ASYMPTOTIC DOMINATION OF SAMPLE MAXIMA

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Abstract: For a given random sample from some underlying multivariate distribution F we consider the domination of the component-wise maxima by some independent random vector $\boldsymbol{W}$ with distribution function G. We show that the probability that certain components of the sample maxima are dominated by the corresponding components of $\boldsymbol{W}$ can be approximated under the assumptions that both F and G are in the max-domain of attraction of some max-stable distribution functions. We study further some basic probabilistic properties of the dominated components of sample maxima by $\boldsymbol{W}$.

Key Words: Max-stable distributions; records; domination of sample maxima; extremal dependence; de Haan representation; infargmax formula;

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1. Introduction

Let $\boldsymbol{Z}_i, i \le n$ be independent d-dimensional random vectors with common continuous distribution function (df) F and denote by $\boldsymbol{M}_n$ their component-wise maxima, i.e., $M_{nj} = \max_{1 \le k \le n} Z_{kj}, j \le d$. If $\boldsymbol{W}$ is another d-dimensional random vector with continuous df G being further independent of $\boldsymbol{M}_n$ the approximation of the probability that at least one component of $\boldsymbol{W}$ dominates the corresponding component of $\boldsymbol{M}_n$ is of interest since it is related to the dependence of the components of $\boldsymbol{M}_n$, see e.g., [1]. In the special case that $\boldsymbol{W}$ has a max-stable df with unit Fréchet marginal df's $\Phi(x) = e^{-1/x}, x > 0$ and $\boldsymbol{M}_n$ has almost surely positive components, we simply have

$$\mathbb{P}\{\exists i \le d : W_i > M_{ni}\} = 1 - \mathbb{P}\{\forall i, 1 \le i \le d : M_{ni} \ge W_i\} = 1 - \mathbb{E}_{\boldsymbol{M}_n}\left\{\exp\Big(-\mathbb{E}_{\boldsymbol{\mathcal{W}}}\left\{\max_{1 \le i \le d} \frac{\mathcal{W}_i}{M_{ni}}\right\}\Big)\right\},$$

where $\boldsymbol{\mathcal{W}} = (\mathcal{W}_1, \ldots, \mathcal{W}_d)$ being independent of $\boldsymbol{M}_n$ is a spectral random vector of G which exists in view of the well-known de Haan representation, see e.g., [2] and (2.1) below. Note that the assumption that W_i has unit Fréchet df implies that $\mathbb{E}\{\mathcal{W}_i\} = 1$.

The above probability is referred to as the marginal domination probability of the sample maxima. If F is also a max-stable df with unit Fréchet marginals, then by definition $\boldsymbol{M}_n/n$ has for any $n > 0$ df F and consequently

$$n[1 - \mathbb{P}\{\forall i,\ 1 \le i \le d : M_{ni} \ge W_i\}] = n\Big[1 - \mathbb{E}_{\boldsymbol{\mathcal{Z}}}\left\{\exp\Big(-\frac{1}{n}\mathbb{E}_{\boldsymbol{\mathcal{W}}}\left\{\max_{1 \le i \le d} \frac{\mathcal{W}_i}{\mathcal{Z}_i}\right\}\Big)\right\}\Big] \sim \mathbb{E}\left\{\max_{1 \le i \le d} \frac{\mathcal{W}_i}{\mathcal{Z}_i}\right\}, \tag{1.1}$$

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where $\sim$ means asymptotic equivalence as $n \to \infty$ and $\boldsymbol{\mathcal{Z}} = (\mathcal{Z}_1, \ldots, \mathcal{Z}_d)$ has df F being further independent of $\boldsymbol{\mathcal{W}}$. Under the above assumptions, we have

$$p_{n,T}(F,G) = \mathbb{P}\{\forall i,\ 1 \le i \le d : W_i > M_{ni}\} \sim \frac{1}{n}\mathbb{E}\left\{\min_{1\le i\le d} \frac{\mathcal{W}_i}{\mathcal{Z}_i}\right\}, \quad T = \{1, \ldots, d\} \tag{1.2}$$

as $n \to \infty$, which follows by (1.1) and the inclusion-exclusion formula or directly by [1][Thm 2.5 and Prop 4.2]. Here $p_{n,T}(F,G)$ is referred to as the probability of the complete domination of sample maxima by $\boldsymbol{W}$. In the particular case that $F = G$ it is related to the probability of observing a multiple maxima or concurrence probability, see [3–9].

Between these two extreme cases, of interest is also to consider the partial domination of the sample maxima. Let therefore below $T \subset \{1, \ldots, d\}$ be non-empty and consider the probability that only the components of $\boldsymbol{W}$ with indices in T dominate $\boldsymbol{M}_n$, i.e.,

$$\mathbb{P}\{\forall i \in T : W_i > M_{ni}, \forall i \in \bar{T} : W_i \le M_{ni}\} =: p_{n,T}(F,G),$$

where $\bar{T} = \{1, \ldots, d\} \setminus T$. Note that $p_{n,T}(F,F)$ relates to the probability of observing a T-record, see [10]. By the continuity of F and G we simply have

$$p_{n,T}(F,G) = \int_{\mathbb{R}^d} \mathbb{P}\{\forall i \in T : W_i > y_i, \forall i \in \bar{T} : W_i \le y_i\}\ dF^n(\boldsymbol{y}),$$

which cannot be evaluated without knowledge of both F and G. In the particular case that F and G are max-stable df's as above, using (1.1) and the inclusion-exclusion formula we obtain

$$\lim_{n\to+\infty} n p_{n,T}(F,G) = \mathbb{E}\left\{\left(\min_{i\in T} \frac{\mathcal{W}_i}{\mathcal{Z}_i} - \max_{i\in\bar{T}} \frac{\mathcal{W}_i}{\mathcal{Z}_i}\right)_+\right\}. \tag{1.3}$$

When $F = G$ the above result is known from [10][Prop 2.2]. Moreover, in the special case that T consists of one element, then the right-hand side of (1.3) is equal to $\mathbb{P}\{C(T) \subset \bar{T}\}$, where $C(T)$ is the tessellation as determined in [11]. If we are not interested on a particular index set T, where the domination of sample maxima by $\boldsymbol{W}$ occurs but simply on the number of components being dominated, i.e., on the random variable (rv)

$$N_n = \sum_{i=1}^{d} \mathbb{1}_{\{W_i > M_{ni}\}}$$

a question of interest is if N_n can be approximated as $n \to \infty$. We have that N_n has the same distribution as

$$\sum_{i=1}^{d} \mathbb{1}_{\{W_i/n > \mathcal{Z}_i\}},$$

provided that F is max-stable as above and $\boldsymbol{\mathcal{Z}}$ has df F being further independent of $\boldsymbol{W}$. Hence if W_i's are unit Fréchet rv's, then

$$\lim_{n\to+\infty} n\mathbb{E}\{N_n\} = \sum_{i=1}^{d} \lim_{n\to+\infty} n\mathbb{P}\{W_i > n\mathcal{Z}_i\} = \sum_{i=1}^{d} \lim_{n\to+\infty} n\left[1 - e^{-\mathbb{E}\left\{\frac{1}{n\mathcal{Z}_i}\right\}}\right] = d.$$

Consequently, the expected number of components of sample maxima being dominated by the components of $\boldsymbol{W}$ decreases as d/n when n goes to infinity. Moreover, the dependence of both $\boldsymbol{W}$ and $\boldsymbol{M}_n$ does not play any role. This is however in general not the case for the expectation of $f(N_n)$, where f is some real-valued function, since the dependence of both $\boldsymbol{M}_n$ and $\boldsymbol{W}$ influence the approximation as we shall show in the next section.

From our discussion above the assumptions that F and G are max-stable df's with unit Fréchet marginals lead to tractable asymptotic formulas for various quantities related to the domination of sample maxima $\boldsymbol{M}_n$ by $\boldsymbol{W}$.

In view of [1] we know that both (1.1) and (1.2) are valid in the more general setup that both F and G are in the max-domain of attraction of some max-stable df's (see next section for details). We shall show in this paper that the same assumptions lead to tractable approximations of both $p_{n,T}(F,G)$ and $\mathbb{E}\{f(N_n)\}$ as $n \to \infty$.

Brief organisation of the paper: Section 2 presents the main results concerning the approximations of the marginal domination probabilities and the expectation of $f(N_n)$. Section 3 is dedicated to properties of $\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}}$ which we call the domination spectral vector. All the proofs are relegated to Section 4.

2. Main Results

We shall recall first some basic properties of max-stable df's, see [2, 12–14] for details. A d.dimensional df $\mathcal{G}$ is max-stable with unit Fréchet marginals if

$$\mathcal{G}^t(tx_1, \ldots, tx_d) = \mathcal{G}(x_1, \ldots, x_d)$$

for any $t > 0, x_i \in (0, \infty), 1 \le i \le d$. In the light of De Haan representation

$$\mathcal{G}(\boldsymbol{x}) = \exp\Big(-\mathbb{E}\{\max_{1 \le j \le d} \mathcal{W}_j/x_j\}\Big), \quad \boldsymbol{x} = (x_1, \ldots, x_d) \in (0,\infty)^d, \tag{2.1}$$

where $\mathcal{W}_j$'s are non-negative rv's with $\mathbb{E}\{\mathcal{W}_j\} = 1, j \le d$ and $\boldsymbol{\mathcal{W}} = (\mathcal{W}_1, \ldots, \mathcal{W}_d)$ is a spectral vector for $\mathcal{G}$ (which is not unique).

In view of multivariate extreme value theory, see e.g., [14] d-dimensional max-stable df's $\mathcal{F}$ are limiting df's of the component-wise maxima of d-dimensional iid random vectors with some df F. In that case, F is said to be in the max-domain of attraction (MDA) of $\mathcal{F}$, abbreviated $F \in MDA(\mathcal{F})$. For simplicity we shall assume throughout in the following that F has marginal df's F_i's such that

$$\lim_{n \to +\infty} F_i^n(nx) = \Phi(x), \quad x \in \mathbb{R} \tag{2.2}$$

for all $i \le d$, where we set $\Phi(x) = 0$ if $x \le 0$. We have thus that $F \in MDA(\mathcal{F})$ if further

$$\lim_{n \to +\infty} \sup_{x_i \in \mathbb{R}, 1 \le i \le d} \Big| F^n(nx_1, \ldots, nx_d) - \mathcal{F}(x_1, \ldots, x_d) \Big| = 0. \tag{2.3}$$

In the following $\mathcal{F}$ is a d-dimensional max-stable df of some random vector $\boldsymbol{\mathcal{Z}}$ with unit Fréchet marginals and $\mathcal{G}$ is another max-stable df with unit Fréchet marginals and spectral random vector $\boldsymbol{\mathcal{W}}$ independent of $\boldsymbol{\mathcal{Z}}$.

Below we extend [15][Prop 1] which considers the case $F = G$.

Proposition 2.1. *If F and G have continuous marginal distributions satisfying (2.2) and $F \in MDA(\mathcal{F}), G \in MDA(\mathcal{G})$, then for any non-empty $T \subset \{1, \ldots, d\}$ we have*

$$\lim_{n \to +\infty} n p_{n,T}(F,G) \quad = \quad \mathbb{E}\left\{ \Big(\min_{i \in T} \mathcal{W}_i/\mathcal{Z}_i - \max_{i \in \bar{T}} \mathcal{W}_i/\mathcal{Z}_i \Big)_+ \right\} =: \lambda_T(\mathcal{F}, \mathcal{G}). \tag{2.4}$$

Remark 2.2. *Define for a non-emtpy index set T the rv $K_n = \sum_{j=1}^n \mathbb{1}_{\{\forall i \in T: W_i > M_{ji}, \forall i \in \bar{T}: W_i \le M_{ji}\}}$. Under the assumptions of Proposition 2.1 we have (see also [16][Corr 3.2]) that*

$$\lim_{n \to +\infty} \frac{\mathbb{E}\{K_n\}}{\ln n} = \lambda_T(\mathcal{F}, \mathcal{G}). \tag{2.5}$$

Example 2.3 ($\mathcal{F}$ comonotonic and $\mathcal{G}$ a product df)**.** *Suppose that $\mathcal{F}$ is comonotonic, i.e., $\mathcal{Z}_1 = \cdots = \mathcal{Z}_d$ almost surely and let $\mathcal{G}$ be a product df with unit Fréchet marginals df's and let N be rv on $\{1,\ldots,d\}$ with $\mathbb{P}\{N=i\} = 1/d, i \le d$. A spectral vector $\boldsymbol{\mathcal{W}}$ for $\mathcal{G}$ can be defined as follows*

$$(\mathcal{W}_1,\ldots,\mathcal{W}_d) = (d\mathbb{1}_{\{N=1\}},\ldots,d\mathbb{1}_{\{N=d\}}).$$

Indeed $\mathbb{E}\{\mathcal{W}_k\} = d\mathbb{P}\{N=k\} = 1$ for any $k \le d$ and

$$\mathbb{E}\{\max_{1\le i\le d} \mathcal{W}_i/x_i\} = \sum_{k=1}^d \mathbb{E}\{\max_{1\le i\le d} \mathcal{W}_i/x_i \mathbb{1}_{\{N=k\}}\} = \sum_{k=1}^d \mathbb{E}\{\mathcal{W}_k/x_k\mathbb{1}_{\{N=k\}}\} = d\sum_{k=1}^d \mathbb{E}\{\mathbb{1}_{\{N=k\}}/x_k\} = \sum_{k=1}^d 1/x_k$$

for any $x_1,\ldots,x_d$ positive. In particular, for a non-empty index set $K \subset \{1,\ldots,d\}$ with m elements we have

$$\mathbb{E}\{\max_{i\in K} \mathcal{W}_i\} = d\sum_{k\in K} \mathbb{E}\{\mathbb{1}_{\{N=k\}}\} = m.$$

Consequently, using further that (see the proof of Proposition 2.1)

$$\lambda_T(\mathcal{F},\mathcal{G}) = \sum_{j=0}^k (-1)^{j+1} \sum_{J\subset T:|J|=j} \mathbb{E}\left\{\max_{i\in J\cup \bar{T}} \frac{\mathcal{W}_i}{\mathcal{Z}_i}\right\}$$

we obtain

$$\lambda_T(\mathcal{F},\mathcal{G}) = \sum_{j=0}^k (-1)^{j+1} \sum_{J\subset T:|J|=j} \mathbb{E}\left\{\max_{i\in J\cup \bar{T}} \mathcal{W}_i\right\} = \sum_{j=0}^k (-1)^{j+1} \sum_{J\subset T:|J|=j} (j+d-k).$$

If $k=d$, then from above

$$\lambda_T(\mathcal{F},\mathcal{G}) = \sum_{j=0}^d (-1)^{j+1} \sum_{J\subset T:|J|=j} j = d(1-1)^{d-1} = 0. \tag{2.6}$$

A direct probabilistic proof of (2.6) follows by the properties of $\boldsymbol{\mathcal{W}}$, namely when $k=d\ge 2$

$$\lambda_T(\mathcal{F},\mathcal{G}) = \mathbb{E}\{\min_{1\le i\le d} \mathcal{W}_i/\mathcal{Z}_i\} = \mathbb{E}\{\min_{1\le i\le d} \mathcal{W}_i\} = d\mathbb{E}\{\min_{1\le i\le d} \mathbb{1}_{\{N=i\}})\} = 0.$$

Now, let us investigate the number N_n of dominations defined as in Introduction by $\sum_{i=1}^d \mathbb{1}_{\{W_i/n>Z_i\}}$. For a given function $f:\{0,\ldots,d\}\to\mathbb{R}$ we shall be concerned with the behaviour of

$$\mathbb{E}\{f(N_n)\} = \sum_{k=0}^d f(k)\mathbb{P}\{N_n=k\}$$

when n tends to $+\infty$. Throughout in the sequel we set

$$\mathcal{D} = \{1,\ldots,d\}.$$

In Proposition 2.4 below, we first express this expectation as a function of minima or maxima of $\mathcal{W}_i/\mathcal{Z}_i$'s.

Proposition 2.4. *If F and G are as in Proposition 2.1, then we have*

$$\lim_{n\to+\infty} n\mathbb{E}\{f(N_n)\} - nf(0) = \sum_{k=1}^d \Delta^k f(0) \sum_{K\subset\mathcal{D},|K|=k} \mathbb{E}\left\{\min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i\right\} \tag{2.7}$$

or alternatively

$$\lim_{n\to+\infty} n\mathbb{E}\{f(N_n)\} - nf(0) = \sum_{k=1}^d (-1)^{k+1}\Delta^k f(d-k) \sum_{K\subset\mathcal{D},|K|=k} \mathbb{E}\left\{\max_{i\in K} \mathcal{W}_i/\mathcal{Z}_i\right\}, \tag{2.8}$$

where Δ is the difference operator, $\Delta f(x) = f(x+1) - f(x)$.

Proposition 2.5. *If F and G are as in Proposition 2.1, then we have*

$$\lim_{n\to+\infty} n\mathbb{E}\left\{f(N_n)\right\} - nf(0) = \sum_{k=1}^{d} g(k)\mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(k)}\right\} = \sum_{k=0}^{d} f(k)\left[(\mathcal{W}/\mathcal{Z})_{(d-k+1)} - (\mathcal{W}/\mathcal{Z})_{(d-k)}\right], \tag{2.9}$$

where $(\mathcal{W}/\mathcal{Z})_{(1)} \le \ldots \le (\mathcal{W}/\mathcal{Z})_{(d)}$ are the order statistics of $\mathcal{W}_i/\mathcal{Z}_i, i \le d$ and $g(k) = f(d-k+1) - f(d-k)$, with the convention $(\mathcal{W}/\mathcal{Z})_{(0)} = (\mathcal{W}/\mathcal{Z})_{(d+1)} = 0$.

Remark 2.6 (retrieving simple cases)**.** *For particular cases of f we have:*

- *From Proposition 2.4, setting $f(x) = \mathbb{1}_{\{x=d\}}$, one can check that $\Delta^k f(0) = 0$ when $k < d$ and $\Delta^d f(0) = 1$, so that Equation (2.7) implies (1.2). Alternatively, by Proposition 2.5 since $g(1) = f(d) - f(d-1) = 1$ and $g(k) = f(d-k+1) - f(d-k) = 0 - 0 = 0$ if $k > 1$ we have that $\lim_{n\to+\infty} n\mathbb{E}\left\{f(N_n)\right\} - nf(0) = \mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(1)}\right\}$.*
- *In view of Proposition 2.4, setting $f(x) = \mathbb{1}_{\{x\ge 1\}}$, $\Delta^k f(d-k) = \sum_{i=0}^{k}\binom{k}{i}(-1)^{k-i} f(d-k+i)$. Thus $\Delta^k f(d-k) = 0$ if $k < d$. If $k = d$, then*

  $$\Delta^k f(d-k) = \Delta^d f(0) = (1-1)^d - (-1)^d = (-1)^{d+1}$$

  and Equation (2.8) implies (1.1). Alternatively, by Proposition 2.5 since if $k < d$, $g(k) = f(d-k+1) - f(d-k) = 1 - 1 = 0$ and $g(d) = f(1) - f(0) = 1$ we obtain $\lim_{n\to+\infty} n\mathbb{E}\left\{f(N_n)\right\} - nf(0) = \mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(d)}\right\}$.
- *By Proposition 2.5, setting $f(x) = x$, we easily retrieve $\lim_{n\to+\infty} n\mathbb{E}\left\{N_n)\right\} = \sum_{k=1}^{d} \mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(k)}\right\} = d$, as seen previously.*

Remark 2.7 (Interpretation of $(\mathcal{W}/\mathcal{Z})_{(j)}$)**.** *Let $f(k) = \mathbb{1}_{\{k\ge d-j+1\}}$, for any $j, k \in \mathcal{D}$. Then $g(k) = f(d-k+1) - f(d-k) = \mathbb{1}_{\{k=j\}}$. In this case, $f(0) = 0$ and $\mathbb{E}\left\{f(N_n)\right\} = \mathbb{P}\left\{N \ge d-j+1\right\}$, thus*

$$\mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(j)}\right\} \quad = \quad \lim_{n\to+\infty} n\mathbb{P}\left\{N_n \ge d-j+1\right\}.$$

3. Domination spectrum

In the previous results, we have considered a particular setting, and we have expressed the domination probability and some expectations relying on number of dominations (see Section 2). We have seen that all these results were expressed as a function of

$$\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}} = \left(\frac{\mathcal{W}_i}{\mathcal{Z}_i}\right)_{i\in\mathcal{D}}.$$

By the definition $\mathcal{W}_i/\mathcal{Z}_i$'s are nonnegative, and are such that, by independence, $\mathbb{E}\left\{\mathcal{W}_i/\mathcal{Z}_i\right\} = \mathbb{E}\left\{\mathcal{W}_i\right\}\mathbb{E}\left\{\frac{1}{\mathcal{Z}_i}\right\} = 1$. Thus in view of the De Haan representation $\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}}$ can be viewed as the spectral random vector of some max-stable d-dimensional distribution. Since $\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}}$ is related to the domination of $\boldsymbol{M}_n$ by $\boldsymbol{W}$, we will refer to it by the term *domination spectrum*. In this section we shall explore some basic properties of the domination spectrum.

Next, assume that $\boldsymbol{\mathcal{W}}$ has a copula $C_{\boldsymbol{\mathcal{W}}}$ and suppose further that $\boldsymbol{\mathcal{Z}}$ has a copula $C_{\boldsymbol{\mathcal{Z}}}$. Note in passing that the latter copula is unique since the marginals of $\boldsymbol{\mathcal{Z}}$ have continuous df.

We shall first study the link between the diagonal sections of both copulas $C_{\boldsymbol{\mathcal{W}}}$ and $C_{\boldsymbol{\mathcal{Z}}}$, defined for all $u \in [0,1]$ by

$$\delta_{\boldsymbol{\mathcal{W}}}(u) = C_{\boldsymbol{\mathcal{W}}}(u,\ldots,u) \quad \text{and} \quad \delta_{\boldsymbol{\mathcal{Z}}}(u) = C_{\boldsymbol{\mathcal{Z}}}(u,\ldots,u).$$

We recall that the diagonal section characterizes uniquely many Archimedean copulas (under a condition that is called Frank's condition, see e.g., [17]), some non-parametric estimators of the generator of an Archimedean copulas directly rely on this diagonal section. We consider here the case where the df of $\boldsymbol{\mathcal{Z}}$ has spectral random vector $\boldsymbol{\mathcal{W}}$. Notice that the upper tail dependence coefficients can be deduced from the regular variation properties of $\delta_{\boldsymbol{\mathcal{Z}}}$ and $\delta_{\boldsymbol{\mathcal{W}}}$, which is straightforward for $\delta_{\boldsymbol{\mathcal{Z}}}$ in the following result.

Proposition 3.1. *Consider a d-dimensional random vector $\boldsymbol{\mathcal{Z}}$ having max-stable df with Fréchet unit marginals and with copula $C_{\boldsymbol{\mathcal{Z}}}$. If the random vector $\boldsymbol{\mathcal{Z}}$ has df $H(\boldsymbol{y}) = \exp(-\mathbb{E}\{\max_{1\le j\le d} \frac{\mathcal{W}_j}{y_j}\})$, where all $\mathcal{W}_j$ are nonnegative rv's with mean 1, then*

$$\delta_{\boldsymbol{\mathcal{Z}}}(u) = u^{r_{\boldsymbol{\mathcal{W}}}} \quad \text{with} \quad r_{\boldsymbol{\mathcal{W}}} = \mathbb{E}\left\{\max_{j\in\mathcal{D}} \mathcal{W}_j\right\}.$$

In particular, when $r_{\boldsymbol{\mathcal{W}}} > 1$, this diagonal section $\delta_{\boldsymbol{\mathcal{Z}}}(u)$ is the one of a Gumbel copula with parameter

$$\theta = \frac{\ln d}{\ln r_{\boldsymbol{\mathcal{W}}}}. \tag{3.1}$$

Furthermore, if the components of $\boldsymbol{\mathcal{W}}$ are identically distributed and if $F_{\mathcal{W}_1}$ is invertible, then we have

$$r_{\boldsymbol{\mathcal{W}}} = \int_0^1 F_{\mathcal{W}_1}^{-1}(s) d\delta_{\boldsymbol{\mathcal{W}}}(s).$$

Example 3.2 (From independence to comonotonicity). *Let $\mathcal{W}_j = Bd\mathbb{1}_{\{I=j\}} + (1-B)\delta_1$, for all $j\in\mathcal{D}$, where I is a uniformely distributed rv's on $\mathcal{D}$, B is a Bernoulli rv with $\mathbb{E}\{B\} = \alpha \in (0,1]$ and δ_1 is a Dirac mass at 1, all these rv's being mutually independent. In this case, $r_{\boldsymbol{\mathcal{W}}} = \mathbb{E}\left\{\max_{j\in\mathcal{D}} \mathcal{W}_j\right\}$ in Proposition 3.1 becomes $r_{\boldsymbol{\mathcal{W}}} = \alpha d + 1 - \alpha$. As a consequence, $\delta_{\boldsymbol{\mathcal{Z}}}$ is the diagonal of a Gumbel copula which goes from the independence ($\alpha = 1$) to the comonotonicity ($\alpha \to 0$), with parameter*

$$\theta = \frac{\ln d}{\ln(1 + \alpha(d-1))}.$$

Furthermore, we have when all $t_j > 0$,

$$\mathbb{E}\left\{\max_{j\in K} \frac{\mathcal{W}_j}{t_j}\right\} = \alpha \sum_{j\in K} \frac{1}{t_j} + (1-\alpha)\frac{1}{\min_{j\in K} t_j}.$$

Let $t > 0$ and suppose that K has cardinal $|K| > 1$. By conditioning over B, we get

$$\mathbb{P}\{\forall i\in K, \mathcal{W}_i/\mathcal{Z}_i > t\} = (1-\alpha)\mathbb{P}\left\{\forall i\in K, \mathcal{Z}_i < 1/t \;\middle|\; B = 0\right\}$$

since $\mathbb{P}\left\{\forall i\in K, \mathcal{W}_i/\mathcal{Z}_i > t \;\middle|\; B = 1\right\} = 0$ when $|K| > 1$, because in this case at least one component $\mathcal{W}_i$, $i \in K$, is zero when $B = 1$. Recall that $\boldsymbol{\mathcal{Z}}$ is independent from $\boldsymbol{\mathcal{W}}$ and B, thus for $t > 0$ and $|K| > 1$

$$\mathbb{P}\left\{\min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i > t\right\} = (1-\alpha)\exp\left(\mathbb{E}\left\{-\max_{j\in K}\frac{\mathcal{W}_j}{(1/t)}\right\}\right) = (1-\alpha)\exp\left(-t(1+\alpha|K|-\alpha)\right).$$

When $|K| = 1$, we show similarly that $\mathbb{P}\{\min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i > t\} = (1-\alpha)\exp(-t) + \alpha\frac{1}{d}\exp(-\frac{t}{d})$. In both cases $|K| = 1$ and $|K| > 1$, the survival function $\mathbb{P}\{\min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i > t\}$ is a linear combination of exponential functions, and thus can be shown to be a discrete mixture of exponential distributions:

$$\begin{cases} \min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i & \stackrel{d}{=} \quad (1-B)\epsilon_{1+\alpha(|K|-1)} + \mathbb{1}_{\{|K|=1\}} B \mathbb{1}_{\{I=1\}}\epsilon_{1/d} \\ \mathbb{E}\left\{\min_{i\in K} \mathcal{W}_i/\mathcal{Z}_i\right\} & = \quad \frac{1-\alpha}{1+\alpha(|K|-1)} + \mathbb{1}_{\{|K|=1\}}\alpha, \end{cases}$$

where B is a Bernoulli r.v. of parameter α, $\epsilon_{1+\alpha(|K|-1)}$ and $\epsilon_{1/d}$ are exponentially distributed r.v. with respective parameters $1+\alpha(|K|-1)$ and $1/d$, I an uniformly distributed r.v. over $\mathcal{D}$, all being mutually independent (for simplicity, we denote $\mathbb{1}_{\{|K|=1\}}$ the variable whose value is 1 if $|K|=1$ or 0 otherwise). Then all results about the limit law of N_n follow immediately, using Equation (2.7) in Proposition 2.4. Notice that one could also determine $r_{(\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}})}$ from this, and by application of Proposition 3.1, assess the dependence structure of the random vector whose spectrum is $(\boldsymbol{\mathcal{W}}/\boldsymbol{\mathcal{Z}})$.

4. Proofs

We first give hereafter some combinatorial results that show how quantities depending on a number of events can be related to quantities involving only intersections or unions of those events. This generalizes inclusion-exclusion formulas that will correspond to very specific functions f and g.

Lemma 4.1 (Inclusion-exclusion relations)**.** *Let $\mathcal{D}=\{1,\dots,d\}$ and let B_i, $i\in\mathcal{D}$ be events. Consider the number of realized events $N=\sum_{i\in\mathcal{D}}\mathbb{1}_{\{B_i\}}$. Then for any function $f:\{0,\dots,d\}\to\mathbb{R}$*

$$\sum_{k=0}^{d} f(k)\mathbb{P}\{N=k\} \;=\; f(0)+\sum_{j=1}^{d} S_j\Delta^j f(0) \;=\; f(0)+\sum_{j=1}^{d}\bar{S}_j(-1)^{j+1}\Delta^j f(d-j) \tag{4.1}$$

and similarly for any function $g:\mathcal{D}\to\mathbb{R}$

$$\sum_{k=0}^{d} g(k)\mathbb{P}\{N\geq k\} \;=\; \sum_{j=1}^{d} S_j\Delta^{j-1}g(1) \;=\; \sum_{j=1}^{d}\bar{S}_j(-1)^{j+1}\Delta^{j-1}g(d-j+1), \tag{4.2}$$

where $S_j=\sum_{J\subset\mathcal{D},|J|=j}\mathbb{P}\left\{\bigcap_{i\in J}B_i\right\}$ and $\bar{S}_j=\sum_{J\subset\mathcal{D},|J|=j}\mathbb{P}\left\{\bigcup_{i\in J}B_i\right\}$.

Proof of Lemma 4.1. The first equality in Equation (4.1) is known in actuarial sciences under the name of Schuette-Nesbitt formula, see [18, section 8.5]. This formula does not require any independence assumption, it is a simple development of $f(N)=(I+\mathbb{1}_{\{B_1\}}\Delta)\cdots(I+\mathbb{1}_{\{B_d\}}\Delta)f(0)$ where I and Δ are the identity and the difference operators respectively. To prove the second equality in Equation (4.1), let us denote $p_J=\mathbb{P}\{\cap_{i\in J}B_i\}$ and $\bar{p}_J=\mathbb{P}\{\cup_{i\in J}B_i\}$. By inclusion-exclusion principle, we get

$$S_k=\sum_{K\subset\mathcal{D},|K|=k}\sum_{j=1}^{k}(-1)^{j+1}\sum_{J\subset K,|J|=j}\bar{p}_J=\sum_{j=1}^{k}(-1)^{j+1}\binom{d-j}{k-j}\bar{S}_j. \tag{4.3}$$

Now using Equation (4.3),

$$\sum_{k=1}^{d}\Delta^k f(0)S_k=\sum_{k=1}^{d}\Delta^k f(0)\sum_{j=1}^{k}(-1)^{j+1}\binom{d-j}{k-j}\bar{S}_j=\sum_{j=1}^{d}\bar{S}_j(-1)^{j+1}\Delta^j(I+\Delta)^{d-j}f(0)\,,$$

and since $(I+\Delta)^{d-j}f(0)=f(d-j)$, the second equality in Equation (4.1) holds. Similarly, the first equality in Equation (4.2) is a known Schuette-Nesbitt formula, see [18, Section 8.5], and one can retrieve the second equality by using Equation (4.3). Alternatively, one can also deduce (4.2) from (4.1) by setting $f(0)=0$ and $g(k)=\Delta f(k-1)$ for all $k\in\mathcal{D}$. The formulas in Lemma 4.1 generalize a very old formula of Waring which give $\mathbb{P}\{N=k\}$, $k\in\mathcal{D}$. They also generalize the classical inclusion exclusion formula which can be retrieved by setting in (4.1) $f(k)=1$ if $k\geq 1$, and $f(k)=0$ otherwise. ◻

Proof of Proposition 2.1. By inclusion-exclusion formula for a given index set $T \subset \{1,\ldots,d\}$ with $k = |T|$ elements we have

$$\begin{aligned}\mathbb{P}\{\forall i \in \bar{T}: W_i \le y_i, \exists i \in T: W_i \le y_i\} &= \sum_{j=1}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=j}\mathbb{P}\{\forall i \in (J\cup\bar{T}): W_i \le \mathcal{W}_i\}\\ &= \sum_{j=1}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=j} G_{J\cup\bar{T}}(\boldsymbol{y}),\end{aligned}$$

where $G_L(\boldsymbol{y}) = \mathbb{P}\{\forall i \in L : W_i \le y_i\}$ is the L-th marginal df of G. In particular, letting $\mathcal{W}_i \to \infty, i \le d$ yields

$$1 = \sum_{j=1}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=j} 1.$$

Consequently, for all $n > 1$

$$\begin{aligned}p_{n,T}(F,G) &= \int_{\mathbb{R}^d}\mathbb{P}\{\forall i \in T: W_i \ge y_i, \forall i \in \bar{T}: W_i < y_i\}\, dF^n(\boldsymbol{y})\\ &= \int_{\mathbb{R}^d}\mathbb{P}\{\forall i \in \bar{T}: W_i \le y_i\}dF^n(\boldsymbol{y}) - \int_{\mathbb{R}^d}\mathbb{P}\{\forall i \in \bar{T}: W_i \le y_i, \exists i \in T: W_i \le y_i\}dF^n(\boldsymbol{y})\\ &= 1 - \int_{\mathbb{R}^d}\sum_{j=1}^{k}(-1)^{j+1}\sum_{J\subset K:|J|=j} G_{J\cup\bar{T}}(\boldsymbol{y})dF^n(\boldsymbol{y}) - \Big(1 - \int_{\mathbb{R}^d} G_{\bar{T}}(\boldsymbol{y})dF^n(\boldsymbol{y})\Big)\\ &= \sum_{j=0}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=i}\int_{\mathbb{R}^d}[1 - G_{J\cup\bar{T}}(\boldsymbol{y})]dF^n(\boldsymbol{y})\\ &= \sum_{j=0}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=j}\int_{\mathbb{R}^{m+i}}[1 - G_{J\cup\bar{T}}(\boldsymbol{y})]dF^n_{J\cup\bar{T}}(\boldsymbol{y}).\end{aligned}$$

In view of [1][Prop 4.2] we obtain

$$\lim_{n\to+\infty} n\int_{\mathbb{R}^{m+|J|}}[1 - G_{J\cup\bar{T}}(\boldsymbol{y})]dF^n_{J\cup\bar{T}}(\boldsymbol{y}) = -\int_{\mathbb{R}^{m+|J|}}\ln Q_{J\cup\bar{T}}(\boldsymbol{y})dH_{J\cup\bar{T}}(\boldsymbol{y}).$$

Further by [1][Thm 2.5 and Prop 4.2]

$$-\int_{\mathbb{R}^{m+|J|}}\ln Q_{J\cup\bar{T}}(\boldsymbol{y})dH_{J\cup\bar{T}}(\boldsymbol{y}) = \mathbb{E}\Big\{\max_{i\in J\cup\bar{T}}\frac{\mathcal{W}_i}{\mathcal{Z}_i}\Big\}.$$

Consequently, we have

$$\lim_{n\to+\infty} np_{n,T}(F,G) = \sum_{j=0}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=j}\mathbb{E}\Big\{\max_{i\in J\cup\bar{T}}\frac{\mathcal{W}_i}{\mathcal{Z}_i}\Big\}.$$

In the light of [10][Lem 1] for given constants $c_1,\ldots,c_d$

$$\sum_{j=0}^{k}(-1)^{j+1}\sum_{J\subset T:|J|=i}\max_{i\in J\cup\bar{T}} c_i = \max\Big(\max_{i\in\bar{T}} c_i, \min_{i\in T} c_i\Big) - \max_{i\in\bar{T}} c_i = \Big(\min_{i\in T} c_i - \max_{i\in\bar{T}} c_i\Big)_+$$

implying the claim.

Alternatively, we have using again inclusion-exclusion formula

$$\begin{aligned}p_{n,T}(F,G) &= \int_{\mathbb{R}^d}\mathbb{P}\{w_i \ge M_i, i \in T, \quad w_i < M_i, i \in \bar{T}\}\, dG(\boldsymbol{w})\\ &= \int_{\mathbb{R}^d}\mathbb{P}\{M_i \le w_i, i \in T\}dG(\boldsymbol{w}) - \int_{\mathbb{R}^d}\mathbb{P}\{M_i \le w_i, i \in T, \exists i \in \bar{T}: M_i \le w_i\}dG(\boldsymbol{w})\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^d} F_T^n(\boldsymbol{w}) dG_T(\boldsymbol{w}) - \int_{\mathbb{R}^d} \sum_{j=1}^{m} (-1)^{j+1} \sum_{J\subset \bar{T}:|J|=j} F_{J\cup T}^n(\boldsymbol{w}) dG(\boldsymbol{w}) \\
&= \sum_{j=0}^{d-k} (-1)^j \sum_{J\subset \bar{T}:|J|=j} \int_{\mathbb{R}^{k+j}} F_{J\cup T}^n(\boldsymbol{w}) dG_{J\cup T}(\boldsymbol{w}).
\end{aligned}
$$

Applying [1][Thm 2.5 and Prop 4.2] we obtain

$$
\lim_{n\to+\infty} n \int_{\mathbb{R}^{k+i}} F_{J\cup T}^n(\boldsymbol{y}) dG_{J\cup T}(\boldsymbol{y}) = \mathbb{E}\Big\{ \min_{i\in J\cup T} \frac{\mathcal{W}_i}{\mathcal{Z}_i} \Big\}
$$

and thus

$$
\mu_T(H,Q) \quad = \quad \sum_{j=0}^{d-k} (-1)^i \sum_{J\subset \bar{T}:|J|=j} \mathbb{E}\Big\{ \min_{i\in J\cup T} \frac{\mathcal{W}_i}{\mathcal{Z}_i} \Big\}. \tag{4.4}
$$

By [10][Lem 1] we obtain further

$$
\mu_T(H,Q) = \mathbb{E}\Big\{ \min_{i\in T} \frac{\mathcal{W}_i}{\mathcal{Z}_i} - \min(\min_{i\in T} \frac{\mathcal{W}_i}{\mathcal{Z}_i}, \max_{i\in \bar{T}} \frac{\mathcal{W}_i}{\mathcal{Z}_i}) \Big\},
$$

hence the proof is complete. □

Proof of Proposition 2.4. In view of the first equality in Equation (4.1)

$$
\mathbb{E}\{f(N_n)\} = f(0) + \sum_{k=1}^{d} \Delta^k f(0) \sum_{K\subset \mathcal{D},|K|=k} \mathbb{P}\{\forall i \in K, W_i \ge M_{ni}\}.
$$

Alternatively, using the second equality in Equation (4.1)

$$
\mathbb{E}\{f(N_n)\} = f(0) + \sum_{k=1}^{d} (-1)^{k+1} \Delta^k f(d-k) \sum_{K\subset \mathcal{D},|K|=k} \mathbb{P}\{\exists i \in K, W_i \ge M_{ni}\}.
$$

Thus using (1.1) establishes the claim. □

Proof of Proposition 2.5. Let us consider $\mathbb{P}\left\{(\mathcal{W}/\mathcal{Z})_{(k)} \le x\right\} = \mathbb{P}\{\text{at least k events } [\mathcal{W}_i/\mathcal{Z}_i \le x] \text{ are realized}, i \in \mathcal{D}\}$. Using the first equality in Equation (4.2), for any function $g : \{1,\dots,d\} \to \mathbb{R}$ we obtain

$$
\sum_{k=1}^{d} g(k) \mathbb{P}\left\{(\mathcal{W}/\mathcal{Z})_{(k)} \le x\right\} \quad = \quad \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K\subset \mathcal{D},|K|=k} \mathbb{P}\{\forall i \in K, \mathcal{W}_i/\mathcal{Z}_i \le x\}
$$

and hence letting $x \to \infty$ we have

$$
\sum_{k=1}^{d} g(k) \quad = \quad \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K\subset \mathcal{D},|K|=k} 1.
$$

Consequently, for any real x

$$
\sum_{k=1}^{d} g(k) \mathbb{P}\left\{(\mathcal{W}/\mathcal{Z})_{(k)} > x\right\} \quad = \quad \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K\subset \mathcal{D},|K|=k} \mathbb{P}\left\{\max_{i\in K} \mathcal{W}_i/\mathcal{Z}_i > x\right\}.
$$

By the assumptions

$$
\mathbb{E}\{\max_{1\le i\le d} \mathcal{W}_i/\mathcal{Z}_i\} \le \sum_{i=1}^{d} \mathbb{E}\{\mathcal{W}_i/\mathcal{Z}_i\} = d,
$$

hence since $\mathcal{W}_i/\mathcal{Z}_i$'s are non-negative it follows that

$$
\sum_{k=1}^{d} g(k) \mathbb{E}\left\{(\mathcal{W}/\mathcal{Z})_{(k)}\right\} \quad = \quad \sum_{k=1}^{d} \Delta^{k-1} g(1) \sum_{K\subset \mathcal{D},|K|=k} \mathbb{E}\left\{\max_{i\in K} \mathcal{W}_i/\mathcal{Z}_i\right\}.
$$

Finally, in order to retrieve Equation (2.8), we must have for any $k \in \{1, \ldots, d\}$

$$\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k)\,.$$

Now, assuming that for all $k \in \{1, \ldots, d\}$, $g(k) = f(d-k+1) - f(d-k) = \Delta f(d-k)$, then denoting by $T = \Delta + I$ the translation operator

$$\Delta^{k-1} g(1) \quad = \quad \sum_{i=0}^{k-1} \binom{k-1}{i} (-1)^{k-1-i} T^{-i} \Delta f(d-1).$$

This implies

$$\Delta^{k-1} g(1) = (-I + T^{-1})^{k-1} \Delta f(d-1) = (-1)^{k-1} (T^{-1}(T-I))^{k-1} \Delta f(d-1)\,.$$

Thus, for all $k \in \{1, \ldots, d\}$ we have

$$\Delta^{k-1} g(1) = (-1)^{k+1} \Delta^k f(d-k)$$

and hence the claim follows. □

Proof of Proposition 3.1. For the first equality, since $\boldsymbol{Z}$ has unit Fréchet marginals for any $u > 0$ we have

$$C_{\boldsymbol{Z}}(u, \ldots, u) = H\left(\frac{1}{-\ln u}, \ldots, \frac{1}{-\ln u}\right) = \exp\left(\mathbb{E}\left\{\max_{1 \le j \le d} \ln(u) \mathcal{W}_j\right\}\right) = u^{\mathbb{E}\left\{\max_{j \in \mathcal{D}} \mathcal{W}_j\right\}}$$

and thus $\delta_{\boldsymbol{Z}}(u) = u^{r_{\boldsymbol{Y}}}$. Since the diagonal section of a d-dimensional Archimedean copula with parameter θ is $u^{d^{1/\theta}}$ we obtain the formula for θ. This is consistent with the fact that the Gumbel copula is an Extreme Value Copula (the only Archimedean one, see [19]).

For the last equality, setting $\mathcal{W}_j = F^{-1}_{\mathcal{W}_1}(U_j)$, we get $\max_{j \in \mathcal{D}} \mathcal{W}_j = \max_{j \in \mathcal{D}} F^{-1}_{\mathcal{W}_1}(U_j)$. Assuming further that all $\mathcal{W}_i$'s have a common df $F_{\mathcal{W}_1}$, then $\max_{j \in \mathcal{D}} F^{-1}_{\mathcal{W}_1}(U_j) = F^{-1}_{\mathcal{W}_1}(\max_{j \in \mathcal{D}}(U_j))$. Using further

$$\mathbb{P}\left\{\max_{j \in \mathcal{D}} U_j \le u\right\} = \mathbb{P}\left\{U_1 \le u, \ldots U_d \le u\right\} = C_{\boldsymbol{Y}}(u, \ldots, u) = \delta_{\boldsymbol{Y}}(u)$$

we get $\mathbb{E}\left\{\max_{j \in \mathcal{D}} \mathcal{W}_j\right\} = \int_0^1 F^{-1}_{\mathcal{W}_1}(s) d\delta_{\boldsymbol{Y}}(s)$. □

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