ASYMPTOTIC DOMINATION OF SAMPLE MAXIMA

ENKELEJD HASHORVA AND DIDIER RULLIÈRE

Abstract: For a given random sample from some underlying multivariate distribution $F$ we consider the domination of the component-wise maxima by some independent random vector $W$ with distribution function $G$. We show that the probability that certain components of the sample maxima are dominated by the corresponding components of $W$ can be approximated under the assumptions that both $F$ and $G$ are in the max-domain of attraction of some max-stable distribution functions. We study further some basic probabilistic properties of the dominated components of sample maxima by $W$.

Key Words: Max-stable distributions; records; domination of sample maxima; extremal dependence; de Haan representation; infargmax formula;

AMS Classification: Primary 60G15; secondary 60G70

1. Introduction

Let $Z_i, i \leq n$ be independent $d$-dimensional random vectors with common continuous distribution function (df) $F$ and denote by $M_n$ their component-wise maxima, i.e., $M_{nj} = \max_{1 \leq k \leq n} Z_{kj}, j \leq d$. If $W$ is another $d$-dimensional random vector with continuous df $G$ being further independent of $M_n$ the approximation of the probability that at least one component of $W$ dominates the corresponding component of $M_n$ is of interest since it is related to the dependence of the components of $M_n$, see e.g., [1]. In the special case that $W$ has a max-stable df with unit Fréchet marginal df’s $\Phi(x) = e^{-1/x}, x > 0$ and $M_n$ has almost surely positive components, we simply have

$$\mathbb{P}\{\exists i \leq d : W_i > M_{ni}\} = 1 - \mathbb{P}\{\forall i, 1 \leq i \leq d : M_{ni} \geq W_i\} = 1 - \mathbb{E}_{M_n}\left\{\exp\left(-\mathbb{E}_W\left\{\max_{1 \leq i \leq d} \frac{W_i}{Z_i}\right\}\right)\right\},$$

where $W = (W_1, \ldots, W_d)$ being independent of $M_n$ is a spectral random vector of $G$ which exists in view of the well-known de Haan representation, see e.g., [2] and (2.1) below. Note that the assumption that $W_i$ has unit Fréchet df implies that $\mathbb{E}\{W_i\} = 1$.

The above probability is referred to as the marginal domination probability of the sample maxima. If $F$ is also a max-stable df with unit Fréchet marginals, then by definition $M_n/n$ has for any $n > 0$ df $F$ and consequently

$$(1.1)\ n[1 - \mathbb{P}\{\forall i, 1 \leq i \leq d : M_{ni} \geq W_i\}] = n\left[1 - \mathbb{E}_Z\left\{\exp\left(-\frac{1}{n}\mathbb{E}_W\left\{\max_{1 \leq i \leq d} \frac{W_i}{Z_i}\right\}\right)\right\}\right] \sim \mathbb{E}\left\{\max_{1 \leq i \leq d} \frac{W_i}{Z_i}\right\},$$

Date: December 24, 2019.
where \( \sim \) means asymptotic equivalence as \( n \to \infty \) and \( \mathcal{Z} = (Z_1, \ldots, Z_d) \) has df \( F \) being further independent of \( \mathcal{W} \). Under the above assumptions, we have

\[
(1.2) \quad p_{n,T}(F, G) = \mathbb{P}\{\forall i, 1 \leq i \leq d : W_i > M_{ni}\} \sim \frac{1}{n} \mathbb{E}\left\{\min_{1 \leq i \leq d} \frac{W_i}{Z_i}\right\}. \quad T = \{1, \ldots, d\}
\]

as \( n \to \infty \), which follows by (1.1) and the inclusion-exclusion formula or directly by [1][Thm 2.5 and Prop 4.2]. Here \( p_{n,T}(F, G) \) is referred to as the probability of the complete domination of sample maxima by \( \mathcal{W} \). In the particular case that \( F = G \) it is related to the probability of observing a multiple maxima or concurrence probability, see [3–9].

Between these two extreme cases, of interest is also to consider the partial domination of the sample maxima. Let therefore below \( T \subset \{1, \ldots, d\} \) be non-empty and consider the probability that only the components of \( \mathcal{W} \) with indices in \( T \) dominate \( M_n \), i.e.,

\[
\mathbb{P}\{\forall i \in T : W_i > M_{ni}, \forall i \in \bar{T} : W_i \leq M_{ni}\} =: p_{n,T}(F, G),
\]

where \( \bar{T} = \{1, \ldots, d\} \setminus T \). Note that \( p_{n,T}(F, F) \) relates to the probability of observing a \( T \)-record, see [10]. By the continuity of \( F \) and \( G \) we simply have

\[
p_{n,T}(F, G) = \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in T : W_i > y_i, \forall i \in \bar{T} : W_i \leq y_i\} \, dF^n(y),
\]

which cannot be evaluated without knowledge of both \( F \) and \( G \). In the particular case that \( F \) and \( G \) are max-stable df’s as above, using (1.1) and the inclusion-exclusion formula we obtain

\[
(1.3) \quad \lim_{n \to +\infty} n p_{n,T}(F, G) = \mathbb{E}\left\{\min_{i \in T} \frac{W_i}{Z_i} - \max_{i \in \bar{T}} \frac{W_i}{Z_i}\right\}^+.
\]

When \( F = G \) the above result is known from [10][Prop 2.2]. Moreover, in the special case that \( T \) consists of one element, then the right-hand side of (1.3) is equal to \( \mathbb{P}\{C(T) \subset \bar{T}\} \), where \( C(T) \) is the tessellation as determined in [11]. If we are not interested on a particular index set \( T \), where the domination of sample maxima by \( \mathcal{W} \) occurs but simply on the number of components being dominated, i.e., on the random variable (rv)

\[
N_n = \sum_{i=1}^d \mathbb{1}_{\{W_i > M_{ni}\}},
\]

a question of interest is if \( N_n \) can be approximated as \( n \to \infty \). We have that \( N_n \) has the same distribution as

\[
\sum_{i=1}^d \mathbb{1}_{\{W_i/n > Z_i\}},
\]

provided that \( F \) is max-stable as above and \( \mathcal{Z} \) has df \( F \) being further independent of \( \mathcal{W} \). Hence if \( W_i \)'s are unit Fréchet rv’s, then

\[
\lim_{n \to +\infty} n \mathbb{E}\{N_n\} = \sum_{i=1}^d \lim_{n \to +\infty} n \mathbb{P}\{W_i > nZ_i\} = \sum_{i=1}^d \lim_{n \to +\infty} n \left[1 - e^{-\mathbb{E}\left\{\frac{1}{Z_i}\right\}}\right] = d.
\]

Consequently, the expected number of components of sample maxima being dominated by the components of \( \mathcal{W} \) decreases as \( d/n \) when \( n \) goes to infinity. Moreover, the dependence of both \( \mathcal{W} \) and \( M_n \) does not play any role. This is however in general not the case for the expectation of \( f(N_n) \), where \( f \) is some real-valued function, since the dependence of both \( M_n \) and \( \mathcal{W} \) influence the approximation as we shall show in the next section.
From our discussion above the assumptions that $F$ and $G$ are max-stable df's with unit Fréchet marginals lead to tractable asymptotic formulas for various quantities related to the domination of sample maxima $M_n$ by $W$.

In view of [1] we know that both (1.1) and (1.2) are valid in the more general setup that both $F$ and $G$ are in the max-domain of attraction of some max-stable df's (see next section for details). We shall show in this paper that the same assumptions lead to tractable approximations of both $p_{n,T}(F,G)$ and $\mathbb{E}\{f(N_n)\}$ as $n \to \infty$.

Brief organisation of the paper: Section 2 presents the main results concerning the approximations of the marginal domination probabilities and the expectation of $f(N_n)$. Section 3 is dedicated to properties of $\mathcal{W}/\mathcal{Z}$ which we call the domination spectral vector. All the proofs are relegated to Section 4.

2. Main Results

We shall recall first some basic properties of max-stable df's, see [2, 12–14] for details. A $d$-dimensional df $G$ is max-stable with unit Fréchet marginals if

$$G^t(tx_1, \ldots, tx_d) = G(x_1, \ldots, x_d)$$

for any $t > 0, x_i \in (0, \infty), 1 \leq i \leq d$. In the light of De Haan representation

$$G(x) = \exp\left(-\mathbb{E}\{\max_{1 \leq j \leq d} W_j/x_j\}\right), \quad x = (x_1, \ldots, x_d) \in (0, \infty)^d,$$

where $W_j$'s are non-negative rv's with $\mathbb{E}\{W_j\} = 1, j \leq d$ and $\mathcal{W} = (W_1, \ldots, W_d)$ is a spectral vector for $G$ (which is not unique).

In view of multivariate extreme value theory, see e.g., [14] $d$-dimensional max-stable df's $\mathcal{F}$ are limiting df's of the component-wise maxima of $d$-dimensional iid random vectors with some df $F$. In that case, $F$ is said to be in the max-domain of attraction (MDA) of $\mathcal{F}$, abbreviated $F \in MDA(\mathcal{F})$. For simplicity we shall assume throughout in the following that $F$ has marginal df's $F_i$'s such that

$$\lim_{n \to +\infty} \frac{F^n_i(nx)}{n} = \Phi(x), \quad x \in \mathbb{R}$$

for all $i \leq d$, where we set $\Phi(x) = 0$ if $x \leq 0$. We have thus that $F \in MDA(\mathcal{F})$ if further

$$\lim_{n \to +\infty} \sup_{x_i \in \mathbb{R}, 1 \leq i \leq d} \left| F^n(x_1, \ldots, nx_d) - \mathcal{F}(x_1, \ldots, x_d) \right| = 0.$$  

In the following $\mathcal{F}$ is a $d$-dimensional max-stable df of some random vector $\mathcal{Z}$ with unit Fréchet marginals and $\mathcal{G}$ is another max-stable df with unit Fréchet marginals and spectral random vector $\mathcal{W}$ independent of $\mathcal{Z}$.

Below we extend [15][Prop 1] which considers the case $F = G$.

**Proposition 2.1.** If $F$ and $G$ have continuous marginal distributions satisfying (2.2) and $F \in MDA(\mathcal{F}), G \in MDA(\mathcal{G})$, then for any non-empty $T \subset \{1, \ldots, d\}$ we have

$$\lim_{n \to +\infty} np_{n,T}(F,G) = \mathbb{E}\left\{\left(\min_{i \in T} W_i/Z_i - \max_{i \in T} W_i/Z_i\right)_{+}\right\} =: \lambda_T(\mathcal{F}, \mathcal{G}).$$

**Remark 2.2.** Define for a non-empty index set $T$ the rv $K_n = \sum_{j=1}^n 1_{\{\forall \epsilon \in T: W_i > M_j, \forall \epsilon \in T: W_i \leq M_j\}}$. Under the assumptions of Proposition 2.1 we have (see also [16][Corr 3.2]) that

$$\lim_{n \to +\infty} \frac{\mathbb{E}\{K_n\}}{\ln n} = \lambda_T(\mathcal{F}, \mathcal{G}).$$
Example 2.3 \((\mathcal{F} \text{ comonotonic and } \mathcal{G} \text{ a product df}). \) Suppose that \(\mathcal{F}\) is comonotonic, i.e., \(Z_1 = \cdots = Z_d\) almost surely and let \(\mathcal{G}\) be a product df with unit Fréchet marginals df’s and let \(N\) be rv on \(\{1, \ldots, d\}\) with \(\mathbb{P}\{N = i\} = 1/d, i \leq d\). A spectral vector \(\mathbf{W}\) for \(\mathcal{G}\) can be defined as follows

\[
(W_1, \ldots, W_d) = (d \mathds{1}_{\{N=1\}}, \ldots, d \mathds{1}_{\{N=d\}}).
\]

Indeed \(\mathbb{E}\{W_k\} = d\mathbb{P}\{N = k\} = 1\) for any \(k \leq d\) and

\[
\mathbb{E}\{\max_{1 \leq i \leq d} W_i / x_i\} = \sum_{k=1}^{d} \mathbb{E}\{\max_{1 \leq i \leq d} W_i / x_k \mathds{1}_{\{N=k\}}\} = \sum_{k=1}^{d} \mathbb{E}\{W_k / x_k \mathds{1}_{\{N=k\}}\} = \sum_{k=1}^{d} \mathbb{E}\{\mathds{1}_{\{N=k\}} / x_k\} = \sum_{k=1}^{d} 1 / x_k
\]

for any \(x_1, \ldots, x_d\) positive. In particular, for a non-empty index set \(K \subset \{1, \ldots, d\}\) with \(m\) elements we have

\[
\mathbb{E}\{\max_{i \in K} W_i\} = d \sum_{k \in K} \mathbb{E}\{\mathds{1}_{\{N=k\}}\} = m.
\]

Consequently, using further that (see the proof of Proposition 2.1)

\[
\lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T: |J| = j} \mathbb{E}\left\{ \max_{i \in J \cup T} \frac{W_i}{Z_i} \right\}
\]

we obtain

\[
\lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T: |J| = j} \mathbb{E}\left\{ \max_{i \in J \cup T} W_i \right\} = \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T: |J| = j} (j + d - k).
\]

If \(k = d\), then from above

\[
\lambda_T(\mathcal{F}, \mathcal{G}) = \sum_{j=0}^{d} (-1)^{j+1} \sum_{J \subset T: |J| = j} j = d(1 - 1)^{d-1} = 0.
\]

A direct probabilistic proof of (2.6) follows by the properties of \(\mathbf{W}\), namely when \(k = d \geq 2\)

\[
\lambda_T(\mathcal{F}, \mathcal{G}) = \mathbb{E}\left\{ \min_{1 \leq i \leq d} W_i / Z_i \right\} = \mathbb{E}\left\{ \min_{1 \leq i \leq d} W_i \right\} = d\mathbb{E}\left\{ \min_{1 \leq i \leq d} \mathds{1}_{\{N=i\}} \right\} = 0.
\]

Now, let us investigate the number \(N_n\) of dominations defined as in Introduction by \(\sum_{i=1}^{d} \mathds{1}_{\{W_i/n > Z_i\}}\).

For a given function \(f : \{0, \ldots, d\} \rightarrow \mathbb{R}\) we shall be concerned with the behaviour of

\[
\mathbb{E}\{f(N_n)\} = \sum_{k=0}^{d} f(k) \mathbb{P}\{N_n = k\}
\]

when \(n\) tends to \(+\infty\). Throughout in the sequel we set

\[
\mathcal{D} = \{1, \ldots, d\}.
\]

In Proposition 2.4 below, we first express this expectation as a function of minima or maxima of \(W_i/Z_i\)’s.

**Proposition 2.4.** If \(F\) and \(G\) are as in Proposition 2.1, then we have

\[
\lim_{n \to +\infty} n \mathbb{E}\{f(N_n)\} - nf(0) = \sum_{k=1}^{d} \Delta^k f(0) \sum_{K \subset \mathcal{D}, |K| = k} \mathbb{E}\left\{ \min_{i \in K} W_i / Z_i \right\}
\]

or alternatively

\[
\lim_{n \to +\infty} n \mathbb{E}\{f(N_n)\} - nf(0) = \sum_{k=1}^{d} (-1)^{k+1} \Delta^k f(d-k) \sum_{K \subset \mathcal{D}, |K| = k} \mathbb{E}\left\{ \max_{i \in K} W_i / Z_i \right\},
\]

where \(\Delta\) is the difference operator, \(\Delta f(x) = f(x+1) - f(x)\).
Proposition 2.5. If $F$ and $G$ are as in Proposition 2.1, then we have

\begin{equation}
\lim_{n \to +\infty} n\mathbb{E} \{ f(N_n) \} - nf(0) = \sum_{k=0}^{d} g(k) \mathbb{E} \left\{ (W/Z)_{(k)} \right\} = \sum_{k=0}^{d} f(k) \left[ (W/Z)_{(d-k+1)} - (W/Z)_{(d-k)} \right],
\end{equation}

where $(W/Z)_{(1)} \leq \ldots \leq (W/Z)_{(d)}$ are the order statistics of $W_i/Z_i$, $i \leq d$ and $g(k) = f(d-k+1) - f(d-k)$, with the convention $(W/Z)_{(0)} = (W/Z)_{(d+1)} = 0$.

Remark 2.6 (retrieving simple cases). For particular cases of $f$ we have:

- From Proposition 2.4, setting $f(x) = 1_{\{x=d\}}$, one can check that $\Delta^k f(0) = 0$ when $k < d$ and $\Delta^d f(0) = 1$, so that Equation (2.7) implies (1.2). Alternatively, by Proposition 2.5 since $g(1) = f(d) - f(d-1) = 1$ and $g(k) = f(d-k+1) - f(d-k) = 0 - 0 = 0$ if $k > 1$ we have that $\lim_{n \to +\infty} n\mathbb{E} \{ f(N_n) \} - nf(0) = \mathbb{E} \left\{ (W/Z)_{(1)} \right\}$.
- In view of Proposition 2.4, setting $f(x) = 1_{\{x \geq 1\}}$, $\Delta^k f(d-k) = \sum_{i=0}^{k} \binom{k}{i} (-1)^{k-i} f(d-k+i)$. Thus $\Delta^k f(d-k) = 0$ if $k < d$. If $k = d$, then

$$\Delta^k f(d-k) = \Delta^d f(0) = (1-1)^d - (-1)^d = (-1)^{d+1}$$

and Equation (2.8) implies (1.1). Alternatively, by Proposition 2.5 since $g(k) = f(d-k+1) - f(d-k) = 1 - 1 = 0$ and $g(d) = f(1) - f(0) = 1$ we obtain $\lim_{n \to +\infty} n\mathbb{E} \{ f(N_n) \} - nf(0) = \mathbb{E} \left\{ (W/Z)_{(d)} \right\}$.
- By Proposition 2.5, setting $f(x) = x$, we easily retrieve $\lim_{n \to +\infty} n\mathbb{E} \{ f(N_n) \} = \sum_{k=1}^{d} \mathbb{E} \left\{ (W/Z)_{(k)} \right\} = d$, as seen previously.

Remark 2.7 (interpretation of $(W/Z)_{(j)}$). Let $f(k) = 1_{\{k \geq d-j+1\}}$, for any $j, k \in \mathcal{D}$. Then $g(k) = f(d-k+1) - f(d-k) = 1_{\{k=j\}}$. In this case, $f(0) = 0$ and $\mathbb{E} \{ f(N_n) \} = \mathbb{P} \{ N \geq d-j+1 \}$, thus

$$\mathbb{E} \left\{ (W/Z)_{(j)} \right\} = \lim_{n \to +\infty} n\mathbb{P} \{ N \geq d-j+1 \}.$$ 

3. Domination spectrum

In the previous results, we have considered a particular setting, and we have expressed the domination probability and some expectations relying on number of dominations (see Section 2). We have seen that all these results were expressed as a function of

$$W/Z = \left( \frac{W_i}{Z_i} \right)_{i \in \mathcal{D}}.$$ 

By the definition $W_i/Z_i$’s are nonnegative, and are such that, by independence, $\mathbb{E} \{ W_i/Z_i \} = \mathbb{E} \{ W_i \} \mathbb{E} \left\{ \frac{1}{Z_i} \right\} = 1$. Thus in view of the De Haan representation $W/Z$ can be viewed as the spectral random vector of some max-stable $d$-dimensional distribution. Since $W/Z$ is related to the domination of $M_n$ by $W$, we will refer to it by the term domination spectrum. In this section we shall explore some basic properties of the domination spectrum.

Next, assume that $W$ has a copula $C_W$ and suppose further that $Z$ has a copula $C_Z$. Note in passing that the latter copula is unique since the marginals of $Z$ have continuous df.

We shall first study the link between the diagonal sections of both copulas $C_W$ and $C_Z$, defined for all $u \in [0,1]$ by

$$\delta_W(u) = C_W(u, \ldots, u) \quad \text{and} \quad \delta_Z(u) = C_Z(u, \ldots, u).$$
We recall that the diagonal section characterizes uniquely many Archimedean copulas (under a condition that is called Frank’s condition, see e.g., [17]), some non-parametric estimators of the generator of an Archimedean copulas directly rely on this diagonal section. We consider here the case where the df of $\mathbf{Z}$ has spectral random vector $\mathbf{W}$. Notice that the upper tail dependence coefficients can be deduced from the regular variation properties of $\delta_{\mathbf{Z}}$ and $\delta_{\mathbf{W}}$, which is straightforward for $\delta_{\mathbf{Z}}$ in the following result.

**Proposition 3.1.** Consider a $d$-dimensional random vector $\mathbf{Z}$ having max-stable df with Fréchet unit marginals and with copula $C_{\mathbf{Z}}$. If the random vector $\mathbf{Z}$ has df $H(y) = \exp(-\mathbb{E}\{\max_{1 \leq j \leq d} W_j/y\})$, where all $W_j$ are nonnegative rv’s with mean 1, then

$$\delta_{\mathbf{Z}}(u) = u^r_{\mathbf{W}} \quad \text{with} \quad r_{\mathbf{W}} = \mathbb{E}\left\{\max_{j \in \mathcal{D}} W_j\right\}.$$ 

In particular, when $r_{\mathbf{W}} > 1$, this diagonal section $\delta_{\mathbf{Z}}(u)$ is the one of a Gumbel copula with parameter

$$\theta = \frac{\ln d}{\ln r_{\mathbf{W}}}.$$ 

Furthermore, if the components of $\mathbf{W}$ are identically distributed and if $F_{W_i}$ is invertible, then we have

$$r_{\mathbf{W}} = \int_0^1 F_{W_i}^{-1}(s) d\delta_{\mathbf{W}}(s).$$

**Example 3.2** (From independence to comonotonicity). Let $W_j = B d I_{(1=j)} + (1-B)\delta_1$, for all $j \in \mathcal{D}$, where $I$ is a uniformly distributed rv’s on $\mathcal{D}$, $B$ is a Bernoulli rv with $\mathbb{E}\{B\} = \alpha \in (0, 1]$ and $\delta_1$ is a Dirac mass at 1, all these rv’s being mutually independent. In this case, $r_{\mathbf{W}} = \mathbb{E}\left\{\max_{j \in \mathcal{D}} W_j\right\}$ in Proposition 3.1 becomes $r_{\mathbf{W}} = \alpha d + 1 - \alpha$. As a consequence, $\delta_{\mathbf{Z}}$ is the diagonal of a Gumbel copula which goes from the independence ($\alpha = 1$) to the comonotonicity ($\alpha \to 0$), with parameter

$$\theta = \frac{\ln d}{\ln (1 + \alpha (d - 1))}.$$ 

Furthermore, we have when all $t_j > 0$,

$$\mathbb{E}\left\{\max_{j \in K} W_j \right\} = \alpha \sum_{j \in K} \frac{1}{t_j} + (1 - \alpha) \frac{1}{\min_{j \in K} t_j}.$$ 

Let $t > 0$ and suppose that $K$ has cardinal $|K| > 1$. By conditioning over $B$, we get

$$\mathbb{P}\left\{\forall i \in K, W_i/Z_i > t \mid B = 1\right\} = (1 - \alpha) \mathbb{P}\left\{\forall i \in K, Z_i < 1/t \mid B = 0\right\}$$

since $\mathbb{P}\left\{\forall i \in K, W_i/Z_i > t \mid B = 1\right\} = 0$ when $|K| > 1$, because in this case at least one component $W_i$, $i \in K$, is zero when $B = 1$. Recall that $\mathbf{Z}$ is independent from $\mathbf{W}$ and $B$, thus for $t > 0$ and $|K| > 1$

$$\mathbb{P}\left\{\min_{i \in K} W_i/Z_i > t\right\} = (1 - \alpha) \exp\left(\mathbb{E}\left\{\max_{j \in K} W_j/(1/t)\right\}\right) = (1 - \alpha) \exp(-t(1 + \alpha |K| - \alpha)).$$

When $|K| = 1$, we show similarly that $\mathbb{P}\left\{\min_{i \in K} W_i/Z_i > t\right\} = (1 - \alpha) \exp(-t) + \alpha^1 \mathbb{E}\left\{1_{|K|=1} B I_{(1=1)}\right\}.$ In both cases $|K| = 1$ and $|K| > 1$, the survival function $\mathbb{P}\left\{\min_{i \in K} W_i/Z_i > t\right\}$ is a linear combination of exponential distributions, and thus can be shown to be a discrete mixture of exponential distributions:

$$\begin{align*}
\mathbb{E}\left\{\min_{i \in K} W_i/Z_i\right\} &\overset{d}{=} (1 - B) e_{1+\alpha(|K|-1)} + \mathbb{1}_{(|K|=1)} B I_{(1=1)} e_{1/d}, \\
\mathbb{P}\left\{\min_{i \in K} W_i/Z_i > t\right\} &\overset{d}{=} \frac{\alpha e_{1+\alpha(|K|-1)} + \mathbb{1}_{(|K|=1)} e_{1/d}}{1+\alpha(|K|-1)}.
\end{align*}$$
where $B$ is a Bernoulli r.v. of parameter $\alpha$, $\epsilon_1 + \alpha(|K| - 1)$ and $1/d$ are exponentially distributed r.v. with respective parameters $1 + \alpha(|K| - 1)$ and $1/d$, $I$ an uniformly distributed r.v. over $\mathcal{D}$, all being mutually independent (for simplicity, we denote $\mathbf{1}_{\{|K| = 1\}}$ the variable whose value is 1 if $|K| = 1$ or 0 otherwise). Then all results about the limit law of $N_n$ follow immediately, using Equation (2.7) in Proposition 2.4. Notice that one could also determine $r_{\mathcal{W}/\mathcal{Z}}$ from this, and by application of Proposition 3.1, assess the dependence structure of the random vector whose spectrum is $(\mathcal{W}/\mathcal{Z})$.

4. Proofs

We first give hereafter some combinatorial results that show how quantities depending on a number of events can be related to quantities involving only intersections or unions of those events. This generalizes inclusion-exclusion formulas that will correspond to very specific functions $f$ and $g$.

**Lemma 4.1** (Inclusion-exclusion relations). Let $\mathcal{D} = \{1, \ldots, d\}$ and let $B_i$, $i \in \mathcal{D}$ be events. Consider the number of realized events $N = \sum_{i \in \mathcal{D}} \mathbf{1}_{\{B_i\}}$. Then for any function $f : \{0, \ldots, d\} \to \mathbb{R}$

\[
(4.1) \quad \sum_{k=0}^{d} f(k) \mathbb{P}\{N = k\} = f(0) + \sum_{j=1}^{d} S_j \Delta^j f(0) = f(0) + \sum_{j=1}^{d} \bar{S}_j (-1)^{j+1} \Delta^j f(d-j)
\]

and similarly for any function $g : \mathcal{D} \to \mathbb{R}$

\[
(4.2) \quad \sum_{k=0}^{d} g(k) \mathbb{P}\{N \geq k\} = \sum_{j=1}^{d} S_j \Delta^{j-1} g(1) = \sum_{j=1}^{d} \bar{S}_j (-1)^{j+1} \Delta^{j-1} g(d-j+1),
\]

where $S_j = \sum_{J \subseteq \mathcal{D}, |J| = j} \mathbb{P}\{\bigcap_{i \in J} B_i\}$ and $\bar{S}_j = \sum_{J \subseteq \mathcal{D}, |J| = j} \mathbb{P}\{\bigcup_{i \in J} B_i\}$.

**Proof of Lemma 4.1.** The first equality in Equation (4.1) is known in actuarial sciences under the name of Schuette-Nesbitt formula, see [18, section 8.5]. This formula does not require any independence assumption, it is a simple development of $f(N) = (I + \mathbf{1}_{\{B_1\}} \Delta) \cdots (I + \mathbf{1}_{\{B_d\}} \Delta)f(0)$ where $I$ and $\Delta$ are the identity and the difference operators respectively. To prove the second equality in Equation (4.1), let us denote $p_j = \mathbb{P}\{\cap_{i \in J} B_i\}$ and $\bar{p}_j = \mathbb{P}\{\cup_{i \in J} B_i\}$. By inclusion-exclusion principle, we get

\[
(4.3) \quad S_k = \sum_{K \subseteq \mathcal{D}, |K| = k} \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \subseteq K, |J| = j} \bar{p}_j = \sum_{j=1}^{k} (-1)^{j+1} \binom{d-j}{k-j} \bar{S}_j,
\]

Now using Equation (4.3),

\[
\sum_{k=1}^{d} \Delta^k f(0) S_k = \sum_{k=1}^{d} \Delta^k f(0) \sum_{j=1}^{k} (-1)^{j+1} \binom{d-j}{k-j} \bar{S}_j = \sum_{j=1}^{d} \bar{S}_j (-1)^{j+1} \Delta^j (I + \Delta)^{d-j} f(0),
\]

and since $(I + \Delta)^{d-j} f(0) = f(d-j)$, the second equality in Equation (4.1) holds. Similarly, the first equality in Equation (4.2) is a known Schuette-Nesbitt formula, see [18, Section 8.5], and one can retrieve the second equality by using Equation (4.3). Alternatively, one can also deduce (4.2) from (4.1) by setting $f(0) = 0$ and $g(k) = \Delta f(k-1)$ for all $k \in \mathcal{D}$. The formulas in Lemma 4.1 generalize a very old formula of Waring which give $\mathbb{P}\{N = k\}$, $k \in \mathcal{D}$. They also generalize the classical inclusion exclusion formula which can be retrieved by setting in (4.1) $f(k) = 1$ if $k \geq 1$, and $f(k) = 0$ otherwise. \qed
Proof of Proposition 2.1. By inclusion-exclusion formula for a given index set \( T \subset \{1, \ldots, d\} \) with \( k = |T| \) elements we have

\[
P\{\forall i \in \bar{T} : W_i \leq y_i, \exists i \in T : W_i \leq y_i\} = \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} P\{\forall i \in (J \cup \bar{T}) : W_i \leq W_i\} = \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} G_{J \cup \bar{T}}(y),
\]

where \( G_L(y) = P\{\forall i \in L : W_i \leq y_i\} \) is the \( L \)-th marginal df of \( G \). In particular, letting \( W_i \to \infty, i \leq d \) yields

\[
1 = \sum_{j=1}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} 1.
\]

Consequently, for all \( n > 1 \)

\[
p_{n,T}(F,G) = \int_{\mathbb{R}^d} P\{\forall i \in T : W_i \geq y_i, \forall i \in \bar{T} : W_i < y_i\} dF^n(y) = \int_{\mathbb{R}^d} P\{\forall i \in \bar{T} : W_i \leq y_i, \exists i \in T : W_i \leq y_i\} dF^n(y) = 1 - \int_{\mathbb{R}^d} \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} G_{J \cup \bar{T}}(y) dF^n(y) = \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} \int_{\mathbb{R}^{m+|J|}} \left[1 - G_{J \cup \bar{T}}(y)\right] dF^n(y).
\]

In view of [1][Prop 4.2] we obtain

\[
\lim_{n \to +\infty} n \int_{\mathbb{R}^{m+|J|}} \left[1 - G_{J \cup \bar{T}}(y)\right] dF^n_{J \cup \bar{T}}(y) = -\int_{\mathbb{R}^{m+1}} \ln Q_{J \cup \bar{T}}(y) dH_{J \cup \bar{T}}(y).
\]

Further by [1][Thm 2.5 and Prop 4.2]

\[
-\int_{\mathbb{R}^{m+|J|}} \ln Q_{J \cup \bar{T}}(y) dH_{J \cup \bar{T}}(y) = \mathbb{E}\left\{\max_{\{i \in J \cup \bar{T}\}} \frac{W_i}{Z_i}\right\}.
\]

Consequently, we have

\[
\lim_{n \to +\infty} np_{n,T}(F,G) = \sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} \mathbb{E}\left\{\max_{\{i \in J \cup \bar{T}\}} \frac{W_i}{Z_i}\right\}.
\]

In the light of [10][Lem 1] for given constants \( c_1, \ldots, c_d \)

\[
\sum_{j=0}^{k} (-1)^{j+1} \sum_{J \subset T : |J| = j} \max_{i \in J \cup \bar{T}} c_i = \max_{i \in T} \left(\max_{i \in \bar{T}} c_i, \min_{i \in \bar{T}} c_i\right) - \max_{i \in T} c_i = \left(\min_{i \in T} c_i - \max_{i \in \bar{T}} c_i\right),
\]

implying the claim.

Alternatively, we have using again inclusion-exclusion formula

\[
p_{n,T}(F,G) = \int_{\mathbb{R}^d} P\{w_i \geq M_i, i \in T, \ w_i < M_i, i \in \bar{T}\} dG(w) = \int_{\mathbb{R}^d} P\{M_i \leq w_i, i \in T\} dG(w) - \int_{\mathbb{R}^d} P\{M_i \leq w_i, i \in T, \ \exists i \in \bar{T} : M_i \leq w_i\} dG(w)
\]
\[
\int_{\mathbb{R}^d} F^n_T(w) dG_T(w) - \int_{\mathbb{R}^d} \sum_{j=1}^{m} (-1)^{j+1} \sum_{J \subseteq T, |J| = j} F^j_{n,T}(w) dG(w) \\
= \sum_{j=0}^{d-k} (-1)^j \sum_{J \subseteq T, |J| = j} \int_{\mathbb{R}^{k+j}} F^n_{J \cup T}(w) dG_{J \cup T}(w).
\]

Applying [1][Thm 2.5 and Prop 4.2] we obtain
\[
\lim_{n \to +\infty} n \int_{\mathbb{R}^{k+i}} F^n_{J \cup T}(y) dG_{J \cup T}(y) = \mathbb{E}\left\{ \min_{i \in J \cup T} \frac{W_i}{Z_i} \right\}
\]
and thus
\[
\mu_T(H, Q) = \sum_{j=0}^{d-k} (-1)^j \sum_{J \subseteq T, |J| = j} \mathbb{E}\left\{ \min_{i \in J \cup T} \frac{W_i}{Z_i} \right\}.
\]

By [10][Lem 1] we obtain further
\[
\mu_T(H, Q) = \mathbb{E}\left\{ \min_{i \in T} \frac{W_i}{Z_i} - \min_{i \in T} \frac{W_i}{Z_i}, \max_{i \in T} \frac{W_i}{Z_i} \right\},
\]
hence the proof is complete. \(\square\)

**Proof of Proposition 2.4.** In view of the first equality in Equation (4.1)
\[
\mathbb{E}\{f(N_n)\} = f(0) + \sum_{k=1}^{d} \Delta^k f(0) \sum_{K \subseteq \mathcal{D}, |K| = k} \mathbb{P}\{\forall i \in K, W_i \geq M_{ni}\}.
\]
Alternatively, using the second equality in Equation (4.1)
\[
\mathbb{E}\{f(N_n)\} = f(0) + \sum_{k=1}^{d} (-1)^{k+1} \Delta^k f(d-k) \sum_{K \subseteq \mathcal{D}, |K| = k} \mathbb{P}\{\exists i \in K, W_i / Z_i \leq x\}
\]
and hence letting \(x \to \infty\) we have
\[
\sum_{k=1}^{d} g(k) = \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K \subseteq \mathcal{D}, |K| = k} \mathbb{P}\{\max_{i \in K} W_i / Z_i > x\}.
\]
Consequently, for any real \(x\)
\[
\sum_{k=1}^{d} g(k) \mathbb{P}\{(W/Z)_{(k)} \leq x\} = \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K \subseteq \mathcal{D}, |K| = k} \mathbb{P}\{\max_{i \in K} W_i / Z_i \geq M_{ni}\}.
\]
By the assumptions
\[
\mathbb{E}\{ \max_{1 \leq i \leq d} W_i / Z_i \} \leq \sum_{i=1}^{d} \mathbb{E}\{W_i / Z_i\} = d,
\]
hence since \(W_i / Z_i\)'s are non-negative it follows that
\[
\sum_{k=1}^{d} g(k) \mathbb{E}\{(W/Z)_{(k)}\} = \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K \subseteq \mathcal{D}, |K| = k} \mathbb{E}\{\max_{i \in K} W_i / Z_i\}.
\]
Finally, in order to retrieve Equation (2.8), we must have for any \( k \in \{1, \ldots, d\} \)
\[
\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k). 
\]

Now, assuming that for all \( k \in \{1, \ldots, d\} \), \( g(k) = f(d-k+1) - f(d-k) = \Delta f(d-k) \), then denoting by \( T = \Delta + I \) the translation operator
\[
\Delta^{k-1} g(1) = \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} T^{-i} \Delta f(d-1).
\]
This implies
\[
\Delta^{k-1} g(1) = (-I + T^{-1})^{k-1} \Delta f(d-1) = (-1)^{k-1} (T^{-1}(T-I))^{k-1} \Delta f(d-1).
\]
Thus, for all \( k \in \{1, \ldots, d\} \) we have
\[
\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k)
\]
and hence the claim follows. \( \square \)

**Proof of Proposition 3.1.** For the first equality, since \( Z \) has unit Fréchet marginals for any \( u > 0 \) we have
\[
C_Z(u, \ldots, u) = H \left( \frac{1}{-\ln u}, \ldots, \frac{1}{-\ln u} \right) = \exp \left( \mathbb{E} \left\{ \max_{1 \leq j \leq d} \ln(u) W_j \right\} \right) = u \mathbb{E} \left\{ \max_{j \in D} W_j \right\}
\]
and thus \( \delta_Z(u) = u^{1/\theta} \). Since the diagonal section of a \( d \)-dimensional Archimedean copula with parameter \( \theta \) is \( u^{d/\theta} \) we obtain the formula for \( \theta \). This is consistent with the fact that the Gumbel copula is an Extreme Value Copula (the only Archimedean one, see [19]).

For the last equality, setting \( W_j = F_{W_j}^{-1}(U_j) \), we get \( \max_{j \in D} W_j = \max_{j \in D} W_j^{-1}(U_j) \). Assuming further that all \( W_i \)'s have a common df \( F_W \), then \( \max_{j \in D} F_{W_j}^{-1}(U_j) = F_{W}^{-1}(\max_{j \in D}(U_j)) \). Using further
\[
P \left\{ \max_{j \in D} U_j \leq u \right\} = P \left\{ U_1 \leq u, \ldots, U_d \leq u \right\} = C_Y(u, \ldots, u) = \delta_Y(u)
\]
we get \( \mathbb{E} \left\{ \max_{j \in D} W_j \right\} = \int_0^1 F_W^{-1}(s) d\delta_Y(s) \). \( \square \)

**Acknowledgments:** EH is partially supported by SNSF Grant 200021-175752/1 and PSG 1250 grant. We also thank the anonymous reviewer for very helpful comments and suggestions.

**References**


Enkelejd Hashorva, Department of Actuarial Science University of Lausanne, UNIL-Dorigny, 1015 Lausanne, Switzerland
E-mail address: Enkelejd.Hashorva@unil.ch

Didier Rullière, Ecole ISFA, LSAF, université Lyon 1, 69366 Lyon, France
E-mail address: didier.rulliere@univ-lyon1.fr