$

By Proposition 3.3.2 and Lemma 3.6.2, $F_{R_{2,k}}(y)$ can be written in two terms as follows:

- the discrete term

$$
\begin{aligned}
F_{S_{1,k},S_{2,k}}(d_1, d_2) &= \xi_1 F_{S^{(1)}_{1,k}}(d_1) F_{S^{(1)}_{2,k}}(d_2) + \xi_2 F_{S^{(2)}_{1,k}}(d_1) F_{S^{(2)}_{2,k}}(d_2) \\
&\quad + \xi_3 F_{S^{(3)}_{1,k}}(d_1) F_{S^{(3)}_{2,k}}(d_2) + \xi_4 F_{S^{(4)}_{1,k}}(d_1) F_{S^{(4)}_{2,k}}(d_2),
\end{aligned}
$$

- the continuous term

$$
\begin{aligned}
&F_{S_{1,k},S_{2,k}}(d_1+y,d_2+y)\\
&=\\
&\xi_1(F_{S^{(1)}_{1,k}}(d_1)F_{S^{(1)}_{2,k}}(d_2+y)+F_{S^{(1)}_{1,k}}(d_1+y)F_{S^{(1)}_{2,k}}(d_2)+F_{S^{(1)}_{1,k}+S^{(1)}_{2,k}}(d_1,d_2,y))\\
&+\xi_2(F_{S^{(2)}_{1,k}}(d_1)F_{S^{(2)}_{2,k}}(d_2+y)+F_{S^{(2)}_{1,k}}(d_1+y)F_{S^{(2)}_{2,k}}(d_2)+F_{S^{(2)}_{1,k}+S^{(2)}_{2,k}}(d_1,d_2,y))\\
&+\xi_3(F_{S^{(3)}_{1,k}}(d_1)F_{S^{(3)}_{2,k}}(d_2+y)+F_{S^{(3)}_{1,k}}(d_1+y)F_{S^{(3)}_{2,k}}(d_2)+F_{S^{(3)}_{1,k}+S^{(3)}_{2,k}}(d_1,d_2,y))\\
&+\xi_4(F_{S^{(4)}_{1,k}}(d_1)F_{S^{(4)}_{2,k}}(d_2+y)+F_{S^{(4)}_{1,k}}(d_1+y)F_{S^{(4)}_{2,k}}(d_2)+F_{S^{(4)}_{1,k}+S^{(4)}_{2,k}}(d_1,d_2,y)).
\end{aligned}
$$

This completes the proof. □

PROOF OF PROPOSITION 3.4.4 In view of (3.11)

$$
\begin{aligned}
TVaR_p(T_{1,k},R_{2,k}) &= \frac{\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}>VaR_{R_{2,k}}(p)\}})}{1-p}\\
&= \frac{1}{1-p}\int_{VaR_{R_{2,k}}(p)}^{\infty}\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}})ds.
\end{aligned}
\tag{3.15}
$$

First, we need to calculate $\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}})$ as follows

$$\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}}) = \int_0^{\infty} uf_{T_{1,k},T_{1,k}+T_{2,k}=s}(u)du.$$

Let

$$f_{S^{(l)}_{1,k}+S^{(l)}_{2,k}}(d_1,d_2,u) := \frac{d}{du}F_{S^{(l)}_{1,k}+S^{(l)}_{2,k}}(d_1,d_2,u), l=1,2,3,4.$$

As in Proposition 3.4.1, one can express $\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}})$ as follows

$$
\begin{aligned}
&\mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}})\\
&=\\
&\xi_1\left(F_{S^{(1)}_{2,k}}(d_2)\int_0^s uf_{S^{(1)}_{1,k}}(d_1+u)du+\int_0^s uf_{S^{(1)}_{1,k}+S^{(1)}_{2,k}}(d_1,d_2,u)du\right)\\
&+\xi_2\left(F_{S^{(2)}_{2,k}}(d_2)\int_0^s uf_{S^{(2)}_{1,k}}(d_1+u)du+\int_0^s uf_{S^{(2)}_{1,k}+S^{(2)}_{2,k}}(d_1,d_2,u)du\right)\\
&+\xi_3\left(F_{S^{(3)}_{2,k}}(d_2)\int_0^s uf_{S^{(3)}_{1,k}}(d_1+u)du+\int_0^s uf_{S^{(3)}_{1,k}+S^{(3)}_{2,k}}(d_1,d_2,u)du\right)\\
&+\xi_4\left(F_{S^{(4)}_{2,k}}(d_2)\int_0^s uf_{S^{(4)}_{1,k}}(d_1+u)du+\int_0^s uf_{S^{(4)}_{1,k}+S^{(4)}_{2,k}}(d_1,d_2,u)du\right)
\end{aligned}
$$

$$= \sum_{l=1}^{4} \xi_l \bigg(F_{S_{2,k}^{(l)}}(d_2) \int_0^s u f_{S_{1,k}^{(l)}}(d_1+u)du + \int_0^s u f_{S_{1,k}^{(l)}+S_{2,k}^{(l)}}(d_1,d_2,u)du \bigg). \tag{3.16}$$

By Lemma 3.6.1, for $X_i \sim ME(\beta, \underset{\sim}{Q}_i)$ and $d_i > 0, i = 1, 2$

$$\int_0^s u f_{X_i}(d_i+u)du = \frac{1}{\beta} \sum_{k=0}^{\infty} (k+1)\Delta_k(d_i, \beta, \underset{\sim}{Q}_i) W_{k+2}(s,\beta) =: U_{X_i}(s, d_i, \beta). \tag{3.17}$$

$$\int_s^{\infty} u f_{X_i}(d_i+u)du = \frac{1}{\beta} \sum_{k=0}^{\infty} (k+1)\Delta_k(d_i, \beta, \underset{\sim}{Q}_i) \overline{W}_{k+2}(s,\beta) =: \overline{U}_{X_i}(s, d_i, \beta).$$

Similarly, by Lemma 3.6.2

$$\begin{aligned} \int_0^s u f_{X_1+X_2}(d_1,d_2,u)du &= \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k+1)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2) W_{k+j+3}(s,\beta) \\ &=: U_{X_1}(s, d_1, d_2, \beta). \end{aligned} \tag{3.18}$$

$$\begin{aligned} \int_s^{\infty} u f_{X_1+X_2}(d_1,d_2,u)du &= \frac{1}{\beta} \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (k+1)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2) \overline{W}_{k+j+3}(s,\beta) \\ &=: \overline{U}_{X_1}(s, d_1, d_2, \beta). \end{aligned}$$

Taking (3.17) and (3.18) into account, one may write (3.16) as follows

$$\mathbb{E}(T_{1,k} \mathbb{1}_{\{R_{2,k}=s\}}) = \sum_{l=1}^{4} \xi_l \bigg(F_{S_{2,k}^{(l)}}(d_2) U_{S_{1,k}^{(l)}}(s, d_1, Z(\beta_{2k})) + U_{S_{1,k}^{(l)}}(s, d_1, d_2, Z(\beta_{2k})) \bigg). \tag{3.19}$$

Therefore, refering to (3.15) (set $x_p := VaR_p(R_{2,k})$)

$$\begin{aligned} TVaR_p(T_{1,k}, R_{2,k}) &= \frac{1}{1-p} \sum_{l=1}^{4} \xi_l \bigg(F_{S_{2,k}^{(l)}}(d_2) \int_{x_p}^{\infty} U_{S_{1,k}^{(l)}}(s, d_1, Z(\beta_{2k}))ds \\ &\quad + \int_{x_p}^{\infty} U_{S_{1,k}^{(l)}}(s, d_1, d_2, Z(\beta_{2k}))ds \bigg). \end{aligned}$$

Hence, the result follows easily. □

Proof of Corollary 3.4.1 The unpaid losses of the ceding insurer line of business is defined as follows

$$U(K_1, K) = \mathbb{E}\left\{(T_{1,k} - K_1)\mathbb{1}_{\{R_{2,k}>K\}}\right\} = \int_K^\infty \mathbb{E}(T_{1,k}\mathbb{1}_{\{R_{2,k}=s\}})ds - K_1\overline{F}_{R_{2,k}}(K).$$

In light of (3.19)

$$\begin{aligned} U(K_1, K) &= \sum_{l=1}^{4} \xi_l \int_K^\infty \Big(F_{S^{(l)}_{2,k}}(d_2) U_{S^{(l)}_{1,k}}(s, d_1, Z(\beta_{2k})) + U_{S^{(l)}_{1,k}}(s, d_1, d_2, Z(\beta_{2k})) \Big) ds \\ &= \sum_{l=1}^{4} \xi_l \Big(F_{S^{(l)}_{2,k}}(d_2) \overline{U}_{S^{(l)}_{1,k}}(K, d_1, Z(\beta_{2k})) + \overline{U}_{S^{(l)}_{1,k}}(K, d_1, d_2, Z(\beta_{2k})) \Big) \\ &\quad - K_1 \overline{F}_{R_{2,k}}(K). \end{aligned}$$

Hence the proof is complete. □

3.6 Appendix

3.6.1 Properties of Mixed Erlang Distribution

Lemma 3.6.1. For a deductible $d > 0$, if $X \sim ME(\beta, \underset{\sim}{V})$ then the df of $Y := (X-d)_+$ is given by

$$F_Y(y) = \begin{cases} F_X(d) & \text{for} \quad y = 0, \\ F_X(d+y) & \text{for} \quad y > 0, \end{cases} \tag{3.20}$$

where

$$F_X(d+y) = \sum_{k=0}^{\infty} \Delta_k(d, \beta, \underset{\sim}{V}) W_{k+1}(y, \beta),$$

with

$$\Delta_k(d, \beta, \underset{\sim}{V}) = \frac{1}{\beta} \sum_{j=0}^{\infty} q_{j+k+1} w_{j+1}(d, \beta). \tag{3.21}$$

Lemma 3.6.2. Let X_1 and X_2 be two independent risks such that $X_i \sim ME(\beta, \underset{\sim}{Q}_i), i = 1, 2$. If $Y_i = (X_i - d_i)_+$ with $d_i > 0, i = 1, 2$ then $R_2 = Y_1 + Y_2$ has a df

$$F_{R_2}(s) = \begin{cases} F_{X_1}(d_1)F_{X_2}(d_2) & \text{for} \quad s = 0, \\ F_{X_1}(d_1)F_{X_2}(s+d_2) + F_{X_2}(d_2)F_{X_1}(s+d_1) + F_{X1+X_2}(d_1, d_2, s) & \text{for} \quad s > 0, \end{cases}$$

where

$$F_{X_1+X_2}(d_1,d_2,s)=\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta,\underset{\sim}{Q}_2)W_{k+j+2}(s,\beta).$$

Remarks 3.6.3. Given the tractable expression of the df of R_2, its VaR at a confidence level $p\in(0,1)$ is the solution of

$$\begin{aligned}&F_{X_1}(d_1)F_{X_2}(d_2)+F_{X_1}(d_1)F_{X_2}(VaR_{R_2}(p)+d_2)\\&+F_{X_2}(d_2)F_{X_1}(VaR_{R_2}(p)+d_1)+F_{T_{2,k}}(VaR_{R_2}(p))=p,\end{aligned}$$

which can be solved numerically. In addition, the TVaR of R_2 at a confidence level $p\in(0,1)$ is given by (set $x_p:=VaR_{R_2}(p)$)

$$\begin{aligned}TVaR_{R_2}(p) &= \frac{1}{1-p}\Big(F_{X_2}(d_2)\overline{U}_{X_1}(x_p,d_1,\beta)\\&\quad+F_{X_1}(d_1)\overline{U}_{X_2}(x_p,d_2,\beta)+\overline{U}_{X_1+X_2}(x_p,d_1,d_2,\beta)\Big),\end{aligned}$$

where

$$\overline{U}_{X_i}(x_p,d_i,\beta) = \frac{1}{\beta}\sum_{k=0}^{\infty}(k+1)\Delta_k(d_i,\beta,\underset{\sim}{Q}_i)\overline{W}_{k+2}(x_p,\beta),$$

$$\begin{aligned}\overline{U}_{X_1+X_2}(x_p,d_1,d_2,\beta) &= \frac{1}{\beta}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}(k+j+2)\Delta_k(d_1,\beta,\underset{\sim}{Q}_1)\\&\quad\times\Delta_j(d_2,\beta,\underset{\sim}{Q}_2)\overline{W}_{k+j+3}(x_p,\beta).\end{aligned}$$

Proof. Since Y_1 and Y_2 are independent risks which have mixed distribution, the df of R_2 can also be expressed as a df of a mixed distribution which depends on the value of s as follows:

- the discrete part of F_{R_2} is obtained for $s=0$, specifically we have

$$\begin{aligned}F_{R_2}(0) &= \mathbb{P}(Y_1+Y_2\leqslant 0)=\mathbb{P}(Y_1+Y_2=0)\\&= \mathbb{P}(Y_1=0,Y_2=0)=F_{X_1}(d_1)F_{X_2}(d_2),\end{aligned}\tag{3.22}$$

- for $s>0$ the continious part of F_{R_2} is given by

$$F_{R_2}(s) = \mathbb{P}(Y_1+Y_2\leqslant s)$$

$$
\begin{aligned}
&= \mathbb{P}(Y_1+Y_2 \leqslant s, Y_1 = 0, 0 < Y_2 \leqslant s) \\
&\quad + \mathbb{P}(Y_1+Y_2 \leqslant s, 0 < Y_1 \leqslant s, Y_2 = 0) \\
&\quad + \mathbb{P}(Y_1+Y_2 \leqslant s, 0 < Y_1 \leqslant s, 0 < Y_2 \leqslant s) \\
&= \mathbb{P}(Y_1 = 0, 0 < Y_2 \leqslant s) + \mathbb{P}(0 < Y_1 \leqslant s, Y_2 = 0) \\
&\quad + \mathbb{P}(Y_1+Y_2 \leqslant s, 0 < Y_1 \leqslant s, 0 < Y_2 \leqslant s) \\
&= F_{X_1}(d_1)F_{X_2}(s+d_2) + F_{X_2}(d_2)F_{X_1}(s+d_1) \\
&\quad + \int_0^s F_{X_1}(s-u+d_1) f_{X_2}(u+d_2)du. \qquad (3.23)
\end{aligned}
$$

Let $F_{T_{2,k}}(s) := \int_0^s F_{X_1}(s-u+d_1)f_{X_2}(u+d_2)du$, by Lemma 3.6.1 this can be written as

$$
\begin{aligned}
F_{T_{2,k}}(s) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty} \Delta_k(d_1,\beta_1,\underset{\sim}{Q}_1)\Delta_j(d_2,\beta_2,\underset{\sim}{Q}_2) \\
&\quad \times \int_0^s W_{k+1}(s-u,\beta) w_{j+1}(u,\beta)du. \qquad (3.24)
\end{aligned}
$$

It can be seen that $\int_0^s W_{k+1}(s-u,\beta)w_{j+1}(u,\beta)du$ is a convolution of two independent Erlang risks with a common scale parameter β, which is again an Erlang risk with shape parameter $k+j+2$ and scale parameter β. Thus combining (3.22), (3.23) and (3.24) the claim follows easily.

Lemma 3.6.4. Let $X \sim ME(\beta, \underset{\sim}{V})$ with pdf $f(x,\beta,\underset{\sim}{V})$, if g is some positive function such that $\mathbb{E}\{g(X)\} < \infty$, then $c(x,\beta,\underset{\sim}{V}) = \frac{g(x)f(x,\beta,\underset{\sim}{V})}{\mathbb{E}\{g(X)\}}$ is again a pdf of mixed Erlang distribution with scale parameter $Z(\beta)$ and mixing weights $\underset{\sim}{\Theta}(\underset{\sim}{V}) = (\theta_1, \theta_2, \ldots)$, with

$$
c(x,\beta,\underset{\sim}{V}) = \sum_{k=1}^{\infty} \theta_k w_k(x, Z(\beta)),
$$

where

- $Z(\beta) = 2\beta$ and $\theta_k = \frac{1}{2^{k-1}} \sum_{j=1}^{k} \binom{k-1}{j-1} q_j \sum_{l=k-j+1}^{\infty} q_l$, for $g(x) = 2\overline{F}(x)$,
- $Z(\beta) = \beta$ and

$$
\theta_k = \begin{cases} 0 & \text{for} \quad k \leqslant t, \\ \dfrac{q_{k-t}\frac{\Gamma(k)}{\Gamma(k-t)}}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+t)}{\Gamma(j)}} & \text{for} \quad k > t, \end{cases}
$$

for $g(x) = x^t$ with $t \in \mathbb{R}$,

- $Z(\beta) = \beta + t$ and $\theta_k = \frac{q_k \overline{\beta}^k}{\sum_{j=1}^{\infty} q_j \overline{\beta}^j}$ with $\overline{\beta} = \frac{\beta}{\beta+t}$, for $g(x) = e^{-tx}$ with $t \in \mathbb{N}$.

Proof. We have

$$c(x, \beta, \underset{\sim}{V}) = \frac{g(x) f(x, \beta, \underset{\sim}{V})}{\mathbb{E}\left\{g(X)\right\}} = \frac{1}{\mathbb{E}\left\{g(X)\right\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} g(x) x^{k-1} e^{-\beta x}. \tag{3.25}$$

For $g(x) = x^t$ one can write (3.25) as follows

$$\begin{aligned} c(x, \beta, \underset{\sim}{V}) &= \frac{1}{\mathbb{E}\left\{X^t\right\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} x^{t+k-1} e^{-\beta x} \\ &= \sum_{k=1}^{\infty} \left(\frac{q_k \frac{\Gamma(k+t)}{\Gamma(k)}}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+t)}{\Gamma(j)}} \right) w_{k+t}(x, \beta) \\ &= \sum_{s=t+1}^{\infty} \left(\frac{q_{s-t} \frac{\Gamma(s)}{\Gamma(s-t)}}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+t)}{\Gamma(j)}} \right) w_s(x, \beta) \\ &= \sum_{s=1}^{\infty} \theta_s w_s(x, \beta), \end{aligned}$$

with

$$\theta_s = \begin{cases} 0 & \text{for} \quad s \leqslant t, \\ \frac{q_{s-t} \frac{\Gamma(s)}{\Gamma(s-t)}}{\sum_{j=1}^{\infty} q_j \frac{\Gamma(j+t)}{\Gamma(j)}} & \text{for} \quad s > t. \end{cases}$$

For $g(x) = e^{-tx}$, (3.25) can be expressed as follows (set $\overline{\beta} := \frac{\beta}{\beta+t}$)

$$\begin{aligned} c(x, \beta, \underset{\sim}{V}) &= \frac{1}{\mathbb{E}\left\{e^{-tX}\right\}} \sum_{k=1}^{\infty} q_k \frac{\beta^k}{(k-1)!} x^{k-1} e^{-(\beta+t)x} \\ &= \sum_{k=1}^{\infty} \left(\frac{q_k \overline{\beta}^k}{\sum_{j=1}^{\infty} q_j \overline{\beta}^j} \right) w_k(x, \beta+t) \\ &= \sum_{k=1}^{\infty} \theta_k w_k(x, \beta+t). \end{aligned}$$

For $g(x) = 2\overline{F}(x)$, see Cossette et al. [20] for the proof.

3.6.2 Pearson's Coefficient of a Bivariate Sarmanov Mixed Erlang Risk

Pearson's coefficient $\rho_{1,2}$ is one of the most commonly used dependence measures between two risks X_1 and X_2. In this regards, X_1 and X_2 are assumed to be linearly correlated. Following (2.5), we show next that when $(X_1, X_2) \sim SME_2(\boldsymbol{\beta}, \utilde{Q}_1, \utilde{Q}_2)$, the closed expressions for $\rho_{1,2}$ depend on the dependence parameter $\alpha_{1,2}$ and the choice of kernel functions as follows:

- for $\phi_i(x_i) = x_i^t - \mathbb{E}\left\{X_i^t\right\}, t > 0$

$$\rho_{1,2}(X_1, X_2) = \frac{\alpha_{1,2}(m_1^{t+1} - m_1^t m_1^1)(m_2^{t+1} - m_2^t m_2^1)}{\sigma_1\sigma_2},$$

  where σ_i is the standard deviation of X_i and

$$m_i^s = \mathbb{E}\left\{X_i^s\right\} = \frac{1}{\beta_i^s}\sum_{j=1}^{\infty} q_{i,j}\frac{(s+j-1)!}{(j-1)!}, i = 1, 2, \text{ with } s > 0,$$

  in particular for $t = 1$

$$\rho_{1,2}(X_1, X_2) = \alpha_{1,2}\sigma_1\sigma_2,$$

- for $\phi_i(x_i) = e^{-tx_i} - \mathbb{E}\left\{e^{-tX_i}\right\}, t > 0$

$$\rho_{1,2}(X_1, X_2) = \frac{\alpha_{1,2}(\eta_{1,t} - \Gamma_{1,t})(\eta_{2,t} - \Gamma_{2,t})}{\sigma_1\sigma_2},$$

  where $\eta_{i,t} = \frac{1}{\beta_i+t}\sum_{j=1}^{\infty} j q_{i,j}\left(\frac{\beta_i}{\beta_i+t}\right)^j$ and $\Gamma_{i,t} = m_i^1 \sum_{j=1}^{\infty} q_{i,j}\left(\frac{\beta_i}{\beta_i+t}\right)^j, i = 1, 2,$

- for $\phi_i(x_i) = 2\overline{F}_i(x_i) - 1$, see Cossette et al. [20]

$$\rho_{1,2}(X_1, X_2) = \frac{\alpha_{1,2}}{\beta_1\beta_2\sigma_1\sigma_2}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} jk(\theta_{1,j} - q_{1,j})(\theta_{2,k} - q_{2,k}),$$

  where $\theta_{1,j}$ and $\theta_{2,k}$ are defined in (3.4).

Chapter 4

On Age Difference in Joint Life Modelling with Life Insurance Applications

This Chapter is based on F. Dufresne, E. Hashorva, G. Ratovomirija and Y. Toukourou: On age Difference in Joint Life Modelling with Life Insurance Applications. *Submitted.*

4.1 Introduction

Insurance and annuity products covering several lives require the modelling of the joint distribution of future lifetimes. Commonly in actuarial practice, the future lifetimes among a group of people are assumed to be independent. This simplifying assumption is not supported by real insurance data as demonstrated by numerous investigations. Joint life annuities issued to married couples offer a very good illustration of this fact. It is well known that husband and wife tend to be exposed to similar risks as they are likely to have the same living habits. For example, Parkes et al. [77] and Ward [93] have brought to light the increased mortality of widowers, often called the *broken heart syndrome*. Many contributions have shown that there could be a significant difference between risk-related quantities, such as risk premiums, evaluated according to dependence or independence assumptions. Denuit and Cornet [23] have measured the effect of lifetime dependencies on the present value of a widow pension benefit. Based on the data collected in cemeteries, not only do their estimation results confirm that the mortality risk depends on the marital status, but also show that the amounts of premium are reduced approximately by

10 per cent compared to model which assumes independence. According to data from a large Canadian insurance company, Frees et al. [39] have demonstrated that there is a strong positive dependence between joint lives. Their estimation results indicate that annuity values are reduced by approximately 5 per cent compared to the model with independence.
Introduced by Sklar [86], copulas have been widely used to model the dependence structure of random vectors. In the particular case of bivariate lifetimes, frailty models can be used to describe the common risk factors between husband and wife. Oakes [76] has shown that the bivariate distributions generated by frailty models are a subclass of Archimedean copulas. This makes this particular copula family very attractive for modelling bivariate lifetimes. We refer to Nelsen [74] for a general introduction to copulas.

The Archimedean copula family has been proved valuable in numerous life insurance applications, see e.g., Frees et al. [39], Brown and Poterba [13], Carriere [16]. In Luciano et al. [65], the marginal distributions and the copula are fitted separately and, the results show that the dependence increases with age.
It is known that the level of association between variables is characterized by the value of the dependence parameter. In this chapter, a special attention is paid to this dependence parameter. Youn and Shemyakin [99] have introduced the age difference between spouses as an argument of the dependence parameter of the copula. In addition, the sign of the age difference is of great interest in our model. More precisely, we presume that the gender of the older member of the couple has an influence on the level of dependence between lifetimes. In order to confirm our hypothesis, four families of Archimedean copulas are discussed namely, Gumbel, Frank, Clayton and Joe copulas, all these under a Gompertz distribution assumption for marginals. The parameter estimations are based on the maximum likelihood approach using data from a large Canadian insurance company, the same set of data used by Frees et al. [39]. Following Joe and Xu [55] and Oakes [76], a two-step technique, where marginals and copula are estimated separately, is applied. The results make clear that the dependence is higher when husband is older than wife.
Once the marginal and copula parameters are estimated, one needs to assess the goodness of fit of the model. For example, the likelihood ratio test is used in Carriere [16] whereas the model of Youn and Shemyakin [99] is based on the Akaike Information Criterion (AIC). In addition to likelihood ratio test, following Gribkova and Lopez [44] and Lawless [61], we implement a whole goodness of fit procedure to validate the model. Based on the Cramèr-von Mises statistics, the Gumbel copula, whose dependence parameter is a function of the age difference and its sign gives

the best fit.
The rest of the chapter is organized as follows. Section 4.2 describes the dataset and provides some key facts that motivate our study. Section 4.3 describes the maximum likelihood procedure used to estimate the marginal distributions. The dependence models are examined in Section 4.4. In a first hand, we describe the copula models whose parameter are estimated. Secondly, a bootstrap algorithm and likelihood ratio test are proposed for assessing the goodness of fit of the model. Considering several products available on the life insurance market, numerical applications with real data, including best estimate of liabilities, risk capital and stop loss premiums are presented in Section 4.5. Section 4.6 concludes the chapter.

4.2 Motivation

As already shown in Maeder [66], being in a married couple can significantly influence the mortality. Moreover, the remaining lifetimes of male and female in the couple are dependent, see e.g., Carriere [16], Frees et al. [39]. In this contribution, we aim at modelling the dependence between the lifetimes of a man and a woman within a married couple. Common dependence measures, which will be used in our study, are: the Pearson's correlation coefficient r, the Kendall's Tau τ, and the Spearman's Rho ρ. In order to develop these aspects, data [1] from a large Canadian life insurance company are used. The dataset contains information from policies that were in force during the observation period, i.e. from December 29, 1988 to December 31, 1993. Thus, we have 14′947 contracts among which 14′889 couples (one male and one female) and the remaining 58 are contracts where annuitants are both male (22 pairs) or both female (36 pairs). The same dataset has been analysed in Frees et al. [39], Carriere [16], Youn and Shemyakin [99], Gribkova and Lopez [44] among others, also in the framework of modelling bivariate lifetime. Since we are interested in the dependence within the couple, we focus our attention on the male-female contracts.
We refer the readers to Frees et al. [39] for the data processing procedure. The dataset is left truncated as the annuitant information is recorded only from the date they enter the study; this means that insured who have died before the beginning of the observation period were not taken into account in the study. The dataset is also right censored in the sense that most of the insured were alive at the end of the study. Considering our sample as described above, some couples having several contracts

[1] I wish to thank the Society of Actuaries, through the courtesy of Edward (Jed) Frees and Emiliano Valdez, for allowing the use of the data in this thesis.

could appear many times. By considering each couple only once, our dataset consists of $11'457$ different couples for which, we can draw the following information:

- the age at the beginning of the observation x_m and x_f for male and female, respectively,
- the lifetimes under the observation period t_m and t_f for male and female, respectively,
- the binary right censoring indicator δ_m and δ_f for male and female, respectively,
- the couple benefit in Canadian Dollar (CAD) amount within a last survivor contract.

The entry age is the age at which, the annuitant enters the study. The lifetime at entry age corresponds to the lapse of time during which the individual was alive over the period of study. Therefore, for a male (resp. female) aged x_m (resp. x_f) at entry and whose data is not censored i.e. $\delta_m = 0$ (resp. $\delta_f = 0$), $x_m + t_m$ (resp. $x_f + t_f$) is the age at death. When the data is right censored i.e. $\delta_m = 1$ (resp. $\delta_f = 1$), the number $x_m + t_m$ (resp. $x_f + t_f$) is the age at the end of the period of study (December 31, 1993). The lifetime is usually equal to 5.055 years corresponding to the duration of the study period; but it is sometimes less as some people may entry later or die before the end of study. Benefit is paid each year until the death of the last survivor. Its value will be used as an input for the applications of the model to insurance products in Section 4.5.2. Some summary statistics of the age distribution of our dataset are displayed in Table 4.1. It can be seen that the average

	Males age		Females age	
Statistics	Entry	Death	Entry	Death
Number	$11'457$	$1'269$	$11'457$	454
Mean	67.87	74.40	64.91	73.81
Std	6.34	7.19	7.16	7.81
Median	67.60	74.10	65.10	73.15
10^{th}*percentile*	60.20	64.00	55.70	64.20
90^{th}*percentile*	75.60	83.50	73.50	84.10

Table 4.1: Summary of the univariate distribution statistics.

entry age is 66.4 for the entire population, 67.9 for males and 64.9 for female; 90% of annuitants are older than 57.9 at entry and males are older than females by 3 years on average. Among the $11'457$ couples considered there are 193 couples where both annuitants are dead. Based on these 193 couples, the empirical dependence measures are displayed in the last row of Table 4.2. The values show that the ages at death of spouses are positively correlated.

		Dependence measures		
	Samples	r	ρ	τ
$x_m \geq x_f$	133	0.90	0.89	0.74
$x_m < x_f$	60	0.89	0.87	0.71
Total	193	0.82	0.80	0.62

Table 4.2: Empirical dependence measures with respect to the gender of the elder partner.

From the existing literature, see e.g., Denuit and Cornet [23], Youn and Shemyakin [99], Denuit et al. [24], Ji and al. [53], Hougaard [52], the dependence within a couple is often influenced by three factors:

- the **common lifestyle** that husband and wife follow, for example their eating habits, this is referred to as the long-term dependence,
- the short-term dependence or the **broken-heart factor** where the death of one would precipitate the death of the partner, often due to the vacuum caused by the passing away of the companion,
- the **common disaster** that affects simultaneously the husband and his wife, as they are likely to be in the same area when a catastrophic event occurs, this dependence factor is considered as the instantaneous dependence.

Based on the common disaster and the broken-heart, Youn and Shemyakin [99] have introduced the *age difference between spouses*. Their results show that the model captures some additional association between lifetime of the spouses that would not be reflected in a model without age difference. It is also observed that, the higher the age difference is, the lower is the dependence. Referring to the same dataset, Table 4.3 confirms their results, with $|d|$ the absolute value of d and $d = x_m - x_f$.

		Dependence measures		
	Samples	r	ρ	τ
$0 \leq \lvert d\rvert < 2$	59	0.98	0.97	0.93
$2 \leq \lvert d\rvert < 4$	53	0.93	0.93	0.78
$\lvert d\rvert \geq 4$	81	0.74	0.68	0.53

Table 4.3: Empirical dependence measures with respect to the age difference.

Our study follows the same lines of idea as these authors. In addition to the age difference, we believe that the gender of the elder partner may have an impact on their lifetimes dependencies. Indeed, the fact that the husband is older than the wife may influence their relationship, and indirectly, the dependence factors cited above. Despite the smallness of the sizes of the uncensored data does not allow us to conclude on the dependence structure of the 11'457 couples, it highlights well our hypothesis which will be verified with the whole dataset. In this regards, the results displayed in Table 4.2 clearly show that the spouse lifetime dependencies are higher when d is positive, i.e. when husband is older than wife. The variable *gender of the elder member* is measured through the sign of the age difference d. Table 4.4 displays the empirical Kendall's τ with respect to the age difference and to the gender of the elder partner. One can notice that the coefficients can vary for more than 30% depending on who is the older member of the couple.

$x_m \geq x_f$	Samples	τ	$x_m < x_f$	Samples	τ
$0 \leq d < 2$	27	0.89	$-2 \leq d < 0$	32	0.88
$2 \leq d < 4$	38	0.84	$-4 \leq d < -2$	15	0.87
$d \geq 4$	68	0.72	$d \geq -4$	13	0.61
Total	133	0.74	Total	60	0.71

Table 4.4: Kendall's Tau correlation coefficients by age and gender of the elder partner.

In what follows, a bivariate lifetime model will verify our hypothesis. To this end, marginal distributions for each of the male and female lifetimes are firstly defined and secondly the copula models are introduced. The estimation methods will be detailed in the Section 4.3 and Section 4.4.

4.3 Marginal Distributions

4.3.1 Background

The lifetime of a newborn shall be modelled by a positive continuous random variable, say X with df F and survival function S. The symbol (x) will be used to denote a live aged x and $T(x) = (X - x)|X > x$ is the remaining lifetime of (x). The actuarial symbols ${}_tp_x$ and ${}_tq_x$ are, respectively, the survival function and the df of $T(x)$. Indeed, the probability, for a live (x), to remain alive t more years is given by

$$_tp_x = \mathbb{P}\left(X > x+t \mid X > x\right) = \frac{\mathbb{P}\left(X > x+t\right)}{\mathbb{P}\left(X > x\right)} = \frac{S\left(x+t\right)}{S\left(x\right)}.$$

When X has a pdf f, then $T(x)$ has a pdf given by

$$f_x\left(t\right) = {}_tp_x\ \mu\left(x+t\right).$$

where $\mu(.)$ is the hasard rate function, also called *force of mortality.*
Several parametric mortality laws such as De Moivre, constant force of mortality, Gompertz, Inverse-Gompertz, Makeham, Gamma, Lognormal and Weibull are used in the literature; see Bowers et al. [12]. The choice of a specific mortality model is determined mainly by the caracteristics of the available data and the objective of the study. It is well known that the De Moivre law and the constant force of mortality assumptions are interesting for theoretical purposes whereas Gompertz and Weibull are more appropriate for fitting real data, especially for population of age over 30. The dataset exploited in this chapter regroups essentially policyholders who are at least middle-aged. That is why, in our study, the interest is on the Gompertz law whose caracteristics are defined as follows

$$\mu\left(x\right) = Bc^x \qquad \text{and} \qquad S\left(x\right) = \exp\left(-\frac{B}{\ln c}\left(c^x - 1\right)\right) \qquad \text{with} \qquad B > 0,\ \ c > 1,\ \ x \geq 0.$$

In addition, Frees et al. [39] and Carriere [16] have shown that the Gompertz mortality law fits our dataset very well, see Figures 4.1-4.2. For estimation purposes the Gompertz law has been reparametrized as follows (see [15])

$$e^{-m/\sigma} = \frac{B}{\ln c} \qquad \text{and} \qquad e^{1/\sigma} = c,$$

from which we obtain

$$\begin{aligned}
\mu\left(x+t\right) &= \frac{1}{\sigma}\exp\left(\frac{x+t-m}{\sigma}\right), \\
{}_tp_x &= \exp\left(e^{\frac{x-m}{\sigma}}\left(1-e^{\frac{t}{\sigma}}\right)\right), \\
f_x(t) &= \exp\left(e^{\frac{x-m}{\sigma}}\left(1-e^{\frac{t}{\sigma}}\right)\right)\frac{1}{\sigma}\exp\left(\frac{x+t-m}{\sigma}\right), \\
F_x(t) &= 1-\exp\left(e^{\frac{x-m}{\sigma}}\left(1-e^{\frac{t}{\sigma}}\right)\right), \qquad (4.1)
\end{aligned}$$

where the mode $m > 0$ and the dispersion parameter $\sigma > 0$ are the new parameters of the distribution.

4.3.2 Maximum Likelihood Procedure

In what follows, we will use the following notation:

- the index j indicates the gender of the individual, i.e. $j = m$ for male and $j = f$ for female.
- $\theta_j = (m_j, \sigma_j)$ denotes the vector of unknown Gompertz parameters for a given gender j,
- n is the total number of couples in our dataset. Hereafter, a couple means a group of two persons of opposite gender that have signed an insurance contract and i is the couple index with $1 \leq i \leq n$,
- for a couple i, t_j^i is the remaining lifetime observed in the collected data. Indeed, for an individual of gender j aged x_j, the remaining lifetime $T_j^i(x)$ is a random variable such that

  $$T_j^i(x_j) = \min\left(t_j^i, B_j^i\right) \qquad \text{and} \qquad \delta_j^i = \mathbf{1}_{\left\{t_j^i \geq B_j^i\right\}},$$

  where B_j^i is a random censoring point of the individual of gender j in the couple i.

Consider a couple i where the male and female were, respectively, aged x_m and x_f at contract initiation date. For each gender $j = m, f$, the contribution to the likelihood is given by

$$L_j^i(\theta_j) = \left[{}_{B_j^i}p_{x_j}(\theta_j)\right]^{\delta_j^i} \left[f_{x_j}^i\left(t_j^i, \theta_j\right)\right]^{1-\delta_j^i}. \tag{4.2}$$

We recall that the dataset is left truncated that is why likelihood function in (4.2) has therefore to be conditional on survival to the entry age x_j, see e.g., Carriere [16]. Therefore, the overall likelihood function can be written as follows

$$L_j(\theta_j) = \prod_{i=1}^{n} L_j^i(\theta_j), \qquad j = m, f. \tag{4.3}$$

By maximizing the likelihood function in (4.3) using our dataset, the MLE estimates of the Gompertz df are displayed in Table 4.5.

$\widehat{\theta}$	Estimate	Std. error
$\widehat{m}_m$	85.472	0.294
$\widehat{m}_f$	91.569	0.623
$\widehat{\sigma}_m$	10.448	0.439
$\widehat{\sigma}_f$	8.134	0.416

Table 4.5: Gompertz parameter estimates.

Standard errors are relatively low and estimation shows that the modal age at death is larger for females than for males. This latter can be explained by the fact that women have a longer life expectancy than men. A good way to analyse how well the model performs is to compare with the *Kaplan-Meier (KM) product-limit estimator* of the dataset. We recall that the KM technique is an approach which consists in estimating non-parametrically the survival function from the empirical data. Figures 4.1-4.2 compare the KM estimator of the survival function to the ones obtained from the Gompertz distribution estimated above. Since almost all the annuitants are older than 40 at entry, all the distributions are conditional on survival to age 40. The survival functions are plotted as a function of age x (for $x = 40$ to $x = 110$). The Gompertz curve is smooth whereas the KM is jagged. The figures clearly show that the estimated Gompertz model is a valid choice for approximating the KM curve.

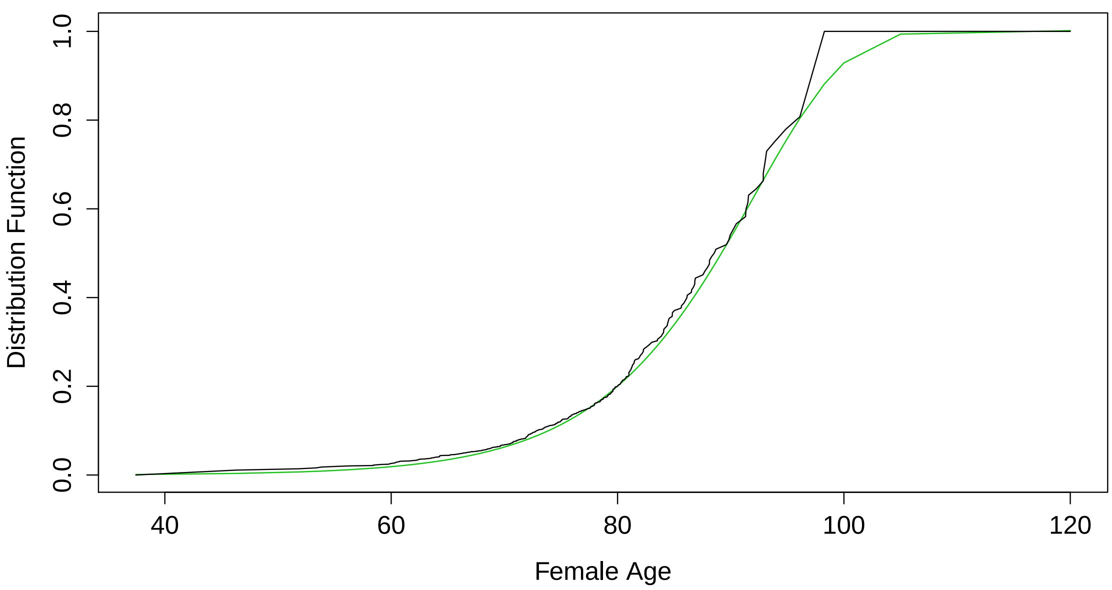


Figure 4.1: Gompertz and Kaplan-Meier fitted female distribution functions

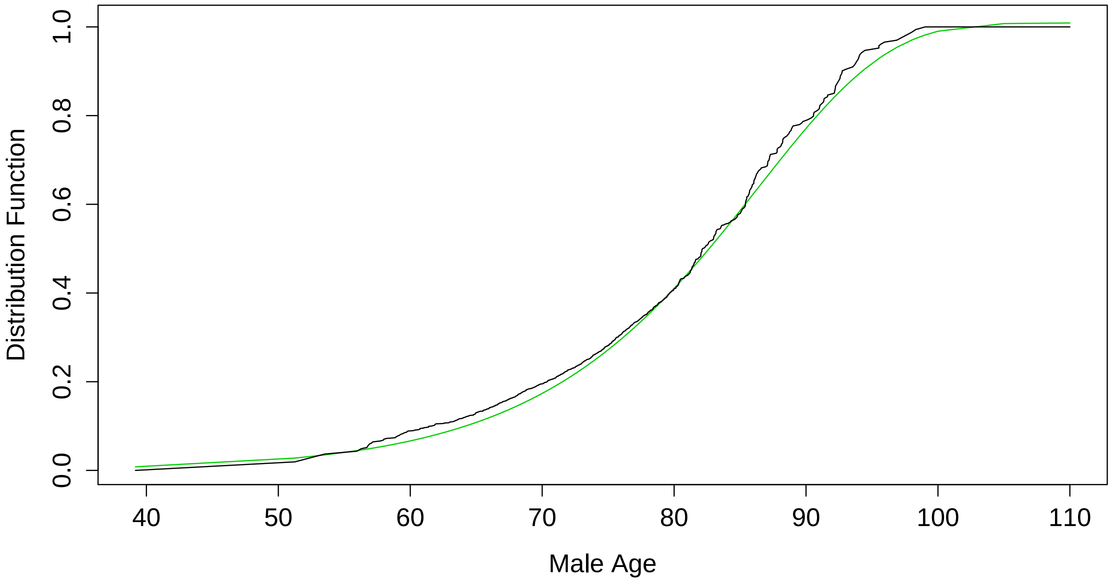


Figure 4.2: Gompertz and Kaplan-Meier fitted male distribution functions

4.4 Dependence Models

4.4.1 Background

Copula models were introduced by Sklar [86] in order to specify the joint df of a random vector by separating the behavior of the marginals and the dependence structure. Without loss of generality, we focus on the bivariate case. We denote by $T(x_m)$ and $T(x_f)$ the future lifetime respectively for man and woman. Following Carriere [16], we couple the lives at the time when they start being observed. Specifically, if $T(x_m)$ and $T(x_f)$ are positive and continuous, there exists a unique copula $C : [0,1]^2 \to [0,1]$ which specifies the joint df of the bivariate random vector $(T(x_m), T(x_f))$ as follows

$$\mathbb{P}(T(x_m) \leqslant t_1, T(x_f) \leqslant t_2) = C\left(\mathbb{P}\left(T(x_m) \leqslant t_1\right), \mathbb{P}\left(T(x_f) \leqslant t_2\right)\right) = C({}_{t_1}q_{x_m}, {}_{t_2}q_{x_f}).$$

Similarly, the survival function of $(T(x_m), T(x_f))$ is written in terms of copulas and marginal survival functions. This is given by

$$\begin{aligned}\mathbb{P}(T(x_m) > t_1, T(x_f) > t_2) &= \widetilde{C}({}_{t_1}p_{x_m}, {}_{t_2}p_{x_f}) \\ &= {}_{t_1}p_{x_m} + {}_{t_2}p_{x_f} - 1 + C({}_{t_1}q_{x_m}, {}_{t_2}q_{x_f}). \qquad (4.4)\end{aligned}$$

A broad range of parametric copulas has been developed in the literature. We refer to Nelsen [74] for a review of the existing copula families. The Archimedean copula family is very popular in life insurance applications, especially due to its flexibility in modelling dependent random lifetimes, see e.g., Fress et al. [39], Youn and Shemyakin [99] . If ϕ is a convex and twice-differentiable strictly increasing function, the df of an Archimedean copula is given by

$$C_\phi(u, v) = \phi^{-1}(\phi(u) + \phi(v)),$$

where $\phi : [0, 1] \to [0, \infty]$ is the generator of the copula satisfying $\phi(1) = 0$ with $u, v \in [0, 1]$. In this chapter, four well known copulas are discussed. Firstly, the Gumbel copula generated by

$$\phi(t) = (-\ln(t))^{-\alpha}, \quad \alpha > 1,$$

which yields the copula

$$C_\alpha(u, v) = \exp\left(-[(-\ln(u))^\alpha + (-\ln(v))^\alpha]^{1/\alpha}\right), \quad \alpha > 1. \qquad (4.5)$$

Secondly, we have the Frank copula

$$C_\alpha(u, v) = -\frac{1}{\alpha}\ln\left(1 + \frac{(e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{(e^{-\alpha} - 1)}\right), \quad \alpha \neq 0, \qquad (4.6)$$

with generator

$$\phi(t) = -\ln\left(\frac{e^{-\alpha t} - 1}{e^{-\alpha} - 1}\right), \quad \alpha \neq 0.$$

Thirdly, the Clayton copula is associated to the generator

$$\phi(t) = t^{-\alpha} - 1, \quad \alpha > 0,$$

and is given by

$$C_\alpha(u, v) = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}, \quad \alpha > 0. \qquad (4.7)$$

Finally, the Joe copula

$$C_\alpha(u,v) = 1 - \Big((1-u)^\alpha + (1-v)^\alpha - (1-u)^\alpha(1-v)^\alpha\Big)^{1/\alpha}, \quad \alpha > 1, \tag{4.8}$$

has generator $\phi(t) = -\ln(1-(1-t)^{-\alpha}), \alpha > 1$.

Clearly, the parameter α in (4.5)-(4.8) determines the dependence level between the two marginal distributions. In our case, that would be the lifetimes of wife and husband. Youn and Shemyakin [99] have utilized a Gumbel copula where the association parameter α depends on d as follows

$$\alpha(d) = 1 + \frac{\beta_0}{1+\beta_2 d^2}, \quad \beta_0, \beta_2 \in \mathbb{R}, \tag{4.9}$$

where $d = x_m - x_f$ with x_m and x_f the ages for male and female, respectively.

In our model for α, in addition to this specification, the gender of the elder partner, represented by the sign of d, is also taken into account. This latter is captured through the second term of the denominator $\beta_1 d$ in equations (4.10) and (4.11). Thus, for our model the copula association parameter for the Frank and the Clayton is expressed by

$$\alpha(d) = \frac{\beta_0}{1+\beta_1 d + \beta_2 |d|}, \quad \beta_0, \beta_1, \beta_2 \in \mathbb{R}. \tag{4.10}$$

Since the copula parameter α in the Gumbel and Joe copulas is restricted to be greater than 1, the corresponding dependence parameter in (4.11) is allowed to have an intercept of 1 and we write

$$\alpha(d) = 1 + \frac{\beta_0}{1+\beta_1 d + \beta_2 |d|}, \quad \beta_0, \beta_1, \beta_2 \in \mathbb{R}. \tag{4.11}$$

It can be seen that if $\beta_1 < 0$, the dependence parameter is lower when husband is younger than wife, i.e. $d < 0$. Also provided that d tends to infinity and in addition if $(\beta_1 < 0, \beta_2 > 0, d < 0)$ or $(\beta_1 > 0, \beta_2 > 0, d > 0)$ or $(\beta_1 < 0, \beta_2 < 0, d > 0)$ or $(\beta_1 < 0, \beta_2 < 0, d > 0)$, the dependence parameter goes to 0 for Frank and Clayton and 1 for the Gumbel copula, thus tending towards the independence assumption. Note in passing that instead of taking d^2 as in equation (4.9), we use $|d|$ in both (4.10) and (4.11) for the representation of the absolute age difference.

4.4.2 Estimation of Parameters

The maximum likelihood procedure has been widely used to fit lifetime data to copula models, see e.g., Lawless [61], Shih and Louis [85], Carriere [16]. A priori, this method consists in estimating jointly the marginal and copula parameters at once. However, given the huge number of parameters to be estimated at the same time, this approach is computationally intensive. Therefore, we adopt a procedure that allows the determination of marginal and copula parameters, separately. In this respect, Joe and Xu [55] have proposed a two step technique which, firstly estimates the marginal parameters $\theta_j, j = m, f$, and the copula parameter $\alpha(d)$ in the second step. This is referred to as the *inference functions for margins* (IFM) method. Specifically, the survival function of each lifetime is evaluated by maximazing the likelihood function in (4.3). For each couple i with x_m^i and x_f^i, let $u_i := {}_{t_m^i}p_{x_m^i}(\widehat{\theta}_m)$ and $v_i := {}_{t_f^i}p_{x_f^i}(\widehat{\theta}_f)$ be the resulting marginal survival functions for male and female, respectively. Considering the right-censoring feature of the two lifetimes as indicated by δ_m^i and δ_f^i, the estimates $\widehat{\alpha(d)}$ of the copula parameters are obtained by maximizing the likelihood function

$$\begin{aligned} L(\alpha(d)) := L(\alpha) \quad &= \prod_{i=1}^{n} \left[\frac{\partial^2 \widetilde{C}_\alpha(u_i, v_i)}{\partial u_i \partial v_i}\right]^{\left(1-\delta_m^i\right)\left(1-\delta_f^i\right)} \left[\frac{\partial \widetilde{C}_\alpha(u_i, v_i)}{\partial u_i)}\right]^{\left(1-\delta_m^i\right)\delta_f^i} \\ &\times \left[\frac{\partial \widetilde{C}_\alpha(u_i, v_i)}{\partial v_i}\right]^{\delta_m^i\left(1-\delta_f^i\right)} \left[\widetilde{C}_\alpha(u_i, v_i)\right]^{\delta_m^i \delta_f^i}. \end{aligned} \tag{4.12}$$

A similar two-step technique, known as the *Omnibus semi-parametric procedure* or the *pseudo-maximum likelihood*, was also introduced by Oakes [76]. In this procedure, the marginal distributions are considered as nuisance parameters of the copula model. The first step consists in estimating the two marginals survival functions non-parametrically using the KM method. After rescaling the resulting estimates by $\frac{n}{n+1}$, we obtain the pseudo-observations $(U_{i,n}, V_{i,n})$ where

$$U_{i,n} = \frac{\widehat{S}_m(x_m^i + t_m^i)}{\widehat{S}_m(x_m^i)} \qquad \text{and} \qquad V_{i,n} = \frac{\widehat{S}_m(x_f^i + t_f^i)}{\widehat{S}_m(x_f^i)}.$$

In the second step, the copula estimation is achieved by maximizing the following function

$$L(\alpha(d)) := L(\alpha) \quad = \prod_{i=1}^{n} \left[\frac{\partial^2 \widetilde{C}_\alpha(U_{i,n}, V_{i,n})}{\partial U_{i,n} \partial V_{i,n}}\right]^{\left(1-\delta_m^i\right)\left(1-\delta_f^i\right)} \left[\frac{\partial \widetilde{C}_\alpha(U_{i,n}, V_{i,n})}{\partial U_{i,n}}\right]^{\left(1-\delta_m^i\right)\delta_f^i}$$

$$\times \left[\frac{\partial \widetilde{C}_\alpha(U_{i,n}, V_{i,n})}{\partial V_{i,n}}\right]^{\delta_m^i\left(1-\delta_f^i\right)} \quad \left[\widetilde{C}_\alpha(U_{i,n}, V_{i,n})\right]^{\delta_m^i\delta_f^i}. \quad (4.13)$$

Genest et al. [40] and Shih and Louis [85] have shown that the stemmed estimators of the copula parameters are consistent and asymptotically normally distributed. Due to their computational advantages, the IFM and the Omnibus approaches are used in our estimations. By comparing the results stemming from the two techniques, we can analyze to which extent a certain copula is a reliable model for bivariate lifetimes within a couple. Table 4.6 and Table 4.7 display the copula estimations based on our dataset. The number in bracket under each estimate represents the standard error of the estimation. The estimated values from the IFM and the omnibus estimations are quite close for the Gumbel, the Frank and the Joe copulas. The important difference observed in the Clayton case indicates that this copula is probably not appropriate for modelling the bivariate lifetimes in our dataset. The negative sign of $\widehat{\beta}_1$ in all cases demonstrates that if husband is older than wife (i.e. $d > 0$), their lifetimes are more likely to be correlated. The positive sign of $\widehat{\beta}_2$ suggests that the higher the age difference is, the lesser is the level of dependence between lifetimes. The parameters $\widehat{\beta}_1$ and $\widehat{\beta}_2$ have opposing effects on $\widehat{\alpha}(d)$. In this regards, since $|\widehat{\beta}_2| > |\widehat{\beta}_1|$, the maximum level of dependence is attained when $d = 0$, i.e. when wife and husband have exactly the same age.

	$\alpha(d)$						α
	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\alpha}(-2)$	$\widehat{\alpha}(0)$	$\widehat{\alpha}(2)$	$\widehat{\alpha}$
Gumbel	1.040 (0.033)	-0.042 (0.010)	0.052 (0.013)	1.875	2.040	2.019	1.955 (0.019)
Frank	7.262 (0.175)	-0.014 (0.006)	0.018 (0.008)	6.826	7.262	7.201	7.026 (0.108)
Clayton	2.249 (0.236)	-0.277 (0.110)	-0.408 (0.123)	0.949	2.249	1.784	1.206 (0.077)
Joe	1.475 (0.046)	-0.054 (0.011)	0.059 (0.014)	2.203	2.475	2.461	2.362 (0.028)

Table 4.6: IFM method: copula parameters estimate $\alpha(d)$ and α.

	$\alpha(d)$						α
	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\widehat{\alpha}(-2)$	$\widehat{\alpha}(0)$	$\widehat{\alpha}(2)$	$\widehat{\alpha}$
Gumbel	1.006	-0.043	0.048	1.851	2.006	1.995	1.936
	(0.032)	(0.009)	(0.012)				(0.019)
Frank	7.004	-0.014	0.016	6.609	7.004	6.978	6.831
	(0.168)	(0.006)	(0.007)				(0.105)
Clayton	1.903	-0.201	0.361	0.936	1.903	1.545	1.120
	(0.199)	(0.091)	(0.104)				(0.069)
Joe	1.440	-0.055	0.056	2.180	2.440	2.438	2.347
	(0.045)	(0.011)	(0.013)				(0.028)

Table 4.7: Omnibus approach: copula parameters estimate $\alpha(d)$ and α.

Our estimate of $\alpha(d)$ under the Gumbel copula is quite similar to the results in the model of Youn and Shemyakin [99] where $\widehat{\beta}_0 = 1.018, \widehat{\beta}_1 = 0$ and $\widehat{\beta}_2 = 0.021$. Column 8 contains the estimation output when the dependence parameter α does not depend on d. When $d = 0$, $\alpha(0) = \beta_0$ (or $1 + \beta_0$ for Gumbel and Joe) and that is equivalent to the case where the dependence parameter is not in function of the age difference.

4.4.3 Goodness of Fit

A goodness of fit procedure is performed in order to assess the robustness of our model. For this purpose, the model, including age difference and gender of the elder member within the couple with $\alpha(d)$, is compared to two other types, namely the one where the copula parameter does not depend on d and the model of Youn and Shemyakin [99]. Many approaches for testing the goodness of fit of copula models are proposed in the litterature, see e.g., Genest et al. [43], Berg [11]. We refer to Genest et al. [43] for an overview of the existing methods. There are several contributions highlighting the properties of the empirical copula, especially when the data are right censored, the contributions of Dabrowska [22], Prentice [79], Gribkova and Lopez [44] are some examples. In our framework, the goodness of fit approach is based on the non parametric copula introduced by Gribkova and Lopez [44] as follows

$$C_n(u_1, u_2) = \frac{1}{n}\sum_{i=1}^{n}(1-\delta_m^i)(1-\delta_f^i)W_{in}\mathbb{1}_{\{T(x_m^i)\leqslant \widehat{F}_{m,n}^{-1}(u_1), T(x_f^i)\leqslant \widehat{F}_{f,n}^{-1}(u_2)\}}, \tag{4.14}$$

where $W_{in} = \frac{1}{S_{B_m}(\max(T_m^i, T_f^i - \epsilon_i)-)}$ and S_{B_m} is the survival function of the right censored random variable B_m that is estimated using KM approach; $\epsilon_i = B_f^i - B_m^i$.

The term $\widehat{F}_{j,n}^{-1}$ is the KM estimator of the quantile function of $T(x_j^i), j = m, f$. The particularity of equation (4.14) is that, the uncensored observations are twice weighted (with $1/n$ and W_{in}) unlike the original empirical copula where the same weight $1/n$ is assigned to each observation. The weight W_{in} is devoted to compensate right censoring. Based on the p-value, the goodness of fit test indicates to which extent a certain parametric copula is close to the empirical copula C_n. We adopt the Cramèr-von Mises statistics to assess the adequacy of the hypothetical copula to the empirical one, namely

$$\mathcal{V}_n = \int_{[0,1]^2} K_n(v) dK_n(v), \tag{4.15}$$

where $K_n(v) = \sqrt{n}(C_n(v) - C_{\widehat{\alpha}(d)}(v))$ is the empirical copula process. Genest et al. [43] have proposed an empirical version of equation (4.15) which is given by

$$\widehat{\mathcal{V}}_n = \sum_{i=1}^{n} (C_n(u_{1i}, u_{2i}) - C_{\widehat{\alpha}(d)}(u_{1i}, u_{2i}))^2. \tag{4.16}$$

The assertion, the bivariate lifetime within the couple is described by the studied copula, is then tested under the null hypothesis H_0. Since the Cramèr-von Mises statistics $\widehat{\mathcal{V}}_n$ does not possess an explicit df, we implement a bootstrap procedure to evaluate the p-value as presented in the following pseudo-algorithm. For some large integer ξ, the following steps are repeated for every $k = 1, \ldots, \xi$:

- **Step 1** Generate lifetimes from the hypothetical copula, i.e. $(U_i^b, V_i^b), i = 1, \ldots, n$ is generated from $C_{\widehat{\alpha}(d)}$. If the IFM method is used to determine $\widehat{\alpha}(d)$, then the two lifetimes are produced from the Gompertz distribution

$$(t_m^{b,i} = F_{x_m}^{-1}(U_i^b, \widehat{\theta}_m), t_f^{b,i} = F_{x_f}^{-1}(V_i^b, \widehat{\theta}_f)),$$

  where $\widehat{\theta}_j,\ j = m, f$ are taken from Table 4.5, while, for the omnibus, the corresponding lifetimes are generated with the KM estimators of the quantile functions of $T\left(x_j\right),\ j = m, f$

$$(t_m^{b,i} = \widehat{F}_{m,n}^{-1}(U_i^b),\ t_f^{b,i} = \widehat{F}_{f,n}^{-1}(V_i^b)).$$

- **Step 2** Generate the censored variables $B_m^{b,i}$ and $B_f^{b,i}, i = 1, \ldots, n$ from the empirical distribution of B_m and B_f respectively.

- **Step 3** Considering the same data as used for the estimation, replicate the insurance portfolio by calculating

$$T^b(x^i_m) = \min(t^{b,i}_m, B^{b,i}_m), \quad \delta^{b,i}_m = \mathbb{1}_{\{t^{b,i}_m \geqslant B^{b,i}_m\}},$$

$$T^b(x^i_f) = \min(t^{b,i}_f, B^{b,i}_f), \quad \delta^{b,i}_f = \mathbb{1}_{\{t^{b,i}_f \geqslant B^{b,i}_f\}},$$

for each couple i of ages x^i_m and x^i_f.

- **Step 4** If the IFM approach is chosen in **Step 1**, the parameters of the marginals and the hypothetical copula parameters are estimated from the bootstrapped data $(T^b(x^i_m), T^b(x^i_f), \delta^{b,i}_m, \delta^{b,i}_f)$ by maximizing (4.2) and (4.12) whereas under the omnibus approach, the hypothetical copula parameters are estimated from the bootstrapped data as well by maximizing equation (4.13).
- **Step 5** Compute the Cramèr-von Mises statistics $\widehat{\mathbb{V}}^b_{n,k}$ using (4.16).
- **Step 6** Evaluate the estimate of the p-value as follows

$$\widehat{p} = \frac{1}{K+1} \sum_{k=1}^{\xi} \mathbb{1}_{\{\widehat{\mathbb{V}}^b_{n,k} \geqslant \widehat{\mathbb{V}}_n\}}.$$

Based on 1000 bootstrap samples, the results of the goodness of fit is summarized in Table 4.8. It can be seen that for both IFM and Omnibus, our model have a greater p-value than the model without age difference, showing that age difference between spouses is an important dependence factor of their joint lifetime. Under the Gumbel model in Youn and Shemyakin [99] where $\beta_1 = 0$, the p-value is evaluated at 0.672. For the Gumbel copula in Table 4.8, the p-value in the model with $\alpha\,(d)$ is slightly higher, strengthening the evidence that the sign of d captures some additional association between spouses.

	IFM		Omnibus	
Copula parameters	α	$\alpha(d)$	α	$\alpha(d)$
Gumbel	0.643	0.675	0.640	0.673
Frank	0.510	0.530	0.520	0.531
Clayton	0.114	0.150	0.115	0.167
Joe	0.319	0.339	0.313	0.327

Table 4.8: Goodness of fit test: p-value of each copula model.

At a critical level of 5%, the three copula families are accepted, even though the Clayton copula performs inadequately. Actually, as pointed out in Gribkova and

Lopez [44], the important percentage of censored data in the sample results in a huge loss of any GoF test. Therefore, these results can not efficiently assess the lifetime dependence within a couple. Nevertheless, the calculated p-values may give an idea about which direction to go. In this regards, since the Gumbel and Frank copulas have the highest p-value, they are good candidates for addressing the dependence of the future lifetimes of husband and wife in this Canadian life insurer portfolio.

Furthermore, since the copula parameter without age difference is nested by the one with age difference, likelihood ratio test can be used to verify if the two parameters β_1 and β_2 in (4.10) and (4.11) are significant. Specifically, introduced by Neyman and Pearson [75] the likelihood ratio test compares two nested hypothesis: the null hypothesis H_0 with constrained parameters and the alternative hypothesis H_1 with unconstrained parameters. Clearly, the model with constraints $\beta_1 = \beta_2 = 0$ in (4.10) and (4.11) corresponds to H_0 and the one with age difference corresponds to H_1. Let L_0, L_1 be the maximum likelihood function values based on H_0 and H_1, respectively, the test statistic is given by

$$\lambda = 2(\ln(L_1) - \ln(L_0)).$$

The null hypothesis is rejected at a significance level α if $\lambda > \chi^2_{r,1-\alpha}$, with r the number of restricted parameters, $r = 2$ in our case, and $\chi^2_{r,1-\alpha}$ is the $1 - \alpha$ quantile of a Chi-squared distribution with r degrees of freedom. At a significance level of 5%, all the test statistics of the copula families presented in Table 4.9 are greater than $\chi^2_{2,0.95} = 5.991$, which means that the null hypothesis is rejected. Thus, the models allowing age difference in the copula parameter give a better fit than the models without age difference. This justifies the significance of β_1 and β_2 in (4.10) and (4.11).

λ	IFM	Omnibus
Gumbel	30.20	30.98
Frank	8.66	8.26
Clayton	56.72	43.94
Joe	41.68	43.62

Table 4.9: Likelihood ratio statistic of each copula model.

4.5 Insurance Applications

4.5.1 Joint Life Insurance Contracts

Multiple life actuarial calculations is common in the insurance practice. Hereafter, (x) stands for the husband aged x whereas (y) is the wife. Considering a couple (xy), $T(xy)$ describes the remaining time until the first death between (x) and (y) and, it is known as the *joint-life status*. Conversely, $T(\overline{xy})$ is the time until death of the *last survivor*. The variables $T(\overline{xy})$ and $T(xy)$ are random and we can write

$$T(xy) = \min(T(x), T(y)) \text{ whereas } T(\overline{xy}) = \max(T(x), T(y)).$$

As in the single life model, the survival probabilities are given by

$$_tp_{xy} = \mathbb{P}(T(xy) > t) \qquad \text{and} \qquad _tp_{\overline{xy}} = \mathbb{P}(T(\overline{xy}) > t). \tag{4.17}$$

Clearly, if $T(x)$ and $T(y)$ are independent, then

$$_tp_{xy} = {_tp_x}\, {_tp_y} \qquad \text{and} \qquad _tp_{\overline{xy}} = 1 - {_tq_x}\, {_tq_y}.$$

The curtate life expectancies, for $T(xy)$ and $T(\overline{xy})$ respectively, are given by

$$e_{xy} = \mathbb{E}(T(xy)) = \sum_{t=1}^{\infty} {_tp_{xy}} \qquad \text{and} \qquad e_{\overline{xy}} = \mathbb{E}(T(\overline{xy})) = \sum_{t=1}^{\infty} {_tp_{\overline{xy}}},$$

with the following relationship

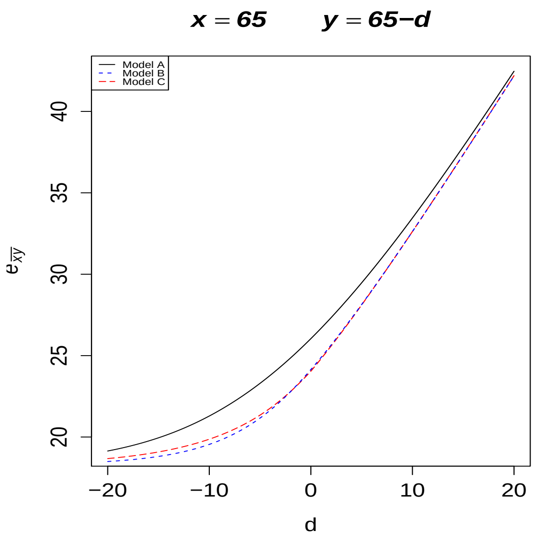
$$e_{\overline{xy}} = e_x + e_y - e_{xy}.$$

Figures 4.3 and 4.4 compare the evolution of $e_{\overline{xy}}$ as a function of the age difference $d = x - y$, under the following models:

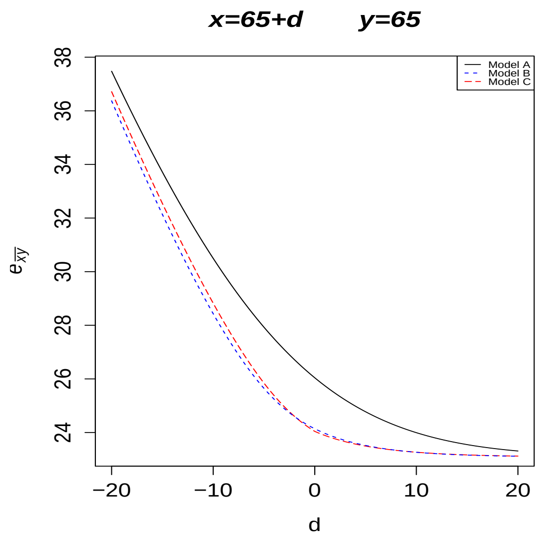
- Model A: $T(x)$ and $T(y)$ are independent;
- Model B: $T(x)$ and $T(y)$ are dependent with a constant copula parameter $\alpha = \alpha_0$;
- Model C: $T(x)$ and $T(y)$ are dependent with a copula parameter $\alpha(d)$ as described in (4.10) and (4.11).

On the left (resp. right), the graphs were constructed under the assumption of $x = 65$ (resp. $y = 65$) for the husband (resp. wife) and the age difference d ranges from -20 to 20 as more than 99% of our portfolio belongs to this interval. The fixed age is set to 65 because this is the retirement age in many countries. The analysis was made under the four families of copula described in Section 4.4. In general, it can be seen that the life expectancy of the last survivor $e_{\overline{xy}}$ increases when $e_{\overline{xy}} = e_{\overline{65:65-d}}$ whereas it decreases when $e_{\overline{xy}} = e_{\overline{65+d:65}}$. This result strengthens the evidence that the sign of d has an effect on annuity values. For example, when $|d| = 10$ under the Gumbel copula,

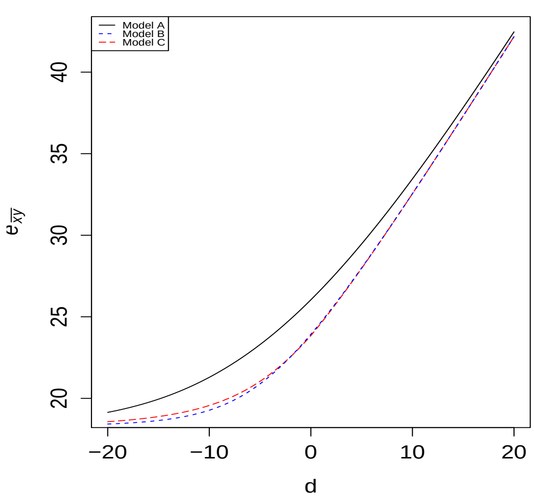
$$e_{\overline{65:55}} = 32.62 \geq e_{\overline{55:65}} = 28.82.$$



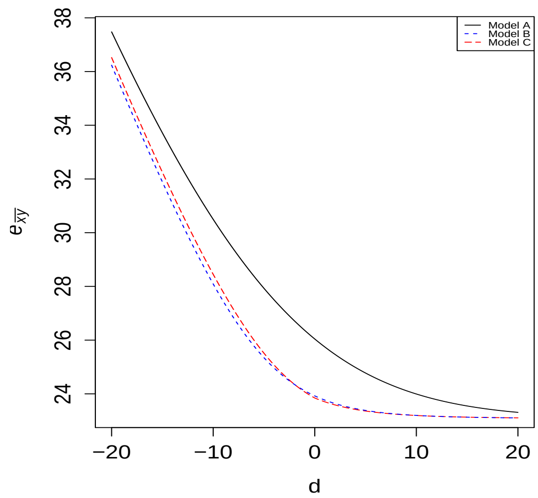
(a) Gumbel copula: $x = 65$



(b) Gumbel copula: $y = 65$

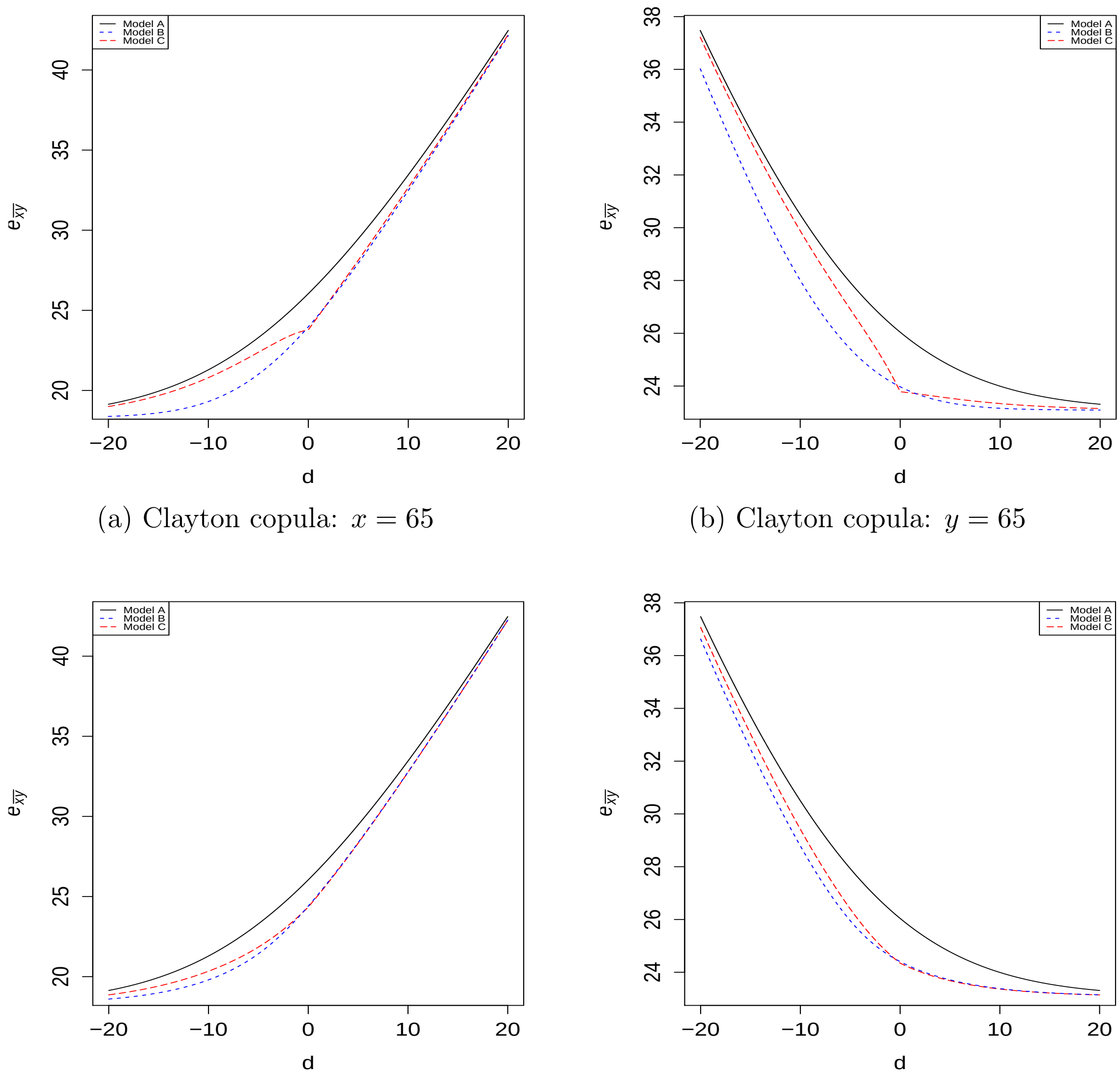


(c) Frank copula: $x = 65$



(d) Frank copula: $y = 65$

Figure 4.3: Comparison of $e_{\overline{xy}}$ under model A, B and C: Gumbel and Frank copulas



(a) Clayton copula: $x = 65$

(b) Clayton copula: $y = 65$

(c) Joe copula: $x = 65$

(d) Joe copula: $y = 65$

Figure 4.4: Comparison of $e_{\overline{xy}}$ under model A, B and C: Clayton and Joe copulas

When comparing the models A, B and C, it can be seen that the life expectancy $e_{\overline{xy}}$ is clearly overvalued under the model A of independence assumption, thus confirming the results obtained in Frees et al. [39], Youn and Shemyakin [99], Denuit and Cornet [23]. Now, let us focus our attention on models B and C considering only Gumbel, Frank and Joe copulas as it has been shown in the previous section that the Clayton copula might not be appropriate for the Canadian insurer's data. In all graphs, the life expectancy is always lower or equal under model B and the rate of decreases may exceed 2%. The largest decrease is observed when $d < 0$, i.e. when husband is younger than wife.

In order to illustrate the importance of these differences, we consider four types of multiple life insurance products. Firstly, Product 1 is the *joint life annuity* which pays benefits until the death of the first of the two annuitants. For a husband (x) and his wife (y) who receive continuously a rate of 1, the present value of future obligations and its expectation are given by

$$\overline{a}_{\overline{T(xy)}|} = \frac{1 - \exp\left(-\delta T\left(xy\right)\right)}{\delta} \qquad \text{and} \qquad \overline{a}_{xy} = \mathbb{E}\left(\overline{a}_{\overline{T(xy)}|}\right),$$

where δ is the constant instantaneous interest rate (also called force of interest). The variable $\overline{a}_{\overline{T(xy)}|}$ can be seen as the insurer liability regarding (xy). Product 2 is the last survivor annuity which pays a certain amount until the time of the second death $T\left(\overline{xy}\right)$. In that case, the present value of future annuities and its expectation are given by

$$\overline{a}_{\overline{T(\overline{xy})}|} = \frac{1 - \exp\left(-\delta T\left(\overline{xy}\right)\right)}{\delta} \qquad \text{and} \qquad \overline{a}_{\overline{xy}} = \mathbb{E}\left(\overline{a}_{\overline{T(\overline{xy})}|}\right).$$

In practice, payments often start at a higher level when both beneficiaries are alive. It drops at a lower level on the death of either and continues until the death of the survivor. This case is emphasized by product 3 where the rate is 1 when both annuitant are alive and reduces to $\frac{2}{3}$ after the first death. Product 3 is actually a combination of the two first annuities. Thus, the insurer liabilities and its expectation are given by

$$V\left(\overline{xy}\right) = \frac{1}{3}\overline{a}_{\overline{T(xy)}|} + \frac{2}{3}\overline{a}_{\overline{T(\overline{xy})}|} \qquad \text{and} \qquad \mathbb{E}\left(V\left(\overline{xy}\right)\right) = V_{\overline{xy}} = \frac{1}{3}\overline{a}_{xy} + \frac{2}{3}\overline{a}_{\overline{xy}},$$

where $\mathbb{E}\left(\overline{a}_{\overline{T(\overline{xy})}|}\right) = \overline{a}_{\overline{xy}}$.

Fourthly, imagine a family or couple whose income is mainly funded by the husband. The family may want to guarantee its source of income for the eventual death of the husband. For this purpose, the couple may buy the so called *reversionary annuity* for which the payments start right after the death of (x) until the death of (y). No payment is made if (y) dies before (x). As for Product 3, the reversionary annuity (Product 4) is also a combination of some specific annuity policies and the total obligations of the insurer and its expectation are computed as follows

$$\overline{a}_{\overline{T(x)|T(y)}|} = \overline{a}_{\overline{T(y)}|} - \overline{a}_{\overline{T(xy)}|} \qquad \text{and} \qquad \overline{a}_{x|y} = \mathbb{E}\left(\overline{a}_{\overline{T(x)|T(y)}|}\right) = \overline{a}_{y} - \overline{a}_{xy}. \tag{4.18}$$

In what follows, considering each of the insurance products 1, 2, 3 and 4, comparison of models A, B and C will be discussed. The analysis will include the valuation of the best estimate (BE) of the aggregate liability of the insurer as well as the quantification of risk capital and stop-loss premiums.

4.5.2 Risk Capital & Stop-loss Premium

In the enterprise risk management framework, insurers are required to hold a certain capital. This amount, known as the *risk capital*, is used as a buffer against unexpected large losses. The value of this capital is quantified in a way that the insurer is able to cover its liabilities with a high probability. For instance, under Solvency II, it is the VaR at a tolerance level of 99.5% of the insurer total liability, while for the Swiss Solvency Test (SST), it is the *Expected Shortfall* (ES) at 99%. Let L be the aggregate liability of the insurer. At a confidence level α, the VaR is given by

$$VaR_L(\alpha) = \inf\left\{l \in \mathbb{R} : \mathbb{P}\left(L \leq l\right) \geq \alpha\right\},$$

whilst the ES is

$$ES_L(\alpha) = \mathbb{E}\left(L | L > VaR_L(\alpha)\right).$$

These risk measures will serve to compare models A, B and C for each type of product. As the insurance portfolio is made of n policyholders, we define

$$L = \sum_{i=1}^{n} L_i,$$

where L_i represents the total amount due to a couple i of (x_i) and (y_i). The dataset used in the calculations is the same as those used for the model estimations and described in Section 4.2. In principle, the couple i receives the amount b_i at the beginning of each year until the death of the last survivor. However, in our applications, b_i will be the continuous benefit rate in CAD for each type of product. For example, in the particular case of Product 3,

$$L_i = b_i V\left(\overline{x_i y_i}\right) = b_i \left(\frac{1}{3}\overline{a}_{\overline{T(x_i,y_i)}|} + \frac{2}{3}\overline{a}_{\overline{T(\overline{x_i,y_i})}|}\right).$$

Since there is no explicit form for the distribution of L, a simulation approach will serve to evaluate the insurer aggregate liability. The pseudo-algorithm used for simulations is presented in the following steps:

- **Step 1** For each couple i, generate (U_i, V_i) from the the copula model (model A or model B or model C).

- **Step 2** For each couple i with x_i and y_i, generate the future lifetime $T(x_i), T(y_i)$

from the Gompertz distribution as follows

$$T(x_i) = F_{x_i}^{-1}(U_i, \widehat{\theta}_m) \qquad \text{and} \qquad T(y_i) = F_{y_i}^{-1}(V_i, \widehat{\theta}_f), \tag{4.19}$$

where $\widehat{\theta}_j,\ j = m, f$ are taken from Table 4.5.

- **Step 3** Evaluate the liability L_i for each couple $i = 1, \ldots, n$.
- **Step 4** Evaluate the aggregate liability of the insurer $L = \sum_{i=1}^{n} L_i$.

Due to its goodness of fit performance, the Gumbel copula will be used in the calculations for Model B and C. Mortality risk is assumed to be the only source of uncertainty and we consider a constant force of interest of $\delta = 1\%$. For each product described in Subsection 4.5.1, Step 1-4 are repeated 1000 times in order to generate the distribution of L. In addition to the risk capital measured as under the Solvency II and the SST framework, the BE of the aggregate liability of the insurer (i.e. $BE = \mathbb{E}\,(L)$), the Coefficient of Variation (CoV) and the stop-Loss premium $SL = \mathbb{E}((L - \zeta)_+)$ are also evaluated, where ζ is the deductible. For the portfolio of Product 1, Product 2, Product 3 and Product 4, the amount of ζ in millions CAD are respectively 4, 4.5, 4.2, 1.7. Results are presented in Table 4.10 − 4.13 according to each product. For the ease of understanding all values have been converted to a per Model A basis (the corresponding amounts are presented in the Appendix). As we could expect, the Model A with independent lifetime assumption misjudges the total liability of the insurer. The highest differences are observable with Product 4 where it reaches 17% for the BE, 25% for the risk capitals and 49% for the stop-loss premiums. By comparing Model B and Model C, the findings tell minor differences. The variation noticed in Figure 4.4 (when $d < 0$) are practically non-existent in the aggregate values for most of the products under investigation. In other words, while the effects of the age difference and its sign are noticeable on the individual liability (see Subsection 4.5.1), the effects on the aggregate liability are merely small. This is due to the law of large number and to the high proportion of couple with $d > 0$ in our portfolio (70%). Actually, the compensation of the positive and negative effects of the age difference on the lifetimes dependency in the whole portfolio mitigates its effects on the aggregate liability. However, it should be noted that the relative difference exceeds 1% for the stop-loss premium in Table 4.13.

Product 1	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.7499	1.0000	1.0000	1.0000
Model B	1.0991	0.7271	1.3032	1.0552	1.0529
Model C	1.0988	0.7272	1.3024	1.0553	1.0529

Table 4.10: Relative BE and risk capital for the joint life annuity portfolio.

Product 2	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.5621	1.0000	1.0000	1.0000
Model B	0.9418	0.6111	0.9152	0.9969	0.9971
Model C	0.9419	0.6109	0.9149	0.9970	0.9972

Table 4.11: Relative BE and risk capital for the last survivor annuity portfolio.

Product 3	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	0.5728	1.0000	1.0000	1.0000
Model B	0.9775	0.6255	1.0415	1.0330	1.0314
Model C	0.9774	0.6255	1.0414	1.0331	1.0316

Table 4.12: Relative BE and risk capital for the last survivor annuity portfolio.

Product 4	BE	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1.0000	1.1877	1.0000	1.0000	1.0000
Model B	0.8248	1.0260	0.5250	0.7365	0.7468
Model C	0.8251	1.0200	0.5199	0.7345	0.7456

Table 4.13: Relative BE and risk capital for the contingent annuity portfolio.

4.6 Conclusion

In this chapter, we propose both parametric and semi-parametric techniques to model bivariate lifetimes commonly seen in the joint life insurance practice. The dependence factors between lifetimes are examined namely the age difference between spouses and the gender of the elder partner in the couple. Using real insurance data, we develop an appropriate estimator of the joint distribution of the lifetimes of spouses with copula models in which the association parameters have been allowed to incorporate the aforementioned dependence factors. A goodness of fit procedure clearly shows that the introduced models outperform the models without age factors. The results of our illustrations, focusing on valuation of joint life insurance products, suggest that lifetimes dependence factors should be taken into account when evaluating the best estimate of the annuity products involving spouses.

Appendix: Measures for the Aggregate Liability of the Insurer

Product 1	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	2'506'318	0.7499	306'254	8'867'772	9'137'217
Model B	2'754'587	0.7271	399'114	9'357'382	9'620'839
Model C	2'753'894	0.7272	398'871	9'358'352	9'620'876

Table 4.14: Risk capital for the joint life annuity portfolio in CAD.

Product 2	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	4'275'139	0.5621	877'391	11'456'016	11'757'270
Model B	4'026'500	0.6111	803'022	11'421'070	11'723'580
Model C	4'026'615	0.6109	802'752	11'422'159	11'724'355

Table 4.15: Risk capital for the last survivor annuity portfolio in CAD.

Product 3	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	3'685'532	0.5728	649'259	10'146'042	10'420'385
Model B	3'602'529	0.6255	676'185	10'481'095	10'748'088
Model C	3'602'375	0.6255	676'123	10'481'720	10'749'342

Table 4.16: Risk capital for the last survivor annuity (Product 3) portfolio in CAD.

Product 4	Mean	CoV	SL	$VaR_L(99.5\%)$	$ES_L(99\%)$
Model A	1'415'591	1.1877	545'202	8'005'644	8'292'725
Model B	1'167'629	1.0260	286'231	5'896'485	6'193'339
Model C	1'167'949	1.0200	283'466	5'880'191	6'182'959

Table 4.17: Risk capital for the life contingent annuity portfolio in CAD.

Chapter 5

New Dependence Models derived from Multivariate Collective Models in Insurance Applications

This Chapter is based on E. Hashorva, G. Ratovomirija and M. Tamraz: On some new Dependence Models derived from Multivariate Collective Models in Insurance Applications, published in the *Scandinavian Actuarial Journal*, *In press.*

5.1 Introduction

Modelling the dependence structure between insurance risks is one of the main tasks of actuaries. For instance, the determination of a risk capital in the risk management framework needed to cover unexpected losses of an insurance portfolio and the allocation of the latter to each line of business is of importance when choosing the best model of dependence for multivariate insurance risks. As discussed in Nelsen [73], copulas are a popular multivariate distribution when modelling the dependency between insurance risks as they separate the marginals from the dependence structure, see Embrechts [32], Genest et al. [41] and references therein. With motivation from Zhang and Lin [100], in this contribution we propose a flexible family of copulas derived from the joint distribution of the largest claim sizes of two insurance portfolios.

Next, in order to introduce our model, we consider the classical collective model over a fixed time period of two insurance portfolios with (X_i, Y_i) modelling the ith claim sizes of both portfolios and N the total number of such claims. If $N = 0$, then there are no claims, so the largest claims in both portfolios are equal to 0. When $N \geq 1$, then $(X_{N:N}, Y_{N:N})$ denotes the maximal claim amounts in both portfolios. Commonly, claim sizes are assumed to be positive, however here we shall simply assume that $(X_i, Y_i), i \geq 1$ are independent with common distribution function (df) G and N is independent of everything else. Such a model is common for proportional reinsurance. In that case $Y_i = cX_i$ with c being a positive constant. Another instance is if X_i's model claim sizes and Y_i's model the expenses related to the settlement of X_i's, see Denuit et al. [26] for statistical treatments and further applications. The df of $(X_{N:N}, Y_{N:N})$ denoted by F^* is given by

$$F^*(x, y) = L_N(-\ln G(x, y)), \quad x, y \geq 0, \tag{5.1}$$

with L_N the Laplace transform of N. Clearly, F^* is a mixture df given by

$$F^*(x, y) = \mathbb{P}\{N = 0\} + \mathbb{P}\{N \geq 1\} F(x, y), \quad x, y \geq 0,$$

where

$$F(x, y) = L_\Lambda(-\ln G(x, y)), \quad x, y \geq 0, \tag{5.2}$$

with $\Lambda = N|N \geq 1$ and L_Λ its Laplace transform.
Since both distributional and asymptotic properties of F^* can be easily derived from those of F, in this chapter we shall focus on F assuming throughout that $\Lambda \geq 1$ is an integer-valued random variable.
When the df G is a product distribution, F above corresponds to the frailty model, see e.g., Denuit et al. [26], whereas the special case that Λ is a shifted geometric random variable is dealt with in Zhang and Lin [100]. We mention three tractable cases for Λ:
Model A: In Zhang and Lin [100], Λ is assumed to have a shifted Geometric distribution with parameter $\theta \in (0, 1)$ which leads to

$$F(x, y) = \frac{\theta G(x, y)}{1 - (1 - \theta)G(x, y)}, \quad x, y \geq 0. \tag{5.3}$$

Model B: Λ has a shifted Poisson distribution with parameter $\theta > 0$, i.e., $\Lambda = 1+K$ with K being a Poisson random variable with mean $\theta > 0$, which implies

$$F(x, y) = G(x, y)e^{-\theta[1-G(x,y)]}, \quad x, y \geq 0. \tag{5.4}$$

Model C: Λ has a truncated Poisson distribution with

$$\mathbb{P}\{\Lambda = k\} = e^{-\theta}\theta^k/(k!(1-e^{-\theta})), k \geq 1,$$

and thus

$$F(x,y) = \frac{e^{-\theta}}{1-e^{-\theta}}[e^{\theta G(x,y)} - 1], \quad x,y \geq 0. \tag{5.5}$$

Since the distributions F and their copulas are indexed by an unknown parameter θ, the new mixture copula family has several interesting properties. In particular, it allows to model highly dependent insurance risks and therefore our model is suitable for numerous insurance applications including risk aggregation, capital allocation and reinsurance premium calculations.

In this contribution we investigate first the basic distributional and extremal properties of F for general Λ. As it will be shown in Section 5.3, interestingly the extremal properties of F are similar to those of G.

With some motivation from Zhang and Lin [100], which investigates **Model A** and its applications, in this chapter, we shall discuss parameter estimation and Monte Carlo simulations for parametric families of bivariate df's induced by F. In particular, we apply our results to actuarial modelling of concrete datasets from actuarial literature. Moreover we shall consider the implications of our findings for a new real dataset from a Swiss insurance company. In several cases **Model B** and **Model C** give both satisfactory fit to the data. For the case of Loss and ALAE dataset we model further the stop loss and the excess of loss reinsurance premium. One of the applications of the joint distribution of the largest claims $(X_{N:N}, Y_{N:N})$ of two insurance portfolios is the analysis of the impact of their sum on the risk profile of the portfolios. Over the last decades, many contributions have been devoted on the study of the influence of the largest claims on aggregate claims, see e.g., Peng [78], Asimit and Chen [4] for an overview of existing contributions on the topic. This analysis is important when designing risk management and reinsurance strategies especially in non proportional reinsurance. Ammeter [3] is one of the first contribution which addressed the impact of the largest claim $X_{N:N}$ on the moments of the total loss of an insurance portfolio $\sum_{i=1}^{N} X_i$, see also Asimit and Chen [4] for recent results. In this chapter we demonstrate by simulation the influence of the sum of the largest claims observed in two insurance portfolios $X_{N:N} + Y_{N:N}$ on the distribution of $S_N = \sum_{i=1}^{N}(X_i + Y_i)$. Moreover, using the covariance capital allocation principle we quantify the impact of $X_{N:N}$ and $Y_{N:N}$ on the total loss S_N. The chapter is organised as follows. We discuss next some basic distributional properties of F. An investigation of the coefficient of upper tail dependence and the max-domain of attractions of F is presented in Section 5.3. Section 5.4 is dedicated to parameter estimation and Monte Carlo simulation with special focus on the cases covered by **Model A-C** above. We present three applications to concrete insurance dataset in Section 5.5. All the proofs are relegated to Appendix.

5.2 Basic Properties of F

Let G denote the df of (X_1, Y_1) and write G_1, G_2 for its marginal df's. Suppose that G_i's are continuous and thus the copula Q of G is unique. For $\Lambda = N|N \geq 1$, we have that the marginal df's of F are

$$F_i(x) = L_\Lambda(-\ln G_i(x_i)), \quad i = 1, 2, x \in \mathbb{R}.$$

Hence, the generalised inverse of F_i is

$$F_i^{-1}(q) = G_i^{-1}(e^{-L_\Lambda^{-1}(q)}), \quad q \in (0, 1),$$

where G_i^{-1} is the generalised inverse of $G_i, i \leq d$. Consequently, since the continuity of G_i's implies that of F_i's, the unique copula C of F is given by

$$\begin{aligned} C(u_1, u_2) &= F(F_1^{-1}(u_1), F_2^{-1}(u_2)) \\ &= L_\Lambda\big(-\ln G(G_1^{-1}(v_1), G_2^{-1}(v_2))\big) \\ &= L_\Lambda\big(-\ln Q(v_1, v_2)\big), \qquad u_1, u_2 \in [0, 1], \end{aligned} \tag{5.6}$$

where we set

$$v_i = e^{-L_\Lambda^{-1}(u_i)}.$$

Remarks 5.2.1. The df of the bivariate copula in (5.6) can be extended to the multivariate case. Let $X_j^{(i)}$ be the j-th claim sizes of the portfolio $i, i = 1, \ldots, d$ and $j = 1, \ldots, N$. Thus, the df of $(X_{N:N}^{(1)}, \ldots, X_{N:N}^{(d)})$ is given by

$$F(z_1, \ldots, z_d) = L_\Lambda\big(-\ln G(z_1, \ldots, z_d)\big), \quad z_1, \ldots, z_d \in \mathbb{R},$$

where G is the df of $(X_1^{(1)}, \ldots, X_1^{(d)})$. Similarly to the bivariate case one may express the copula of F as follows

$$C(u_1, \ldots, u_d) = L_\Lambda\big(-\ln Q(v_1, \ldots, v_d)\big), \qquad u_1, \ldots, u_d \in [0, 1],$$

where Q is the copula of G. Without loss of generality, we present in the rest of the chapter the results for the bivariate case.

Next, if G has a pdf g, then Q has a pdf q given by

$$q(u_1, u_2) = \frac{g(G_1^{-1}(u_1), G_2^{-1}(u_2))}{g_1(G_1^{-1}(u_1))g_2(G_2^{-1}(u_2))}, \quad u_1, u_2 \in [0, 1],$$

with g_1, g_2 the marginal pdf's. Consequently, the pdf c of C is given by (set $t = -\ln Q(v_1, v_2)$)

$$\begin{aligned} c(u_1, u_2) &= \frac{\frac{\partial v_1}{\partial u_1}\frac{\partial v_2}{\partial u_2}}{Q^2(v_1, v_2)}\Bigg(\big(L'_\Lambda(t) + L''_\Lambda(t)\big)\frac{\partial Q(v_1, v_2)}{\partial v_1}\frac{\partial Q(v_1, v_2)}{\partial v_2} \\ &\quad -L'_\Lambda(t)Q(v_1, v_2)q(v_1, v_2)\Bigg), \end{aligned} \tag{5.7}$$

where $L'_\Lambda(s) = -\Lambda e^{-s\Lambda}$ and $L''_\Lambda(s) = \mathbb{E}\left\{\Lambda^2 e^{-s\Lambda}\right\}$. The explicit form of c for tractable copulas Q and Laplace transform L_Λ is useful for the pseudo-likelihood method of parameter estimation treated in Section 5.4.

To this end, we briefly discuss the correlation order and its implication for the dependence exhibited by F. Clearly, for any x, y non-negative

$$F(x, y) \le G(x, y).$$

Consequently, in view of the correlation order, see e.g., Denuit et al. [26] we have that Kendall's tau $\tau(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda})$, Spearman's rank correlation $\rho_S(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda})$ and the correlation coefficient $\rho(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda})$ (when it is defined) are bounded by the same dependence measures calculated to (X_1, Y_1) with df G, respectively.

Moreover, if $\mathbb{E}\{\Lambda\} < \infty$, then by applying Jensen's inequality (recall $\Lambda \ge 1$ almost surely) for any x, y non-negative

$$G^a(x, y) \le G^{\mathbb{E}\{\Lambda\}}(x, y) = e^{\mathbb{E}\{\Lambda\}\ln G(x,y)} \le \mathbb{E}\left\{e^{\Lambda \ln G(x,y)}\right\} \le F(x, y), \tag{5.8}$$

with a the smallest integer larger than $\mathbb{E}\{\Lambda\}$. Since G^a is a df, say of (S, T), then again the correlation order implies that $\tau(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda}) \ge \tau(S, T)$, and similar bounds hold for Spearman's rank correlation and the correlation coefficient. In the following we shall write also $\tau(C)$ and $\tau(Q)$ (if $a = 1$) instead of $\tau(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda})$ and $\tau(S, T)$, respectively. Similarly, we denote $\rho_S(C)$ and $\rho_S(Q)$ instead of $\rho_S(X_{\Lambda:\Lambda}, Y_{\Lambda:\Lambda})$ and $\rho_S(S, T)$, respectively.

5.3 Extremal Properties of F

In this section, we investigate the extremal properties of F and its copula. Assume that $\Lambda = \Lambda_n$ depends on n and write C_n instead of C. Suppose for simplicity that $\mathbb{E}\{\Lambda_n\} = n$ and G has unit Fréchet margins. Assume additionally the following convergence in probability

$$\frac{\Lambda_n}{n} \xrightarrow{p} 1, \quad n \to \infty. \tag{5.9}$$

The above conditions can be easily verified in concrete examples, in particular it holds if $\Lambda_n = n$ almost surely.
In order to understand the dependence of C_n, we can calculate Kendall's tau $\tau(C_n)$ as $n \to \infty$. For instance, as shown in the simulation results in Table 5.1, if the copula Q of G has a coefficient of upper tail dependence $\mu_Q = 0$, then $\lim_{n\to\infty} \tau(C_n) = 0$. Note that by definition if μ_Q exists, then it is calculated by

$$\mu_Q \quad = \quad 2 - \lim_{u \downarrow 0} u^{-1}[1 - Q(1-u, 1-u)] \in [0,1]. \tag{5.10}$$

The following result establishes the convergence of both Kendall's tau for C_n and Spearman's rank correlation $\rho_S(C_n)$ to the corresponding measures of dependence with respect to an extreme value copula Q_A which approximates Q, i.e.,

$$\lim_{n\to\infty} \sup_{u_1,u_2\in[0,1]} \left| (Q(u_1^{1/n}, u_2^{1/n}))^n - Q_A(u_1,u_2) \right| = 0, \tag{5.11}$$

where

$$Q_A(u_1,u_2) = (u_1u_2)^{A(y/(x+y))}, \quad x = \ln u_1, y = \ln u_2 \tag{5.12}$$

for any $(u_1,u_2) \in (0,1]^2 \setminus (1,1)$, with $A : [0,1] \to [1/2, 1]$ a convex function which satisfies

$$\max(t, 1-t) \le A(t) \le 1, \quad \forall t \in [0,1]. \tag{5.13}$$

In the literature, see e.g., Folk et al. [36], Molchanov [70], Bücher and Segers [14], Aulbach et al. [5, 6], A is referred to as the Pickands dependence function.

Proposition 5.3.1. If the copula Q satisfies (5.11) and further (5.9) holds, then

$$\lim_{n\to\infty} \tau(C_n) = \tau(Q_A), \quad \lim_{n\to\infty} \rho_S(C_n) = \rho_S(Q_A). \tag{5.14}$$

If Q_A is different from the independence copula, and therefore $A(t) < 1$ for any $t \in (0,1)$, then we have (see e.g., Molchanov [70])

$$\tau(Q_A) = \int_0^1 \frac{t(1-t)}{A(t)} dA'(t), \quad \rho_S(Q_A) = 12 \int_0^1 \frac{1}{(1+A(t))^2} dt - 3. \tag{5.15}$$

To illustrate the results stated above, we compare by simulations the dependence properties of both C and Q. To this end, we simulate random samples from both copulas and compute the empirical dependence measures. Specifically, we generate a random sample from C in which Step 1-Step 4 in Subsection 5.4.2 are repeated $10'000$ times. Also, we simulate Λ from **Model B** and two cases of Q namely, a Gumbel copula with parameter 10 and a Clayton copula with parameter 10. Table 5.1 describes the simulated empirical Kendall's tau and Spearman's rho for the random samples generated from C and Q.

	Q: Gumbel copula with $\alpha = 10$				Q: Clayton copula with $\alpha = 10$			
$\mathbb{E}\{\Lambda\}$	$\tau(C)$	$\tau(Q)$	$\rho_S(C)$	$\rho_S(Q)$	$\tau(C)$	$\tau(Q)$	$\rho_S(C)$	$\rho_S(Q)$
10	0.9059	0.9022	0.9871	0.9862	0.3533	0.8343	0.5030	0.9588
100	0.8980	0.9002	0.9848	0.9854	0.0518	0.8348	0.0775	0.9589
1'000	0.9007	0.9004	0.9856	0.9856	0.0043	0.8334	0.0064	0.9577
10'000	0.9016	0.9018	0.9857	0.9859	0.0019	0.8324	0.0027	0.9573
100'000	0.8997	0.8996	0.9851	0.9854	-0.0104	0.8316	-0.0156	0.9569

Table 5.1: Empirical Kendall's Tau and Spearman's rho according to $\mathbb{E}\{\Lambda\}$.

The table above shows that for the Gumbel copula case, the level of dependence of a bivariate risk governed by C is lower or approximately equal to the one corresponding to Q when $\mathbb{E}\{\Lambda\}$ increases. For the case of Clayton copula, the bigger $\mathbb{E}\{\Lambda\}$, the weaker the dependence associated with C. In particular, for a copula Q with no upper tail dependence, Clayton copula in our example, it can be seen that when $\mathbb{E}\{\Lambda\}$ increases, C tends to the independence copula. However, when Q is an extreme value copula, Gumbel copula in our illustration, the rate of decrease in the level of dependence with respect to $\mathbb{E}\{\Lambda\}$ is small. These empirical findings are due to the correlation order demonstrated in (5.8). To verify the results obtained from simulations, we show that, under (5.15), for $\alpha = 10$, we obtain $\tau(Q_A) = 0.9$ and $\rho_S(Q_A) = 0.9855$ for the Gumbel copula which are in line with the simulation results presented in Table 5.1.

It should be noted that for the Gumbel copula, the Pickands dependence function can be written as follows

$$A(t) = (t^{1/\alpha} + (1-t)^{1/\alpha})^{\alpha}, \quad t \in (0,1), \alpha \in (0,1)$$

leading to a closed form for $\tau(Q_A)$ given by

$$\tau(Q_A) = 1 - \frac{1}{\alpha}.$$

Also, it is well-known that for Clayton copula (5.11) holds with Q_A being the independence copula, hence for this case by (5.14) we have $\lim_{n\to\infty} \tau(C_n) = 0$, which confirms the findings in Table 5.1.

This section is concerned with the extremal properties of the df F introduced in (5.1) in terms of G and Λ. The natural question which we want to answer here is whether the extremal properties of G and F are the same. Therefore, we shall assume that G is in the max-domain of attraction of some max-stable bivariate distribution H. Without loss of generality we shall assume that H has unit Fréchet marginal df's. Hence, our assumption is that

$$\lim_{n\to\infty} G^n(nx, ny) = H(x,y), \quad x, y \in [0, \infty). \tag{5.16}$$

The max-stability of H and the fact that its marginal df's are unit Fréchet imply

$$H^t(tx, ty) = H(x,y), \quad \forall x, y, t \in (0, \infty) \tag{5.17}$$

see e.g., Falk et al. [36]. In case Λ is a shifted geometric random variable as in **Model A**, then the above assumptions imply for any x, y non-negative (set $q := 1 - \theta$)

$$\begin{aligned} n[1 - F(nx, ny)] &= n\Big[1 - \frac{\theta G(nx, ny)}{1 - qG(nx, ny)}\Big] \\ &= n\frac{1 - G(nx, ny)}{1 - qG(nx, ny)} \\ &\to -\frac{1}{\theta}\ln H(x,y), \quad n \to \infty. \end{aligned}$$

Hence

$$\lim_{n\to\infty} F^n(nx, ny) = H^{1/\theta}(x,y) \tag{5.18}$$

or equivalently, using (5.17)

$$\lim_{n\to\infty} F^n(nx/\theta, ny/\theta) = H^{1/\theta}(x/\theta, y/\theta) = H(x,y), \quad x, y \in (0,\infty) \tag{5.19}$$

and thus F is also in the same max-domain of attraction as G.

Our result below shows that the extremal properties of G are preserved for the general case when $\mathbb{E}\{\Lambda\}$ is finite. This assumption is natural in collective models, since otherwise we cannot insure such portfolios.

Proposition 5.3.2. If $\mathbb{E}\{\Lambda\}$ is finite then $\mu_Q = \mu_C$. Moreover, if (5.16) holds, then

$$\lim_{n\to\infty} F^n(a_n x, a_n y) \;=\; H(x,y), \quad x, y \in (0,\infty), \tag{5.20}$$

where $a_n = \mathbb{E}\{\Lambda\} n$.

Remarks 5.3.3. i) It is well-known, see e.g., Falk et al. [36] that if G is in the max-domain of attraction of H, then the coefficient of upper tail dependence μ_Q of G with copula Q exists and

$$\mu_Q \;=\; 2 + \ln H(1,1) = 2 - 2A(1/2).$$

By the above proposition, F is also in the max-domain of attraction of H, and thus

$$\mu_C \;=\; 2 - 2A(1/2) = \mu_Q \in [0,1]. \tag{5.21}$$

ii) Although F and G are in the same max-domain of attraction, the above proposition shows that the normalising constant $a_n = \mathbb{E}\{\Lambda\} n$ for F is different that for G (here $a_n = n$) if $\mathbb{E}\{\Lambda\} \neq 1$.

5.4 Parameter Estimation & Monte Carlo Simulations

5.4.1 Parameter Estimation

This section focuses on the estimation of the parameters of the new copula C i.e., θ of N and α of the copula Q. Hereafter, we denote $\Theta = (\theta, \alpha)$. There are three widely used methods for the estimation of the copula parameters. The classical one is the maximum likelihood estimation (MLE). Another popular method is the inference function for margins (IFM), which is a step-wise parametric method. First, the parameters of the marginal df's are estimated and then the copula parameter Θ

are obtained by maximizing the likelihood function of the copula with the marginal parameters replaced by their first-stage estimators. Typically, the success of this method depends upon finding appropriate parametric models for the marginals, see Kim et al. [57].

Finally, the pseudo-maximum likelihood (PML) method, introduced by Oakes [76] consists also of two steps. In the first step, the marginal df's are estimated non-parametrically. The copula parameters are determined in the second step by maximizing the pseudo log-likelihood function. Specifically, let $X \sim G_1$ and $Y \sim G_2$ where G_1 and G_2 are the unknown marginals df's of X and Y. For instance, if the data is not censored, a commonly used non-parametric estimator of G_1 and G_2 is their sample empirical distributions which are specified as follows

$$\widehat{G_1}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \leq x), \quad \widehat{G_2}(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(Y_i \leq y). \tag{5.22}$$

Therefore, in order to estimate the parameter Θ, we maximize the following pseudo log-likelihood function

$$l(\Theta) = \sum_{i=1}^{n} \ln c_\Theta(U_{1i}, U_{2i}), \quad U_{1i} = \frac{n}{n+1}\widehat{G_1}(x_i), \quad U_{2i} = \frac{n}{n+1}\widehat{G_2}(y_i), \tag{5.23}$$

where c_Θ denotes the pdf of the copula. This rescaling is used to avoid difficulties arising from the unboundedness of the pseudo log-likelihood function in (5.23) as $\widehat{G_1}(x_i)$ or $\widehat{G_2}(y_i)$ tends to 1, see Genest et al. [40].

Kim et al. [57] show in a recent simulation study that the PML approach is better than the well-known IFM and MLE methods when the marginal df's are unknown, which is almost always the case in practice. Moreover, it is shown in Genest et al. [40] that the resulting estimators from the PML approach are consistent and asymptotically normally distributed.

Therefore, for our study, we shall use the PML method for the estimation of Θ which takes into account the empirical counterparts of the marginal df's to find the parameter estimators.

As described in the Introduction, we consider three types of distributions for the random variable Λ:

- **Model A:** Λ follows a shifted Geometric distribution with parameter $\theta \in (0, 1)$.

  The pdf of the Geometric copula is given by

$$c_\Theta(u_1,u_2) \;=\; W(v_1,v_2)\left(\frac{(1-(1-\theta)v_1)^2(1-(1-\theta)v_2)^2}{\theta(1-(1-\theta)Q_\alpha(v_1,v_2))^3}\right), \quad (5.24)$$

where $v_i = \frac{u_i}{\theta+(1-\theta)u_i}, i = 1,2$ and

$$\begin{aligned} W(v_1,v_2) \;&=\; (1-(1-\theta)Q_\alpha(v_1,v_2))\Big(\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1 \partial v_2}\Big)\Big) \\ &+2(1-\theta)\Big(\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\Big), \end{aligned}$$

which yields the following pseudo log-likelihood function

$$\begin{aligned} l(\Theta) \;&=\; \sum_{i=1}^{n}\Big(2\ln(1-(1-\theta)v_{1i}) + 2\ln(1-(1-\theta)v_{2i}) - \ln(\theta) \\ &-3\ln(1-(1-\theta)Q_\alpha(v_{1i},v_{2i})) + \ln W(v_{1i},v_{2i})\Big). \end{aligned} \quad (5.25)$$

- **Model B:** Λ follows a Shifted Poisson distribution with parameter $\theta > 0$. The pdf of the shifted Poisson copula is of the form

$$c_\Theta(u_1,u_2) \;=\; W(v_1,v_2)\left(\frac{e^{\theta(Q_\alpha(v_1,v_2)+1-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)}\right), \quad (5.26)$$

where $v_j = f^{-1}(u_j)$ with $f(x) = x\exp(\theta(x-1))$ and

$$\begin{aligned} W(v_1,v_2) \;&=\; (1+\theta Q_\alpha(v_1,v_2))\Big(\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1 \partial v_2}\Big) \\ &+\theta(2+\theta Q_\alpha(v_1,v_2))\Big(\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\Big). \end{aligned}$$

The corresponding pseudo log-likelihood of the above copula is thus given by

$$\begin{aligned} l(\Theta) \;&=\; \sum_{i=1}^{n}\Big(\theta(Q_\alpha(v_{1i},v_{2i}) + 1 - v_{1i} - v_{2i}) - \ln(1+\theta v_{1i}) \\ &- \ln(1+\theta v_{2i}) + \ln W(v_{1i},v_{2i})\Big). \end{aligned}$$

- **Model C:** Λ follows a Truncated Poisson distribution with parameter $\theta > 0$.

The joint density of the truncated Poisson copula is given by

$$c_\Theta(u_1, u_2) \;=\; \frac{1}{\theta}(1-e^{-\theta})W(v_1, v_2)e^{\theta(1-v_1-v_2+Q_\alpha(v_1,v_2))}, \tag{5.27}$$

where

$$W(v_1, v_2) = \theta\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_2} + \frac{\partial^2 Q_\alpha(v_1, v_2)}{\partial v_1 \partial v_2},$$

$$v_j = \frac{1}{\theta}\ln\left(1 + \frac{u_j(1-e^{-\theta})}{e^{-\theta}}\right), \quad j = 1, 2.$$

The resulting pseudo log-likelihood of the above copula can be written as follows

$$\begin{aligned} l(\Theta) \;=\; & \sum_{i=1}^{n}\Big(\ln\Big(\frac{1-e^{-\theta}}{\theta}\Big) + \theta(1 - v_{1i} - v_{2i}) \\ & +\theta Q_\alpha(v_{1i}, v_{2i}) + \ln W(v_{1i}, v_{2i})\Big). \end{aligned} \tag{5.28}$$

Remarks 5.4.1. The copula C_θ of Model A and Model B include the corresponding original copula Q. In particular, if $\theta = 1$ the pdf c_θ in (5.24) becomes the pdf of the original copula Q, see e.g., Zhang and Lin [100], while the copula C_θ of Model B reduces to the original copula Q when $\theta = 0$.

Next, we generate random samples from the proposed copula models C.

5.4.2 Monte Carlo Simulations

Based on the distributional properties of F derived in Section 5.2, we have the following pseudo-algorithm for the simulation procedure which depends on the choice of Λ and Q:

- **Step 1:** Generate a value λ from Λ.
- **Step 2:** Generate λ random samples $(U_{1,i}, U_{2,i}), i = 1, \ldots, \lambda$, from the original copula Q.
- **Step 3:** Calculate (M_1, M_2) as follows

$$M_j = \max_{i=1,\ldots,\lambda} U_{j,i}, \quad j = 1, 2.$$

- **Step 4:** Return (V_1, V_2), such that

$$V_j = L_\Lambda(-\ln M_j), \quad j = 1, 2.$$

Simulation results are important for exploring the dependence of F. The simulation results in the table below complete those presented already in Table 5.1. In this regard, we generate random samples from the Joe copula with parameter $\alpha = 10$.

	Q: Joe copula with $\alpha = 10$			
$\mathbb{E}\{\Lambda\}$	$\tau(C)$	$\tau(Q)$	$\rho_S(C)$	$\rho_S(Q)$
10	0.8982	0.8194	0.9849	0.9504
100	0.9005	0.8190	0.9857	0.9509
1'000	0.8997	0.8164	0.9855	0.9492
10'000	0.9004	0.8209	0.9857	0.9520
100'000	0.8999	0.8206	0.9852	0.9513

Table 5.2: Empirical Kendall's Tau and Spearman's rho according to $\mathbb{E}\{\Lambda\}$.

For the Joe copula, the Pickands dependence function can be written as follows

$$A(t) = 1 - ((\psi_1(1-t))^{-\alpha} + (\psi_2 t)^{-\alpha})^{-\frac{1}{\alpha}},$$

where $\psi_1, \psi_2 \leq 1$, $t \in (0,1)$ and $\alpha \in (0,1)$.
By using (5.15) and for $\alpha = 10$ and $\psi_1 = \psi_2 = 1$, we obtain $\tau(Q_A) = 0.9066$ and $\rho_S(Q_A) = 0.9874$ which are in line with the simulation results observed in Table 5.2 for $\tau(C)$ and $\rho_S(C)$ as $\mathbb{E}\{\Lambda\}$ increases.

Another benefit of our simulation algorithm is that we can assess the accuracy of our estimation method proposed above. Therefore, we simulate random samples of size n from the copula C with different distributions for Λ: **Model A**, **Model B** and **Model C** and two types of copula for Q: the Gumbel copula and the Joe copula. Hereof, the parameters θ of Λ and α of Q are estimated from the dataset described in Subsection 5.5.1 and are presented in Table 5.3 .

	Q: Joe copula		Q: Gumbel copula	
Model for Λ	θ	α	θ	α
Model A	0.3254	2.3727	0.7630	2.2758
Model B	0.9537	2.6634	0.1490	2.3276
Model C	1.8660	2.5885	0.3133	2.3240

Table 5.3: Parameters used for sampling from C.

	Model A				Model B				Model C			
n	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.
100	0.2461	-24%	2.2255	-6%	1.0765	13%	2.6597	0%	1.7400	-7%	2.2535	-13%
1'000	0.3353	3%	2.3262	-2%	0.9906	4%	2.6999	1%	1.9238	3%	2.6491	2%
10'000	0.3304	2%	2.3260	-2%	0.9795	3%	2.6651	0%	1.8996	2%	2.5999	0%
100'000	0.3285	1%	2.3462	-1%	0.9541	0%	2.6600	0%	1.8721	0%	2.5877	0%

Table 5.4: Parameters used for sampling from C where Q is the Joe Copula.

	Model A				Model B				Model C			
n	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.	$\widehat{\theta}$	Diff.	$\widehat{\alpha}$	Diff.
100	0.9565	25%	2.3595	4%	0.1712	15%	2.4308	4%	0.3164	1%	2.3675	2%
1'000	0.7376	-3%	2.3076	1%	0.1563	5%	2.3458	1%	0.3084	-2%	2.3126	0%
10'000	0.7660	0%	2.3083	1%	0.1545	4%	2.3476	1%	0.3136	0%	2.3185	0%
100'000	0.7596	0%	2.2639	-1%	0.1506	1%	2.3232	0%	0.3279	5%	2.3063	-1%

Table 5.5: Parameters used for sampling from C where Q is the Gumbel Copula.

It can be seen from Table 5.4 and Table 5.5 that the estimated parameters from the simulated samples tend to the true value of the parameters as the sample size n increases, thus indicating the accuracy of our proposed models.

5.4.3 Influence of $X_{N:N} + Y_{N:N}$ on Total Loss

In this subsection, we focus on the distribution of the aggregate claim of two insurance portfolios by excluding the largest claim of each portfolio. Specifically, we analyse the aggregate influence of $M_N := X_{N:N} + Y_{N:N}$ on some risk measures of the total loss $S_N = \sum_{i=1}^{N}(X_i + Y_i)$. Moreover, by considering the joint distribution of $(X_{N:N}, Y_{N:N})$ we quantify the individual impact of $X_{N:N}$ and $Y_{N:N}$ on the distribution of S_N. Let S_N^* be the aggregate claim excluding the largest claims, based on some risk measure $\rho(.)$ and suppose that X_i, Y_i's have a finite second moment, the influence of the largest claims on the aggregate claim is evaluated as follows

$$I^* = \rho(S_N) - \rho(S_N^*).$$

By the covariance capital allocation principle, the contribution of $X_{N:N}$ on the change of the distribution of S_N is given by

$$I(X_{N:N}, M_N) = \frac{cov(X_{N:N}, M_N)}{var(M_N)} I^*.$$

To illustrate our results we have implemented the following simulation pseudo-algorithm:

- **Step 1:** Generate the number of claims N from Λ.
- **Step 2:** Generate N random samples $(u_{1,i}, u_{2,i}), i = 1, \ldots, N$, from the original copula Q.
- **Step 3:** For each portfolio, simulate N claim sizes by using the inverse method as follows
$$X_i^b = F_1^{-1}(u_{1,i}), \quad Y_i^b = F_2^{-1}(u_{2,i}), i = 1, \ldots, N,$$
where $F_i, i = 1, 2$, is the df of X and Y, respectively.
- **Step 4:** Evaluate the total loss with and without the largest claims, respectively
$$S_N^b = \sum_{i=1}^{N}(X_i^b + Y_i^b), \quad S_N^{*b}.$$

To obtain the simulated distribution of S_N and S_N^* **Step 1-4** are repeated B times. The results presented in Table 5.6 is in million and is obtained from the following assumptions:

- number of simulations $B = 100'000$,
- the original copula is a Gumbel copula with dependence parameter $\alpha = 2.324$,
- the number of claims follows the Shifted Poisson (**Model B**) with parameter $\theta = 1000$,
- the claim sizes are Pareto distributed as follows

$$X_i \sim Pareto(10000, 2.2), \quad Y_i \sim Pareto(50000, 2.5).$$

Risk measures (ρ)	S_N	S_N^*	I^*	I^* (in %)	$I(X_{N:N}, M_N)$	$I(Y_{N:N}, M_N)$
Mean	101.77	100.21	1.57	1.54	0.38	1.19
Standard deviation	4.41	3.91	0.50	11.24	0.15	0.35
VaR (99 %)	112.75	109.65	3.10	2.74	0.90	2.20
TVaR (99 %)	117.08	111.03	6.05	5.17	1.75	4.30

Table 5.6: Influence of the largest claims on the total loss.

It can be seen that a significant proportion of the aggregate claims is consumed by $X_{N:N} + Y_{N:N}$. For instance, based on the standard deviation as risk measure, 11.24% of the total loss is driven by the largest claims. In this regards, $X_{N:N}$

has more important contribution to I^* than $Y_{N:N}$. This result is helpful for the insurance company when choosing the appropriate reinsurance treaty in the sense that the main source of volatility of the correlated portfolios is quantified.

5.5 Real Insurance Data Applications

In this section, we illustrate the applications of the new copula families in the modelling of three real insurance data. Specifically, we shall consider four copula families for Q_α: Gumbel, Frank, Student and Joe and three mixture copulas in which Λ with parameter θ follows one of the three distributions: Shifted Geometric, Shifted Poisson and Truncated Poisson. The AIC criteria is used to assess the quality of each model fit relative to each of the other models.

5.5.1 Loss ALAE from Accident Insurance

We shall model real insurance data from a large insurance company operating in Switzerland. The dataset consists of 33'258 accident insurance losses and their corresponding allocated loss adjustment expenses (ALAE) which includes mainly the cost of medical consultancy and legal fees. The observation period encompasses the claims occuring during the accident period 1986-2014[1].

Let X_i be the i^{th} loss observed and Y_i its corresponding ALAE.
Some statistics on the data are summarised in Table 5.7.

	Loss	ALAE
Min	10	1
Q1	13'637	263
Q2	32'477	563
Q3	95'880	1'509
Max	133'578'900	2'733'282
No. Obs.	33'258	33'258
Mean	292'715	5'990
Std	2'188'622	42'186

Table 5.7: Statistics for Loss ALAE data from accident line.

[1] dataset can be downloaded here `http://dx.doi.org/10.13140/RG.2.1.1830.2481`

The scatterplot of (ALAE, loss) on a log scale is depicted in Figure 5.1. It can be seen that large values of loss is likely to be associated with large values of ALAE. In addition, the empirical estimator of some dependence measures in Table 5.8 suggests a positive dependence between X_i and Y_i. For instance, the empirical estimator of the upper tail dependence of 0.6869 indicates that there is a strong dependence in the tail of the distribution of X_i and Y_i.

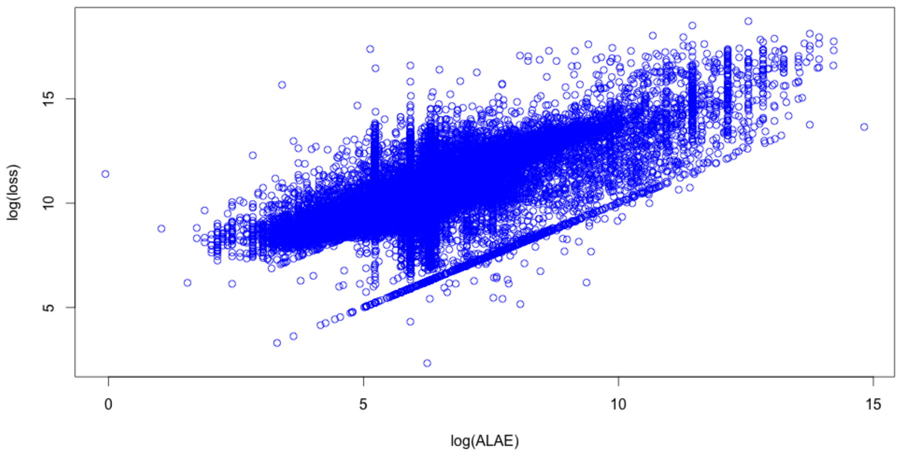


Figure 5.1: Scatterplot for log ALAE and log Loss: accident insurance data.

Pearson's Correlation	0.7460
Spearman's Rho	0.7465
Kendall's Tau	0.6012
Upper tail dependence	0.6869

Table 5.8: Empirical dependence measures for Loss ALAE data from accident line.

Referring to the marginal's estimator in (5.22), the estimation results for each copula model are found by maximizing (5.23) and are summarized in Table 5.9 below.

Model	θ	α	m	AIC
Gumbel	-	2.3876	-	-32'073
Gumbel Geometric	0.7630	2.2758	-	-32'128
Gumbel Truncated Poisson	0.3133	2.3240	-	-32'104
Gumbel Shifted Poisson	0.1490	2.3276	-	-32'059
Frank	-	8.0774	-	-30'137
Frank Geometric	0.9999	8.0772	-	-30'134
Frank Truncated Poisson	0.0001	8.0773	-	-30'135
Frank Shifted Poisson	0.0001	8.0773	-	-30'135
Student	-	0.8142	1.9805	-32'909
Student Geometric	0.1137	0.5492	1.9992	-38'088
Student Truncated Poisson	0.0001	0.7841	9.6744	-28'672
Student Shifted Poisson	0.0001	0.7885	8.7113	-29'042
Joe	-	3.0967	-	-30'655
Joe Geometric	0.3254	2.3727	-	-33'015
Joe Truncated Poisson	1.8660	2.5885	-	-32'578
Joe Shifted Poisson	0.9537	2.6634	-	-32'411

Table 5.9: Copula families parameters estimates.

It can be seen that the model which best fits the data is the Student Geometric copula followed by the Joe Geometric copula. We note in passing that the Student copula Q_α has an additional parameter m which is the degree of freedom.

5.5.2 Loss ALAE from General Liability Insurance

This dataset describes the general liability claims associated with their ALAE retrieved from the Insurance Services Office available in the R package. In this respect, the sample consists of 1'466 uncensored data points and 34 censored observations. We refer to [27] for more details on the description of the data. Let X_i be the i^{th} loss observed and Y_i the ALAE associated to the settlement of X_i. Each loss is associated with a maximum insured claim amount (policy limit) M. Thus, the loss variable X_i is censored when it exceeds the policy limit M. We define the censored indicator of the loss variable by

$$\delta_i = \begin{cases} 1 & \text{if} \quad X_i \leqslant M, \\ 0 & \text{if} \quad X_i > M, i = 1, \ldots, 1'500. \end{cases}$$

Next, we shall use the Kaplan-Meir estimator $\widehat{G}_X$ to estimate G_1 and the empirical distribution $\widehat{G}_Y$ for G_2 as in (5.22). In particular, the corresponding pseudo log-likelihood function is given by

$$l(\Theta) = \sum_{i=1}^{n}\Big(\delta_i \ln(c_\Theta(U_{1i}, U_{2i}) + (1-\delta_i)\ln\Big(1 - \frac{C_\Theta(U_{1i}, U_{2i})}{\partial U_{2i}}\Big)\Big), \tag{5.29}$$

where $U_{1i} = \frac{n}{n+1}\widehat{G}_X(x_i)$ and $U_{2i} = \frac{n}{n+1}\widehat{G}_Y(y_i)$ for $i = 1, \dots, n$, see Denuit et al. [27]. By maximizing (5.29), the resulting estimators of Θ for the considered copula models are presented in Table 5.10.

Model	θ	α	m	AIC
Gumbel	-	1.4284	-	-210.18
Gumbel Geometric	0.5425	1.3127	-	-278.23
Gumbel Truncated Poisson	0.0001	1.4422	-	-360.49
Gumbel Shifted Poisson	0.1410	1.4083	-	-361.20
Frank	-	3.0440	-	-321.44
Frank Geometric	0.7800	2.7464	-	-174.40
Frank Truncated Poisson	0.0001	3.0375	-	-306.40
Frank Shifted Poisson	0.0001	3.0375	-	-306.41
Student	-	0.4642	10.0006	-180.99
Student Geometric	0.7095	0.4252	9.1897	-228.82
Student Truncated Poisson	1	0.4094	13.9922	-271.40
Student Shifted Poisson	1	0.4016	13.9983	-295.42
Joe	-	1.6183	-	-179.00
Joe Geometric	0.4379	1.3864	-	-292.41
Joe Truncated Poisson	0.0607	1.6356	-	-331.21
Joe Shifted Poisson	0.8075	1.4629	-	-361.76

Table 5.10: Copula families parameters estimates.

Since the Joe Shifted Poisson copula has the the smallest AIC it represents the best model for describing the dependence in the dataset followed by the Gumbel Shifted Poisson copula.

5.5.3 Danish Fire Insurance Data

The corresponding dataset describes the Danish fire insurance claims collected from the Copenhagen Reinsurance Company for the period 1980-1990. It can be retrieved from the following website: *www.ma.hw.ac.uk/* $\sim$ *mcneil/*. This dataset has first been considered by Embrechts et al. [34] (Example 6.2.9) and explored by Haug et al. [50]. It consists of three components: loss to buildings, loss to contents and loss to profit. However, in this case, we model the dependence between the first two components. The total number of observations is of 1'501. We only consider the observations where both components are non-null. As indicated by the empirical dependence measures in Table 5.11, the level of dependence between these two losses is low.

Pearson's Correlation	0.1413
Spearman's Rho	0.1417
Kendall's Tau	0.0856
Upper tail dependence	0.1998

Table 5.11: Dependence measures for the Danish fire insurance.

The estimation results for each copula is summarized in Table 5.12 below.

Model	θ	α	m	AIC
Gumbel	-	1.1762	-	-133.18
Gumbel Geometric	0.9999	1.1762	-	-131.17
Gumbel Truncated Poisson	0.0001	1.1762	-	-131.18
Gumbel Shifted Poisson	0.0001	1.1762	-	-131.17
Frank	-	0.8807	-	-29.12
Frank Geometric	0.9999	0.8804	-	-27.12
Frank Truncated Poisson	0.0001	0.8806	-	-27.12
Frank Shifted Poisson	0.0001	0.8805	-	-27.12
Student	-	0.1574	9.5998	-47.86
Student Geometric	0.9999	0.1576	10.0063	-45.84
Student Truncated Poisson	0.0001	0.1570	9.0048	-45.81
Student Shifted Poisson	0.0001	0.1562	8.9833	-45.42
Joe	-	1.3585	-	-204.85
Joe Geometric	0.9999	1.3585	-	-202.83
Joe Truncated Poisson	0.0001	1.3585	-	-202.84
Joe Shifted Poisson	0.0001	1.3585	-	-202.83

Table 5.12: Copula families parameters estimates.

It can be seen that the model that best fits the data is the Joe copula followed by the Joe Truncated Poisson copula. The Frank mixture copulas and Student mixture copulas are not a good fit for the data as their AIC is higher by far compared to the Gumbel and Joe mixture copulas families.

5.5.4 Reinsurance Premiums

In this section, we examine the effects of the dependence structure on reinsurance premiums by using the proposed copula models. In practice, it is well known that insurance risks dependency has an impact on reinsurance. For instance, Dhaene and Goovaerts [28] have shown that stop loss premium is greater under the dependence assumption than under the independence case. In what follows, we consider the insurance claims data described in Subsection 5.5.1 where we denote X the loss variable, Y the associated ALAE and K the number of claims for the next accident year. In addition, two types of reinsurance treaties are analyzed namely:

- Excess-of-loss reinsurance, where the claims from Y_i 's are attributed proportionally to the insurer and the reinsurer. For a given observation (X_i, Y_i) the payment for the reinsurer is described as follows, see Cebrian et al. [17]

$$g(X_i, Y_i, r) = \begin{cases} 0 & \text{if} \quad X_i \leqslant r, \\ X_i - r + \left(\frac{X_i - r}{X_i}\right) Y_i & \text{if} \quad X_i > r \end{cases}$$

  leading to a reinsurance premium of the form

$$\kappa(r) = \mathbb{E}\left\{K\right\} \mathbb{E}\left\{g(X_i, Y_i, r)\right\}, \tag{5.30}$$

  where $r > 0$ is the retention level.

- Stop loss reinsurance, where the premium is given by

$$\pi(d) = \mathbb{E}\left\{\left(\sum_{i=1}^{K}(X_i + Y_i) - d\right)_+\right\} \tag{5.31}$$

  and d is a positive deductible.

In order to calculate the reinsurance premiums defined above, Monte Carlo simulations have been implemented. Hereof, we assume that K is Poisson distributed with a mean of 156.2, representing the expected number of claims estimated by the insurance company. Additionally, we use the empirical distributions of X_i and

Y_i for the simulation of the claims amount. Regarding the dependence model, the following copulas are considered: independent copula, Joe copula, Geometric Joe Copula, Truncated Poisson Joe copula and the Shifted Poisson Joe copula where the parameters are summarized in Table 5.9. The following steps summarize the implemented pseudo-algorithm:

- **Step 1:** Generate the number of claims $K \sim Poisson(156.2)$.
- **Step 2:** Simulate $(u_i, v_i), i = 1, \ldots, K$ from the considered copula C.
- **Step 3:** Generate the loss and ALAE claims as follows

$$(x_i = \widehat{F}_{X,n}^{-1}(u_i), y_i = \widehat{F}_{Y,n}^{-1}(v_i)), i = 1, \ldots, K,$$

  where $\widehat{F}_{X,n}^{-1}$ and $\widehat{F}_{Y,n}^{-1}$ are the inverse of the empirical df of X and Y respectively, with

$$\widehat{F}_{X,n}(x) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(X_i \le x), \quad \widehat{F}_{Y,n}(y) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}(Y_i \le y).$$

- **Step 4:** Calculate the reinsurance premiums $\kappa^b(r)$ and $\pi^b(d)$ as in (5.30) and (5.31) respectively.
- **Step 5: Step 1 -Step 4** are repeated B times and the estimators of the reinsurance premiums are given by

$$\widehat{\kappa}(r) = \frac{1}{B}\sum_{b=1}^{B} \kappa^b(r), \quad \widehat{\pi}(d) = \frac{1}{B}\sum_{b=1}^{B} \pi^b(d).$$

The estimation results presented in Table 5.13 are obtained from repeating **Step 1 -Step 4** 100'000 times. These amounts are expressed in CHF million.

	$\widehat{\kappa}(r)$			$\widehat{\pi}(d)$		
Copula model	$r = 1$	$r = 5$	$r = 10$	$d = 10$	$d = 20$	$d = 30$
Independent	13.1137	6.5692	3.0971	14.7530	7.5145	3.5738
Joe	13.6950	6.7776	3.1619	15.1056	7.7691	3.8233
Joe Geometric	13.4776	6.7365	3.1319	14.9250	7.6797	3.7177
Joe Truncated Poisson	13.4483	6.7183	3.1241	14.8975	7.6698	3.6702
Joe Shifted Poisson	13.4038	6.6789	3.1081	14.8016	7.6266	3.6493

Table 5.13: Reinsurance premiums with respect to copula models.

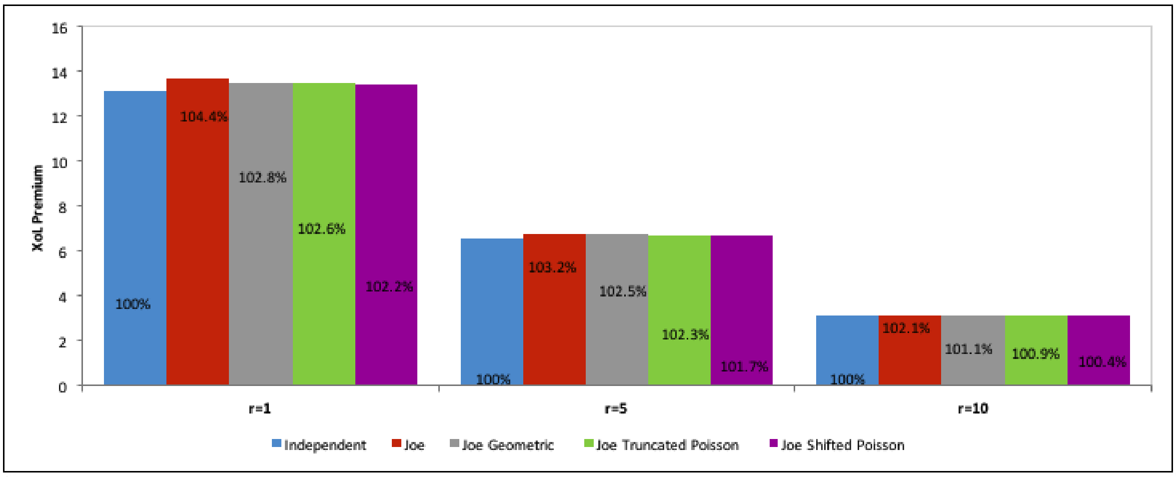


Figure 5.2: Comparison of Excess of Loss premiums with respect to different copula models.

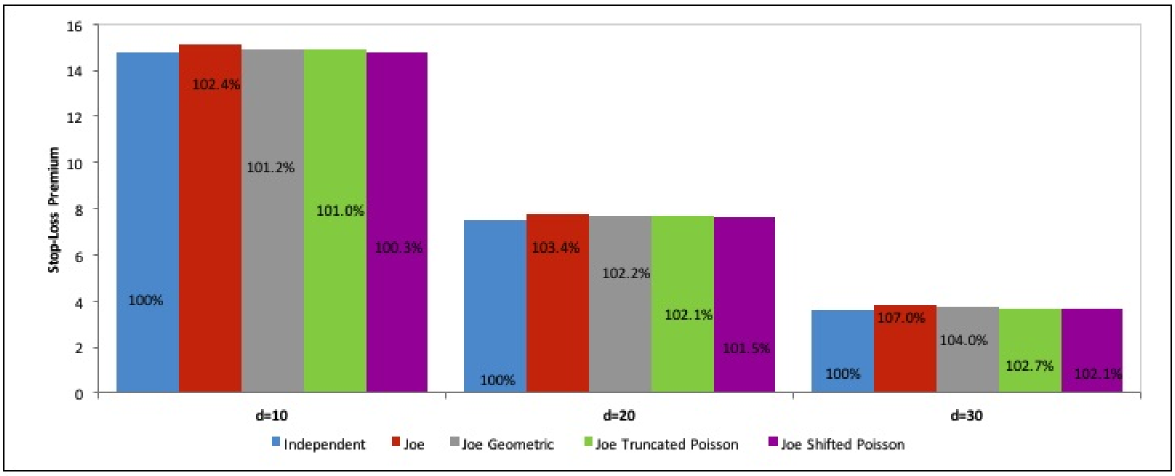


Figure 5.3: Comparison of Stop-Loss premiums with respect to different copula models.

Table 5.13 and Figure 5.2-5.3 show that the reinsurance premiums $\widehat{\kappa}(r)$ and $\widehat{\pi}(d)$ are lower under the independence hypothesis. Hence, the portfolio is less risky when the loss variable X_i and the ALAE variable Y_i are assumed to be independent. Furthermore, when the retention limit r increases for the excess of loss treaty, the reinsurance premiums estimates $\widehat{\kappa}(r)$ under the copula models tend to the estimated values under the independence assumption. Conversely, for the stop loss treaty, the higher the deductible d the higher the deviation from the independence hypothesis. Furthermore, by comparing the results for each copula model, it can be seen that the Joe copula generates the highest reinsurance premiums. This result is expected given that the strongest dependence structure is obtained under the Joe copula. On the other hand, the weakest dependence model for this data is observed under the Joe Shifted Poisson copula as the reinsurance premiums $\widehat{\kappa}(r)$ and $\widehat{\pi}(d)$ are the smallest for different values of r and d.

5.6 Appendix

5.6.1 Proofs

Derivation of (5.26)-(5.27): We show first (5.7). The corresponding joint density c of the df C is given by

$$c(u_1, u_2) = \frac{\partial C(u_1, u_2)}{\partial u_1 \partial u_2} = \frac{\partial L_\Lambda(-\ln Q_\alpha(v_1, v_2))}{\partial u_1 \partial u_2}, \tag{5.32}$$

where

$$C(v_1, v_2) = L_\Lambda(-\ln Q_\alpha(v_1, v_2)), \quad v_i = e^{-L_\Lambda^{-1}(u_i)}, \quad i = 1, 2.$$

In view of (5.32), the partial derivative of C with respect to u_1 is

$$\frac{\partial L_\Lambda(-\ln Q_\alpha(v_1, v_2))}{\partial u_1} = \frac{1}{Q_\alpha(v_1, v_2)} L'_\Lambda(-\ln Q_\alpha(v_1, v_2)) \frac{-\partial Q_\alpha(v_1, v_2)}{\partial v_1} \frac{\partial v_1}{\partial u_1}$$

leading to

$$\begin{aligned}
c(u_1, u_2) &= \frac{\partial}{\partial v_2}\left(\frac{\partial L_\Lambda(-\ln Q_\alpha(v_1, v_2))}{\partial u_1}\right)\frac{\partial v_1}{\partial u_1}\frac{\partial v_2}{\partial u_2} \\
&= \frac{\partial v_2}{\partial u_2}\left(L''_\Lambda(-\ln Q_\alpha(v_1, v_2))\frac{\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}}{Q_\alpha^2(v_1, v_2)}\right. \\
&\quad \left. + L'_\Lambda(-\ln Q_\alpha(v_1, v_2))\frac{\frac{-\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1 \partial v_2} Q_\alpha(v_1, v_2) + \frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}}{Q_\alpha^2(v_1, v_2)}\right) \\
&= \frac{\frac{\partial v_1}{\partial u_1}\frac{\partial v_2}{\partial u_2}}{Q_\alpha^2(v_1, v_2)}\left(\Big(L''_\Lambda(-\ln Q_\alpha(v_1, v_2))\right. \\
&\quad + L'_\Lambda(-\ln Q_\alpha(v_1, v_2))\Big)\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_2} \\
&\quad \left. - L'_\Lambda(-\ln Q_\alpha(v_1, v_2)) Q_\alpha(v_1, v_2)\frac{\partial^2 Q_\alpha(v_1, v_2)}{\partial v_1 \partial v_2}\right).
\end{aligned}$$

We derive next the pdf c_Θ in (5.26): In this case, Λ follows a shifted Poisson distribution. In view of (5.7), we need to compute at first the following components:

$$\begin{aligned}
L'_\Lambda(-\ln Q_\alpha(v_1,v_2)) &= -e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2)),\\
L''_\Lambda(-\ln Q_\alpha(v_1,v_2)) &= e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)(1+3\theta Q_\alpha(v_1,v_2)+\theta^2 Q_\alpha^2(v_1,v_2)),
\end{aligned}$$

where for $i=1,2$, $v_i = e^{-L_\Lambda^{-1}(u_i)}$ which implies $u_i = v_i e^{-\theta(1-v_i)}$ and thus $\frac{\partial v_i}{\partial u_i} = \frac{e^{-\theta(1-v_i)}}{1+\theta v_i}$. By replacing these components into (5.7), we have

$$\begin{aligned}
c_\Theta(u_1,u_2) &= \frac{1}{Q_\alpha(v_1,v_2)^2}\frac{e^{\theta(2-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)}\\
&\quad\times\Bigg(\Big(e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)(1+3\theta Q_\alpha(v_1,v_2)+\theta^2 Q_\alpha^2(v_1,v_2))\\
&\quad - e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2))\Big)\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\\
&\quad + e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2))Q_\alpha(v_1,v_2)\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\Bigg)\\
&= \frac{1}{Q_\alpha(v_1,v_2)^2}\frac{e^{\theta(2-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)}\Bigg(e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)\\
&\quad\times\Big(1+3\theta Q_\alpha(v_1,v_2)+\theta^2 Q_\alpha^2(v_1,v_2)-1-\theta Q_\alpha(v_1,v_2)\Big)\\
&\quad\times\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\\
&\quad + e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha^2(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2))\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\Bigg)\\
&= \frac{1}{Q(v_1,v_2)^2}\frac{e^{\theta(2-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)}\\
&\quad\times\Bigg(e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha(v_1,v_2)\Big(2\theta Q_\alpha(v_1,v_2)+\theta^2 Q_\alpha^2(v_1,v_2)\Big)\\
&\quad\times\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\\
&\quad + e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha^2(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2))\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\Bigg)\\
&= \frac{1}{Q(v_1,v_2)^2}\frac{e^{\theta(2-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)}\\
&\quad\times\Bigg(e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha^2(v_1,v_2)\Big(2\theta+\theta^2 Q_\alpha(v_1,v_2)\Big)\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\\
&\quad + e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha^2(v_1,v_2)(1+\theta Q_\alpha(v_1,v_2))\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{e^{-\theta(1-Q_\alpha(v_1,v_2))}Q_\alpha^2(v_1,v_2)}{Q(v_1,v_2)^2}\frac{e^{\theta(2-v_1-v_2)}}{(1+\theta v_1)(1+\theta v_2)} \\
&\quad \times\left(\theta\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}\Big(2+\theta Q_\alpha(v_1,v_2)\Big)\right. \\
&\quad \left.+(1+\theta Q_\alpha(v_1,v_2))\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1 \partial v_2}\right) \\
&= \frac{e^{\theta(2-v_1-v_2)}e^{\theta(Q_\alpha(v_1,v_2)-1)}}{(1+\theta v_1)(1+\theta v_2)}\left(\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}(1+\theta Q_\alpha(v_1,v_2))\right. \\
&\quad \left.+\theta\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}(2+\theta Q_\alpha(v_1,v_2))\right).
\end{aligned}
$$

Next, we show (5.27): Since Λ follows a truncated Poisson distribution, in light of (5.7), the joint density c_Θ is expressed in terms of (set $\eta_\theta = e^{-\theta}/(1-e^{-\theta})$)

$$
\begin{aligned}
L'_\Lambda(-\ln Q_\alpha(v_1,v_2)) &= -\eta_\theta\theta Q_\alpha(v_1,v_2)e^{\theta Q_\alpha(v_1,v_2)}, \\
L''_\Lambda(-\ln Q_\alpha(v_1,v_2)) &= \eta_\theta\theta Q_\alpha(v_1,v_2)e^{\theta Q_\alpha(v_1,v_2)}(1+\theta Q_\alpha(v_1,v_2)),
\end{aligned}
$$

where for $i=1,2$, $v_i = e^{-L_\Lambda^{-1}(u_i)}$ and $u_i = \frac{e^{-\theta}}{1-e^{-\theta}}(v_i^\theta - 1)$ with $\frac{\partial v_i}{\partial u_i} = \frac{1-e^{-\theta}}{\theta}e^{\theta(1-v_i)}$. By substituting the above components in the joint density expressed in (5.7), we obtain

$$
\begin{aligned}
c_\Theta(u_1,u_2) &= \left(\frac{1-e^{-\theta}}{\theta}\right)^2\frac{e^{\theta(2-v_1-v_2)}}{Q_\alpha^2(v_1,v_2)}\left(\left(\eta_\theta\theta Q_\alpha(v_1,v_2)e^{\theta Q_\alpha(v_1,v_2)}(1+\theta Q_\alpha(v_1,v_2))\right.\right. \\
&\quad \left.-\eta_\theta\theta Q_\alpha(v_1,v_2)e^{\theta Q_\alpha(v_1,v_2)}\right)\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2} \\
&\quad \left.+\eta_\theta\theta Q_\alpha(v_1,v_2)e^{\theta Q_\alpha(v_1,v_2)}Q_\alpha(v_1,v_2)\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\right) \\
&= (1-e^{-\theta})e^{\theta[1-v_1-v_2+Q_\alpha(v_1,v_2)]} \\
&\quad \times\left(\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_1}\frac{\partial Q_\alpha(v_1,v_2)}{\partial v_2}+\frac{1}{\theta}\frac{\partial^2 Q_\alpha(v_1,v_2)}{\partial v_1\partial v_2}\right).
\end{aligned}
$$

□PROOF OF PROPOSITION 5.3.1 Since G has Fréchet marginals, by assumption (5.11), we have that

$$\lim_{n\to\infty} G^n(nx, ny) = \mathcal{G}(x,y), \quad x,y\in(0,\infty),$$

where $\mathcal{G}$ has copula Q_A and thus $\tau(\mathcal{G}) = \tau(Q_A)$. We have thus with $F_n(x,y) =$

$\mathbb{E}\left\{G^{\Lambda_n}(x,y)\right\}$ using further (5.9)

$$\lim_{n\to\infty} F_n(nx_n, ny_n) = \lim_{n\to\infty} \mathbb{E}\left\{G^{n\frac{\Lambda_n}{n}}(nx_n, ny_n)\right\} = \mathcal{G}(x,y), \quad x, y \in (0,\infty) \quad (5.33)$$

for any x_n, y_n such that $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$. Consequently,

$$\begin{aligned} \tau(C_n) &= 4\int_{(0,\infty)^2} F_n(x,y)\, dF_n(x,y) - 1 \\ &= 4\int_{(0,\infty)^2} F_n(nx,ny)\, dF_n(nx,ny) - 1 \\ &\to 4\int_{(0,\infty)^2} \mathcal{G}(x,y)\, d\mathcal{G}(x,y) - 1, \quad n\to\infty \\ &= \tau(\mathcal{G}), \end{aligned}$$

where the convergence above follows by Lemma 4.2 in Hashorva [46] (see also Resnick and Zeber [81] and Kulik and Soulier [60] for more general results). Next, the convergence in (5.33) implies

$$\lim_{n\to\infty} F_{ni}(ns_n) = \lim_{n\to\infty} \mathbb{E}\left\{G_i^{n\frac{\Lambda_n}{n}}(ns_n)\right\} = \mathcal{G}_i(s), \quad s\in(0,\infty), i = 1,2$$

for any $s_n, n \geq 1$ such that $\lim_{n\to\infty} s_n = s$, where $F_{ni}, G_i, \mathcal{G}_i$ is the ith marginal df of F_n, G, and $\mathcal{G}$, respectively. Hence, with similar arguments as above, we have

$$\begin{aligned} \rho_S(C_n) &= 12\int_{(0,\infty)^2} F_n(x,y)\, dF_{n1}(x) dF_{n2}(y) - 3 \\ &= 12\int_{(0,\infty)^2} F_n(nx,ny)\, dF_{n1}(nx) dF_{n2}(nx) - 3 \\ &\to 12\int_{(0,\infty)^2} \mathcal{G}(x,y)\, d\mathcal{G}_1(x) d\mathcal{G}_2(x) - 3, \quad n\to\infty \\ &= \rho_S(\mathcal{G}) \end{aligned}$$

establishing the proof. $\square$

Proof of Proposition 5.3.2 For $v = e^{-L_\Lambda^{-1}(1-u)}$ we have

$$1 - L_\Lambda(-\ln v) \sim u, \quad u\downarrow 0, \quad \lim_{u\downarrow 0} v = 1.$$

By the assumption that $\mathbb{E}\{\Lambda\}$ is finite we have

$$1 - L_\Lambda(t) \sim -L'_\Lambda(0)t = \mathbb{E}\{\Lambda\} t, \quad t\to 0. \quad (5.34)$$

Since further

$$\mu_Q = 2 - \lim_{u \downarrow 0} \frac{Q(1-u, 1-u)}{u} = 2 - \lim_{v \uparrow 1} \frac{\ln Q(v,v)}{\ln v}$$

and $\lim_{v\uparrow 1} Q(v,v) = 1$, then using (5.6) and (5.34) we obtain

$$\begin{aligned}
\mu_C &= 2 - \lim_{u \downarrow 0} u^{-1}[1 - C(1-u, 1-u))] \\
&= 2 - \lim_{u \downarrow 0} u^{-1}\Big[1 - L_\Lambda\big(-\ln Q(v,v)\big)\Big] \\
&= 2 - \lim_{u \downarrow 0} \frac{1 - L_\Lambda\big(-\ln Q(v,v)\big)}{1 - L_\Lambda(-\ln v)} \\
&= 2 - \lim_{v \uparrow 1} \frac{\ln Q(v,v)}{\ln v} \\
&= 2 - [2 - \mu_Q] = \mu_Q,
\end{aligned}$$

hence the first claim follows. Next, in view of (5.16) we have

$$\lim_{n\to\infty} n[1 - G(nx, ny)] = -\ln H(x,y), \quad x, y \in (0, \infty),$$

hence as $n \to \infty$

$$n[1 - G(nx, ny)] \sim \frac{1 - G(nx, ny)}{1 - G(n,n)} \sim -\ln H(x,y), \quad x, y \in (0, \infty).$$

Let $a_n, n \geq 1$ be non-negative constants such that $\lim_{n\to\infty} a_n = \infty$. By the above and (5.34)

$$n[1 - F(a_n x, a_n y)] \;=\; n[1 - L_\Lambda(-\ln G(a_n x, a_n y))] \sim \mathbb{E}\{\Lambda\}\, n(-\ln G(a_n x, a_n y))$$

as $n \to \infty$. Setting now $a_n = \mathbb{E}\{\Lambda\}\, n$ we have thus as $n \to \infty$

$$\begin{aligned}
n[1 - F(a_n x, a_n y)] &\sim a_n \frac{1 - F(a_n x, a_n y)}{\mathbb{E}\{\Lambda\}} \\
&= a_n \frac{1 - L_\Lambda(-\ln G(a_n x, a_n y))}{\mathbb{E}\{\Lambda\}} \\
&\sim a_n(-\ln G(a_n x, a_n y)) \\
&\sim a_n[1 - G(a_n x, a_n y)[\\
&\sim \mathbb{E}\{\Lambda\} \Big(-\ln H(x\mathbb{E}\{\Lambda\}, y\mathbb{E}\{\Lambda\})\Big) \\
&= -\ln H(x,y)
\end{aligned}$$

establishing the proof. □

For our study, we consider several copula families for Q_α, which are described hereafter.

5.6.2 Gumbel Copula

The df of a Gumbel copula with a dependence parameter $\alpha \geq 1$ is given by

$$Q_\alpha(v_1, v_2) = \exp\Big(-\Big((-\ln v_1)^\alpha + (-\ln v_2)^\alpha\Big)^{\frac{1}{\alpha}}\Big)$$

by differentiating $Q_\alpha(v_1, v_2)$ with respect to v_1 we have

$$\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1} = \frac{1}{v_1}(-\ln v_1)^{\alpha-1}\Bigg((-\ln v_1)^\alpha + (-\ln v_2)^\alpha\Bigg)^{\frac{1}{\alpha-1}} e^{-\Big((-\ln v_1)^\alpha + (-\ln v_2)^\alpha\Big)^{\frac{1}{\alpha}}},$$

and the corresponding joint density is expressed as follows

$$q_\alpha(v_1, v_2) \quad = \quad \frac{(-\ln v_1)^{\alpha-1}(-\ln v_2)^{\alpha-1}}{v_1 v_2}\Big(a^{\frac{2}{\alpha}-2} + (\alpha-1)a^{\frac{1}{\alpha}-2}\Big)e^{-a^{\frac{1}{\alpha}}},$$

where $a = (-\ln v_1)^\alpha + (-\ln v_2)^\alpha$.

5.6.3 Frank Copula

The df of a Frank copula with a dependence parameter $\alpha \neq 0$ is of the form

$$Q_\alpha(v_1, v_2) = \frac{-1}{\alpha}\ln\Big(1 + \frac{(e^{-\alpha v_1} - 1)(e^{-\alpha v_2} - 1)}{e^{-\alpha} - 1}\Big),$$

which yields the partial derivative of $Q_\alpha(v_1, v_2)$ with respect to v_1 as follows

$$\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1} = \frac{e^{-\alpha v_1}(e^{-\alpha v_2} - 1)}{(e^{-\alpha} - 1) + (e^{-\alpha v_1} - 1)(e^{-\alpha v_2} - 1)}$$

and the associated pdf is given by

$$q_\alpha(v_1, v_2) \quad = \quad \frac{\alpha(1 - e^{-\alpha})e^{-\alpha(v_1+v_2)}}{\Big((1 - e^{-\alpha}) - (1 - e^{-\alpha v_1})(1 - e^{-\alpha v_2})\Big)^2}.$$

5.6.4 Joe copula

The Joe copula with dependence parameter $\alpha \geq 1$ has df

$$Q_\alpha(v_1, v_2) = 1 - \Big((1-v_1)^\alpha + (1-v_2)^\alpha - (1-v_1)^\alpha(1-v_2)^\alpha\Big)^{\frac{1}{\alpha}}.$$

Deriving $Q_\alpha(v_1, v_2)$ with respect to v_1 we obtain

$$\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1} = (1-v_1)^{\alpha-1}(1-(1-v_2)^\alpha)\Big((1-v_1)^\alpha + (1-v_2)^\alpha - (1-v_1)^\alpha(1-v_2)^\alpha\Big)^{\frac{1}{\alpha}-1}.$$

The associated pdf is obtained by differentiating $Q_\alpha(v_1, v_2)$ with respect to v_1 and v_2 leading to

$$q_\alpha(v_1, v_2) = (1-v_1)^{\alpha-1}(1-v_2)^{\alpha-1}\Big(\alpha - 1 + b\Big)b^{\frac{1}{\alpha}-2},$$

where $b = (1-v_2)^\alpha - (1-v_1)^\alpha(1-v_2)^\alpha$.

5.6.5 Student Copula

Let t_m be the df of a Student random variable with degree of freedom m and write t_m^{-1} for its inverse. The df of the Student copula, with correlation $\alpha \in (-1, 1)$ and degree of freedom $m > 0$ can be expressed as follows

$$\begin{aligned} Q_{\alpha,m}(v_1, v_2) &= t_{\alpha,m}(t_m^{-1}(v_1), t_m^{-1}(v_2)) \\ &= \int_{-\infty}^{t_m^{-1}(v_1)} \int_{-\infty}^{t_m^{-1}(v_2)} \frac{1}{\sqrt{2\pi(1-\alpha^2)}}\Big(1 + \frac{s^2 - 2\alpha st + t^2}{m(1-\alpha^2)}\Big)^{-(m+2)/2} dsdt. \end{aligned}$$

Its partial derivative with respect to v_1 is given by

$$\frac{\partial Q_\alpha(v_1, v_2)}{\partial v_1} = t_{m+1}\left(\frac{t_m^{-1}(v_2) - \alpha t_m^{-1}(v_1)}{\sqrt{\frac{(m+(t_m^{-1}(v_1))^2)(1-\alpha^2)}{m+1}}}\right),$$

whereas the corresponding pdf is

$$\begin{aligned} q_{\alpha,m}(v_1, v_2) &= \frac{1}{2\pi\sqrt{1-\alpha^2}} \frac{1}{k(t_m^{-1}(v_1))k(t_m^{-1}(v_2))} \\ &\quad \times \Big(1 + \frac{t_m^{-1}(v_1)^2 + t_m^{-1}(v_2)^2 - 2\alpha t_m^{-1}(v_1)t_m^{-1}(v_2)}{m(1-\alpha^2)}\Big)^{-\frac{m+2}{2}}, \end{aligned}$$

where for $i = 1, 2$

$$k(t_m^{-1}(v_i)) = \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m}{2})\sqrt{\pi m}}\Big(1 + \frac{t_m^{-1}(v_i)^2}{m}\Big)^{-\frac{m+1}{2}}.$$

Bibliography

[1] A. Abdallah, J.P. Boucher, H. Cossette, and J. Trufin. Sarmanov family of bivariate distributions for multivariate loss reserving analysis. 2015.

[2] S. Alink, M. Löwe, and M.V. Wüthrich. Diversification of aggregate dependent risks. *Insurance: Mathematics and Economics*, 35(1):77–95, 2004.

[3] H. Ammeter. Note concerning the distribution function of the total loss excluding the largest individual claims. *Astin Bulletin*, 3(02):132–143, 1964.

[4] A.V. Asimit and Y. Chen. Asymptotic results for conditional measures of association of a random sum. *Insurance Math. Econom.*, 60:11–18, 2015. ISSN 0167-6687. doi: 10.1016/j.insmatheco.2014.10.012. URL `http://dx.doi.org/10.1016/j.insmatheco.2014.10.012`.

[5] S. Aulbach, M. Falk, M. Hofmann, and M. Zott. Max-stable processes and the functional D-norm revisited. *Extremes*, 18(2):191–212, 2015. ISSN 1386-1999. doi: 10.1007/s10687-014-0210-0. URL `http://dx.doi.org/10.1007/s10687-014-0210-0`.

[6] S. Aulbach, M. Falk, and M. Zott. The space of D-norms revisited. *Extremes*, 18(1):85–97, 2015. ISSN 1386-1999. doi: 10.1007/s10687-014-0204-y. URL `http://dx.doi.org/10.1007/s10687-014-0204-y`.

[7] A. Badescu, L. Gong, X.S. Lin, and D. Tang. Modeling correlated frequencies with application in operational risk management. *Journal of Operational Risk*, forthcoming.

[8] Z. Bahraoui, C. Bolancé, E. Pelican, and R. Vernic. On the bivariate Sarmanov distribution and copula. An application on insurance data using truncated marginal distribution. *SORT-Statistics and Operations Research Transactions*, 39(2):209–230, 2015.

[9] I. Bairamov, S. Kotz, and M. Bekçi. New generalized Farlie-Gumbel-Morgenstern distributions and concomitants of order statistics. *Journal of Applied Statistics*, 28(5):521–536, 2001. ISSN 0266-4763. doi: 10.1080/02664760120047861. URL `http://dx.doi.org/10.1080/02664760120047861`.

[10] N. Balakrishnan, N.L. Johnson, and S. Kotz. Distributions in statistics: Discrete multivariate distributions, 1998.

[11] D. Berg. Copula goodness-of-fit testing: an overview and power comparison. *The European Journal of Finance*, 15(7-8):675–701, 2009.

[12] N.L. Bowers, H.U. Gerber, J.C. Hickman, D.A. Jones, and C.J. Nesbitt. *Actuarial mathematics*, volume 2. Society of Actuaries Itasca, Ill., 1986.

[13] J.R. Brown and J.M. Poterba. Joint life annuities and annuity demand by married couples. Technical report, National bureau of economic research, 1999.

[14] A. Bücher and J. Segers. Extreme value copula estimation based on block maxima of a multivariate stationary time series. *Extremes*, 17(3):495–528, 2014. ISSN 1386-1999. doi: 10.1007/s10687-014-0195-8. URL `http://dx.doi.org/10.1007/s10687-014-0195-8`.

[15] J.F. Carriere. An investigation of the Gompertz law of mortality. *Actuarial Research Clearing House*, 2:161–177, 1994.

[16] J.F. Carriere. Bivariate survival models for coupled lives. *Scandinavian Actuarial Journal*, 2000(1):17–32, 2000.

[17] A.C. Cebrian, M. Denuit, P. Lambert, et al. Analysis of bivariate tail dependence using extreme value copulas: An application to the soa medical large claims database. *Belgian Actuarial Journal*, 3(1):33–41, 2003.

[18] C. Constantinescu, E. Hashorva, and L. Ji. Archimedean copulas in finite and infinite dimensions—with application to ruin problems. *Insurance Math. Econom.*, 49(3):487–495, 2011. ISSN 0167-6687. doi: 10.1016/j.insmatheco.2011.08.006. URL `http://dx.doi.org/10.1016/j.insmatheco.2011.08.006`.

[19] H. Cossette, M. Mailhot, and E. Marceau. TVaR-based capital allocation for multivariate compound distributions with positive continuous claim amounts. *Insurance: Mathematics & Economics*, 50(2):247–256, 2012. ISSN 0167-6687.

doi: 10.1016/j.insmatheco.2011.11.006. URL `http://dx.doi.org/10.1016/j.insmatheco.2011.11.006`.

[20] H. Cossette, M.-P. Côté, E. Marceau, and K. Moutanabbir. Multivariate distribution defined with Farlie-Gumbel-Morgenstern copula and mixed Erlang marginals: Aggregation and capital allocation. *Insurance: Mathematics & Economics*, 52:560–572, 2013.

[21] J.D. Cummins. Allocation of capital in the insurance industry. *Risk Management and Insurance Review*, 3(1):7–27, 2000.

[22] D.M. Dabrowska. Kaplan-meier estimate on the plane. *The Annals of Statistics*, pages 1475–1489, 1988.

[23] M. Denuit and A. Cornet. Multilife premium calculation with dependent future lifetimes. *Journal of Actuarial Practice*, 7:147–171, 1999.

[24] M. Denuit, J. Dhaene, C. Le Bailly de Tilleghem, and S. Teghem. Measuring the impact of dependence among insured lifelengths. *Belgian Actuarial Bulletin*, 1(1):18–39, 2001.

[25] M. Denuit, J. Dhaene, M. Goovaerts, and R. Kaas. *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Wiley, 2005.

[26] M. Denuit, J. Dhaene, M. Goovaerts, and R. Kass. *Actuarial Theory for Dependent Risks: Measures, Orders and Models*. Wiley, 2005.

[27] M. Denuit, O. Purcaruff, and I. Van Keilegorni. Bivariate archimedean copula models for censored data in non-life insurance. Technical report, Université catholique de Louvain, 2006.

[28] J. Dhaene and M.J. Goovaerts. Dependency of risks and stop-loss order. *Astin Bulletin*, 26(02):201–212, 1996.

[29] J. Dhaene, A. Tsanakas, E. A. Valdez, and S. Vanduffel. Optimal capital allocation principles. *Journal of Risk and Insurance*, 79(1):1–28, 2012.

[30] D.C.M. Dickson. Some explicit solutions for the joint density of the time of ruin and the deficit at ruin. *Astin Bulletin*, 38(1):259–276, 2008. ISSN 0515-0361. doi: 10.2143/AST.38.1.2030413. URL `http://dx.doi.org/10.2143/AST.38.1.2030413`.

[31] D.C.M. Dickson and G.E. Willmot. The density of the time to ruin in the classical Poisson risk model. *Astin Bulletin*, 35(1):45–60, 2005. ISSN 0515-0361. doi: 10.2143/AST.35.1.583165. URL `http://dx.doi.org/10.2143/AST.35.1.583165`.

[32] P. Embrechts. Copulas: a personal view. *Journal of Risk and Insurance*, 76 (3):639–650, 2009.

[33] P. Embrechts and G. Puccetti. Aggregating risk capital, with an application to operational risk. *The Geneva Risk and Insurance Review*, 31(2):71–90, 2006.

[34] P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling extremal events*, volume 33 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, 1997. For insurance and finance.

[35] P. Embrechts, E. Hashorva, and T. Mikosch. Aggregation of log-linear risks. *Journal of Applied Probability*, 51:203–212, 2014.

[36] M. Falk, J. Hüsler, and R.-D. Reiss. Laws of small numbers: extremes and rare events. In *DMV Seminar*, volume 23, page 3rd edn. Birkhäuser, Basel, 2010.

[37] D.J.G. Farlie. The performance of some correlation coefficients for a general bivariate distribution. *Biometrika*, 47:307–323, 1960. ISSN 0006-3444.

[38] E.W. Frees and E. A. Valdez. Understanding relationships using copulas. *North American actuarial journal*, 2(1):1–25, 1998.

[39] E.W. Frees, J.F. Carriere, and E. Valdez. Annuity valuation with dependent mortality. *Journal of Risk and Insurance*, pages 229–261, 1996.

[40] C. Genest, K. Ghoudi, and L.P. Rivest. A semiparametric estimation procedure of dependence parameters in multivariate families of distributions. *Biometrika*, 82(3):543–552, 1995.

[41] C. Genest, M. Gendron, and M. Bourdeau-Brien. The advent of copulas in finance. *The European Journal of Finance*, 15(7-8):609–618, 2009.

[42] C. Genest, H.U. Gerber, M.J. Goovaerts, and R.J.A. Laeven. Editorial to the special issue on modeling and measurement of multivariate risk in insurance and finance. *Insurance: Mathematics and Economics*, 44(2):143–145, 2009.

[43] C. Genest, B. Rémillard, and D. Beaudoin. Goodness-of-fit tests for copulas: A review and a power study. *Insurance: Mathematics and economics*, 44(2): 199–213, 2009.

[44] S. Gribkova and O. Lopez. Non-parametric copula estimation under bivariate censoring. *Scandinavian Journal of Statistics*, 2015.

[45] E.J. Gumbel. Bivariate exponential distributions. *Journal of the American Statistical Association*, 55:698–707, 1960. ISSN 0162-1459.

[46] E. Hashorva. Extremes of conditioned elliptical random vectors. *J. Multivariate Anal.*, 98(8):1583–1591, 2007. ISSN 0047-259X.

[47] E. Hashorva and D. Kortschak. Tail asymptotics of random sum and maximum of log-normal risks. *Statist. Probab. Lett.*, 87:167–174, 2014. ISSN 0167-7152. doi: 10.1016/j.spl.2014.01.018. URL `http://dx.doi.org/10.1016/j.spl.2014.01.018`.

[48] E. Hashorva and J. Li. Tail behaviour of weighted sums of order statistics of dependent risks. *Stoch. Models, in press*, 31(1):–, 2015.

[49] E. Hashorva and G. Ratovomirija. On Sarmanov mixed Erlang risks in insurance applications. *ASTIN Bulletin*, 45:175–205, 1 2015. ISSN 1783-1350. doi: 10.1017/asb.2014.24. URL `http://journals.cambridge.org/article_S0515036114000245`.

[50] S. Haug, C. Klüppelberg, and L. Peng. Statistical models and methods for dependence in insurance data. *Journal of the Korean Statistical Society*, 40 (2):125–139, 2011.

[51] A. Hernández-Bastida and M.P. Fernández-Sánchez. A sarmanov family with beta and gamma marginal distributions: an application to the Bayes premium in a collective risk model. *Statistical Methods & Applications*, 21(4):391–409, 2012.

[52] P. Hougaard. *Analysis of multivariate survival data*. Springer Science & Business Media, 2012.

[53] M. Ji, M. Hardy, and J.S. Li. Markovian approaches to joint-life mortality. *North American Actuarial Journal*, 15(3):357–376, 2011.

[54] H. Joe. *Multivariate Models and Multivariate Dependence Concepts*. Chapman & Hall/CRC Monographs on Statistics & Applied Probability. Taylor & Francis, 1997. ISBN 9780412073311.

[55] H. Joe and J.J. Xu. The estimation method of inference functions for margins for multivariate models. Technical report, Technical report, 1996.

[56] N.L. Johnson, S. Kotz, and N. Balakrishnan. *Continuous multivariate distributions, volume 1, models and applications*, volume 59. New York: John Wiley & Sons, 2002.

[57] G. Kim, M.J. Silvapulle, and P. Silvapulle. Comparison of semiparametric and parametric methods for estimating copulas. *Computational Statistics and Data Analysis*, 51(6):2836–2850, 2007.

[58] S.A. Klugman, H.H. Panjer, and G. E. Willmot. *Loss models: From data to decisions*. Wiley Series in Probability and Statistics. John Wiley & Sons Inc., Hoboken, NJ, third edition, 2008. ISBN 978-0-470-18781-4. doi: 10.1002/9780470391341. URL `http://dx.doi.org/10.1002/9780470391341`.

[59] S. Kotz, N. Balakrishnan, and N.L. Johnson. *Continuous multivariate distributions. Vol. 1*. Wiley Series in Probability and Statistics: Applied Probability and Statistics. Wiley-Interscience, New York, second edition, 2000. Models and applications.

[60] R. Kulik and P. Soulier. Heavy tailed time series with extremal independence. *Extremes*, 18(2):273–299, 2015. ISSN 1386-1999. doi: 10.1007/s10687-014-0213-x. URL `http://dx.doi.org/10.1007/s10687-014-0213-x`.

[61] J.F. Lawless. *Statistical models and methods for lifetime data*, volume 362. John Wiley & Sons, 2011.

[62] M.L.T. Lee. Properties and applications of the sarmanov family of bivariate distributions. *Communications in Statistics-Theory and Methods*, 25(6):1207–1222, 1996.

[63] S.C.K. Lee and X.S. Lin. Modeling and evaluating insurance losses via mixtures of Erlang distributions. *North American Actuarial Journal*, 14(1):107–130, 2010. ISSN 1092-0277. doi: 10.1080/10920277.2010.10597580. URL `http://dx.doi.org/10.1080/10920277.2010.10597580`.

[64] S.C.K. Lee and X.S. Lin. Modeling dependent risks with multivariate Erlang mixtures. *Astin Bulletin*, 42(1):153–180, 2012. ISSN 0515-0361.

[65] E. Luciano, J. Spreeuw, and E. Vigna. Modelling stochastic mortality for dependent lives. *Insurance: Mathematics and Economics*, 43(2):234–244, 2008.

[66] Ph. Maeder. La construction des tables de mortalite du tarif collectif 1995 de l'UPAV. *Insurance Mathematics and Economics*, 3(18):226, 1996.

[67] D.D. Mari and S. Kotz. *Correlation and dependence*, volume 518. World Scientific, 2001.

[68] A.J. McNeil, R. Frey, and P. Embrechts. *Quantitative risk management: Concepts, Techniques and Tools*. Princeton Series in Finance. Princeton University Press, Princeton, NJ, 2005. ISBN 0-691-12255-5.

[69] G.G. Meyers, F. L. Klinker, and D.A. Lalonde. The aggregation and correlation of reinsurance exposure. *Casualty Actuarial Society Forum*, 2003.

[70] I. Molchanov. Convex geometry of max-stable distributions. *Extremes*, 11 (3):235–259, 2008. ISSN 1386-1999. doi: 10.1007/s10687-008-0055-5. URL `http://dx.doi.org/10.1007/s10687-008-0055-5`.

[71] D. Morgenstern. Einfache Beispiele zweidimensionaler Verteilungen. *Mitteilingsblatt für Mathematische Statistik*, 8:234–235, 1956.

[72] S.C. Myers and J.A. Read Jr. Capital allocation for insurance companies. *Journal of Risk and Insurance*, pages 545–580, 2001.

[73] R.B. Nelsen. *An introduction to copulas*, volume 139 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 1999. ISBN 0-387-98623-5.

[74] R.B. Nelsen. *An introduction to copulas*. Springer Science & Business Media, 2007.

[75] J. Neyman and E.S. Pearson. On the problem of the most efficient tests of statistical inference. *Biometrika A*, 20:175–240, 1933.

[76] D. Oakes. Bivariate survival models induced by frailties. *Journal of the American Statistical Association*, 84(406):487–493, 1989.

[77] C.M. Parkes, B. Benjamin, and R.G. Fitzgerald. Broken heart: A statistical study of increased mortality among widowers. *Journal of Occupational and Environmental Medicine*, 12(4):143, 1970.

[78] L. Peng. Joint tail of ECOMOR and LCR reinsurance treaties. *Insurance Math. Econom.*, 58:116–120, 2014. ISSN 0167-6687. doi: 10.1016/j.insmatheco.2014.06.013. URL `http://dx.doi.org/10.1016/j.insmatheco.2014.06.013`.

[79] R.L. Prentice, F. Zoe Moodie, and J. Wu. Hazard-based nonparametric survivor function estimation. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 66(2):305–319, 2004.

[80] G. Ratovomirija. On mixed Erlang reinsurance risk: aggregation, capital allocation and default risk. *European Actuarial Journal*, 6(1):149–175, 2016. ISSN 2190-9741. doi: 10.1007/s13385-016-0124-0. URL `http://dx.doi.org/10.1007/s13385-016-0124-0`.

[81] S.I. Resnick and D. Zeber. Asymptotics of Markov kernels and the tail chain. *Adv. in Appl. Probab.*, 45(1):186–213, 2013. ISSN 0001-8678. URL `http://projecteuclid.org/euclid.aap/1363354108`.

[82] M. Rosenblatt. Remarks on a multivariate transformation. *Annals of Mathematical Statistics*, 23:470–472, 1952. ISSN 0003-4851.

[83] J.M. Sarabia and E. Gómez-Déniz. Multivariate Poisson-Beta distributions with applications. *Communications in Statistics-Theory and Methods*, 40(6):1093–1108, 2011.

[84] O.V. Sarmanov. Generalized normal correlation and two-dimensional Fréchet classes. *Doklady Akademii Nauk SSSR*, 168:32–35, 1966. ISSN 0002-3264.

[85] J.H. Shih and T.A. Louis. Inferences on the association parameter in copula models for bivariate survival data. *Biometrics*, pages 1384–1399, 1995.

[86] M. Sklar. *Fonctions de répartition à n dimensions et leurs marges*. Université Paris 8, 1959.

[87] A. Tang and E.A. Valdez. Economic capital and the aggregation of risks using copulas. In *Proceedings of the 28th International Congress of Actuaries, Paris France*, 2006.

[88] D. Tasche. Allocating portfolio economic capital to sub-portfolios. *Economic Capital: A Practitioner's Guide, Risk Books*, pages 275–302, 2004.

[89] D. Tasche. Measuring sectoral diversification in an asymptotic multi-factor framework. *arXiv preprint physics/0505142*, 2005.

[90] R. Verbelen, K. Antonio, L. Gong, A. Badescu, and X.S. Lin. Fitting mixtures of Erlangs to censored and truncated data using the EM algorithm. *Working paper*, 2014.

[91] R. Vernic. On the distribution of a sum of Sarmanov distributed random variables. *Journal of Theoretical Probability*, pages 1–25, 2014.

[92] R. Vernic. Capital allocation for Sarmanov's class of distributions. *Available at SSRN 2709591*, 2015.

[93] A.W. Ward. Mortality of bereavement. *British Medical Journal*, 1(6011): 700–702, 1976.

[94] G.E. Willmot and X.S. Lin. Risk modeling with the mixed Erlang distribution. *Applied Stochastic Models in Business and Industry*, 27(1):8–22, 2011. ISSN 1524-1904. doi: 10.1002/asmb.838. URL `http://dx.doi.org/10.1002/asmb.838`.

[95] G.E. Willmot and J.K. Woo. On the class of Erlang mixtures with risk theoretic applications. *North American Actuarial Journal*, 11(2):99–115, 2007. ISSN 1092-0277. doi: 10.1080/10920277.2007.10597450. URL `http://dx.doi.org/10.1080/10920277.2007.10597450`.

[96] G.E. Willmot and J.K. Woo. On some properties of a class of multivariate Erlang mixtures with insurance applications. *ASTIN Bulletin*, 45:151–173, 1 2015. ISSN 1783-1350. doi: 10.1017/asb.2014.23. URL `http://journals.cambridge.org/article_S0515036114000233`.

[97] Y. Yang and E. Hashorva. Extremes and products of multivariate AC-product risks. *Insurance: Mathematics & Economics*, 52(2):312–319, 2013. ISSN 0167-6687. doi: 10.1016/j.insmatheco.2013.01.005. URL `http://dx.doi.org/10.1016/j.insmatheco.2013.01.005`.

[98] Y. Yang and Y. Wang. Tail behavior of the product of two dependent random variables with applications to risk theory. *Extremes*, 16(1):55–74, 2013. ISSN 1386-1999. doi: 10.1007/s10687-012-0153-2. URL `http://dx.doi.org/10.1007/s10687-012-0153-2`.

[99] H. Youn and A. Shemyakin. Statistical aspects of joint life insurance pricing. *1999 Proceedings of the Business and Statistics Section of the American Statistical Association*, 34:38, 1999.

[100] K. Zhang and J. Lin. A new class of copulas involved geometric distribution: Estimation and applications. *Insurance Math. Econom.*, 66:1–10, 2016.