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Symplectic analysis of time-frequency spaces

Elena Cordero[a,*], Gianluca Giacchi[b,c,d,e]

[a] Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino, Italy
[b] Università di Bologna, Dipartimento di Matematica, Piazza di Porta San Donato 5, 40126 Bologna, Italy
[c] Institute of Systems Engineering, School of Engineering, HES-SO Valais-Wallis, Rue de l'Industrie 21, 1950 Sion, Switzerland
[d] Lausanne University Hospital and University of Lausanne, Lausanne, Department of Diagnostic and Interventional Radiology, Rue du Bugnon 46, Lausanne 1011, Switzerland
[e] The Sense Innovation and Research Center, Avenue de Provence 82 1007, Lausanne and Ch. de l'Agasse 5, 1950 Sion, Switzerland

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A B S T R A C T

We present a different symplectic point of view in the definition of weighted modulation spaces $M^{p,q}_m(\mathbb{R}^d)$ and weighted Wiener amalgam spaces $W(\mathcal{F}L^p_{m_1}, L^q_{m_2})(\mathbb{R}^d)$. All the classical time-frequency representations, such as the short-time Fourier transform (STFT), the τ-Wigner distributions and the ambiguity function, can be written as metaplectic Wigner distributions $\mu(\mathcal{A})(f \otimes \bar{g})$, where $\mu(\mathcal{A})$ is the metaplectic operator and $\mathcal{A}$ is the associated symplectic matrix. Namely, time-frequency representations can be represented as images of metaplectic operators, which become the real protagonists of time-frequency analysis. In [13], the authors suggest that any metaplectic Wigner distribution that satisfies the so-called *shift-invertibility condition* can replace the STFT in the definition of modulation spaces. In this work, we prove that shift-invertibility alone is not sufficient, but it has to be complemented by an upper-triangularity condition for this characterization to hold, whereas a lower-triangularity property comes into play for Wiener amalgam spaces. The shift-invertibility property is necessary: Rihaczek and conjugate Rihaczek distributions are not shift-invertible and they fail the characterization of the above spaces. We also exhibit examples of shift-invertible distributions without upper-triangularity condition which do not define modulation spaces. Finally, we provide new families of time-frequency representations that characterize modulation spaces, with the purpose of replacing the time-frequency shifts with other atoms that allow to decompose signals differently, with possible new outcomes in applications.

R É S U M É

Nous présentons un point de vue symplectique différent dans la définition des espaces de modulation pondérés $M^{p,q}_m(\mathbb{R}^d)$ et des espaces de Wiener pondérés $W(\mathcal{F}L^p_{m_1}, L^q_{m_2})(\mathbb{R}^d)$. Toutes les représentations classiques temps-fréquence, telles que la transformée de Fourier à court terme (STFT), les distributions de Wigner τ et la fonction d'ambiguité, peuvent être écrites comme des distributions de Wigner

* Corresponding author.
E-mail addresses: elena.cordero@unito.it (E. Cordero), gianluca.giacchi2@unibo.it (G. Giacchi).

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métaplectiques $\mu(\mathcal{A})(f \otimes \bar{g})$, o ú $\mu(\mathcal{A})$ est l'opérateur métaplectique et $\mathcal{A}$ est la matrice symplectique associé. Autrement dit, les représentations temps-fréquence peuvent être représentées comme des images d'opérateurs métaplectiques, qui deviennent les véritables protagonistes de l'analyse temps-fréquence. Dans [13], les auteurs suggérent que toute distribution de Wigner métaplectique qui satisfait la soi-disant *condition d'inversibilité de translation* peut remplacer la STFT dans la définition des espaces de modulation. Dans ce travail, nous prouvons que l'inversibilité de translation seule n'est pas suffisante, mais qu'elle doit être complétée par une condition de triangularité supérieure pour que cette caractérisation soit valable, tandis qu'une propriété de triangularité inférieure entre en jeu pour les espaces de Wiener amalgamés. La propriété d'inversibilité de translation est nécessaire : les distributions de Rihaczek et les distributions de Rihaczek conjuguées ne sont pas inversibles et elles ne satisfont pas la caractérisation des espaces ci-dessus. Nous présentons également des exemples de distributions inversibles de translation sans condition de triangularité supérieure qui ne définissent pas d'espaces de modulation. Enfin, nous proposons de nouvelles familles de représentations temps-fréquence qui caractérisent les espaces de modulation, dans le but de remplacer les décalages temps-fréquence par d'autres atomes qui permettent de décomposer les signaux différemment, avec de nouveaux résultats possibles dans les applications.

1. Introduction

The modulation and Wiener amalgam spaces were introduced by H. Feichtinger in 1983 in his pioneering work [17] and they started to become popular in the early 2000s in many different frameworks. In fact, they were successfully applied in the study of pseudodifferential and Fourier integral operators, PDE's, quantum mechanics, signal processing. Nowadays, the rich literature on these spaces witnesses their importance: see, e.g., the very partial list of works [4,5,7,9,31,33,35–37], as well as the textbooks [3,11,26,25,38].

The modulation spaces $M^{p,q}_m(\mathbb{R}^d)$ are classically defined in terms of the short-time Fourier transform (STFT), i.e.,

$$V_g f(x,\xi) = \int_{\mathbb{R}^d} f(t)\bar{g}(t-x)e^{-2\pi i\xi\cdot t}dt, \qquad f \in L^2(\mathbb{R}^d),\ x,\xi \in \mathbb{R}^d, \tag{1}$$

where $g \in L^2(\mathbb{R}^d) \setminus \{0\}$ is a so-called window function and the definition is extended to $(f,g) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ as in Section 2.3. Namely, for a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$f \in M^{p,q}_m(\mathbb{R}^d) \quad \Longleftrightarrow \quad V_g f \in L^{p,q}_m(\mathbb{R}^{2d}),$$

for an arbitrary fixed window g. The identity $V_g f(x,\xi) = \mathcal{F}(f \cdot \bar{g}(\cdot - x))(\xi)$ justifies the choice of the STFT as the time-frequency representation used to define modulation spaces. In fact, it states that the STFT can be used to measure the local frequency content of signals in terms of weighted mixed norm spaces (see Section 2.4 below).

Apart of its interpretation, there is no reason why the STFT shall serve as the leading time-frequency representation in the definition of modulation spaces. Actually, there are many reasons that make it unsuitable in many contexts, such as the theory of pseudodifferential operators and quantum mechanics, cf. [25,34].

In [25] M. De Gosson proved that the (cross-)Wigner distribution, defined for all $f,g \in L^2(\mathbb{R}^d)$ as

$$W(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\bar{g}(x-\frac{t}{2})e^{-2\pi i\xi\cdot t}dt, \qquad x,\xi \in \mathbb{R}^d, \tag{2}$$

can be used to define modulation spaces. Namely,

$$f \in M^{p,q}_m(\mathbb{R}^d) \quad \Leftrightarrow \quad W(f,g) \in L^{p,q}_m(\mathbb{R}^{2d}).$$

Later, in [12], the above characterization was extended to (cross-)τ-Wigner distributions

$$W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\tau t)\bar{g}(x-(1-\tau)t)e^{-2\pi i\xi\cdot t}dt, \qquad x,\xi \in \mathbb{R}^d,$$

with $\tau \in \mathbb{R}\setminus\{0,1\}$. The cases $\tau = 0,1$ correspond to the so-called (cross-) Rihaczek and conjugate Rihaczek distributions, respectively. Their explicit expressions

$$W_0(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}e^{-2\pi i\xi\cdot x} \quad \textit{and} \quad W_1(f,g)(x,\xi) = \hat{f}(\xi)\overline{g(x)}e^{2\pi i\xi\cdot x},$$

$x,\xi \in \mathbb{R}^d$, reveal that

$$W_0(f,g) \in L^{p,q}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad f \in L^p(\mathbb{R}^d),$$

as well as

$$W_1(f,g) \in L^{p,q}(\mathbb{R}^{2d}) \quad \Leftrightarrow \quad f \in \mathcal{F}L^p(\mathbb{R}^d).$$

The lowest common denominator of these time-frequency representations is that they can all be written as

$$W_{\mathcal{A}}(f,g) = \mu(\mathcal{A})(f\otimes\bar{g}), \tag{3}$$

where $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$ is a so-called metaplectic operator and $\mathcal{A} \in Sp(2d,\mathbb{R})$ is the unique associated symplectic matrix, we refer to Section 2.6 for the precise definitions. In fact,

$$V_g f = \mu(A_{ST})(f\otimes\bar{g}) \quad \text{and} \quad W_\tau(f,g) = \mu(A_\tau)(f\otimes\bar{g}),$$

where

$$A_{ST} = \begin{pmatrix} I_{d\times d} & -I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & I_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & -I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix} \tag{4}$$

and

$$A_\tau = \begin{pmatrix} (1-\tau)I_{d\times d} & \tau I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & \tau I_{d\times d} & -(1-\tau)I_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & I_{d\times d} \\ -I_{d\times d} & I_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{5}$$

Using (3) plenty of new time-frequency representations, that we call *metaplectic Wigner distributions*, can be defined in terms of metaplectic operators, cf. [8,12,13,24]. The question becomes for which $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$ or, equivalently, for which $\mathcal{A} \in Sp(2d,\mathbb{R})$, the following property holds: for a moderate weight function m, $0 < p,q \leq \infty$, and a fixed non-zero $g \in \mathcal{S}(\mathbb{R}^d)$,

$$f \in M^{p,q}_m(\mathbb{R}^d) \quad \Leftrightarrow \quad W_{\mathcal{A}}(f,g) \in L^{p,q}_m(\mathbb{R}^{2d}), \quad f \in \mathcal{S}'(\mathbb{R}^d). \tag{6}$$

A partial answer is given in [8,13], where the authors proved that for distributions of the form $W_{\mathcal{A}}(f,g) = \mathcal{F}_2\mathfrak{T}_L(f\otimes\bar{g})$ (see Example 2.2 below for their definition), the characterization property holds. In [13] it is conjectured that the characterization (6) should hold for $W_{\mathcal{A}}$ satisfying the so-called *shift-invertibility* property. Namely, for every $W_{\mathcal{A}}$ the following equality holds

$$|W_{\mathcal{A}}(\pi(y,\eta)f,g)| = |W_{\mathcal{A}}(f,g)(\cdot - E_{\mathcal{A}}(y,\eta))|, \qquad (y,\eta)\in\mathbb{R}^{2d}, \tag{7}$$

for a suitable matrix $E_{\mathcal{A}}\in\mathbb{R}^{2d\times 2d}$, where $\pi(y,\eta)f(t) = e^{2\pi i\eta\cdot t}f(t-y)$ (see (8) ahead). $W_{\mathcal{A}}$ is called *shift-invertible* if $E_{\mathcal{A}}\in GL(2d,\mathbb{R})$ (that is, $E_{\mathcal{A}}$ is invertible).

The main results of this work solve this conjecture for the Banach setting $1\le p,q\le\infty$. We summarize them is the following theorem.

Theorem 1.1. *Let $1\le p,q\le\infty$, m be a v-moderate weight, g a fixed non-zero window function in $\mathcal{S}(\mathbb{R}^d)$. Consider a metaplectic operator $\mu(\mathcal{A})\in Mp(2d,\mathbb{R})$ and let $\mathcal{A}$ be the unique symplectic matrix associated to $\mu(\mathcal{A})$. The following statements hold.*
(i) If $m\circ E_{\mathcal{A}}^{-1}\asymp m$ and $E_{\mathcal{A}}\in GL(2d,\mathbb{R})$ in (7) is upper triangular, then (6) holds with equivalence of norms.
(ii) If $(v\otimes v)\circ\mathcal{A}^{-1}\asymp v\otimes v$, then the window class in (i) can be enlarged to $M_v^1(\mathbb{R}^d)\setminus\{0\}$.
(iii) If $1\le p=q\le\infty$ and $E_{\mathcal{A}}\in GL(2d,\mathbb{R})$, then (6) holds with equivalence of norms.

The core of Theorem 1.1 is that shift-invertibility alone is not sufficient to characterize modulation spaces. This is not surprising: as it is observed in [22], $E_{\mathcal{A}}$ has to be upper triangular, other than invertible, for the operator $f\mapsto f(E_{\mathcal{A}}\cdot)$ to preserve the $L^{p,q}_m$ spaces. Nevertheless, we claim that the conditions on $E_{\mathcal{A}}$ stated in Theorem 1.1 are fundamental to characterize modulation spaces. To support this thesis, we stress that the Rihaczek distributions are examples of non shift-invertible metaplectic Wigner distributions for which Theorem 1.1 fails, see Remark 3.8 (a) in the sequel.

Furthermore, we provide examples of shift-invertible metaplectic Wigner distributions which characterize modulation spaces if and only if the matrix $E_{\mathcal{A}}$ is upper triangular, see Example 4.1 below (see also Remark 3.8).

A relevant contribution of this work consists of constructing explicit examples of metaplectic Wigner distributions that can be used to characterize modulation spaces, some of them extend the representations studied in [40,41] (see also references therein). Our leading idea is to substitute the time-frequency shifts in (1) and (2) with new *time-frequency atoms*. Namely, we replace the chirp $e^{2\pi i\xi\cdot t}$ with the more general one $\Phi_C(\xi,t) = e^{i\pi(\xi,t)^T\cdot C(\xi,t)^T}$, with $C\in\mathbb{R}^{2d\times 2d}$ symmetric matrix. The importance of these examples is that different atoms provide alternative ways to decompose signals into fundamental time-frequency functions, yielding to important applications in many branches of engineering, learning theory and signal analysis. In particular, discretization of time-frequency representations under this point of view could entail consequences in frame theory, phase retrival, and maybe in other aspects of signal processing, producing advances in these frameworks.

Besides the applications above, the metaplectic approach to time-frequency representations carry a high potential in many other situations were time-frequency representations play a crucial role, see e.g., [1,6,15, 29,30]. Finally, observe that a first attempt to generalize the τ-Wigner distributions is contained in the work [2], see also [14].

Outline. Section 2 contains preliminaries and notation. The main results are exposed in Section 3 whereas Section 4 exhibits the most relevant examples. In the Appendix A we extend some of the results in [22] to general invertible matrices and to the quasi-Banach setting. In the Appendix B we compute the matrices associated to tensor products of metaplectic operators.

2. Preliminaries

Notation. We denote $t^2 = t \cdot t$, $t \in \mathbb{R}^d$, and $xy = x \cdot y$ (scalar product on $\mathbb{R}^d$). The space $\mathcal{S}(\mathbb{R}^d)$ is the Schwartz class whereas $\mathcal{S}'(\mathbb{R}^d)$ the space of temperate distributions. The brackets $\langle f, g\rangle$ denote the extension to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ of the inner product $\langle f, g\rangle = \int f(t)\overline{g(t)}dt$ on $L^2(\mathbb{R}^d)$ (conjugate-linear in the second component). We write a point in the phase space (or time-frequency space) as $z = (x, \xi) \in \mathbb{R}^{2d}$, and the corresponding phase-space shift (time-frequency shift) acts on a function or distribution as

$$\pi(z)f(t) = e^{2\pi i\xi\cdot t}f(t-x), \quad t \in \mathbb{R}^d. \tag{8}$$

$\mathcal{C}_0^\infty(\mathbb{R}^{2d})$ denotes the space of smooth functions with compact support. The notation $f \lesssim g$ means that $f(x) \le Cg(x)$ for all x. If $g \lesssim f \lesssim g$ or, equivalently, $f \lesssim g \lesssim f$, we write $f \asymp g$. For two measurable functions $f, g : \mathbb{R}^d \to \mathbb{C}$, we set $f \otimes g(x, y) := f(x)g(y)$. If X, Y are vector spaces, $X \otimes Y$ is the unique completion of $\operatorname{span}\{x \otimes y : x \in X, y \in Y\}$. If $X(\mathbb{R}^d) = L^2(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$, the set $\operatorname{span}\{f \otimes g : f, g \in X(\mathbb{R}^d)\}$ is dense in $X(\mathbb{R}^{2d})$. Thus, for all $f, g \in \mathcal{S}'(\mathbb{R}^d)$ the operator $f \otimes g \in \mathcal{S}'(\mathbb{R}^{2d})$ characterized by its action on $\varphi \otimes \psi \in \mathcal{S}(\mathbb{R}^{2d})$ by

$$\langle f \otimes g, \varphi \otimes \psi\rangle = \langle f, \varphi\rangle\langle g, \psi\rangle$$

extends uniquely to a tempered distribution of $\mathcal{S}'(\mathbb{R}^{2d})$. The subspace $\operatorname{span}\{f \otimes g : f, g \in \mathcal{S}'(\mathbb{R}^d)\}$ is dense in $\mathcal{S}'(\mathbb{R}^{2d})$.

2.1. Weighted mixed norm spaces

We denote by v a continuous, positive, even, submultiplicative weight function on $\mathbb{R}^{2d}$, i.e., $v(z_1 + z_2) \le v(z_1)v(z_2)$, for all $z_1, z_2 \in \mathbb{R}^{2d}$. Observe that since v is even, positive and submultiplicative, it follows that $v(z) \ge 1$ for all $z \in \mathbb{R}^{2d}$. We say that $w \in \mathcal{M}_v(\mathbb{R}^{2d})$ if w is a positive, continuous, even weight function on $\mathbb{R}^{2d}$ that is *v-moderate*: $w(z_1 + z_2) \le Cv(z_1)w(z_2)$ for all $z_1, z_2 \in \mathbb{R}^{2d}$. A fundamental example is the polynomial weight

$$v_s(z) = (1 + |z|)^s, \quad s \in \mathbb{R}, \quad z \in \mathbb{R}^{2d}. \tag{9}$$

Two weights m_1, m_2 are equivalent if $m_1 \asymp m_2$. For example, $v_s(z) \asymp (1 + |z|^2)^{s/2}$.

If $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $0 < p, q \le \infty$ and $f : \mathbb{R}^{2d} \to \mathbb{C}$ measurable, we set

$$\|f\|_{L^{p,q}_m} := \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|f(x,y)|^p m(x,y)^p\right)^{q/p} dy\right)^{1/q} = \|y \mapsto \|f(\cdot,y)m(\cdot,y)\|_p\|_q,$$

with the obvious adjustments when $\min\{p, q\} = \infty$. The space of measurable functions f having $\|f\|_{L^{p,q}_m} < \infty$ is denoted by $L^{p,q}_m(\mathbb{R}^{2d})$. If $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $1 \le p, q \le \infty$, then $L^{p,q}_m(\mathbb{R}^{2d}) * L^1_v(\mathbb{R}^{2d}) \hookrightarrow L^{p,q}_m(\mathbb{R}^{2d})$.

2.2. Fourier transform

In this work, the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$ is defined as

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi\cdot x}dx, \qquad \xi \in \mathbb{R}^d.$$

If $f \in \mathcal{S}'(\mathbb{R}^d)$, the Fourier transform of f is defined by duality as the tempered distribution characterized by

$$\langle \hat{f}, \hat{\varphi} \rangle = \langle f, \varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

We denote with $\mathcal{F}f := \hat{f}$ the Fourier transform operator. It is a surjective automorphism of $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, as well as a surjective isometry of $L^2(\mathbb{R}^d)$.

If $1 \le j \le d$, the *partial Fourier transform* with respect to the jth coordinate is defined as

$$\mathcal{F}_j f(t_1, \ldots, t_{j-1}, \xi_j, t_{j+1} \ldots, t_d) = \int_{-\infty}^{\infty} f(t_1, \ldots, t_d) e^{-2\pi i t_j \xi_j} dt_j, \quad f \in L^1(\mathbb{R}^d). \tag{10}$$

Analogously, the definition is transported on $\mathcal{S}'(\mathbb{R}^d)$ in terms of antilinear duality pairing: for all $f \in \mathcal{S}'(\mathbb{R}^d)$,

$$\langle \mathcal{F}_j f, \varphi \rangle := \langle f, \mathcal{F}_j^{-1} \varphi \rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Observe that $\mathcal{F}_j \mathcal{F}_k = \mathcal{F}_k \mathcal{F}_j$ for all $1 \le j, k \le d$. In particular,

$$\mathcal{F} = \mathcal{F}_{\sigma(1)} \circ \ldots \circ \mathcal{F}_{\sigma(d)} \tag{11}$$

holds that for all permutations $\sigma \in Sym(\{1, \ldots, d\})$. Finally, for all $1 \le j \le d$,

$$\mathcal{F}_j^2 f(x_1, \ldots, x_d) = f(x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_d).$$

2.3. Time-frequency analysis tools

The *short-time Fourier transform* of $f \in L^2(\mathbb{R}^d)$ with respect to the window $g \in L^2(\mathbb{R}^d)$ is the time-frequency representation defined in (1). Its definition extends to $(f, g) \in \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d)$ by antilinear duality as $V_g f(x, \xi) = \langle f, \pi(x, \xi) g \rangle$. Among all the equivalent definitions of $V_g f$, we recall that

$$V_g f = \mathcal{F}_2 \mathfrak{T}_L (f \otimes \bar{g}),$$

where $\mathfrak{T}_L F(x, y) = F(y, y-x)$ and $\mathcal{F}_2$ is the partial Fourier transform with respect to the second coordinate, cf. Example 2.2 below. This equality allows to extend the definition of $V_g f$ up to $\mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}'(\mathbb{R}^d)$.

We recall the fundamental identity of time-frequency analysis:

$$V_g f(x, \xi) = e^{-2\pi i x \cdot \xi} V_{\hat{g}} \hat{f}(J(x, \xi)), \qquad (x, \xi) \in \mathbb{R}^{2d}, \tag{12}$$

where the symplectic matrix J is defined by

$$J = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & 0_{d\times d} \end{pmatrix}. \tag{13}$$

Here $I_{d\times d} \in \mathbb{R}^{d\times d}$ is the identity matrix and $0_{d\times d}$ is the matrix of $\mathbb{R}^{d\times d}$ having all zero entries. The reproducing formula for the STFT reads as follows: for all $g, \gamma \in L^2(\mathbb{R}^d)$ such that $\langle g, \gamma \rangle \neq 0$,

$$f(t) = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(x, \xi) \pi(x, \xi) \gamma(t) dx d\xi, \tag{14}$$

where the identity holds in $L^2(\mathbb{R}^d)$ as a vector-valued integral in a weak sense (see, e.g., [11, Subsection 1.2.4]).

In high-dimensional complex features information processing τ-Wigner distributions ($\tau \in \mathbb{R}$) play a crucial role [39]. They are defined as

$$W_\tau(f,g)(x,\xi) = \int_{\mathbb{R}^d} f(x+\tau t)\overline{g(x-(1-\tau)t)}e^{-2\pi i\xi\cdot t}dt, \qquad x,\xi \in \mathbb{R}^d, \tag{15}$$

for $f,g \in L^2(\mathbb{R}^d)$. The cases $\tau = 0$ and $\tau = 1$ are the so-called (cross-)*Rihaczek distribution*

$$W_0(f,g)(x,\xi) = f(x)\overline{\hat{g}(\xi)}e^{-2\pi i\xi\cdot x}, \quad x,\xi \in \mathbb{R}^d, \tag{16}$$

and (cross-)*conjugate Rihaczek distribution*

$$W_1(f,g)(x,\xi) = \hat{f}(\xi)\overline{g(x)}e^{2\pi i\xi\cdot x}, \quad x,\xi \in \mathbb{R}^d. \tag{17}$$

Observe that $W_\tau f(x,\xi) = \mathcal{F}_2\mathfrak{T}_{L_\tau}(f \otimes \bar{g})$, where for any F on $\mathbb{R}^{2d}$,

$$\mathfrak{T}_{L_\tau}F(x,y) = F(x+\tau y, x-(1-\tau)y), \quad x,y \in \mathbb{R}^d.$$

2.4. Modulation spaces [3,17,18,26,23,28]

From now on, we work with weights in the class $\mathcal{M}_v(\mathbb{R}^{2d})$ of v-moderate weights, defined in Subsection 2.1.

Fix $0 < p,q \le \infty$, $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. The *modulation space* $M^{p,q}_m(\mathbb{R}^d)$ is classically defined as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{M^{p,q}_m} := \|V_g f\|_{L^{p,q}_m} < \infty.$$

If $\min\{p,q\} \ge 1$, the quantity $\|\cdot\|_{M^{p,q}_m}$ defines a norm, otherwise a quasi-norm. Different windows give rise to equivalent (quasi-)norms. Modulation spaces are (quasi-)Banach spaces and the following continuous inclusions hold:
if $0 < p_1 \le p_2 \le \infty$, $0 < q_1 \le q_2 \le \infty$ and $m_1, m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$ satisfy $m_2 \lesssim m_1$:

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow M^{p_1,q_1}_{m_1}(\mathbb{R}^d) \hookrightarrow M^{p_2,q_2}_{m_2}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

In particular, $M^1_v(\mathbb{R}^d) \hookrightarrow M^{p,q}_m(\mathbb{R}^d)$ whenever $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ and $\min\{p,q\} \ge 1$. We will also use the inclusion $M^1_{m\otimes 1}(\mathbb{R}^{2d}) \hookrightarrow L^1_m(\mathbb{R}^{2d})$. We denote with $\mathcal{M}^{p,q}_m(\mathbb{R}^d)$ the closure of $\mathcal{S}(\mathbb{R}^d)$ in $M^{p,q}_m(\mathbb{R}^d)$, which coincides with the latter whenever $p,q < \infty$. Moreover, if $1 \le p,q < \infty$, $(M^{p,q}_m(\mathbb{R}^d))' = M^{p',q'}_{1/m}(\mathbb{R}^d)$, where p' and q' denote the Lebesgue conjugate exponents of p and q respectively. Finally, if $m_1 \asymp m_2$, then $M^{p,q}_{m_1}(\mathbb{R}^d) = M^{p,q}_{m_2}(\mathbb{R}^d)$ for all p,q.

2.5. Wiener amalgam spaces [19,20,32]

For $0 < p,q \le \infty$, $m_1, m_2 \in \mathcal{M}_v(\mathbb{R}^{2d})$, the *Wiener amalgam space* $W(\mathcal{F}L^p_{m_1}, L^q_{m_2})(\mathbb{R}^d)$, is defined as the space of tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$ such that for some (hence, all) window $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$,

$$\|f\|_{W(\mathcal{F}L^p_{m_1}, L^q_{m_2})} := \|x \mapsto m_2(x)\|V_g f(x,\cdot)m_1\|_p\|_q < \infty.$$

Using (12), we have that $\|f\|_{W(\mathcal{F}L^p_{m_1},L^q_{m_2})} = \|\hat{f}\|_{M^{p,q}_{m_1\otimes m_2}}$, so that

$$\mathcal{F}M^{p,q}_{m_1\otimes m_2}(\mathbb{R}^d) = W(\mathcal{F}L^p_{m_2}, L^q_{m_1})(\mathbb{R}^d).$$

Also, for $p = q$,

$$W(\mathcal{F}L^p_{m_1}, L^p_{m_2})(\mathbb{R}^d) = M^p_{m_1\otimes m_2}(\mathbb{R}^d). \tag{18}$$

2.6. The symplectic group $Sp(d,\mathbb{R})$ and the metaplectic operators

A matrix $\mathcal{A} \in \mathbb{R}^{2d\times 2d}$ is symplectic, we write $\mathcal{A} \in Sp(d,\mathbb{R})$, if $\mathcal{A}^T J\mathcal{A} = J$, where the matrix J is defined in (13). We represent $\mathcal{A}$ as a block matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{19}$$

If $\mathcal{A} \in Sp(d,\mathbb{R})$, then $\det(\mathcal{A}) = 1$. The matrix $\mathcal{A} \in Sp(d,\mathbb{R})$ with block decomposition (19) is called *free* if $\det B \neq 0$.

For $L \in GL(d,\mathbb{R})$ and $C \in \mathbb{R}^{d\times d}$, C symmetric, we define

$$\mathcal{D}_L := \begin{pmatrix} L^{-1} & 0_{d\times d} \\ 0_{d\times d} & L^T \end{pmatrix} \qquad \text{and} \qquad V_C := \begin{pmatrix} I_{d\times d} & 0 \\ C & I_{d\times d} \end{pmatrix}. \tag{20}$$

J and the matrices in the form V_C (C symmetric) and $\mathcal{D}_L$ (L invertible) generate the group $Sp(d,\mathbb{R})$.

Let ρ be the Schrödinger representation of the Heisenberg group, that is

$$\rho(x,\xi,\tau) = e^{2\pi i\tau}e^{-\pi i\xi\cdot x}\pi(x,\xi),$$

for all $x,\xi \in \mathbb{R}^d$, $\tau \in \mathbb{R}$. For all $\mathcal{A} \in Sp(d,\mathbb{R})$, $\rho_{\mathcal{A}}(x,\xi,\tau) := \rho(\mathcal{A}(x,\xi),\tau)$ defines another representation of the Heisenberg group that is equivalent to ρ, i.e. there exists a unitary operator $\mu(\mathcal{A}) : L^2(\mathbb{R}^{2d}) \to L^2(\mathbb{R}^{2d})$ such that

$$\mu(\mathcal{A})\rho(x,\xi,\tau)\mu(\mathcal{A})^{-1} = \rho(\mathcal{A}(x,\xi),\tau), \qquad x,\xi \in \mathbb{R}^d,\ \tau \in \mathbb{R}. \tag{21}$$

This operator is not unique, but if $\mu'(\mathcal{A})$ is another unitary operator satisfying (21), then $\mu'(\mathcal{A}) = c\mu(\mathcal{A})$, for some unitary constant $c \in \mathbb{C}$, $|c| = 1$. The set $\{\mu(\mathcal{A}) : \mathcal{A} \in Sp(d,\mathbb{R})\}$ is a group under composition and it admits a subgroup that contains exactly two operators for each $\mathcal{A} \in Sp(d,\mathbb{R})$. This subgroup is called **metaplectic group**, denoted by $Mp(d,\mathbb{R})$. It is a realization of the two-fold cover of $Sp(d,\mathbb{R})$ and the projection

$$\pi^{Mp} : Mp(d,\mathbb{R}) \to Sp(d,\mathbb{R}) \tag{22}$$

is a group homomorphism with kernel $\ker(\pi^{Mp}) = \{-id_{L^2}, id_{L^2}\}$.

Proposition 2.1. *[21, Proposition 4.27] The operator $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$ maps $\mathcal{S}(\mathbb{R}^d)$ isomorphically to $\mathcal{S}(\mathbb{R}^d)$ and it extends to an isomorphism on $\mathcal{S}'(\mathbb{R}^d)$.*

For $C \in \mathbb{R}^{d\times d}$, define

$$\Phi_C(t) = e^{\pi it\cdot Ct}, \quad t \in \mathbb{R}^d. \tag{23}$$

If we add the assumptions C symmetric and invertible, then we can compute explicitly its Fourier transform, that is

$$\widehat{\Phi_C} = |\det(C)|\,\Phi_{-C^{-1}}. \tag{24}$$

Example 2.2. For particular choices of $\mathcal{A} \in Sp(d,\mathbb{R})$, $\mu(\mathcal{A})$ is known. Let J, $\mathcal{D}_L$ and V_C be defined as in (13) and (20), respectively. Then, up to a sign,

(i) $\mu(J)f = \mathcal{F}f$,
(ii) $\mu(\mathcal{D}_L)f = \mathfrak{T}_L f = |\det(L)|^{1/2} f(L\cdot)$,
(iii) $\mu(V_C)f = \Phi_C f$,
(iv) $\mu(V_C^T)f = \widehat{\Phi_{-C}} * f = \mathcal{F}\Phi_{-C}\mathcal{F}^{-1}$,
(v) if $\mathcal{A}_{FT2} \in Sp(2d,\mathbb{R})$ is the $4d \times 4d$ matrix with block decomposition

$$\mathcal{A}_{FT2} := \begin{pmatrix} I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & I_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & -I_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix},$$

then,

$$\mu(\mathcal{A}_{FT2})F(x,\xi) = \mathcal{F}_2 F(x,\xi), \quad x,\xi \in \mathbb{R}^d,$$

the partial Fourier transform w.r.t. the second variable.
(vi) Assume $\mathcal{A} \in Sp(d,\mathbb{R})$ with block decomposition (19). Then,
if $\mathcal{A}$ is free:

$$\mu(\mathcal{A})f(x) = (\det(B))^{-1/2}\Phi_{-DB^{-1}}(x)\int_{\mathbb{R}^d} e^{2\pi i y\cdot B^{-1}x}\Phi_{-B^{-1}A}(y)f(y)dy, \tag{25}$$

if $\det A \neq 0$,

$$\mu(\mathcal{A})f(x) = (\det(A))^{-1/2}\Phi_{-CA^{-1}}(x)\int_{\mathbb{R}^d} e^{2\pi i\xi\cdot A^{-1}x}\Phi_{-A^{-1}B}(\xi)\hat{f}(\xi)d\xi. \tag{26}$$

Other important symplectic matrices are the so-called quasi-permutation matrices [16,22].

Definition 2.3. For $1 \leq j \leq d$, the symplectic interchange matrix $\Pi_j \in Sp(d,\mathbb{R})$ is the matrix obtained interchanging the columns j and $j+d$ of the $2d$-by-$2d$ identity matrix and multiplying the jth column of the resulting matrix by -1.

The corresponding metaplectic operators are the partial Fourier transforms, as we can see below.

Example 2.4. Let $\mathcal{F}_j$, $1 \leq j \leq d$, be the partial Fourier transform w.r.t. the *jth* coordinate defined in (10). Then

$$\mathcal{F}_j\rho(x,\xi,\tau)\mathcal{F}_j^{-1} = \rho(\Pi_j(x,\xi),\tau), \quad x,\xi \in \mathbb{R}^d,\ \tau \in \mathbb{R}. \tag{27}$$

In fact, take any $f \in L^1(\mathbb{R}^d)$ and compute $\mathcal{F}_j\rho(x,\xi,\tau)\mathcal{F}_j^{-1}f$ as follows:

$$
\begin{aligned}
&\mathcal{F}_j\rho(x,\xi,\tau)\mathcal{F}_j^{-1}f(t_1,\dots,t_d)\\
&\quad= e^{2\pi i\tau}e^{-\pi i x\cdot\xi}e^{2\pi i\sum_{k\neq j}t_k\cdot\xi_k}\int_{\mathbb{R}} e^{-2\pi t_j\zeta_j}e^{2\pi i\zeta_j\xi_j}\mathcal{F}_j^{-1}f(t_1-x_1,\dots,\zeta_j-x_j,\dots,t_d-x_d)\,d\zeta_j\\
&\quad= e^{2\pi i\tau}e^{-\pi i x\cdot\xi}e^{2\pi i\sum_{k\neq j}t_k\cdot\xi_k}\int_{\mathbb{R}} e^{-2\pi i(u_j+x_j)(t_j-\xi_j)}\mathcal{F}_j^{-1}f(t_1-x_1,\dots,u_j,\dots,t_d-x_d)\,du_j\\
&\quad= e^{2\pi i\tau}e^{-\pi i x\cdot\xi}e^{2\pi i\sum_{k\neq j}t_k\cdot\xi_k}e^{2\pi i t_j(-x_j)}e^{2\pi i x_j\xi_j}\int_{\mathbb{R}} e^{-2\pi u_j(\xi_j-\zeta_j)}\\
&\qquad\qquad\times\quad \mathcal{F}_j^{-1}f(t_1-x_1,\dots,u_j,\dots,t_d-x_d)\,du_j\\
&\quad= \rho(\Pi_j(x,\xi),\tau)f(t_1,\dots,t_d).
\end{aligned}
$$

Observe also that $\prod_j \Pi_j = J$, in line with (11).

3. Shift-invertibility and modulation spaces

In this section we present the features of metaplectic operators that guarantee the representations of modulation and Wiener amalgam spaces by metaplectic operators. We first need to recall the definition of metaplectic Wigner distributions and their main properties.

Metaplectic Wigner distributions. Metaplectic Wigner distributions were introduced and developed in [8,12,13], see also [40,41]. The reader may recognize that Definition 3.1 below generalizes the STFT and the τ-Wigner distributions (among others).

Definition 3.1. Let $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$. The (cross-)**metaplectic Wigner distribution** with matrix $\mathcal{A}$ (or $\mathcal{A}$-Wigner distribution) is defined as

$$
W_{\mathcal{A}}(f,g)(x,\xi) = \mu(\mathcal{A})(f\otimes\bar{g})(x,\xi), \quad f,g\in L^2(\mathbb{R}^d).
$$

The following results are direct consequences of Proposition 2.1.

Proposition 3.2. *Let $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$.*
(i) $W_{\mathcal{A}} : L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^{2d})$ is continuous.
(ii) $W_{\mathcal{A}} : \mathcal{S}(\mathbb{R}^d)\times \mathcal{S}(\mathbb{R}^d)\to \mathcal{S}(\mathbb{R}^{2d})$ is continuous.
(iii) $W_{\mathcal{A}} : \mathcal{S}'(\mathbb{R}^d)\times \mathcal{S}'(\mathbb{R}^d)\to \mathcal{S}'(\mathbb{R}^{2d})$ is continuous.

We observe that [13, Theorem 2.7] and [13, Proposition 2.18] extend to tempered distributions.

Corollary 3.3. *Let $\mu(\mathcal{A}) \in Mp(2d,\mathbb{R})$. Consider $g_1\in\mathcal{S}'(\mathbb{R}^d)$, $g_2\in\mathcal{S}(\mathbb{R}^d)$ with $\langle g_1,g_2\rangle\neq 0$. Then, for all $f\in\mathcal{S}'(\mathbb{R}^d)$,*

$$
f(x) = \frac{1}{\langle g_2,g_1\rangle}\int_{\mathbb{R}^d}\mu(\mathcal{A})^{-1}W_{\mathcal{A}}(f,g_1)(x,\xi)g_2(\xi)d\xi, \tag{28}
$$

with equality in $\mathcal{S}'(\mathbb{R}^d)$, the integral being meant in the weak sense.

Proof. It is the same as in [13, Theorem 2.7]. ∎

Corollary 3.4. *Fix $g_3 \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$. Under the assumptions of Corollary 3.3, for any $f \in \mathcal{S}'(\mathbb{R}^d)$,*

$$V_{g_3} f(w) = \frac{1}{\langle g_2, g_1 \rangle} \langle W_{\mathcal{A}}(f, g_1), W_{\mathcal{A}}(\pi(w)g_3, g_2) \rangle, \qquad w \in \mathbb{R}^{2d},$$

where $\langle \cdot, \cdot \rangle$ is the antilinear duality paring between $\mathcal{S}'(\mathbb{R}^{2d})$ and $\mathcal{S}(\mathbb{R}^{2d})$.

Proof. It goes as in [13, Proposition 2.18], using Proposition 3.2 (ii) and (iii). ■

Consider $\mu(\mathcal{A}) \in Mp(2d, \mathbb{R})$, $\mathcal{A} \in Sp(2d, \mathbb{R})$ having block decomposition

$$\mathcal{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{pmatrix}, \tag{29}$$

where the sub-matrices A_{ij} are of size $d \times d$. Then it was shown in [13] that

$$|W_{\mathcal{A}}(\pi(w)f, g)| = |\pi(E_{\mathcal{A}} w, F_{\mathcal{A}} w) W_{\mathcal{A}}(f, g)|, \quad w \in \mathbb{R}^{2d}, \quad \forall f, g \in L^2(\mathbb{R}^d), \tag{30}$$

where the matrices $E_{\mathcal{A}}$ and $F_{\mathcal{A}}$ are given by

$$E_{\mathcal{A}} = \begin{pmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{pmatrix} \qquad \text{and} \qquad F_{\mathcal{A}} = \begin{pmatrix} A_{31} & A_{33} \\ A_{41} & A_{43} \end{pmatrix}. \tag{31}$$

Definition 3.5. Under the notation above, we say that $W_{\mathcal{A}}$ (or, by abuse, $\mathcal{A}$) is **shift-invertible** if $E_{\mathcal{A}} \in GL(2d, \mathbb{R})$.

We need the following representation formula.

Lemma 3.6. *Let $\mu(\mathcal{A}) \in Mp(2d, \mathbb{R})$, $\gamma, g \in \mathcal{S}(\mathbb{R}^d)$ be such that $\langle \gamma, g \rangle \neq 0$ and $f \in \mathcal{S}'(\mathbb{R}^d)$. Then,*

$$W_{\mathcal{A}}(f, g) = \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(w) W_{\mathcal{A}}(\pi(w)\gamma, g) dw \tag{32}$$

with equality in $\mathcal{S}'(\mathbb{R}^{2d})$, the integral being intended in the weak sense.

Proof. Take any $\varphi \in \mathcal{S}(\mathbb{R}^{2d})$ and use the definition of vector-valued integral in a weak sense (see, e.g., [11, Section 1.2.4] or [26, pag. 43]), which entails

$$\begin{aligned} \langle W_{\mathcal{A}}(f, g)(z), \varphi \rangle &= \langle f \otimes \bar{g}, \mu(\mathcal{A})^{-1} \varphi \rangle \\ &= \langle \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(w) [(\pi(w)\gamma) \otimes \bar{g}] dw, \mu(\mathcal{A})^{-1} \varphi \rangle \\ &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(w) \langle [(\pi(w)\gamma) \otimes \bar{g}], \mu(\mathcal{A})^{-1} \varphi \rangle dw \\ &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(w) \langle \mu(\mathcal{A})(\pi(w)\gamma \otimes \bar{g}), \varphi \rangle dw \\ &= \frac{1}{\langle \gamma, g \rangle} \int_{\mathbb{R}^{2d}} V_g f(w) \langle W_{\mathcal{A}}(\pi(w)\gamma, g)(z), \varphi \rangle dw. \end{aligned}$$

Therefore,

$$W_{\mathcal{A}}(f,g)=\frac{1}{\langle\gamma,g\rangle}\int_{\mathbb{R}^{2d}}V_gf(w)W_{\mathcal{A}}(\pi(w)\gamma,g)dw$$

with equality in $\mathcal{S}'(\mathbb{R}^{2d})$. ∎

Theorem 3.7. *Let $W_{\mathcal{A}}$ be shift-invertible with $E_{\mathcal{A}}$ upper-triangular. Fix a non-zero window function $g\in\mathcal{S}(\mathbb{R}^d)$. For $m\in\mathcal{M}_v(\mathbb{R}^{2d})$ with $m\asymp m\circ E_{\mathcal{A}}^{-1}$, $1\le p,q\le\infty$,*

$$f\in M^{p,q}_m(\mathbb{R}^d)\qquad\Leftrightarrow\qquad W_{\mathcal{A}}(f,g)\in L^{p,q}_m(\mathbb{R}^{2d}),\tag{33}$$

with equivalence of norms.

Proof. $\Rightarrow$. Assume $f\in M^{p,q}_m(\mathbb{R}^d)$. For any $\gamma\in\mathcal{S}(\mathbb{R}^d)$ such that $\langle\gamma,g\rangle\neq0$, the inversion formula for the STFT (cf. Theorem 2.3.7 in [11]) reads

$$f=\frac{1}{\langle\gamma,g\rangle}\int_{\mathbb{R}^{2d}}V_gf(w)\pi(w)\gamma dw.$$

Multiplying both sides of the above equality by $\bar{g}(z_2)$, for any $z=(z_1,z_2)\in\mathbb{R}^{2d}$, we can write

$$f(z_1)\bar{g}(z_2)=(f\otimes\bar{g})(z)=\frac{1}{\langle\gamma,g\rangle}\int_{\mathbb{R}^{2d}}V_gf(w)[(\pi(w)\gamma)\otimes\bar{g}](z)dw.$$

Applying $\mu(\mathcal{A})$ to $(f\otimes\bar{g})$ we obtain $\mu(\mathcal{A})(f\otimes\bar{g})=W_{\mathcal{A}}(f,g)\in\mathcal{S}'(\mathbb{R}^{2d})$. Using Lemma 3.6, we get:

$$W_{\mathcal{A}}(f,g)=\frac{1}{\langle\gamma,g\rangle}\int_{\mathbb{R}^{2d}}V_gf(w)W_{\mathcal{A}}(\pi(w)\gamma,g)dw,$$

with equality holding in $\mathcal{S}'(\mathbb{R}^{2d})$.

Now, if $f\in M^{p,q}_m(\mathbb{R}^d)$, the integral on the right-hand side is absolutely convergent as we shall see presently. For any $z\in\mathbb{R}^{2d}$,

$$\begin{aligned}|W_{\mathcal{A}}(f,g)(z)|&\le\frac{1}{|\langle\gamma,g\rangle|}\int_{\mathbb{R}^{2d}}|V_gf(w)||W_{\mathcal{A}}(\gamma,g)(z-E_{\mathcal{A}}w)|dw\\&=\frac{|\det(E_{\mathcal{A}})|^{-1}}{|\langle\gamma,g\rangle|}\int_{\mathbb{R}^{2d}}|V_gf(E_{\mathcal{A}}^{-1}u)||W_{\mathcal{A}}(\gamma,g)(z-u)|du\\&=\frac{|\det(E_{\mathcal{A}})|^{-1}}{|\langle\gamma,g\rangle|}|V_gf\circ E_{\mathcal{A}}^{-1}|*|W_{\mathcal{A}}(\gamma,g)|(z).\end{aligned}\tag{34}$$

Since $\gamma,g\in\mathcal{S}(\mathbb{R}^d)$, $W_{\mathcal{A}}(\gamma,g)\in\mathcal{S}(\mathbb{R}^{2d})\subset L^1_v(\mathbb{R}^{2d})$. Moreover, by Theorem A.2 and Theorem A.3 both applied with $S=E_{\mathcal{A}}^{-1}$, we have that $V_gf\circ E_{\mathcal{A}}^{-1}\in L^{p,q}_m(\mathbb{R}^{2d})$. Young's convolution inequality applied to (34) entails

$$\|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m}\lesssim\|V_gf\|_{L^{p,q}_m}\|W_{\mathcal{A}}(\gamma,g)\|_{L^1_v}<\infty.$$

$\Leftarrow$. Assume that $W_{\mathcal{A}}(f,g)\in L^{p,q}_m(\mathbb{R}^{2d})$. Using Corollary 3.4 with $g_3=g_1=g$, $g_2=\gamma$, for any $w\in\mathbb{R}^{2d}$,

$$
\begin{aligned}
|V_g f(w)| &\lesssim \frac{1}{|\langle \gamma, g\rangle|} |\langle W_{\mathcal{A}}(f,g), W_{\mathcal{A}}(\pi(w)g,\gamma)\rangle| \\
&\lesssim \int_{\mathbb{R}^{2d}} |W_{\mathcal{A}}(f,g)(u)||W_{\mathcal{A}}(\pi(w)g,\gamma)(u)|du \\
&\lesssim \int_{\mathbb{R}^{2d}} |W_{\mathcal{A}}(f,g)(u)||W_{\mathcal{A}}(g,\gamma)(u - E_{\mathcal{A}}w)|du \\
&\lesssim \int_{\mathbb{R}^{2d}} |W_{\mathcal{A}}(f,g)(u)||[W_{\mathcal{A}}(g,\gamma)]^*(E_{\mathcal{A}}w - u)|du \\
&= |W_{\mathcal{A}}(f,g)| * |[W_{\mathcal{A}}(g,\gamma)]^*|(E_{\mathcal{A}}w).
\end{aligned} \tag{35}
$$

Applying Theorem A.2 and Theorem A.3 with $S = E_{\mathcal{A}}$, we obtain

$$
\begin{aligned}
\|f\|_{M^{p,q}_m} \asymp \|V_g f\|_{L^{p,q}_m} &\lesssim \||W_{\mathcal{A}}(f,g)| * |[W_{\mathcal{A}}(g,\gamma)]^*|\|_{L^{p,q}_m} \\
&\lesssim \|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m} \|W_{\mathcal{A}}(g,\gamma)\|_{L^1_v} < \infty,
\end{aligned}
$$

since we considered an even submultiplicative weight v. ■

Remark 3.8. Theorem 3.7 is sharp. Namely, if either $E_{\mathcal{A}}$ is not shift-invertible or $E_{\mathcal{A}}$ is not upper triangular, $W_{\mathcal{A}}$ may not characterize modulation spaces. We provide two counterexamples.
(a) If $E_{\mathcal{A}}$ is not shift-invertible, then $W_{\mathcal{A}}$ may not characterize modulation spaces. Let W_0 be the (cross-)Rihaczek distribution defined in (16). Obviously, for every $f \in L^p(\mathbb{R}^d)$ and $g \in \mathcal{S}(\mathbb{R}^d)$, we obtain $\|W_0(f,g)\|_{L^{p,q}} = \|f\|_p \|\hat{g}\|_q$. This means that the $L^{p,q}$-norm of W_0 is not equivalent to the modulation norm in general. Observe that the corresponding matrix E_{A_0} is not shift-invertible. In fact,

$$
E_{A_0} = \begin{pmatrix} I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} \end{pmatrix}
$$

is not invertible. Similarly, the (cross-)conjugate-Rihaczek distribution W_1 in (17) is not shift-invertible and does not characterize modulation spaces [12, Remark 3.7].
(b) If $E_{\mathcal{A}}$ is not upper-triangular, then $W_{\mathcal{A}}$ may not characterize modulation spaces. Let $C \in \mathbb{R}^{2d\times 2d} \setminus \{0_{2d\times 2d}\}$ be any symmetric matrix. Then, up to a sign,

$$
V_g(\mu(V_C)f) = \mu(A_{ST})(\mu(V_C)f \otimes \bar{g}) = \mu(A_{ST}V_{\tilde{C}})(f \otimes \bar{g}),
$$

where

$$
V_{\tilde{C}} = \begin{pmatrix} I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ C & 0_{d\times d} & I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & I_{d\times d} \end{pmatrix},
$$

see formula (B.4) in the Appendix B. Let $\mathcal{A} := A_{ST}V_{\tilde{C}}$ It is easy to verify that

$$
E_{\mathcal{A}} = \begin{pmatrix} I_{d\times d} & 0_{d\times d} \\ C & I_{d\times d} \end{pmatrix},
$$

which is always invertible and lower-triangular. The metaplectic operator $\mu(V_C)$ is unbounded on $M^{p,q}(\mathbb{R}^d)$, $1 \le p, q \le \infty$, $p \ne q$, cf. [10, Proposition 7.1]. Namely, if $f \in M^{p,q}(\mathbb{R}^d)$, $\mu(V_C)f \notin M^{p,q}(\mathbb{R}^d)$ for $p \ne q$ and, consequently, $\mu(\mathcal{A})(f \otimes \bar{g}) \notin L^{p,q}(\mathbb{R}^{2d})$. Observe that a similar result with different methods is obtained in [22, Theorem 3.3].

As a byproduct of the proof to Theorem 3.7 we obtain new properties for shift-invertible representations $W_\mathcal{A}$, see ahead.

Corollary 3.9. *Let $\mu(\mathcal{A}) \in Mp(2d, \mathbb{R})$ with $W_\mathcal{A}$ shift-invertible. Then, for all $f, g \in L^2(\mathbb{R}^d)$, we have $W_\mathcal{A}(f, g) \in L^\infty(\mathbb{R}^{2d})$ and it is everywhere defined.*

Proof. If $f \in L^2(\mathbb{R}^d)$ and $g, \gamma \in \mathcal{S}(\mathbb{R}^d)$, the inequality (34) holds pointwise (take $p = q = 2$, $m = 1$). Also, if $g \in L^2(\mathbb{R}^d)$ the right hand-side of (34) is also well defined for all $z \in \mathbb{R}^{2d}$, since $W_\mathcal{A}$ maps $L^2(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^{2d})$. By Young's inequality,

$$\|W_\mathcal{A}(f, g)\|_{L^\infty(\mathbb{R}^{2d})} \lesssim \|V_g f \circ E_\mathcal{A}^{-1}\|_{L^2(\mathbb{R}^{2d})} \|W_\mathcal{A}(\gamma, g)\|_{L^2(\mathbb{R}^{2d})} = \|f\|_2 \|g\|_2^2 \|\gamma\|_2 < \infty.$$

Hence, $W_\mathcal{A}(f, g) \in L^\infty(\mathbb{R}^{2d})$ and $W_\mathcal{A}(f, g)(z)$ is well defined for all $z \in \mathbb{R}^{2d}$. ■

If we limit to the case $p = q$, then $\mathfrak{T}_S : L^p(\mathbb{R}^{2d}) \to L^p(\mathbb{R}^{2d})$ is bounded for all $S \in GL(2d, \mathbb{R})$, without any further assumption on its triangularity. In this case, arguing as above, but using Theorem A.1, we obtain the following result.

Theorem 3.10. *Let $W_\mathcal{A}$ be shift-invertible and $m \in \mathcal{M}_v(\mathbb{R}^{2d})$ with $m \asymp m \circ E_\mathcal{A}^{-1}$. For $1 \le p \le \infty$ and $g \in \mathcal{S}(\mathbb{R}^d) \setminus \{0\}$, we have*

$$\|f\|_{M^p_m} \asymp \|W_\mathcal{A}(f, g)\|_{L^p_m}.$$

Corollary 3.11. *Under the assumptions of Theorem 3.7, assume that $(v \otimes v) \circ \mathcal{A}^{-1} \asymp v \otimes v$, then the window class can be enlarged to $M^1_v(\mathbb{R}^d)$.*

Proof. By Theorem 3.10, if $\gamma \in \mathcal{S}(\mathbb{R}^d)$ and $g \in M^1_v(\mathbb{R}^d)$, $W_\mathcal{A}(g, \gamma) \in L^1_v(\mathbb{R}^{2d})$, so that

$$|W_\mathcal{A}(f, g)(z)| \lesssim \frac{1}{|\langle \gamma, g\rangle|} |\det(E_\mathcal{A})|^{-1} |V_g f \circ E_\mathcal{A}^{-1}| * |W_\mathcal{A}(\gamma, g)|(z) \tag{36}$$

is well defined by (34) provided that $W_\mathcal{A}(\gamma, g) \in L^1_v(\mathbb{R}^{2d})$.

By [13, Proposition 2.4],

$$W_\mathcal{A}(\gamma, g) = W_{\tilde{\mathcal{A}}}(\bar{g}, \bar{\gamma}) = \mu(\mathcal{A})\mu(\mathcal{D}_L)(\bar{g} \otimes \gamma),$$

with $\tilde{\mathcal{A}} = \mathcal{A}\mathcal{D}_L$. Now, $\bar{g} \otimes \gamma \in M^1_v(\mathbb{R}^d) \otimes \mathcal{S}(\mathbb{R}^d) \subset M^1_{v\otimes v}(\mathbb{R}^{2d})$ and $\mu(\mathcal{D}_L)(\bar{g} \otimes \gamma) = \gamma \otimes \bar{g}$, so that $\mu(\mathcal{D}_L) : M^1_{v\otimes v}(\mathbb{R}^{2d}) \to M^1_{v\otimes v}(\mathbb{R}^{2d})$. Indeed,

$$\|\mu(\mathcal{D}_L)(\bar{g} \otimes \gamma)\|_{M^1_{v\otimes v}} \asymp \|\gamma\|_{M^1_v} \|g\|_{M^1_v}.$$

On the other hand, $\mu(\mathcal{A}) : M^1_{v\otimes v}(\mathbb{R}^{2d}) \to M^1_{v\otimes v}(\mathbb{R}^{2d})$ by [22, Theorem 3.2] and [22, Corollary 4.5]. Moreover, $M^1_{v\otimes v}(\mathbb{R}^{2d}) \hookrightarrow M^1_{v\otimes 1}(\mathbb{R}^{2d}) \hookrightarrow L^1_v(\mathbb{R}^{2d})$, since

$$v(z) \le v(z)v(w)$$

for all $z, w \in \mathbb{R}^{2d}$. Hence,

$$\|W_{\mathcal{A}}(\gamma,g)\|_{L^1_v} = \|W_{\tilde{\mathcal{A}}}(\bar{g},\bar{\gamma})\|_{L^1_v} \leq \|W_{\tilde{\mathcal{A}}}(\bar{g},\bar{\gamma})\|_{M^1_{v\otimes 1}} \leq \|W_{\tilde{\mathcal{A}}}(\bar{g},\bar{\gamma})\|_{M^1_{v\otimes v}}$$
$$\lesssim_{\mathcal{A}} \|\mu(\mathcal{D}_L)\bar{g}\otimes\gamma\|_{M^1_{v\otimes v}} \asymp \|\bar{g}\|_{M^1_v}\|\gamma\|_{M^1_v} \asymp_{\mathcal{A},\gamma} \|g\|_{M^1_v} < \infty.$$

Going back to (36), we obtain

$$\|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m} \lesssim \|V_g f\|_{L^{p,q}_m}\|W_{\mathcal{A}}(\gamma,g)\|_{L^1_v} \asymp_{\mathcal{A},\gamma,g} \|f\|_{M^{p,q}_m}.$$

Whence, if $g\in M^1_v(\mathbb{R}^d)$ and $f\in M^{p,q}_m(\mathbb{R}^d)$, the $\mathcal{A}$-Wigner transform $W_{\mathcal{A}}(f,g)$ is in $L^{p,q}_m(\mathbb{R}^{2d})$, with $\|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m} \lesssim \|f\|_{M^{p,q}_m}$.

Vice versa, we have shown that if $g\in M^1_v(\mathbb{R}^d)$ and $f\in M^{p,q}_m(\mathbb{R}^d)$, the $\mathcal{A}$-Wigner $W_{\mathcal{A}}(f,g)$ is in $L^{p,q}_m(\mathbb{R}^{2d})$. By (35), for all $w\in\mathbb{R}^{2d}$,

$$|V_g f(w)| \lesssim |W_{\mathcal{A}}(f,g)| * |[W_{\mathcal{A}}(g,\gamma)]^*|(E_{\mathcal{A}}w),$$

and Young's inequality gives

$$\|f\|_{M^{p,q}_m} \lesssim_{\mathcal{A},g,\gamma} \|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m}.$$

In conclusion, $\|f\|_{M^{p,q}_m} \asymp \|W_{\mathcal{A}}(f,g)\|_{L^{p,q}_m}$, with $g\in M^1_v(\mathbb{R}^d)$. ∎

Another consequence of Theorem 3.7 is the characterization of Wiener amalgam spaces $W(\mathcal{F}L^p_{m_1}, L^q_{m_2})(\mathbb{R}^d)$.

Corollary 3.12. *Let $\mu(\mathcal{A})\in Mp(2d,\mathbb{R})$ be such that $W_{\mathcal{A}}$ is shift-invertible and $1\leq p,q\leq\infty$. Let $m_1,m_2\in\mathcal{M}_v(\mathbb{R}^d)$ be such that $m_2\asymp\mathcal{I}m_2$, being $\mathcal{I}m_2(x)=m_2(-x)$, and $\mathcal{A}=\pi^{Mp}(\mu(\mathcal{A}))$ having block decomposition in (29). Fix $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$ and define*

$$\tilde{E}_{\mathcal{A}} = JE_{\mathcal{A}}J. \tag{37}$$

If $m_1\otimes m_2\asymp(m_1\otimes m_2)\circ\tilde{E}_{\mathcal{A}}^{-1}$ and $E_{\mathcal{A}}$ is lower triangular, then

$$\|f\|_{W(\mathcal{F}L^p_{m_1},L^q_{m_2})} \asymp \Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|W_{\mathcal{A}}(f,g)(x,\xi)|^p m_1(\xi)^p d\xi\Big)^{q/p} m_2(x)^q dx\Big)^{1/q},$$

with the analogous for $\max\{p,q\}=\infty$.

Proof. Assume that $\max\{p,q\}<\infty$. We use (12). Let $f\in\mathcal{S}'(\mathbb{R}^d)$, $g\in\mathcal{S}(\mathbb{R}^d)\setminus\{0\}$. Then,

$$\Big(\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|W_{\mathcal{A}}(f,g)(x,\xi)|^p m_1(\xi)^p d\xi\Big)^{q/p} m_2(x)^q dx\Big)^{1/q}$$
$$= \left(\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|W_{\mathcal{A}}(f,g)(J^{-1}(\xi,-x))|^p m_1(\xi)^p d\xi\right)^{q/p} m_2(x)^q dx\right)^{1/q}.$$

Now, $|W_{\mathcal{A}}(f,g)\circ J^{-1}| = |\mu(\mathcal{D}_{J^{-1}})\mu(\mathcal{A})(f\otimes\bar{g})| = |\mu(\mathcal{A}_0)(f\otimes\bar{g})| = |W_{\mathcal{A}_0}(f,g)|$, where

$$\pi^{Mp}(\mu(\mathcal{A}_0)) = \mathcal{A}_0 := \mathcal{D}_{J^{-1}}\mathcal{A} = \begin{pmatrix} A_{21} & A_{22} & A_{23} & A_{24} \\ -A_{11} & -A_{12} & -A_{13} & -A_{14} \\ A_{41} & A_{42} & A_{43} & A_{44} \\ -A_{31} & -A_{32} & -A_{33} & -A_{34} \end{pmatrix}.$$

By [13, Proposition 2.7], $|W_{\mathcal{A}_0}(f,g)| = |W_{\tilde{\mathcal{A}}_0}(\hat{f},\hat{g})|$, where

$$\tilde{\mathcal{A}}_0 = \begin{pmatrix} -A_{23} & A_{24} & A_{21} & -A_{22} \\ A_{13} & -A_{14} & -A_{11} & A_{12} \\ -A_{43} & A_{44} & A_{41} & -A_{42} \\ A_{33} & -A_{34} & -A_{31} & A_{32} \end{pmatrix}.$$

Hence, using that $\mathcal{I}m_2 \asymp m_2$,

$$\begin{aligned}
&\Big(\int\limits_{\mathbb{R}^d}\Big(\int\limits_{\mathbb{R}^d}|W_{\mathcal{A}}(f,g)(x,\xi)|^p m_1(\xi)^p d\xi\Big)^{q/p} m_2(x)^q dx\Big)^{1/q}\\
&\asymp \left(\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|W_{\tilde{\mathcal{A}}_0}(\hat{f},\hat{g})(\xi,-x)|^p m_1(\xi)^p d\xi\right)^{q/p} m_2(x)^q dx\right)^{1/q}\\
&= \left(\int\limits_{\mathbb{R}^d}\left(\int\limits_{\mathbb{R}^d}|W_{\tilde{\mathcal{A}}_0}(\hat{f},\hat{g})(\xi,x)|^p m_1(\xi)^p d\xi\right)^{q/p} \mathcal{I}m_2(x)^q dx\right)^{1/q}\\
&\asymp \|W_{\tilde{\mathcal{A}}_0}(\hat{f},\hat{g})\|_{L^{p,q}_{m_1\otimes m_2}}.
\end{aligned}$$

Observe that $\tilde{E}_{\mathcal{A}} = E_{\tilde{\mathcal{A}}_0}$. Since $E_{\mathcal{A}}$ is invertible and lower triangular the matrix $\tilde{E}_{\mathcal{A}}$ in (37) is obviously invertible (and upper triangular). Hence, using the assumption $m_1 \otimes m_2 \asymp (m_1\otimes m_2)\circ E^{-1}_{\tilde{\mathcal{A}}_0}$, we have

$$\|W_{\tilde{\mathcal{A}}_0}(\hat{f},\hat{g})\|_{L^{p,q}_{m_1\otimes m_2}} \asymp \|\hat{f}\|_{M^{p,q}_{m_1\otimes m_2}} = \|f\|_{W(\mathcal{F}L^p_{m_1},L^q_{m_2})}.$$

The same argument also proves the case $\max\{p,q\}=\infty$, simply replacing the corresponding integrals with the essential supremums. ■

Remark 3.13. Because of (18), Corollary 3.12 is significant only for $p \neq q$. For $p = q$ we refer to Theorem 3.10 with $m = m_1 \otimes m_2$.

4. Examples

We exhibit a manifold of new metaplectic Wigner distributions which may find application in time-frequency analysis, signal processing, quantum mechanics and pseudodifferential theory.

Example 4.1. This example generalizes the STFT by applying a metaplectic operator either on the window function g or on the function f as follows. First, consider the matrix

$$L = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ -I_{d\times d} & I_{d\times d} \end{pmatrix},$$

which allows to rewrite the STFT $V_g f$ as composition of the metaplectic operators $V_g f(x,\xi) = \mathcal{F}_2\mathfrak{T}_L(f\otimes \bar{g})(x,\xi)$.

(i) We may act on the window g by replacing $\bar{g}$ with $\mu(\mathcal{A}')\bar{g}$, $\mu(\mathcal{A}') \in Mp(d,\mathbb{R})$. Namely, we consider the time-frequency representation

$$\mathcal{U}_g f(x,\xi) = \mathcal{F}_2\mathfrak{T}_L(f\otimes \mu(\mathcal{A}')\bar{g})(x,\xi).$$

Denoting

$$\mathcal{A}' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}, \tag{38}$$

by (B.4),

$$f \otimes \mu(\mathcal{A}')\bar{g} = \mu(\mathcal{A}'')(f \otimes \bar{g}),$$

with

$$\mathcal{A}'' = \begin{pmatrix} I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & A' & 0_{d\times d} & B' \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & C' & 0_{d\times d} & D' \end{pmatrix},$$

so that $\mathcal{U}_g f = W_{\mathcal{A}}(f,g)$ with

$$\mathcal{A} = \begin{pmatrix} I_{d\times d} & -A' & 0_{d\times d} & -B' \\ 0_{d\times d} & C' & I_{d\times d} & D' \\ 0_{d\times d} & -C' & 0_{d\times d} & -D' \\ -I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \end{pmatrix},$$

which is always shift-invertible with $E_{\mathcal{A}}$ diagonal. This is not surprising, since $\mu(\mathcal{A}')\bar{g} \in \mathcal{S}(\mathbb{R}^d)$ for $g \in \mathcal{S}(\mathbb{R}^d)$ and different windows in $\mathcal{S}(\mathbb{R}^d)$ yield equivalent norms.

(ii) A more interesting example comes out by applying $\mu(\mathcal{A}')$, with $\mathcal{A}' \in Sp(d,\mathbb{R})$ having block decomposition in (38), to the function f. Namely, we consider

$$\widetilde{\mathcal{U}}_g f(x,\xi) = \mathcal{F}_2\mathfrak{T}_L(\mu(\mathcal{A}')f \otimes \bar{g})(x,\xi).$$

Then $\widetilde{\mathcal{U}}_g f = W_{\mathcal{A}}(f,g)$ with

$$\mathcal{A} = \begin{pmatrix} A' & -I_{d\times d} & B' & 0_{d\times d} \\ C' & 0_{d\times d} & D' & I_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & -I_{d\times d} \\ -A' & 0_{d\times d} & -B' & 0_{d\times d} \end{pmatrix},$$

and

$$E_{\mathcal{A}} = \mathcal{A}' = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix},$$

in (38). This $W_{\mathcal{A}}$ characterizes modulation spaces if and only if the symplectic matrix $\mathcal{A}'$ is upper triangular, since $\mu(\mathcal{A}') : M^{p,q}(\mathbb{R}^d) \to M^{p,q}(\mathbb{R}^d)$, $p \neq q$, if and only if $\mathcal{A}'$ is an upper block triangular matrix [22].

Observe that these time-frequency representations find applications in signal processing, see Zhang et al. [40,41].

Example 4.2. (i). For $z = (x,\xi) \in \mathbb{R}^{2d}$, the time-frequency shift $\pi(z)$ can be written as follows: $\pi(z)g(t) = \Phi_{\tilde{I}}(\xi,t)T_x g(t)$, where $\tilde{I} \in \mathbb{R}^{2d\times 2d}$ is the symmetric matrix

$$\tilde{I} = \begin{pmatrix} 0_{d\times d} & I_{d\times d} \\ I_{d\times d} & 0_{d\times d} \end{pmatrix}.$$

Thus, we can define a *generalized STFT* replacing the time-frequency atoms $\pi(z)g$ with the more general atoms $\varsigma(x,\xi) := \Phi_C(\xi,\cdot)T_x$, $x,\xi \in \mathbb{R}^d$, where Φ_C is the chirp function related to the symmetric matrix

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{pmatrix}$$

(hence $C_{11}^T = C_{11}$, $C_{22}^T = C_{22}$). Namely, we may define the *generalized STFT* $\mathcal{V}_{g,C}f$ as

$$\mathcal{V}_{g,C}f(x,\xi) = |\det(C_{12})|^{1/2} e^{-i\pi C_{11}\xi\cdot\xi} \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-i\pi C_{22}t\cdot t} e^{-2\pi i C_{12}^T \xi\cdot t} dt = \langle f, \varsigma(x,\xi)g\rangle,$$

$f,g \in L^2(\mathbb{R}^d)$. Observe that, if $C_{12} \in GL(d,\mathbb{R})$, then $\mathcal{V}_{g,C}f = W_{\mathcal{A}}(f,g)$, with

$$\mathcal{A} = \begin{pmatrix} I_{d\times d} & -I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ -C_{12}^{-T}C_{22} & 0_{d\times d} & C_{12}^{-T} & C_{12}^{-T} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & -I_{d\times d} \\ -C_{12} + C_{11}C_{12}^{-T}C_{22} & 0_{d\times d} & -C_{11}C_{12}^{-T} & -C_{11}C_{12}^{-T} \end{pmatrix},$$

which is always shift-invertible, but unless $C_{22} \neq 0_{d\times d}$, $E_{\mathcal{A}}$ is lower triangular.

(ii) For $\tau \in \mathbb{R}$ consider the τ-Wigner distribution defined in (15) and replace the Gabor atoms $\pi(x,\xi)$ with the more general chirp functions Φ_C as before to obtain

$$\mathcal{W}_{\tau,C}(f,g)(x,\xi) = |\det(C_{12})|^{1/2} e^{-i\pi C_{11}\xi\cdot\xi} \int_{\mathbb{R}^d} f(x+\tau t)\overline{g(x-(1-\tau)t)} e^{-i\pi C_{22}t\cdot t} e^{-2\pi i C_{12}^T\xi\cdot t} dt,$$

$f,g \in L^2(\mathbb{R}^d)$. Again, if $C_{12} \in GL(d,\mathbb{R})$, then $\mathcal{W}_{\tau,C}(f,g) = W_{\mathcal{A}}(f,g)$ with

$$\mathcal{A} = \begin{pmatrix} (1-\tau)I_{d\times d} & \tau I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ -C_{12}^{-T}C_{22} & -C_{12}^{-T}C_{22} & \tau C_{12}^{-T} & -(1-\tau)C_{12}^{-T} \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & I_{d\times d} \\ -C_{12} + C_{11}C_{12}^{-T}C_{22} & -C_{12} + C_{11}C_{12}^{-T}C_{22} & -\tau C_{11}C_{12}^{-T} & (1-\tau)C_{11}C_{12}^{-T} \end{pmatrix}.$$

This matrix is shift-invertible if and only if $\tau \neq 0, 1$, and in this case $E_{\mathcal{A}}$ is upper-triangular if and only if $C_{22} = 0_{d\times d}$.

Example 4.3. Every $\mathcal{A} \in Sp(2d,\mathbb{R})$ can be written as $\Pi_{\mathcal{J}} V_Q \mathcal{D}_L V_{-P}^T$, where $L \in GL(2d,\mathbb{R})$, the matrices $Q,P \in \mathbb{R}^{2d\times 2d}$ are symmetric and, if $1 \leq k \leq 2d$, $1 \leq j_1,\dots,j_k \leq 2d$ and $\mathcal{J} = \{j_1,\dots,j_k\}$, $\Pi_{\mathcal{J}} = \Pi_{j_1}\dots\Pi_{j_k}$ is the matrix associated to the partial Fourier transform $\mathcal{F}_{\mathcal{J}} := \mathcal{F}_{j_1}\dots\mathcal{F}_{j_k}$, cf. Example 2.4. Set

$$Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}, \ P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}, \ L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix} \text{ and } L^{-1} = \begin{pmatrix} L'_{11} & L'_{12} \\ L'_{21} & L'_{22} \end{pmatrix}.$$

A direct computation shows

$$V_Q\mathcal{D}_L V_{-P}^T = \begin{pmatrix} L'_{11} & L'_{12} & -L'_{11}P_{11}^T - L'_{12}P_{12}^T & -L'_{11}P_{12} - L'_{12}P_{22}^T \\ L'_{21} & L'_{22} & -L'_{21}P_{11}^T - L'_{22}P_{12}^T & -L'_{21}P_{12} - L'_{22}P_{22}^T \\ M_{11} & M_{12} & N_{11} & N_{12} \\ M_{21} & M_{22} & N_{21} & N_{22} \end{pmatrix}.$$

In what follows the explicit expressions of $M_{11}, M_{12}, M_{21}, M_{22}, N_{11}, N_{12}, N_{21}, N_{22} \in \mathbb{R}^{d\times d}$ are irrelevant. We consider the case $\mathcal{J} = \{d+1,\dots,2d\}$, i.e., $\mathcal{F}_{\mathcal{J}} = \mathcal{F}_2$. The effect of left-multiplying $V_Q\mathcal{D}_L V_{-P}^T$ by $\mathcal{A}_{FT2}$ is to swap the second column blocks of $V_Q\mathcal{D}_L V_{-P}^T$ with the fourth, up to change the sign of the latter. Hence, the matrix $E_{\mathcal{A}}$ associated to

$$W_{\mathcal{A}}(f,g) = \mathcal{F}_2(\Phi_Q \cdot (\mathcal{F}^{-1}\Phi_P * (f \otimes \bar{g})))$$

is

$$E_{\mathcal{A}} = \begin{pmatrix} L'_{11} & -L'_{11}P_{11}^T - L'_{12}P_{12}^T \\ L'_{21} & -L'_{21}P_{11}^T - L'_{22}P_{12}^T \end{pmatrix}.$$

This matrix is upper triangular if and only if $L'_{21} = 0_{d\times d}$ or, equivalently, if and only if L is upper triangular. In this case, we also can compute explicitly L^{-1} in terms of the blocks of L. Namely,

$$L = \begin{pmatrix} L_{11} & L_{12} \\ 0_{d\times d} & L_{22} \end{pmatrix} \in GL(2d,\mathbb{R}) \Rightarrow L^{-1} = \begin{pmatrix} L_{11}^{-1} & -L_{11}^{-1}L_{12}L_{22}^{-1} \\ 0_{d\times d} & L_{22}^{-1} \end{pmatrix}.$$

So, the corresponding $E_{\mathcal{A}}$ is invertible if and only if $L_{22}^{-1}P_{12}^T \in GL(d,\mathbb{R})$, i.e., if and only if $P_{12} \in GL(d,\mathbb{R})$. In conclusion, any metaplectic Wigner distribution of the form

$$W_{\mathcal{A}}(f,g)(x,\xi) = \mathcal{F}_2(\Phi_Q \cdot (\mathcal{F}^{-1}\Phi_P * (f \otimes \bar{g})))(x,\xi)$$

with $P, Q \in \mathbb{R}^{2d\times 2d}$ symmetric, $P_{12} \in GL(d,\mathbb{R})$, and $L \in GL(2d,\mathbb{R})$ upper triangular, can be used to define modulation spaces.

Appendix A

In Appendix A we generalize the results in [22] to the quasi-Banach setting. Also, we observe that [22, Corollary 4.2] holds for general invertible matrices. For $S \in GL(2d,\mathbb{R})$, recall the definition of the metaplectic operator

$$\mathfrak{T}_S f(z) = |\det(S)|^{\frac{1}{2}} f(Sz), \quad z \in \mathbb{R}^{2d},$$

defined in Example 2.2 (ii).

Theorem A.1. *Let $S \in GL(2d,\mathbb{R})$ and $0 < p \le \infty$. The mapping $\mathfrak{T}_S : L^p(\mathbb{R}^{2d}) \to L^p(\mathbb{R}^{2d})$ is everywhere defined and bounded with $\|\mathfrak{T}_S\|_{L^p\to L^p} = |\det(S)|^{\frac{1}{2}-\frac{1}{p}}$. We use the convention $1/\infty = 0$.*

Proof. Trivially, if $0 < p < \infty$ and $f \in L^p(\mathbb{R}^{2d})$,

$$\|\mathfrak{T}_S f\|_{L^p(\mathbb{R}^{2d})} = \|f \circ S\|_{L^p(\mathbb{R}^{2d})} = |\det(S)|^{\frac{1}{2}-\frac{1}{p}} \|f\|_{L^p(\mathbb{R}^{2d})}.$$

Also, $\|\mathfrak{T}_S f\|_{L^\infty(\mathbb{R}^{2d})} = |\det(S)|^{1/2}\|f\|_{L^\infty(\mathbb{R}^{2d})}$. ∎

Theorem A.2. *Consider $A, D \in GL(d,\mathbb{R})$, $B \in \mathbb{R}^{d\times d}$ and $0 < p, q \le \infty$. Define*

$$S = \begin{pmatrix} A & B \\ 0_{d\times d} & D \end{pmatrix}.$$

The mapping $\mathfrak{T}_S : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ is an isomorphism with bounded inverse $\mathfrak{T}_{S^{-1}}$.

Proof. Let $f \in L^{p,q}(\mathbb{R}^{2d})$. Then,

$$\begin{aligned}\|\mathfrak{T}_S f\|_{L^{p,q}(\mathbb{R}^{2d})} &= \|\xi \mapsto |\det(S)|^{1/2}\|f(A\cdot+B\xi, D\xi)\|_{L^p(\mathbb{R}^d)}\|_{L^q(\mathbb{R}^d)}\\ &= \|\xi \mapsto |\det(S)|^{1/2}\|f(A\cdot, D\xi)\|_{L^p(\mathbb{R}^d)}\|_{L^q(\mathbb{R}^d)}\\ &= \|\xi \mapsto |\det(S)|^{1/2}|\det(A)|^{-1/p}\|f(\cdot, D\xi)\|_{L^p(\mathbb{R}^d)}\|_{L^q(\mathbb{R}^d)}\\ &= |\det(S)|^{1/2}|\det(A)|^{-1/p}|\det(D)|^{-1/q}\|\xi \mapsto \|f(\cdot,\xi)\|_{L^p(\mathbb{R}^d)}\|_{L^q(\mathbb{R}^d)}\\ &= |\det(A)|^{\frac{1}{2}-\frac{1}{p}}|\det(D)|^{\frac{1}{2}-\frac{1}{q}}\|f\|_{L^{p,q}(\mathbb{R}^{2d})},\end{aligned}$$

where $1/\infty = 0$. Observe that $\mathfrak{T}_S^{-1} = \mathfrak{T}_{S^{-1}}$. It remains to prove that $\mathfrak{T}_{S^{-1}}$ is also bounded. Since A (or, equivalently, D) is invertible, this follows by

$$S^{-1} = \begin{pmatrix} A^{-1} & -A^{-1}BD^{-1} \\ 0_{d\times d} & D^{-1}\end{pmatrix}$$

and by the first part of the statement. ■

Theorem A.3. *Let $m \in \mathcal{M}_v(\mathbb{R}^{2d})$, $S \in GL(2d,\mathbb{R})$ and $0 < p, q \le \infty$. Consider the operator*

$$(\mathfrak{T}_S)_m : f \in L^{p,q}_m(\mathbb{R}^{2d}) \mapsto |\det(S)|^{1/2} f\circ S.$$

If $m\circ S \asymp m$, then $\mathfrak{T}_S : L^{p,q}(\mathbb{R}^{2d}) \to L^{p,q}(\mathbb{R}^{2d})$ is bounded if and only if $(\mathfrak{T}_S)_m : L^{p,q}_m(\mathbb{R}^{2d}) \to L^{p,q}_m(\mathbb{R}^{2d})$ is bounded.

Proof. Observe that the condition $m\circ S \asymp m$ is equivalent to $m\circ S^{-1} \asymp m$. Assume that $\mathfrak{T}_S$ is bounded on $L^{p,q}(\mathbb{R}^{2d})$ and consider $f \in L^{p,q}_m(\mathbb{R}^{2d})$. Then, $fm \in L^{p,q}(\mathbb{R}^{2d})$ and

$$\begin{aligned}\|\mathfrak{T}_S f\|_{L^{p,q}_m(\mathbb{R}^{2d})} &= \|\mathfrak{T}_S f\cdot m\|_{L^{p,q}(\mathbb{R}^{2d})} = \|\mathfrak{T}_S(f\cdot(m\circ S^{-1}))\|_{L^{p,q}(\mathbb{R}^{2d})}\\ &\lesssim \|f\cdot(m\circ S^{-1})\|_{L^{p,q}(\mathbb{R}^{2d})} = \left\|\left(f\frac{m\circ S^{-1}}{m}\right)m\right\|_{L^{p,q}(\mathbb{R}^{2d})}\\ &= \left\|f\frac{m\circ S^{-1}}{m}\right\|_{L^{p,q}_m(\mathbb{R}^{2d})} \lesssim \|f\|_{L^{p,q}_m(\mathbb{R}^{2d})}.\end{aligned}$$

For the converse, assume that $(\mathfrak{T}_S)_m : L^{p,q}_m(\mathbb{R}^{2d}) \to L^{p,q}_m(\mathbb{R}^{2d})$ is bounded and take $f \in L^{p,q}(\mathbb{R}^{2d})$. Then, $f/m \in L^{p,q}_m(\mathbb{R}^{2d})$ and

$$\begin{aligned}\|\mathfrak{T}_S f\|_{L^{p,q}(\mathbb{R}^{2d})} &\lesssim \left\|\mathfrak{T}_S f\left(\frac{m}{m\circ S}\right)\right\|_{L^{p,q}(\mathbb{R}^{2d})} \asymp \left\|\mathfrak{T}_S\left(\frac{f}{m}\right)\right\|_{L^{p,q}(\mathbb{R}^{2d})}\\ &= \left\|\mathfrak{T}_S\left(\frac{f}{m}\right)\right\|_{L^{p,q}_m(\mathbb{R}^{2d})} \lesssim \|f/m\|_{L^{p,q}_m(\mathbb{R}^{2d})} = \|f\|_{L^{p,q}(\mathbb{R}^{2d})}. \quad ■\end{aligned}$$

Appendix B

In this Appendix we study tensor products of metaplectic operators and refer to [27] for the theory of tensor products of HIlbert spaces. We are interested in proving the following result.

Theorem B.1. *Let $\mu(\mathcal{A}), \mu(\mathcal{B}) \in Mp(d,\mathbb{R})$ with $\mathcal{A} = \pi^{Mp}(\mu(\mathcal{A}))$ and $\mathcal{B} = \pi^{Mp}(\mu(\mathcal{B}))$ having block decompositions*

$$\mathcal{A}=\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \ \textit{and}\ \mathcal{B}=\begin{pmatrix} E & F \\ G & H \end{pmatrix}.$$

Then, the bilinear operator $S: L^2(\mathbb{R}^d)\times L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^{2d})$ *defined for all* $f,g\in L^2(\mathbb{R}^d)$ *as*

$$S(f,g)=\mu(\mathcal{A})f\otimes\mu(\mathcal{B})g$$

extends uniquely to a metaplectic operator $\mu(\mathcal{C})\in Mp(2d,\mathbb{R})$ *with* $\mathcal{C}=\pi^{Mp}(\mu(\mathcal{C}))$ *having block decomposition*

$$\mathcal{C}=\begin{pmatrix} A & 0_{d\times d} & B & 0_{d\times d} \\ 0_{d\times d} & E & 0_{d\times d} & F \\ C & 0_{d\times d} & D & 0_{d\times d} \\ 0_{d\times d} & G & 0_{d\times d} & H \end{pmatrix}.$$

Proof. By [27, Proposition 2.6.6], there exists a unique linear mapping $T: L^2(\mathbb{R}^d)\otimes L^2(\mathbb{R}^d)=L^2(\mathbb{R}^{2d})\to L^2(\mathbb{R}^{2d})$ satisfying

$$T(f\otimes g)=\mu(\mathcal{A})f\otimes\mu(\mathcal{B})g, \qquad f,g\in L^2(\mathbb{R}^d).$$

By [27, Proposition 2.6.12], this extension is also bounded. Moreover, T is invertible because $\mu(\mathcal{A})$ and $\mu(\mathcal{B})$ are. In particular, T is surjective. To prove that T is a metaplectic operator, it remains to check that T preserves the L^2 inner product and that

$$T\rho(z,\tau)=\rho(\mathcal{C}z,\tau)T, \qquad z\in\mathbb{R}^{4d},\ \tau\in\mathbb{R}. \tag{B.1}$$

For all $f,g,\varphi,\psi\in L^2(\mathbb{R}^d)$,

$$\begin{aligned}\langle T(f\otimes g),T(\varphi\otimes\psi)\rangle &= \langle\mu(\mathcal{A})f,\mu(\mathcal{A})\varphi\rangle\langle\mu(\mathcal{B})g,\mu(\mathcal{B})\psi\rangle=\langle f,\varphi\rangle\langle g,\psi\rangle\\ &=\langle f\otimes g,\varphi\otimes\psi\rangle.\end{aligned}$$

If $\Phi\in L^2(\mathbb{R}^{2d})$, $\Phi=\sum_{j=1}^\infty c_j\varphi_j\otimes\psi_j$, with the sequence $(c_j)_j\subseteq\mathbb{C}$ vanishing definitely,

$$\begin{aligned}\langle T(f\otimes g),T\Phi\rangle &= \sum_j \bar{c}_j\langle T(f\otimes g),T(\varphi_j\otimes\psi_j)\rangle=\sum_j\bar{c}_j\langle f\otimes g,\varphi_j\otimes\psi_j\rangle\\ &=\langle f\otimes g,\Phi\rangle.\end{aligned}$$

Now, consider $\Phi\in L^2(\mathbb{R}^{2d})$ and $(\Phi_j)_j\subseteq\mathrm{span}\{\varphi\otimes\psi:\varphi,\psi\in L^2(\mathbb{R}^d)\}$ satisfying $\lim_{j\to+\infty}\|\Phi-\Phi_j\|_{L^2(\mathbb{R}^{2d})}=0$. Then, by the continuity of T and of the inner product,

$$\begin{aligned}\langle T(f\otimes g),T\Phi\rangle &= \langle T(f\otimes g),T(\lim_{j\to+\infty}\Phi_j)\rangle=\lim_{j\to+\infty}\langle T(f\otimes g),T\Phi_j\rangle\\ &=\lim_{j\to+\infty}\langle f\otimes g,\Phi_j\rangle=\langle f\otimes g,\Phi\rangle.\end{aligned}$$

So, we proved that

$$\langle TF,T\Phi\rangle=\langle F,\Phi\rangle \tag{B.2}$$

holds for all $F=f\otimes g$, $f,g\in L^2(\mathbb{R}^d)$ and all $\Phi\in L^2(\mathbb{R}^{2d})$. The same argument applied to the first component of the inner product shows that (B.2) holds for all $F\in L^2(\mathbb{R}^{2d})$ as well. So, T is surjective and preserves the inner product, hence it is unitary. It remains to prove (B.1), which states that T is a metaplectic operator with $\pi^{Mp}(T) = \mathcal{C}$.

For, consider $f, g \in L^2(\mathbb{R}^d)$, $\tau \in \mathbb{R}$, $z = (x_1, x_2, \xi_1, \xi_2) \in \mathbb{R}^{4d}$ and $z_j = (x_j, \xi_j) \in \mathbb{R}^{2d}$ $(j = 1, 2)$. First, observe that

$$\rho(z,\tau)(f \otimes g) = e^{-2\pi i\tau}\rho(z_1,\tau)f \otimes \rho(z_2,\tau)g \tag{B.3}$$

and

$$\pi(\mathcal{C}z)(f \otimes g) = \pi(\mathcal{A}z_1)f \otimes \pi(\mathcal{B}z_2)g,$$

so that

$$\begin{aligned}
\rho(\mathcal{C}z,\tau)T(f\otimes g) &= e^{2\pi i\tau}e^{-i\pi(Ax_1+B\xi_1)(Cx_1+D\xi_1)}e^{-i\pi(Ex_2+F\xi_2)(Gx_2+H\xi_2)} \\
&\qquad \times \pi(\mathcal{C}z)(\mu(\mathcal{A})f \otimes \mu(\mathcal{B})g) \\
&= e^{2\pi i\tau}e^{-i\pi(Ax_1+B\xi_1)(Cx_1+D\xi_1)}e^{-i\pi(Ex_2+F\xi_2)(Gx_2+H\xi_2)} \\
&\qquad \times \pi(\mathcal{A}z_1)\mu(\mathcal{A})f \otimes \pi(\mathcal{B}z_2)\mu(\mathcal{B})g \\
&= e^{-2\pi i\tau}\rho(\mathcal{A}z_1,\tau)\mu(\mathcal{A})f \otimes \rho(\mathcal{B}z_2,\tau)\mu(\mathcal{B})g \\
&= e^{-2\pi i\tau}\mu(\mathcal{A})\rho(z_1,\tau)f \otimes \mu(\mathcal{B})\rho(z_2,\tau)g \\
&= e^{-2\pi i\tau}T(\rho(z_1,\tau)f \otimes \rho(z_2,\tau)g) \\
&= T\rho(z,\tau)(f \otimes g)
\end{aligned}$$

and (B.1) follows for tensor products. Next, if $F = \sum_{j=1}^{\infty} c_j f_j \otimes g_j$, $(c_j)_j \subseteq \mathbb{C}$ definitely zero, $f_j, g_j \in L^2(\mathbb{R}^d)$ $(j = 1, 2, \ldots)$,

$$\begin{aligned}
\rho(\mathcal{C}z,\tau)TF &= \rho(\mathcal{C}z,\tau)T(\sum_j c_j f_j \otimes g_j) = \rho(\mathcal{C}z,\tau)\sum_j c_j T(f_j \otimes g_j) \\
&= \sum_j c_j \rho(\mathcal{C}z,\tau)T(f_j \otimes g_j) = \sum_j c_j T\rho(z,\tau)(f_j \otimes g_j) = T\rho(z,\tau)F,
\end{aligned}$$

and the assertion follows by a standard density argument. ■

Remark B.2. Under the same notation as in Theorem B.1, if $\mathcal{A} = I_{d\times d}$, then

$$f \otimes \mu(\mathcal{B})g = \mu(\mathcal{C}_1)(f \otimes g), \tag{B.4}$$

where

$$\mathcal{C}_1 = \begin{pmatrix} I_{d\times d} & 0_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & E & 0_{d\times d} & F \\ 0_{d\times d} & 0_{d\times d} & I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & G & 0_{d\times d} & H \end{pmatrix}.$$

If $\mathcal{B} = I_{d\times d}$, we infer

$$\mu(\mathcal{A})f \otimes g = \mu(\mathcal{C}_2)(f \otimes g),$$

where

$$\mathcal{C}_2 = \begin{pmatrix} A & 0_{d\times d} & B & 0_{d\times d} \\ 0_{d\times d} & I_{d\times d} & 0_{d\times d} & 0_{d\times d} \\ C & 0_{d\times d} & D & 0_{d\times d} \\ 0_{d\times d} & 0_{d\times d} & 0_{d\times d} & I_{d\times d} \end{pmatrix}.$$

Observe that $\mathcal{C} = \mathcal{C}_1\mathcal{C}_2 = \mathcal{C}_2\mathcal{C}_1$.

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