Why Does Implied Risk Aversion Smile?

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Abstract

Implied risk aversion estimates reported in the literature are strongly U-shaped. This paper explores different potential explanations for these "smile" patterns: (i) preference aggregation, both with and without stochastic volatility and jumps in returns, (ii) misestimation of investors' beliefs caused by stochastic volatility, jumps or a Peso problem, and (iii) heterogeneous beliefs. The results reveal that preference aggregation and misestimation of investors' beliefs caused by stochastic volatility and jumps are unlikely to be the explanation for the smile. Although a Peso problem can account for the smile, the required probability of a market crash is unrealistically large. Heterogeneous beliefs cause sizable distortions in implied risk aversion, but the degree of heterogeneity required to explain the smile is implausibly large.

JEL Classification: G12, G13

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In a representative agent economy, equilibrium asset prices reflect the agent's preferences and beliefs. Rubinstein (1994) shows that any two of the following imply the third: (i) the representative agent's preferences, (ii) his subjective probability assessments, and (iii) the state-price density. Therefore, essentially any state-price density can be reconciled with the distribution of asset prices by using an appropriate set of preferences for the representative agent. Building on this insight, a number of papers have derived estimates of the representative agent's risk aversion from the state-price density and the subjective probability density.

Aït-Sahalia and Lo (2000), Jackwerth (2000), and Rosenberg and Engle (2002) report implied risk aversion estimates that have similar shapes, with some differences. They all find that implied risk aversion varies strongly across S&P 500 index values and is U-shaped around the futures price. Jackwerth (2000) and Rosenberg and Engle (2002) even obtain negative risk aversion estimates for a range of index values centered around the futures price.

The aim of this paper is to explore different potential explanations for the implied risk aversion smile by investigating the properties of implied risk aversion estimators in different settings within the standard consumption-based framework commonly used in the implied risk aversion literature. Before going into the details of these potential explanations, it is worth reflecting on the advantages and limitations of this framework. Its limitations are well-known: in addition to the assumptions of standard consumption-based asset pricing models – an exchange economy with complete, frictionless markets and a single consumption good – this framework assumes that the stock index is a good proxy for the aggregate endowment. These assumptions are restrictive. For instance, the stochastic processes driving index returns and aggregate consumption are different: stock indices exhibit stochastic volatility, while aggregate consumption is almost homoscedastic [Lochstoer (2004)]. In addition, numerous papers argue that it is necessary to relax the assumptions of complete, frictionless markets and a single consumption good in order to reconcile the stochastic processes driving aggregate consumption and asset prices. [1] It is therefore important to keep in mind that the conclusions drawn below are valid only within the framework considered, and that the limitations of the standard consumption-based framework constitute alternative potential explanations for the implied risk aversion smile. For example, if the stock index is not a good proxy for the aggregate endowment, then preferences inferred on the basis of stock indices will be distorted. Similarly, market incompleteness – such as the existence of non-tradable background risk, borrowing and/or short-selling constraints – may affect individual agents' behavior in such a way that market prices suggest oddly-behaved preferences unless the precise nature of incompleteness is modeled explicitly and its effect on individual behavior accounted for. [2]

In spite of its limitations, the standard consumption-based framework is appealing to explore potential explanations for the implied risk aversion smile. Its main advantage is that the link between the representative agent's preferences, his beliefs and the state-price density naturally suggests a number of potential explanations. The first possibility is that the *aggregation of investors' heterogeneous preferences* could lead to an oddly-behaved economy-wide risk aversion function. This aggregation problem could be compounded by the presence of stochastic volatility and jumps in returns.

Second, *misestimation of investors' beliefs* could distort implied risk aversion estimates. Since agents' beliefs are unobservable, a long tradition has emerged in financial economics of estimating them using historical return distributions. As noted by Brown and Jackwerth (2004), the problem with this approach is that such estimates are backward-looking, while investor beliefs are by definition forward-looking. Accordingly, such estimates are inaccurate whenever the return process is time-varying – for instance, because of stochastic volatility and jumps – and whenever agents expect return realizations that are absent from historical return realizations, i.e., in the presence of a Peso problem. In addition, beliefs estimates based on historical returns typically ignore the possibility of heterogeneous beliefs, whose existence could distort implied risk aversion estimates.

Third, *misestimation of the state-price density* could distort implied risk aversion estimates. However, the estimation of state-price densities does not suffer from the same pitfalls as the estimation of beliefs because state-price densities are forward-looking estimates obtained from *observed forward-looking* variables, namely traded option prices. [3] Moreover, they are *unique* market prices, irrespective of whether investors have homogeneous or heterogeneous beliefs or preferences.

Although the accuracy of non-parametric estimators is sometimes criticized, especially in regions with few observations [Aït-Sahalia and Duarte (2003)], the state-price density estimation methodology seems unlikely to be the cause of the implied risk aversion smile, for three reasons. First, although they each use a somewhat different methodology, Aït-Sahalia and Lo (2000), Jackwerth (2000), and Rosenberg and Engle (2002) all obtain similar results. Second, the most puzzling part of the implied risk aversion patterns – the negative risk aversion estimates – arises in the range of index values centered around the futures price, i.e., in the region where options are most liquid and the largest number of observations is available. Third, although the reliability of nonparametric estimators has been questioned in the literature, Bliss and Panigirtzoglou (2000) show that the smoothed implied volatility smile method – which is used in Aït-Sahalia and Lo (2000) – is actually quite stable.

Since state-price density estimates are quite reliable, this paper focuses on *preference aggregation* and *beliefs* as potential explanations for the implied risk aversion smile. More specifically, it investigates to what extent the following factors can account for the smile within the standard consumption-based framework:

1. The aggregation of heterogeneous preferences among agents, both with and without stochastic volatility and jumps;
2. Misestimation of agents' beliefs due to stochastic volatility, jumps or a Peso problem; and
3. Heterogeneous beliefs among agents.

The results reveal that most of the properties of individual agents' risk aversion functions carry over to implied risk aversion. Preference aggregation therefore seems unlikely to be the explanation for the implied risk aversion smile. This result also holds true in the presence of stochastic volatility and jumps in returns.

Implied risk aversion estimates are found to be very sensitive to beliefs estimates, warranting a detailed investigation of the potential causes of beliefs estimation errors. In order to assess whether beliefs estimation errors caused by stochastic volatility and jumps can explain the smile, the risk aversion function implicit in the Pan (2002) stochastic volatility and jumps model is computed. Although it does not smile, the resulting risk aversion function varies strongly with the stock index level and is negative in high return states, suggesting that misestimation of beliefs caused by stochastic volatility and jumps as captured in the Pan (2002) model is unlikely to be the explanation for the smile. The beliefs misestimation pattern implied by the formal link between beliefs estimation errors and risk aversion estimation errors indicates that beliefs estimates based on historical returns overestimate the probability of very high return realizations and underestimate the probability of very low return realizations – a pattern that would typically arise in the presence of a Peso problem. However, in order to reproduce the semi-parametric state-price density of Aït-Sahalia and Lo (2000), the perceived probability of a market crash must be unrealistically large. Thus, although a Peso problem can contribute to explaining the smile, it appears unlikely to be its only cause.

The presence of heterogeneous beliefs is shown to cause significant distortions in implied risk aversion estimates if heterogeneity is not accounted for explicitly in the estimation process. This result holds *even if* implied risk aversion is estimated using the weighted average of investors' beliefs. However, fitting a heterogeneous-beliefs state-price density with three groups of investors to the semi-parametric state-price density of Aït-Sahalia and Lo (2000) reveals that two groups of pessimistic investors with an implausibly large degree of pessimism are required to generate the fat left tail of the empirical state-price density and explain the implied risk aversion smile.

Thus, for plausible parameter values, none of the three potential explanations considered in this paper is able to account for the implied risk aversion smile within the standard consumption-based framework commonly used in the implied risk aversion literature. In order to explain the smile, it therefore seems necessary to go beyond the standard consumption-based framework and analyze the impact of factors such as market incompleteness, market frictions, and the fact that the stock index may not be a good proxy for the aggregate endowment.

The paper is organized as follows. Section 1 presents the heterogeneous-beliefs model used in the analysis. Section 2 characterizes the properties of implied risk aversion in different settings in order to explore the three potential explanations for the implied risk aversion smile. Section 3 concludes.

1 The Model

In order to study the properties of implied risk aversion estimates in a setting with heterogeneous preferences and beliefs, we consider a continuous-time incomplete information exchange economy with a finite horizon $[0, T]$. Our economy is similar to that used by Basak (2005) in his analysis of asset pricing with heterogeneous beliefs, and the following description is therefore similar to the one in his paper. There is a single consumption good, and markets are complete. There is a large number of price-taking risk-averse agents indexed by $i = 1, \ldots, I$. These investors may have heterogeneous beliefs, preferences, and endowments. This setting generalizes the representative agent setting commonly used in the implied risk aversion literature by explicitly allowing for heterogeneity among agents.

1.1 Information structure and investor beliefs

The information structure is similar to that in Basak (2005), except that the possibility of jumps in the aggregate endowment is allowed for. The uncertainty in the economy is represented by a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The complete information filtration is denoted $\{\mathcal{F}_t\} = \sigma(B, N)$ where B is a standard Brownian motion and N a Poisson counter. Investors commonly observe the aggregate endowment c_t, which is exogenous and bounded away from zero. However, they have incomplete (but symmetric) information on its dynamics. The aggregate endowment $c_t \gg 0$ follows

$$dc_t = \mu_t dt + \sigma_t dB_t + dJ_t \tag{1}$$

where B_t is an $\{\mathcal{F}_t\}$-standard Brownian motion and J_t a $\{\mathcal{F}_t\}$-pure jump process. Jumps have probability distribution ν_t and arrival intensity η_t. [4] The coefficients μ_t, σ_t, ν_t and η_t can be random, and are assumed to satisfy appropriate regularity conditions so that all processes and expectations considered below are well-defined. The mean growth μ_t, the volatility σ_t, the jump arrival intensity η_t and the jump size distribution ν_t are all assumed to be $\{\mathcal{F}_t\}$-adapted.

Investors observe c_t continuously. They have the incomplete information filtration $\mathcal{F}_t^c \subset \mathcal{F}_t$, $t \in [0, T]$, where $\{\mathcal{F}_t^c\}$ denotes the filtration generated by c_t, $\mathcal{F}_t^c = \sigma(c_u : u \leq t)$. Because of continuous observation, they know the time and magnitude of jump realizations. They are thus able to deduce σ_t from the quadratic variation of c_t. However, they can only draw inferences about μ_t, η_t and ν_t. Investors have equivalent probability measures $\mathcal{P}^i$, which are also equivalent to the complete information measure $\mathcal{P}$, and may have heterogeneous prior beliefs about μ_0, η_0 and ν_0. As they observe realizations of the aggregate endowment c_t, investors learn. They update their beliefs in a Bayesian fashion, via $\mu_t^i = E^i(\mu_t|\mathcal{F}_t^c)$, $\eta_t^i = E^i(\eta_t|\mathcal{F}_t^c)$ and $\nu_t^i = E^i(\nu_t|\mathcal{F}_t^c)$. Due to their different priors, investors may draw different inferences about these parameters at all times. Thus, our setting is quite general, allowing for disagreement among investors about μ_t, η_t and ν_t. As is common in incomplete information models, disagreement among investors arises because of different prior beliefs. In these models, although investors learn and their perceived processes typically become more similar as time passes, different priors are generally sufficient to guarantee disagreement among investors at all times. [5]

We shall not provide a detailed exposition of investors' inference process, as it is not necessary to derive the properties of equilibrium prices and implied risk aversion. [6] For our purposes, it is sufficient to recognize that given his initial beliefs and the inference he performs, each investor i perceives c_t to follow a process of the form

$$dc_t = \mu_t^i dt + \sigma_t dB_t^i + dJ_t^i \tag{2}$$

where μ_t^i denotes the investor-specific drift estimate, the (known) diffusion parameter σ_t is common across investors, $dB_t^i = dB_t + \frac{\mu_t - \mu_t^i}{\sigma_t} dt$ denotes the investor-specific Brownian innovation and dJ_t^i a jump component. Although jump realizations are common across investors, we use the superscript i to emphasize that investors may disagree as to the arrival rate of jumps, η_t^i, and the distribution of their size, ν_t^i. Effectively, each investor is endowed with the probability space $(\Omega, \mathcal{F}^i, \mathcal{P}^i)$ and the filtration $\{\mathcal{F}_t^i\} = \{\mathcal{F}_t^c\}$. Different investors i and j's innovations are related by

$$dB_t^j = dB_t^i + \frac{\mu_t^i - \mu_t^j}{\sigma_t} dt \tag{3}$$

In our setting, at any given time t, investors thus have individual probability measures about the realizations of $\omega \in \Omega$ at each future time s, $\mathcal{P}^i(\omega, s)$. In what follows, we will be particularly interested in investors' beliefs about the distribution of the aggregate endowment c at some future time s, given its value at current time t. We will capture these beliefs through the conditional density function $p_{t,s}^i(c_s)$. We use the subscripts t and s to emphasize that at any current time t, there is one such conditional density for each future point in time s considered.

1.2 Securities markets

Trading can take place continuously, and we assume that a sufficient number of securities is available so that the market is dynamically complete. Hence, there exists a (unique) state-price density process for each investor, $\xi_t^i = \xi^i(\omega, t)$, which represents the Arrow-Debreu price per unit probability as perceived by investor i, $\mathcal{P}^i$, of a unit of consumption in state $\omega \in \Omega$ at time t, with the obvious property that $\xi_0^i = 1$.

1.3 Investor endowments, preferences and optimization

Each investor i lives until time T and is endowed with an endowment stream e_t^i, with $\sum_{i=1}^{I} e_t^i = c_t$ for all t. Since markets are complete, this endowment stream is equivalent to an initial wealth of $W_0^i = E^i \left(\int_0^T \xi_t^i e_t^i dt \right)$, where E^i denotes expectation taken with respect to investor i's beliefs $\mathcal{P}^i$. Each investor seeks to maximize his smooth-additive lifetime utility of consumption conditional on his beliefs $\mathcal{P}^i$,

$$U^i(c^i) = E^i \left(\int_0^T u^i(c_t^i, t) dt \right) \tag{4}$$

where the utility function $u^i(c_t^i, t)$ is assumed to be twice continuously differentiable, strictly increasing and strictly concave. This maximization is subject to the constraint of budget feasibility,

$$E^i \left(\int_0^T \xi_t^i c_t^i dt \right) \leq W_0^i \tag{5}$$

Using the Lagrange multiplier $\lambda_i > 0$ that ensures that the above budget constraint holds at the optimum, this maximization problem can be rewritten as the unconstrained problem

$$\max_{\{c^i\}} E^i \left(\int_0^T \left(u^i(c_t^i, t) - \lambda_i \xi_t^i c_t^i \right) dt \right) \tag{6}$$

Maximizing time by time and state by state yields the first-order condition

$$u_c^i(c_t^i, t) = \lambda_i \xi_t^i \tag{7}$$

which uniquely identifies the investor's optimal consumption c_t^i as a function of his individual state-price deflator $\xi_t^i = \xi^i(\omega, t)$.

1.4 Equilibrium

Under market completeness, one can view each investor as purchasing $c_t^i(\xi_t^i)$ units of the consumption good given the realized value of his individual state-price deflator

ξ_t^i at time t. We therefore define equilibrium in this economy as a collection of consumption processes $\{c^i\}$ and state-price deflator processes $\{\xi^i\}$ such that (1) individual investors' consumption processes are optimal given their individual state-price deflators ξ_t^i (which themselves depend on their beliefs $\mathcal{P}^i$ and state prices, both of which are exogenous from each investor's perspective), and (2) markets clear, i.e., $\sum_{i=1}^I c_t^i = c_t$ at all times. In our complete markets setting, securities market clearing is implied by goods market clearing and need therefore not be considered separately.

As noted in Basak (2005), who uses a similar equilibrium definition, equilibrium in our setting requires that the price system perceived by investors to clear the market at a given time and state does actually clear the market if that state is realized at that time, i.e., investors' expectations are rational and self-fulfilled in equilibrium.

1.5 Equilibrium allocations and prices

With complete markets, any equilibrium allocation $\{c^i\}$, $i = 1, \ldots, I$, must solve the central planning problem

$$\max_{\{c^i\}} \sum_{i=1}^{I} \kappa_i E^i \left(\int_0^T u^i(c_t^i, t) dt \right) \quad \text{s.t.} \quad \sum_{i=1}^{I} c_t^i \leq c_t \ \forall t \tag{8}$$

for some appropriate set of weights $\{\kappa_i\}$. This expression can be rewritten as

$$\max_{\{c^i\}} E^1 \left(\sum_{i=1}^{I} \kappa_i \int_0^T u^i(c_t^i, t) \zeta_t^i dt \right) \quad \text{s.t.} \quad \sum_{i=1}^{I} c_t^i \leq c_t \ \forall t \tag{9}$$

where the random variable $\zeta_t^i = d\mathcal{P}^i / d\mathcal{P}^1$, $\zeta_0^i = 1$, is the Radon-Nikodym derivative of investor i's beliefs with respect to investor 1's beliefs, which is used as reference without loss of generality. The first-order conditions for a maximum are

$$\kappa_i \zeta_t^i u_c^i(c_t^i, t) = \lambda_t \tag{10}$$

for all agents i, where λ_t is the multiplier that ensures that the aggregate budget constraint is met in the realized state ω at time t. Note from (10) that each agent's consumption is a function of aggregate consumption c_t, of ζ_t^i, and of λ_t. Hence, λ_t itself will be a function of c_t and the ζ_t^is, $\lambda_t = \lambda_t(c_t, \zeta_t^2, \ldots, \zeta_t^I)$, and each agent's optimal consumption can be written as

$$c_t^i = c_t^i(c_t, \zeta_t^i, \lambda_t(c_t, \zeta_t^2, \ldots, \zeta_t^I)) = c_t^i(c_t, \zeta_t^2, \ldots, \zeta_t^I) \tag{11}$$

Thus, individual optimal consumption depends not only on the aggregate endowment c_t, but also on the Radon-Nikodym derivatives of the different agents' beliefs $d\mathcal{P}^i$ with respect to agent 1's beliefs $d\mathcal{P}^1$. Since ζ_t^i is in general path-dependent, i.e., a function of ω, so will be individual agents' consumption, $c_t^i = c_t^i(\omega)$. Using the first-order condition from the individual agent optimization, $u_c^i(c_t^i, t) = \lambda_i \xi_t^i$, agent i's state-price deflator can be expressed as a function of the aggregate endowment and the ζ_t^is, $\xi_t^i = \xi_t^i(c_t, \zeta_t^2, \ldots, \zeta_t^I)$. Therefore, ξ_t^i is a function of ω, not just of c_t. This is intuitive: ξ_t^i represents the price of a unit of consumption in state ω per unit of probability as perceived by agent i, $\mathcal{P}^i$. That price itself, however, reflects the beliefs of all agents in the economy.

For our upcoming analysis of implied risk aversion, it is useful to consider the price at time t of a claim that pays one unit of consumption if the aggregate endowment c_s falls in some interval $\mathcal{A}$ at some future time s, irrespective of the remainder of the state of the economy (in other words, that claim is not a "pure" state-contingent claim). For the agent's consumption plan to be optimal, that price, denoted $\mathcal{Q}_{t,s}(\mathcal{A})$, must be given by

$$\mathcal{Q}_{t,s}(\mathcal{A}) = \frac{1}{u_c^i(c_t^i, t)} \mathcal{P}^i(c_s \in \mathcal{A}) E^i(u_c^i(c_s^i(c_s, \zeta_s^2, \ldots, \zeta_s^I), s) | c_s \in \mathcal{A}) \tag{12}$$

In words, the equilibrium price of the claim must be proportional to the agent's perceived probability that c_s falls in the interval $\mathcal{A}$ considered, $\mathcal{P}^i(c_s \in \mathcal{A})$, times his *expected* marginal utility if this happens, $E^i(u_c^i(c_s^i(\cdot), s) | c_s \in \mathcal{A})$. The relationship is cast in terms of *expected* marginal utility because the agent's optimal consumption depends not only on the aggregate endowment c_s, but also on the Radon-Nikodym derivatives $\{\zeta_s^i\}$.

In formal computations, we will use the state-price density

$$q_{t,s}(c_s) = \frac{1}{u_c^i(c_t^i, t)} p_{t,s}^i(c_s) E^i(u_c^i(c_s^i(c_s, \zeta_s^2, \dots, \zeta_s^I), s)|c_s) \tag{13}$$

where $p_{t,s}^i(c_s)$ denotes the density of c_s as perceived by agent i at time t. Taking the logarithmic derivative of this expression with respect to c_s yields

$$\frac{q_{t,s}'(c_s)}{q_{t,s}(c_s)} = \frac{p_{t,s}^{i\prime}(c_s)}{p_{t,s}^i(c_s)} + \frac{\frac{d}{dc_s} E^i\left(u_c^i(c_s^i(\cdot), s)\middle|c_s\right)}{E^i(u_c^i(c_s^i(\cdot), s)|c_s)} \tag{14}$$

This expression provides a link between state prices $q_{t,s}$, the agent's beliefs $p_{t,s}^i$, and the rate of change of his expected marginal utility with respect to shifts in the aggregate endowment c_s. Again, the expectation is required because under heterogeneous beliefs, there is no one-to-one correspondence between individual consumption and the aggregate endowment.

1.6 Implied risk aversion

Having described equilibrium prices in this economy, we can investigate the properties of implied risk aversion estimates. In doing so, we maintain the assumption, conventional in the implied risk aversion literature, that the stock index, whose price is denoted S, is a good proxy for the aggregate endowment, and therefore let $c_s = S$. To simplify notation, we drop time subscripts and let $P_i(S)$ denote agent i's perceived distribution at time t of S at some future time s, and $Q(S)$ be the price at time t of a claim that pays one unit of the consumption good if the index achieves a certain value S at time s.

For the reasons discussed in the introduction, assume that although the state-price density Q can be estimated accurately, individual agents' beliefs P_i cannot. Rather, suppose that it is only possible to obtain a single beliefs estimate $\hat{P}(S)$. The implied absolute risk aversion estimator $\alpha(S)$ is given by

$$\alpha(S) = \frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{Q'(S)}{Q(S)} \tag{15}$$

Substituting (14) with $c_s = S$ into (15) yields

$$\alpha(S) = -\frac{\frac{d}{dS}E^i\left(u_c^i(c_s^i(\cdot), s)|S\right)}{E^i(u_c^i(c_s^i(\cdot), s)|S)} + \left(\frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P_i'(S)}{P_i(S)}\right) \tag{16}$$

Thus, the implied risk aversion estimate $\alpha(S)$ equals minus the rate of change in each agent's expected marginal utility with respect to shifts in the aggregate endowment S, plus an estimation error arising from the fact that his beliefs cannot be estimated perfectly. Note that in our heterogeneous-beliefs setting, there is *no direct correspondence* between the agent's risk aversion and the implied risk aversion estimate. This suggests that implied risk aversion estimates may deviate significantly from agents' true preferences in the presence of heterogeneous beliefs. The properties of the implied risk aversion error will be investigated in detail in Section 2.3.

1.7 The special case of homogeneous beliefs

It is important to note that the absence of a direct correspondence between agents' preferences and implied risk aversion is solely driven by heterogeneity in beliefs, and *not* by the nature of the stochastic process followed by the aggregate endowment (such as the presence of stochastic volatility or jumps), nor by the fact that investors have incomplete information. This section characterizes the link between agents' preferences and implied risk aversion estimates under homogeneous beliefs. This relationship will be useful for the investigation of the properties of implied risk aversion in the different homogeneous-beliefs settings considered in Section 2.

Under homogeneous beliefs, $\zeta_t^i = 1$ for all i, and the equilibrium allocations and prices can be determined from the simpler central planning problem

$$\max_{\{c^i\}} E\left(\sum_{i=1}^{I} \kappa_i \int_0^T u^i(c_t^i, t)dt\right) \quad \text{s.t.} \quad \sum_{i=1}^{I} c_t^i \leq c_t \ \forall t \tag{17}$$

where E denotes expectation with respect to investors' common beliefs. The first-order conditions for a maximum read

$$\kappa_i u_c^i(c_t^i, t) = \lambda_t \tag{18}$$

and λ_t is therefore a function of the aggregate endowment c_t only, $\lambda_t = \lambda_t(c_t)$. Accordingly, each agent's optimal consumption is a function of the aggregate endowment only as well, $c_t^i = c_t^i(c_t)$. Using the first-order condition from the individual agent optimization, $u_c^i(c_t^i, t) = \lambda_i \xi_t^i$, the (common) state-price deflator of all agents is also function of the aggregate endowment only, i.e., one has $\xi_t^i = \xi(c_t, t)$ for all i.

The state-price density $q_{t,s}(c_s)$ is given by

$$q_{t,s}(c_s) = \frac{1}{u_c^i(c_t^i, t)} p_{t,s}(c_s) E\left(u_c^i(c_s^i(c_s), s)|c_s\right) = \frac{1}{u_c^i(c_t^i(c_t), t)} p_{t,s}(c_s) u_c^i(c_s^i(c_s), s) \quad (19)$$

where $p_{t,s}(c_s)$ denotes investors' common assessment at time t of the physical density of c_s at time s and the second equality follows from the fact that each agent's consumption is a function of the aggregate endowment c_s alone. Hence, under homogeneous beliefs, c_s fully describes the state of the economy. State prices $q_{t,s}(c_s)$ are a function of c_s only and tie directly to agents' marginal utility for a given level of c_s, not just to its conditional expectation.

Taking the logarithmic derivative of this expression with respect to c_s and rearranging yields

$$\frac{p'_{t,s}(c_s)}{p_{t,s}(c_s)} - \frac{q'_{t,s}(c_s)}{q_{t,s}(c_s)} = -\frac{u_{cc}^i(c_s^i(c_s), s)}{u_c^i(c_s^i(c_s), s)} \frac{dc_s^i(c_s)}{dc_s} \quad (20)$$

This expression provides the link between each agent's degree of absolute risk aversion $-u_{cc}^i(c_s^i(c_s), s)/u_c^i(c_s^i(c_s), s)$, common beliefs about the future value of the aggregate endowment $p_{t,s}(c_s)$, and state prices $q_{t,s}(c_s)$.

These results have important implications for the properties of the implied risk aversion estimator (15). Setting $P_i(S) = P(S)$ for all i and substituting (20) with $c_s = S$ into (15), one has

$$\alpha(S) = -\frac{u_{cc}^i(c_s^i, s)}{u_c^i(c_s^i, s)} \frac{dc_s^i}{dS} + \left(\frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)}\right) \quad (21)$$

Thus, under homogeneous beliefs, the implied risk aversion estimate $\alpha(S)$ is equal to each individual agent's actual risk aversion, scaled by the sensitivity of his consumption to shifts in the index price dc_s^i/dS, plus an estimation error arising from the fact that agents' (common) beliefs cannot be estimated perfectly.

The sensitivity of individual agents' consumption to shifts in the index price dc_s^i/dS is unobservable. In order to express implied risk aversion in terms of the observable aggregate endowment S, one can resort to an equilibrium argument. Solving (21) for dc_s^i/dS, aggregating across agents, requiring market clearing, $\sum_{i=1}^I dc_s^i/dS = 1$, and rearranging allows writing the implied risk aversion estimator as

$$\alpha(S) = \frac{1}{\sum_{i=1}^I \frac{1}{\alpha_i(c_s^i)}} + \left(\frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)} \right) \tag{22}$$

where $\alpha_i(c_s^i) \equiv -u_{cc}^i(c_s^i, s)/u_c^i(c_s^i, s)$ denotes investor i's degree of absolute risk aversion on his optimal consumption path. Under homogeneous beliefs, implied absolute risk aversion $\alpha(S)$ is thus the harmonic sum of individual investors' absolute risk aversion, plus an adjustment term that depends on the divergence between investors' actual and estimated beliefs, P and $\hat{P}$.

2 Properties of Implied Risk Aversion Estimators

This section investigates the properties of implied risk aversion estimators in three different settings in order to explore the three potential explanations for the implied risk aversion smile mentioned in the introduction. Section 2.1 investigates *preference aggregation*, Section 2.2 *beliefs misestimation*, and Section 2.3 *heterogeneous beliefs*.

2.1 Preference aggregation

This section investigates preference aggregation as a potential explanation for the implied risk aversion smile by analyzing the properties of the implied risk aversion estimator in a special case of the framework presented above in which investors have homogeneous beliefs and these can be estimated accurately. In this setting, the following holds (proofs of all propositions in this section are in Appendix A):

Proposition 1: Implied absolute risk aversion $\alpha(S)$ is the harmonic sum of individual agents' absolute risk aversion on their optimal consumption path, $\alpha_i(c_s^i)$,

$$\alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_s^i)}} \tag{23}$$

Equation (23) is the well-known result that risk tolerance is additive across agents [Wilson (1968)]. Implied risk aversion is thus *exact* and accurately reflects the economy-wide risk aversion. The obvious implication for empirical implied risk aversion estimates is that if all agents are risk-averse, then implied risk aversion is strictly positive. Equation (23) also implies that if all agents have constant absolute risk aversion (CARA) utility, then implied risk aversion displays constant absolute risk aversion as well.

Additional properties of individual agents' risk aversion functions carry over to implied risk aversion:

Proposition 2: If all agents have increasing (decreasing) absolute risk aversion, implied risk aversion also displays increasing (decreasing) absolute risk aversion.

Proposition 2 implies that if all agents have CRRA utility, which exhibits decreasing absolute risk aversion, then implied absolute risk aversion is also decreasing. For the special case of CRRA utility, however, even stronger properties can be established on the basis of the following result, originally derived by Benninga and Mayshar (2000) in a somewhat different setting:

Proposition 3: Implied *relative* risk aversion $\rho(S) = S\alpha(S)$ is a harmonic weighted average of individual agents' relative risk aversion, with the weights in this average given by each agent's share of aggregate consumption,

$$\rho(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\rho_i(c_s^i)} \frac{c_s^i}{S}} \tag{24}$$

where $\rho_i(c_s^i) \equiv -u_{cc}^i(c_s^i, s)c_s^i/u_c^i(c_s^i, s)$ denotes agent i's relative risk aversion on his optimal consumption path.

Benninga and Mayshar (2000) also show that if agents have heterogeneous, CRRA preferences, then the economy-wide relative risk aversion will be *decreasing* in the aggregate endowment. This is so because as the aggregate endowment increases, relatively less risk averse agents' share of aggregate consumption increases, driving down the average in (24).

The results in this section have the following implications for implied risk aversion estimates. First, if individual agents are risk averse, then the negative implied risk aversion estimates reported in Jackwerth (2000) and Rosenberg and Engle (2002) cannot be caused by preference aggregation. Second, if agents have nonincreasing absolute risk aversion [as postulated by Arrow (1970), and supported by everyday observation] or constant relative risk aversion, then the U-shaped estimates obtained by Jackwerth (2000), Aït-Sahalia and Lo (2000) and Rosenberg and Engle (2002) cannot be attributed to preference aggregation either. [7] Third, rather than contributing to explaining the smile, heterogeneous preferences make it (more specifically, the sizable increase in implied risk aversion at high index levels) even more puzzling. To see this, suppose that some (or all) agents have increasing relative risk aversion at high index levels. Risk-sharing causes agents with relatively low risk aversion to have a large share of aggregate consumption at high index levels, thus driving down implied risk aversion. Thus, with heterogeneous preferences, the pronounced smile patterns reported in the literature could only arise if a very significant portion of agents had strongly increasing risk aversion at high index levels.

Summarizing, preference aggregation cannot account for the implied risk aversion smile – if anything, the effect of risk-sharing among agents with heterogeneous preferences on implied risk aversion makes the smile even more puzzling. This result also holds in the presence of stochastic volatility and jumps in returns. Indeed, in this case, both prices $Q(S)$ and beliefs $P(S)$ reflect the presence of stochastic volatility and jumps, and implied risk aversion estimates obtained by comparing the two accurately reflect agents' risk aversion.

As mentioned in the introduction, these conclusions hold true only in the framework that we consider, i.e., assuming an exchange economy with complete, frictionless markets and a single consumption good in which the stock index is a good proxy for the aggregate endowment. In the context of an incomplete markets economy, for example, heterogeneous preferences may be important to explain the implied risk aversion smile. It is also worth noting that given that aggregate consumption is almost homoscedastic [Lochstoer (2004)], our framework is unlikely to be able to generate the level of stochastic volatility present in stock indices. Stochastic volatility and jumps could well be important to explain the smile in the context of a more general framework allowing for market incompleteness, for example. However, even within our framework, *misestimation* of beliefs caused by the presence of stochastic volatility and jumps could cause implied risk aversion to smile. This potential explanation is investigated in the next section.

2.2 Misestimation of agents' beliefs

This section investigates misestimation of agents' beliefs as a potential explanation for the smile. The analysis is performed in a setting in which agents have homogeneous beliefs but these beliefs cannot be estimated accurately. A useful result for this analysis is the relationship between beliefs misestimation and the implied risk aversion estimation error. From Section 1.7, in particular equation (22), one has:

Proposition 4: Suppose that agents have homogeneous beliefs $P(S)$ but that these are inaccurately estimated to be $\hat{P}(S)$. Then, implied risk aversion $\alpha(S)$ is given by

$$\alpha(S) = \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_s^i)}} + \left(\frac{\hat{P}'(S)}{\hat{P}(S)} - \frac{P'(S)}{P(S)} \right) \equiv \frac{1}{\sum_{i=1}^{I} \frac{1}{\alpha_i(c_s^i)}} + \epsilon(S) \tag{25}$$

where $\epsilon(S) \equiv (\hat{P}'(S)/\hat{P}(S) - P'(S)/P(S))$ denotes the implied risk aversion estimation error.

Equation (25) explicitly relates the implied risk aversion estimation error ϵ to the divergence between agents' actual and estimated beliefs, P and $\hat{P}$. Simple computations reveal that implied risk aversion estimates are very sensitive to the underlying beliefs estimates. The implied risk aversion literature uses beliefs estimates based on historical returns. [8] As mentioned in the introduction, such estimates will be inaccurate whenever the return process is time-varying – for instance, because of stochastic volatility and jumps – and in the presence of a Peso problem. The remainder of this section considers whether these two sources of beliefs misestimation can explain the implied risk aversion smile.

2.2.1 Misestimation caused by stochastic volatility and jumps

The recent literature provides ample evidence that an accurate description of actual asset markets needs to incorporate stochastic volatility and jumps [see, for example, Aït-Sahalia (2002), Andersen, Benzoni and Lund (2002), Bates (2000), Chernov et al. (2003), and Pan (2002)]. A natural approach to assess whether beliefs misestimation caused by stochastic volatility and jumps can explain the smile is to investigate the risk aversion patterns implicit in stochastic volatility and jumps models that specify both P and Q, thus ensuring that both P and Q contain the effect of stochastic volatility and jumps.

A qualification is in order. Since these risk aversion patterns are computed in the context of a particular stochastic volatility and jumps model, they do not allow drawing general conclusions about the ability or inability of stochastic volatility and jumps models to explain the implied risk aversion smile. The finding that the preferences implicit in a given model are not well-behaved has three possible interpretations. First, it can be viewed as evidence that the assumptions required for these computations to be valid (complete, frictionless markets with a single consumption good in which the stock index is a good proxy for the aggregate endowment) are too limiting. Second, it can mean that the particular stochastic volatility and jumps model used to obtain the densities P and Q is misspecified, i.e., unable to adequately reconcile index return dynamics and option prices. Finally, it can mean that the preferences required for this reconciliation are not well-behaved, i.e., interpreted as evidence that beliefs misestimation caused by stochastic volatility and jumps cannot account for the implied risk aversion smile.

Given this qualification, a comprehensive analysis of the preferences implicit in the numerous stochastic volatility and jumps models available in the literature is beyond the scope of this paper. This section focuses on the implied risk aversion estimates for the Pan (2002) model, which was selected for three reasons. First, it is quite general, allowing for both stochastic volatility and jumps in returns. [9] Second, it provides closed-form expressions for the transforms of both P and Q. Third, it is calibrated on the basis of both index returns and option prices, allowing the risk premia for both diffusion and jump risk to be estimated, and making the model appealing for an investigation of the implied risk aversion smile, which ultimately arises from the difficulty in reconciling index returns and option prices.

In order to obtain the implied risk aversion patterns for the Pan (2002) model, the physical density P and the state-price density Q are first computed by transform inversion using the transforms and the parameter values provided in Pan (2002) (the computations are detailed in Appendix B). In these computations, the initial values of the short interest rate and dividend yield are taken to be equal to their estimated long-term means of 0.058 and 0.025, respectively, as reported in Table 6 of Pan (2002). Five initial volatility parameter values are used: the estimated long-term mean of 0.0153, as well as the mean plus or minus one or two standard deviations [reported to be 0.0029 in Table 6 of Pan (2002)]. [10] The resulting densities for time horizons of 6 months and 1 year are depicted in panels A and B of Figure 1, respectively.

In a second step, the local relative risk aversion implied by these density estimates is computed using $\rho(S) = S\,(P'(S)/P(S) - Q'(S)/Q(S))$. As can be seen in panels C and D of Figure 1, when stochastic volatility and jumps are accounted for, implied risk aversion does not smile. However, it exhibits considerable variation and is negative in high return states. [11] Furthermore, the implied risk aversion pattern is remarkably consistent across initial volatility values. Thus, the Pan (2002) model does not lead to well-behaved preferences and cannot explain the implied risk aversion smile. Although the results in this section cannot be generalized, they suggest that stochastic volatility and jump models have difficulties in accounting for the smile. [12]

2.2.2 Beliefs estimation errors suggested by implied risk aversion

Since misestimation of beliefs caused by stochastic volatility and jumps seems unable to explain the implied risk aversion smile, it is natural to investigate what beliefs estimates based on historical returns are missing. This can be done by assuming reasonable values for agents' actual risk aversion, computing the resulting implied risk aversion estimation error, and deriving the corresponding beliefs estimation error. Indeed, as shown in Appendix A, the following holds:

Proposition 5: Let $\epsilon(S)$ denote the implied risk aversion estimation error. Under homogeneous beliefs, investors' beliefs $P(S)$ and their estimate $\hat{P}(S)$ are related by

$$\frac{\hat{P}(S)}{P(S)} = \gamma \exp\left(\int_{\underline{S}}^{S} \epsilon(z)dz\right) \tag{26}$$

for some arbitrary reference point $\underline{S}$ and some constant γ that ensures that $P(S)$ integrates to 1.

The density misestimation factor (26) measures the extent to which estimated beliefs $\hat{P}$ under- or overestimate agents' actual beliefs P based on a comparison of actual and implied risk aversion. As a numerical illustration, suppose that implied risk aversion is quadratic, reaches a minimum of -15 at a gross return level of 1 and a value of zero at return levels of 0.97 and 1.03, the basic picture that emerges from Figure 3, Panel D of Jackwerth (2000), and assume that the true coefficient of absolute risk aversion is 4. Although it seems restrictive, this assumption is innocuous. Similar patterns would arise with any assumed actual risk aversion function as long as it exhibits less curvature than the implied risk aversion function and the functions cross twice. Given the strong curvature of the implied risk aversion function reported by Jackwerth (2000), this would be the case for a very wide set of investor preferences. [13]

Panel A of Figure 2 shows the assumed implied and actual risk aversion functions. Panel B reports the corresponding density misestimation factor (26) for $\gamma = 1$ and a reference level of $\underline{S} = 1$. The beliefs estimation error pattern suggests that historical return frequency distributions underestimate agents' assessment of the probability of very low gross index returns (Brown and Jackwerth's "crash-o-phobia"), moderately overestimate the probability of returns slightly below 1, moderately underestimate the probability of returns slightly above 1 and significantly overestimate the probability of very high returns. A natural cause of this pattern is a Peso problem, as suggested, for example, by Aït-Sahalia, Wang and Yared (2001) and Brown and Jackwerth (2004).

2.2.3 Can a Peso problem explain the smile?

The beliefs misestimation pattern reported in Figure 2 reveals that a Peso problem can in principle account for the implied risk aversion smile. In order to assess the plausibility of this explanation, this section estimates the frequency and magnitude of jumps implied by the empirical state-price density.

In a setting without assumptions on investor preferences, Aït-Sahalia, Wang and Yared (2001) find that the incorporation of a jump term capturing the Peso problem (with a jump size of -10% and a jump intensity of once every three years) produces an improvement towards reconciling the cross-sectional state-price density (estimated from option prices) and the time-series state-price density (estimated from index futures returns), but is not sufficient to explain the magnitude of the dispersion of the excess skewness and excess kurtosis of the former relative to the latter. For the case of logarithmic utility, Bates (2000) finds that a Peso problem alone cannot explain the divergence between state-price densities and observed market returns. He shows that the risk-neutral probability of observing at least one weekly move of 10% in magnitude over the period 1988-1993 is 90%, but that none was observed. He notes, however, that more extreme risk aversion could make the Peso problem explanation viable, and that whether this can be achieved under plausible levels of risk aversion is an open question.

This section quantifies the frequency and magnitude of jumps implied by the empirical state-price density, assuming a particular functional form for the market-wide utility function, but, in contrast to the setting analyzed by Bates (2000), treating the risk aversion coefficient as a free parameter. [14] Assume that aggregate investor preferences are of the CRRA type,

$$u(S) = \frac{S^{1-\beta}}{1-\beta} \tag{27}$$

where β denotes the relative risk aversion coefficient. To model the possibility of jumps, assume that beliefs are given by a mixture of lognormal distributions,

$$P(S) = \lambda P_1(S) + (1-\lambda)P_2(S) \tag{28}$$

Heuristically, P_1 can be thought of as agents' beliefs conditional on no jump occurring, and P_2 as beliefs conditional on a jump occurring, with $1-\lambda$ corresponding to the probability of a jump. [15]

Given the assumed investor preferences, the state-price density has the form

$$Q(S) = \gamma S^{-\beta} P(S) \tag{29}$$

for some constant γ. Beliefs are estimated as follows. First, the state-price density is estimated using the semi-parametric approach and the data of Aït-Sahalia and Lo (2000). The data consists of 14,431 S&P 500 index option prices for the period January 4, 1993 to December 31, 1993. The semi-parametric approach involves regressing option implied volatility nonparametrically on moneyness and time to expiration and estimating the state-price density as the second derivative of the Black-Scholes option pricing formula with respect to the strike price, using the nonparametric volatility estimate as an input. As recommended in their paper, the kernel functions and bandwidth values are chosen so as to optimize the properties of the state-price density estimator (Gaussian kernel functions and bandwidths of 0.040 and 20.52 for moneyness and time to expiration, respectively). The density is computed for values of S between 300 and 600, with a step size of 0.2, yielding 1501 data points.

Equation (29) is then calibrated to the empirical density using nonlinear least squares, yielding the parameter values $\gamma = 1.0348$, $\lambda = 0.5861$, $\beta = 0.9158$, $\mu_1 = 0.0366$, $\mu_2 = -0.1288$, $\sigma_1 = 0.0760$ and $\sigma_2 = 0.1294$, where the distribution parameters are reported as annual figures. [16] Thus, relaxing Bates' (2000) logarithmic utility assumption does not change his basic conclusion that a Peso problem alone cannot reconcile the state-price density with historical index returns. Indeed, the estimates indicate that agents' utility is almost logarithmic, that the expected return conditional on no jump is about 3.7% and that conditional on a jump −12.9%. Thus, in order to generate the fat left tail of the empirical state-price density, agents must expect a crash of an average magnitude of about 3.7% + 12.9% = 16.6% to occur with a probability of $1 - \lambda = 42.2\%$.

The semi-parametric density and the fitted mixture state-price density are depicted in Figure 3 which also shows, for comparison, the Black-Scholes state-price density that best fits the semi-parametric density. Although the mixture density – in contrast to the Black-Scholes density – captures the essential features of the empirical density pretty well, especially its thick left tail, the required crash probability of 42.2% seems implausibly large: as noted by Bates (2000), if the true probability of a market crash were that high, then crashes would show up frequently in the data, and there would be no reason to expect that historical returns underestimate the probability of their occurrence. Thus, although a Peso problem can contribute to explaining the smile, it appears unlikely to be its only cause.

2.3 Heterogeneous beliefs

The recent literature provides evidence that heterogeneous beliefs play an important role for equilibrium prices. For example, Anderson, Ghysels and Juergens (2005) show that heterogeneous beliefs among investors – as measured by the dispersion of analysts' earnings forecasts – is a priced risk factor and a good out-of-sample predictor of future returns and return volatility. It is therefore natural to investigate the consequences of heterogeneity for implied risk aversion estimates.

Recall from the discussion in Section 1.6 (in particular equation (16)) that under heterogeneous beliefs, there is no direct correspondence between individual agents' actual risk aversion and implied risk aversion. Hence, in principle, implied risk aversion can have almost any shape and may differ significantly from agents' actual preferences. Although (16) reveals that the implied risk aversion pattern depends on agents' true preferences, on the nature of the heterogeneity in beliefs among them, and on how estimated beliefs $\hat{P}$ compare to individual agents' actual beliefs P_i, it is difficult to determine the precise impact of these three factors in the general case, i.e., without making any assumptions on investor beliefs and preferences.

This section uses the special case of CRRA preferences in order to disentangle these three components, allowing to gain additional insights into how the presence of heterogeneous beliefs causes implied risk aversion to deviate from agents' preferences. A simple parametric example is then used to assess the magnitude of the distortion in implied risk aversion caused by heterogeneous beliefs. This analysis reveals that if heterogeneous beliefs are not accounted for explicitly, implied risk aversion and preferences may differ significantly. A further specialization to the logarithmic utility case is then used to derive the distortion in implied risk aversion caused by heterogeneous beliefs in closed form, allowing its properties to be investigated. Finally, a heterogeneous-beliefs state-price density is calibrated to the data in order to assess the degree of heterogeneity suggested by the empirical state-price density and the plausibility of the heterogeneous beliefs explanation for the smile.

2.3.1 A simple CRRA framework

To set the stage for the subsequent analysis, let c_t denote the aggregate endowment, and suppose that there are I groups of investors indexed by $i = 1, \ldots, I$. Assume that these investors have identical CRRA preferences with relative risk aversion β,

$$u^i(c_t^i, t) = \frac{\left(c_t^i\right)^{1-\beta}}{1-\beta} \tag{30}$$

Let $\{\kappa_i\}$ denote the initial weights of the individual groups in the economy. For ease of presentation and without loss of generality, we let group 1 be the reference group for our analysis and set $\kappa_1 = 1$. Then, the central planning problem with heterogeneous beliefs (9) becomes

$$\max_{\{c^i\}} E^1 \left(\int_0^T \left(\frac{\left(c_t^1\right)^{1-\beta}}{1-\beta} + \sum_{i=2}^{I} \kappa_i \zeta_t^i \frac{\left(c_t^i\right)^{1-\beta}}{1-\beta} \right) dt \right) \quad \text{s.t.} \quad \sum_{i=1}^{I} c_t^i = c_t \ \forall t \tag{31}$$

where $\zeta_t^i = d\mathcal{P}^i / d\mathcal{P}^1$, $\zeta_0^i = 1$ denotes the Radon-Nikodym derivative of group i agents' beliefs with respect to the beliefs of the reference group 1. Maximizing time by time and state by state, the equilibrium consumption of the individual groups is given by [see, for example, Basak (2005), and Buraschi and Jiltsov (2005), for the case of two groups]

$$c_t^1 = c_t \frac{1}{1 + \sum_{i=2}^{I} \left(\kappa_i \zeta_t^i\right)^{1/\beta}} \tag{32}$$

and

$$c_t^j = c_t \frac{\left(\kappa_j \zeta_t^j\right)^{1/\beta}}{1 + \sum_{i=2}^{I} \left(\kappa_i \zeta_t^i\right)^{1/\beta}}, \ j \neq 1 \tag{33}$$

Using (13), the state-price density is given by

$$q_{t,s}(c_s) = \left(\frac{c_t}{c_s}\right)^{\beta} p_{t,s}^1(c_s) E_t^1 \left(\left(\frac{1 + \sum_{i=2}^{I} (\kappa_i \zeta_s^i)^{1/\beta}}{1 + \sum_{i=2}^{I} (\kappa_i \zeta_t^i)^{1/\beta}} \right)^{\beta} \middle| c_s \right) \tag{34}$$

where $p_{t,s}^1(c_s)$ denotes the density of the aggregate endowment at time s as perceived by group 1 agents at time t. Hence, the state-price density $q_{t,s}(c_s)$ is completely specified once the distribution of c_s and that of the ζ_s^is conditional on c_s from group 1 agents' perspective are known.

2.3.2 The implied risk aversion error caused by heterogeneous beliefs

Let $\hat{p}_{t,s}(c_s)$ denote the estimated beliefs used in the implied risk aversion estimation process. In order to determine the relationship between agents' preferences and implied risk aversion, substitute the state-price density (34) into the implied risk aversion estimator $\alpha(c_s) = \hat{p}'_{t,s}(c_s)/\hat{p}_{t,s}(c_s) - q'_{t,s}(c_s)/q_{t,s}(c_s)$ to obtain [17]

$$\alpha(c_s) = \frac{\beta}{c_s} + \left(\frac{\hat{p}'_{t,s}(c_s)}{\hat{p}_{t,s}(c_s)} - \frac{p^{1\prime}_{t,s}(c_s)}{p^1_{t,s}(c_s)}\right) + \frac{\frac{d}{dc_s}E^1_t\left(\left(1+\sum_{i=2}^{I}(\kappa_i\zeta^i_s)^{1/\beta}\right)^{\beta}\Big|c_s\right)}{E^1_t\left(\left(1+\sum_{i=2}^{I}(\kappa_i\zeta^i_s)^{1/\beta}\right)^{\beta}\Big|c_s\right)} \qquad (35)$$

Thus, under CRRA preferences and heterogeneous beliefs, implied absolute risk aversion is the sum of three components: (i) agents' actual absolute risk aversion, β/c_s, (ii) an implied risk aversion estimation error driven by the divergence between estimated beliefs and the actual beliefs of the reference group 1, $\hat{p}'_{t,s}(c_s)/\hat{p}_{t,s}(c_s) - p^{1\prime}_{t,s}(c_s)/p^1_{t,s}(c_s)$, and (iii) a term that captures the distortion in implied risk aversion introduced by the presence of heterogeneous beliefs. Observe that this distortion itself depends on investors' relative risk aversion β; this is so because preferences affect the way that heterogeneous beliefs impact equilibrium prices. Because of the third term in (35), in the presence of heterogeneous beliefs, implied risk aversion estimates will differ from agents' true preferences even if the beliefs of the reference group 1 are estimated accurately, $\hat{p}_{t,s}(c_s) = p^1_{t,s}(c_s)$.

In fact, unless the effect of heterogeneity is modeled explicitly, implied risk aversion will not reflect true preferences even if *all* groups' beliefs are estimated accurately and implied risk aversion is estimated on the basis of average beliefs weighted using $\kappa_i\zeta^i_t$, $\overline{p}_{t,s}(c_s) = \left(p^1_{t,s}(c_s) + \sum_{i=2}^{I}\kappa_i\zeta^i_t p^i_{t,s}(c_s)\right) / \left(1+\sum_{i=2}^{I}\kappa_i\zeta^i_t\right)$. [18] To see this, note that the (homogeneous-beliefs) state-price density obtained in this fashion, $\overline{q}_{t,s}(c_s)$, reads

$$\overline{q}_{t,s}(c_s) = \left(\frac{c_t}{c_s}\right)^{\beta}\overline{p}_{t,s}(c_s) = \left(\frac{c_t}{c_s}\right)^{\beta}\frac{p^1_{t,s}(c_s) + \sum_{i=2}^{I}\kappa_i\zeta^i_t p^i_{t,s}(c_s)}{1+\sum_{i=2}^{I}\kappa_i\zeta^i_t} \qquad (36)$$

and differs from the actual density under heterogeneous beliefs, (34). Most would agree that obtaining such average beliefs estimates would be a significant achievement for any empiricist. Yet, contrasting (34) and (36) reveals that knowledge of these average beliefs would not be sufficient to be able to accurately infer investors' preferences from implied risk aversion estimates – heterogeneity must be accounted for explicitly.

2.3.3 The sensitivity of implied risk aversion to heterogeneous beliefs

Further insights into the pattern and magnitude of the distortion in implied risk aversion caused by the presence of heterogeneous beliefs can be obtained by specifying the aggregate endowment process and agents' beliefs more precisely. Assume that the aggregate endowment has dynamics

$$\frac{dc_t}{c_t} = \mu_t dt + \sigma dB_t \tag{37}$$

where the drift μ_t follows

$$d\mu_t = \sigma_\mu dB_t^\mu \tag{38}$$

Suppose that the different groups of investors do not observe the drift, but estimate it based on their prior information and the path of the aggregate endowment. Each group i starts with a drift estimate μ_0^i and rationally revises this estimate using

$$d\mu_t^i = \frac{V_t}{\sigma} dB_t^i \tag{39}$$

where V_t is the variance of the estimate of μ_t and

$$dB_t^i = \frac{1}{\sigma}\left(\frac{dc_t}{c_t} - \mu_t^i dt\right) \tag{40}$$

denotes the investor-specific Brownian innovation. For simplicity, assume further that the estimation variance V_t is equal to its long-run mean, $V = \sigma\sigma_\mu$.

As shown in Appendix C, given the assumed processes, group i agents perceive c_s to be lognormal with $E_t^i(\ln(c_s)) = \mu_c^i$ and $\mathrm{Var}_t^i(\ln(c_s)) = \sigma_c^2$. Moreover, group 1 agents perceive ζ_s^i to be lognormal with $E_t^1(\ln(\zeta_s^i)) = \ln(\zeta_t^i) + \mu_{\zeta^i}$, $\mathrm{Var}_t^1(\ln(\zeta_s^i)) = \sigma_{\zeta^i}^2$ and $\mathrm{Cov}_t^1(\ln(c_s), \ln(\zeta_s^i)) = \sigma_{c\zeta^i}$. Hence, using the fact that for any two groups i and j, $\ln(\zeta_s^i)$ and $\ln(\zeta_s^j)$ are perfectly correlated with correlation coefficient $\rho_{\zeta^i\zeta^j} =$

$\text{sign}(\overline{\mu}_t^i \overline{\mu}_t^j)$, where $\overline{\mu}_t^i \equiv (\mu_t^1 - \mu_t^i)/\sigma$ denotes the normalized divergence in beliefs about μ_t between group i and group 1 at time t, and letting $\rho_{c\zeta^i} \equiv \sigma_{c\zeta^i}/(\sigma_c \sigma_{\zeta^i})$, the state-price density (34) can be computed as

$$
\begin{aligned}
q_{t,s}(c_s) &= \left(\frac{c_t}{c_s}\right)^{\beta} \frac{\exp\left(-\frac{(\ln(c_s)-\mu_c^1)^2}{2\sigma_c^2}\right)}{c_s \sigma_c \sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{z^2}{2}\right)}{\sqrt{2\pi}} \\
&\times \left(\frac{1 + \sum_{i=2}^{I} \left(\kappa_i \zeta_t^i \exp\left(\mu_{\zeta^i} + \frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1) + \text{sign}(\overline{\mu}_t^i)\sigma_{\zeta^i}\sqrt{1-\rho_{c\zeta^i}^2}\, z \right) \right)^{1/\beta}}{1 + \sum_{i=2}^{I} (\kappa_i \zeta_t^i)^{1/\beta}} \right)^{\beta} dz
\end{aligned}
\tag{41}
$$

Given the values of the model parameters β, $\{\kappa_i\}$ $\{\mu_t^i\}$, σ and V and initial values c_t and $\{\zeta_t^i\}$, all the parameters in (41) are known, and the state-price density can be obtained by numerically integrating out z.

In order to assess the magnitude of the distortion in implied risk aversion caused by heterogeneous beliefs, consider a numerical example based on (41). Suppose that there are only two groups, that $c_t = 1$, $\zeta_t^2 = 1$, $\kappa_2 = 1$ (implying that the weights of both groups are equal), $\beta = 2$, $\sigma = 0.1$ and $V = 0.01$ and that the degree of heterogeneity in beliefs is small: group 1 agents believe the expected return to be $\mu_1 = 0.05$, while group 2 agents consider it to be slightly lower, $\mu_2 = 0.04$. Panel A of Figure 4 shows the beliefs of both groups, $p_{t,s}^1(c_s)$ and $p_{t,s}^2(c_s)$, average beliefs $\overline{p}_{t,s}(c_s) = (p_{t,s}^1(c_s) + \kappa_2 \zeta_t^2 p_{t,s}^2(c_s))/(1 + \kappa_2 \zeta_t^2)$, and the heterogeneous-beliefs state-price density $q_{t,s}(c_s)$. Panel B depicts the implied relative risk aversion estimates $\rho(c_s) = c_s(\hat{p}'_{t,s}(c_s)/\hat{p}_{t,s}(c_s) - q'_{t,s}(c_s)/q_{t,s}(c_s))$ obtained using either group's beliefs or average beliefs, i.e., without accounting explicitly for the effect of heterogeneous beliefs. Observe that in spite of the small degree of heterogeneity assumed in this example, when implied risk aversion is estimated based on either group's beliefs, it is significantly distorted away from agents' (assumed) true relative risk aversion of 2. When it is estimated based on the more optimistic group 1's beliefs, implied risk aversion exceeds the actual one, with values of about 2.3. Conversely, when it is estimated based on the more pessimistic group 2's beliefs, it lies below the actual one, with values of about 1.5. When it is estimated based on average beliefs, however, implied risk aversion is quite close to true risk aversion.

Figure 5 reports the results of similar computations for a setting with large heterogeneity in beliefs. The parameter values are the same as those used for Figure 4, except that group 2 agents believe expected returns to be $\mu_2 = -0.1$. As in Figure 4, panel A shows the beliefs of both groups, average beliefs, and the state-price density, while panel B depicts implied relative risk aversion obtained using either group's beliefs or average beliefs. In this setting with high heterogeneity, implied risk aversion is significantly distorted away from agents' (assumed) true relative risk aversion of 2 not only when it is estimated using either group's beliefs, but also when it is estimated using average beliefs. Estimating implied risk aversion based on the more optimistic group 1's beliefs would induce one to infer a risk aversion exceeding the actual one, with values ranging from 4 to about 10. Estimating it using the more pessimistic group 2's beliefs would induce one to underestimate risk aversion to such an extent that investors appear to be risk-loving, with a (local) relative risk aversion coefficient ranging from about -8 to -2. Finally, estimating implied risk aversion using average beliefs would induce one to conclude that investors are risk-averse in some return states and risk-loving in others, with a risk aversion coefficient ranging from -2 to about 4. None of these estimates come close to matching investors' actual risk aversion of 2.

The fact that the simple examples in Figures 4 and 5 do not reproduce the implied risk aversion smile found in the literature should not be interpreted as meaning that heterogeneous beliefs are unable to account for it. Indeed, these implied risk aversion patterns are computed under fairly restrictive assumptions, not just about investor preferences, but especially about estimated beliefs. Of course, there is no reason to expect that estimated beliefs have anything to do with either group's beliefs, nor with average beliefs. The point of Figures 4 and 5 is that heterogeneous beliefs, if not accounted for explicitly, can cause implied risk aversion estimates to deviate significantly from actual preferences even if the beliefs of the different groups are estimated accurately.

2.3.4 Closed-form expressions for the implied risk aversion estimation error

In the special case of logarithmic utility ($\beta = 1$), the implied risk aversion estimation error can be computed in closed form both when implied risk aversion is estimated using some group's beliefs and using average beliefs, thus allowing a more detailed investigation of its properties. Indeed, with logarithmic preferences, the integral in (41) can be evaluated algebraically and the state-price density is given by

$$q_{t,s}(c_s) = \frac{c_t}{c_s} p^1_{t,s}(c_s) \frac{1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma^2_c} (\ln(c_s) - \mu^1_c) - \frac{1}{2} \frac{\sigma^2_{c\zeta^i}}{\sigma^2_c} \right)}{1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t} \tag{42}$$

where $p^1_{t,s}(c_s) = \frac{\exp\left(-(\ln(c_s) - \mu^1_c)^2/(2\sigma^2_c)\right)}{c_s \sigma_c \sqrt{2\pi}}$ denotes group 1 agents' beliefs and we have used the fact that $\exp(\mu_{\zeta^i} + \sigma^2_{\zeta^i}/2) = 1$ since ζ^i is an exponential martingale. [19]

The implied risk aversion obtained using group 1 agents' beliefs, i.e., by setting $\hat{p}_{t,s}(c_s) = p^1_{t,s}(c_s)$, is given by

$$\begin{aligned} \alpha(c_s) &= \frac{1}{c_s} - \frac{d \ln\left(1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma^2_c} (\ln(c_s) - \mu^1_c) - \frac{1}{2} \frac{\sigma^2_{c\zeta^i}}{\sigma^2_c} \right)\right)}{dc_s} \\ &= \frac{1}{c_s} - \frac{1}{c_s \sigma^2_c} \frac{\sum_{i=2}^{I} \sigma_{c\zeta^i} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma^2_c} (\ln(c_s) - \mu^1_c) - \frac{1}{2} \frac{\sigma^2_{c\zeta^i}}{\sigma^2_c} \right)}{1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma^2_c} (\ln(c_s) - \mu^1_c) - \frac{1}{2} \frac{\sigma^2_{c\zeta^i}}{\sigma^2_c} \right)} \end{aligned} \tag{43}$$

Since agents' true risk aversion is $1/c_s$, the implied risk aversion estimation error is the second term appearing in each line of (43). This error has the following properties. First, since a weighted sum of exponentials has at most one minimum, the first line in (43) implies that the error is monotonic in the aggregate endowment c_s. Thus, under logarithmic preferences and heterogeneous lognormal beliefs, if implied risk aversion is estimated on the basis of some (in fact, *any*) investor's beliefs, heterogeneous beliefs cannot cause implied risk aversion to smile.

Second, the error is strictly decreasing and its sign depends on whether the reference group (on whose beliefs the implied risk aversion estimate is based) is the most optimistic or pessimistic in the economy or has intermediate beliefs. To see this, observe that the sign of the error is the opposite of that of $\sum_{i=2}^{I} \sigma_{c\zeta^i} \kappa_i \zeta_t^i$ $\exp\left(\frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1) - \frac{1}{2}\frac{\sigma^2_{c\zeta^i}}{\sigma_c^2}\right)$, which itself depends on the sign of the $\sigma_{c\zeta^i}$s, the covariances between the aggregate endowment and the density processes. From (82) in Appendix C, the sign of $\sigma_{c\zeta^i}$ is the opposite of that of $\overline{\mu}_t^i = (\mu_t^1 - \mu_t^i)/\sigma$. Hence, if implied risk aversion is estimated based on the most optimistic group's beliefs (i.e., if the reference group 1 is the most optimistic group in the economy), $\overline{\mu}_t^i > 0$ for all i, $\sigma_{c\zeta^i} < 0$ for all i, and implied risk aversion exceeds true risk aversion for all values of the aggregate endowment. In addition, the error tends to 0 for large values of c_s, and is therefore strictly decreasing. Conversely, if implied risk aversion is estimated using the most pessimistic group's beliefs, it understates true risk aversion for all values of c_s, tends to 0 for small values of c_s, and is therefore strictly decreasing as well. Finally, if group 1 is neither the most optimistic nor the most pessimistic, the error is positive for some values of the aggregate endowment and negative for others. More precisely, since the terms with $\sigma_{c\zeta^i} > 0$ tend to 0 for small values of c_s and those with $\sigma_{c\zeta^i} < 0$ do so for large values of c_s, the error is positive for small c_s and negative for large c_s, i.e., is again strictly decreasing.

Figure 6 illustrates these findings in a setting with three groups of investors. Panel A shows the beliefs of the three groups, average beliefs, and the state-price density under heterogeneous beliefs (42) for the parameter values $c_t = 1$, $\zeta_t^2 = \zeta_t^3 = 1$, $\kappa_2 = \kappa_3 = 1$, $\beta = 1$, $\mu_1 = 0.05$, $\mu_2 = -0.1$, $\mu_3 = 0.20$, $\sigma = 0.1$ and $V = 0.01$. Panel B shows that implied relative risk aversion exceeds true risk aversion when it is estimated based on the most optimistic group 3's beliefs and is lower than true risk aversion when it is estimated using the most pessimistic group 2's beliefs. When it is estimated based on the intermediate group 1's beliefs, implied risk aversion overstates true risk aversion for small values of the aggregate endowment and understates it for large ones. In all three cases, the implied risk aversion error is strictly decreasing.

Consider now the properties of implied risk aversion when it is estimated using average beliefs, i.e., setting $\hat{p}_{t,s}(c_s) = \left(p^1_{t,s}(c_s) + \sum_{i=2}^{I} \kappa_i \zeta^i_t p^i_{t,s}(c_s)\right) / \left(1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t\right)$. In this case,

$$
\begin{aligned}
\alpha(c_s) \quad = \quad & \frac{1}{c_s} - \frac{1}{c_s\sigma_c^2} \frac{\sum_{i=2}^{I} \sigma_{c\zeta^i} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1) - \frac{1}{2}\frac{\sigma^2_{c\zeta^i}}{\sigma_c^2}\right)}{1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t \exp\left(\frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1) - \frac{1}{2}\frac{\sigma^2_{c\zeta^i}}{\sigma_c^2}\right)} \\
& + \frac{1}{c_s\sigma_c^2} \frac{\sum_{i=2}^{I} (\mu_c^1 - \mu_c^i) \kappa_i \zeta^i_t \exp\left(\frac{\ln(c_s)(\mu_c^1 - \mu_c^i)}{\sigma_c^2} + \frac{(\mu_c^1 - \mu_c^i)(\mu_c^1 + \mu_c^i)}{2\sigma_c^2}\right)}{1 + \sum_{i=2}^{I} \kappa_i \zeta^i_t \exp\left(\frac{\ln(c_s)(\mu_c^1 - \mu_c^i)}{\sigma_c^2} + \frac{(\mu_c^1 - \mu_c^i)(\mu_c^1 + \mu_c^i)}{2\sigma_c^2}\right)}
\end{aligned} \tag{44}
$$

The implied risk aversion error can be decomposed into two components. The first is the error that would arise if implied risk aversion were estimated based on the reference group 1's beliefs. The second is an error caused by the divergence between group 1's and average beliefs. Although both components have a similar formulaic structure, their properties are exactly opposite, both in terms of sign and slope. For example, if group 1 is the most optimistic, the error that would arise if implied risk aversion were estimated based on group 1's beliefs is positive and monotone decreasing, while the error corresponding to the difference between group 1's and average beliefs is negative and monotone increasing. Similarly, if group 1 is the most pessimistic, the first component is negative and monotone decreasing, whereas the second is positive and monotone increasing. Finally, if group 1 has intermediate beliefs, the first component is monotone decreasing, with positive values for low c_s and negative ones for large c_s, whereas the second is monotone increasing, with negative values for low c_s and positive ones for large c_s. The net effect of both components is difficult to ascertain algebraically. In the example considered in Figure 6, for instance, the second component dominates the first one, so that overall, the implied risk aversion error is monotone increasing, with negative values for low c_s and positive ones for high c_s.

2.3.5 Heterogeneity suggested by empirical state-price densities

The above analysis reveals that heterogeneous beliefs affect the state-price density in nontrivial forms and can cause sizable distortions in implied risk aversion estimates. This section quantifies the degree of heterogeneity suggested by the empirical state-price density in order to assess the plausibility of the heterogeneous beliefs explanation for the implied risk aversion smile.

The analysis is performed by fitting the heterogeneous-beliefs density (41) with three groups of investors to the empirical semi-parametric state-price density estimated by Aït-Sahalia and Lo (2000). As in the calibration of the mixture density in Figure 3, the empirical density is computed for values of S between 300 and 600, with a step size of 0.2, yielding 1501 data points, and the calibration performed using nonlinear least squares. The estimated parameter values are $\kappa_2 = 0.1182$, $\kappa_3 = 0.0157$, $\beta = 2.9824$, $\mu_1 = 0.0792$, $\mu_2 = -0.2262$, $\mu_3 = -0.5650$, $\sigma = 0.0672$ and $V = 1.019\text{E-}4$, where distribution parameters are again reported in annual terms. These estimates suggest that a fraction $1/(1+\kappa_2+\kappa_3) = 0.8819$ of agents estimate expected returns on the index to be about 7.9%, a fraction $\kappa_2/(1+\kappa_2+\kappa_3) = 0.1043$ of pessimistic agents estimate them to be about -23%, and a fraction $\kappa_3/(1+\kappa_2+\kappa_3) = 0.0138$ of extremely pessimistic agents estimate them to be about -56%. Thus, in order to generate the fat left tail of the empirical state-price density, an extremely large degree of pessimism is required.

The fitted heterogeneous-beliefs state-price density is depicted in Figure 7, which also shows, for comparison, the Black-Scholes state-price density that best fits the semi-parametric density. In contrast to the Black-Scholes density, the heterogeneous-beliefs density fits the empirical density almost perfectly, even in the left tail. Figure 8 shows the implied risk aversion estimates that would be obtained by comparing the heterogeneous-beliefs density and either group's beliefs or average beliefs in a format similar to Figures 4, 5 and 6. Observe that although they do not exhibit a straight smile, the implied risk aversion estimates are significantly negative for a wide range of index values and exhibit considerable variation across index values. Moreover, throughout the range of index values considered, implied risk aversion differs significantly from agents' estimated degree of relative risk aversion if heterogeneous beliefs are taken into account, $\beta = 2.9824$. However, the extreme degree of pessimism required to generate the empirical state-price density casts doubts on the plausibility of the heterogeneous beliefs explanation for the implied risk aversion smile.

In order to understand why an extremely large level of pessimism is necessary to generate the empirical state-price density and explain the implied risk aversion smile, it is useful to consider the results obtained when fitting a homogeneous-beliefs state-price density in which agents' beliefs are given by a mixture of three lognormal densities (which amounts to allowing for two different kinds of jumps), $P = (1-\lambda_2-\lambda_3)P_1+\lambda_2 P_2+\lambda_3 P_3$. Performing this estimation yields an almost perfect fit and the parameter values $\lambda_2 = 0.3917$, $\lambda_3 = 0.2076$, $\beta = 0.5802$, $\mu_1 = 0.0604$, $\mu_2 = -0.0637$, $\mu_3 = -0.2152$, $\sigma_1 = 0.0662$, $\sigma_2 = 0.0909$ and $\sigma_3 = 0.1555$. [20]

Observe that the fitted mixture homogeneous-beliefs density has return standard deviations for the jump cases (σ_2 and σ_3) that far exceed the standard deviation in the case of no jump, σ_1. These high standard deviations allow the mixture density to capture the fat left tail of the empirical state-price density. In contrast, the heterogeneous-beliefs density does not have different standard deviations available to do so – in the heterogeneous-beliefs model considered above, the standard deviation of returns as perceived by each group is identical. In order to capture the fat left tail of the empirical state-price density, the heterogeneous-beliefs density therefore needs to use the additional source of risk it has available – the variability in the Radon-Nikodym derivatives ζ^i. As shown in Appendix C, the standard deviation of ζ^i is proportional to the divergence in beliefs between the different groups. The sizable difference in standard deviations between the jump and the no-jump cases for the fitted mixture density therefore translates to a high degree of pessimism for the estimated heterogeneous-beliefs density. In order to quantify the impact of the variability in the Radon-Nikodym derivatives on the state-price density, Figure 7 also reports the homogeneous-beliefs state-price density obtained using the average beliefs of the three groups. Observe that the difference between this density and the heterogeneous-beliefs density – which measures the impact of the variability in the

Radon-Nikodym derivatives – is sizable.

Thus, an extreme degree of pessimism is required to generate the empirical state-price density because our heterogeneous-beliefs model assumes a lognormal setting in which the standard deviation perceived by each group is the same. In addition, since actual asset returns exhibit stochastic volatility and jumps, the model attributes the portion of the excess skewness and kurtosis in the empirical state-price density that is actually caused by stochastic volatility and jumps to heterogeneous beliefs. Presumably, a heterogeneous-beliefs model allowing for different perceived standard deviations among agents and for beliefs that are not lognormal by introducing stochastic volatility and jumps in returns would produce a more plausible degree of pessimism.

3 Conclusion

This paper explores different potential explanations for the implied risk aversion smile by investigating the properties of implied risk aversion estimators in different settings within the standard consumption-based framework commonly used in the implied risk aversion literature. Three potential explanations are investigated: (i) the aggregation of heterogeneous preferences among agents, both with and without stochastic volatility and jumps, (ii) misestimation of investors' beliefs due to stochastic volatility, jumps or a Peso problem, and (iii) heterogeneous beliefs.

The analysis reveals that if agents' beliefs are homogeneous and can be estimated accurately, implied risk aversion inherits most of the properties of agents' utility functions. More specifically, if all agents are risk averse, then implied risk aversion is strictly positive. Moreover, if all agents exhibit constant (decreasing, increasing) absolute risk aversion, so does implied risk aversion. Preference aggregation therefore seems unlikely to be the explanation for the smile. This result also holds true in the presence of stochastic volatility and jumps in returns.

Analyzing misestimation of investors' beliefs as a potential explanation reveals that implied risk aversion estimates are very sensitive to beliefs estimation errors. Misestimation of beliefs caused by stochastic volatility and jumps as captured in the Pan (2002) model seems unlikely to be able to account for the smile. The beliefs misestimation patterns that can be inferred from the implied risk aversion estimates found in the literature suggest a Peso problem explanation. However, the perceived probability of a market crash required to generate the fat left tail of the empirical state-price density appears unreasonably high.

Heterogeneous beliefs cause significant distortions in implied risk aversion estimates if heterogeneity is not accounted for explicitly in the estimation process. However, fitting a simple model with three groups of CRRA investors with heterogeneous, lognormal beliefs to the empirical state-price density reveals that two groups of pessimistic investors with an implausibly large degree of pessimism are required to explain the implied risk aversion smile.

Thus, for plausible parameter values, none of the three potential explanations considered in the paper is able to account for the implied risk aversion smile within the standard consumption-based framework. In order to explain the smile, it therefore seems necessary to go beyond the standard consumption-based framework and analyze the impact of factors such as market incompleteness, market frictions, and the fact that the stock index may not be a good proxy for the aggregate endowment.

Figure Legends

Figure 1: State-price density, statistical density and implied risk aversion with stochastic volatility and jumps. State-price density and statistical density obtained by inverting the transforms of the stochastic volatility and jumps model of Pan (2002), and the corresponding implied risk aversion. The computations are performed for time horizons of 6 months (panels A and C) and one year (panels B and D). Five initial volatility parameter values are used: the estimated long-term mean (corresponding to the middle curve in each set), as well as the mean plus or minus one or two standard deviations. Implied risk aversion is remarkably consistent across these different initial values; in all five cases, it exhibits considerable variation and is negative in high return states.

Figure 2: Link between beliefs estimation errors and risk aversion estimation errors. Beliefs estimation errors can be derived from implied risk aversion and assumptions about actual risk aversion using (26). The U-shaped implied risk aversion patterns reported in the literature (panel A) suggest that historical return frequency distributions underestimate agents' assessment of very low index returns and overestimate the probability of very high index returns (panel B).

Figure 3: A Peso problem: CRRA state-price density with a mixture of lognormal distributions. Fitting the state-price density with beliefs given by a mixture of lognormal distributions, (29), closely reproduces the semi-parametrically estimated state-price density of Aït-Sahalia and Lo (2000). The Black-Scholes state-price density, on the other hand, misses some salient features of the data.

Figure 4: Heterogeneous beliefs and implied risk aversion – small heterogeneity. Panel A shows the beliefs of both groups of investors, average beliefs, and the state-price density under heterogeneous beliefs (41) for the parameter values $c_t = 1$, $\zeta_t^2 = 1$, $\kappa_2 = 1$, $\beta = 2$, $\mu_1 = 0.05$, $\mu_2 = 0.04$, $\sigma = 0.1$ and $V = 0.01$. Panel B reports the implied relative risk aversion estimated using either group's beliefs or using average beliefs. When implied relative risk aversion is estimated based on

either group's beliefs, it is significantly distorted away from agents' true relative risk aversion (2 in this example) by the presence of heterogeneous beliefs, even though the degree of heterogeneity in beliefs is small. When it is estimated based on average beliefs, however, implied risk aversion is quite close to true risk aversion.

Figure 5: Heterogeneous beliefs and implied risk aversion – large heterogeneity. Panel A shows the beliefs of both groups of investors, average beliefs, and the state-price density under heterogeneous beliefs (41) for the parameter values $c_t = 1$, $\zeta_t^2 = 1$, $\kappa_2 = 1$, $\beta = 2$, $\mu_1 = 0.05$, $\mu_2 = -0.1$, $\sigma = 0.1$ and $V = 0.01$. Panel B reveals that implied relative risk aversion is significantly distorted away from agents' true relative risk aversion (2 in this example) by the presence of heterogeneous beliefs. This happens regardless of whether it is estimated using either group's beliefs or using average beliefs.

Figure 6: Heterogeneous beliefs and implied risk aversion – logarithmic utility, three groups. Panel A shows the beliefs of the three groups of investors, average beliefs, and the state-price density under heterogeneous beliefs (42) for the parameter values $c_t = 1$, $\zeta_t^2 = \zeta_t^3 = 1$, $\kappa_2 = \kappa_3 = 1$, $\beta = 1$, $\mu_1 = 0.05$, $\mu_2 = -0.1$, $\mu_3 = 0.20$, $\sigma = 0.1$ and $V = 0.01$. Panel B shows that implied relative risk aversion exceeds true risk aversion when it is estimated based on the most optimistic group 3's beliefs and is lower than true risk aversion when it is estimated using the most pessimistic group 2's beliefs. When it is estimated based on the intermediate group 1's beliefs, it overstates true risk aversion for small values of the aggregate endowment and understates it for large ones.

Figure 7: CRRA state-price density with heterogeneous beliefs. Calibrating the heterogeneous-beliefs state-price density (41) with three groups of investors with lognormal beliefs to the semi-parametrically estimated state-price density of Aït-Sahalia and Lo (2000) yields an almost perfect fit. The Black-Scholes state-price density, on the other hand, misses some salient features of the data. The figure also shows the homogeneous-beliefs state-price density obtained using the average beliefs of the three groups.

Figure 8. CRRA state-price density with heterogeneous beliefs: density and implied risk aversion estimates. Panel A shows the beliefs of the three groups of investors, average beliefs, and the state-price density under heterogeneous beliefs obtained by fitting the heterogeneous-beliefs state-price density (41) with three groups of investors to the semi-parametrically estimated density of Aït-Sahalia and Lo (2000). Panel B shows that implied relative risk aversion is distorted away from agents' degree of relative risk aversion (estimated to be 2.98 in this example) by the presence of heterogeneous beliefs. This happens regardless of whether it is estimated using either group's beliefs or using average beliefs.

Appendix

A Proof of Propositions

Proof of Proposition 1: With homogeneous beliefs and accurate estimation, $\hat{P}'(S)/\hat{P}(S) = P'(S)/P(S)$. Substitution in the implied risk aversion estimator (22) yields (23).

Proof of Proposition 2: Differentiating (23) with respect to S yields

$$\alpha'(S) = -\frac{\sum_{i=1}^{I} -\frac{\alpha_i'(c_s^i)}{\alpha_i^2(c_s^i)}\frac{dc_s^i}{dS}}{\left(\sum_{i=1}^{I}\frac{1}{\alpha_i(c_s^i)}\right)^2} = \frac{\sum_{i=1}^{I} \frac{\alpha_i'(c_s^i)}{\alpha_i^2(c_s^i)}\frac{dc_s^i}{dS}}{\left(\sum_{i=1}^{I}\frac{1}{\alpha_i(c_s^i)}\right)^2} = \alpha^2(S)\sum_{i=1}^{I}\frac{\alpha_i'(c_s^i)}{\alpha_i^2(c_s^i)}\frac{dc_s^i}{dS} \tag{45}$$

Since all agents are risk-averse and have homogeneous beliefs, risk-sharing among them implies that $dc_s^i/dS > 0$ for all i. Therefore, (45) is positive whenever $\alpha_i'(c_s^i) > 0$ for all i and negative whenever $\alpha_i'(c_s^i) < 0$ for all i, establishing the result.

Proof of Proposition 3: Relative risk aversion is given by $\rho(S) = S\alpha(S)$. Using (23) then yields

$$\rho(S) = S\frac{1}{\sum_{i=1}^{I}\frac{1}{-u_{cc}^i(c_s^i,s)/u_c^i(c_s^i,s)}} = \frac{1}{\sum_{i=1}^{I}\frac{1}{-u_{cc}^i(c_s^i,s)c_s^i/u_c^i(c_s^i,s)}\frac{c_s^i}{S}} = \frac{1}{\sum_{i=1}^{I}\frac{1}{\rho_i(c_s^i)}\frac{c_s^i}{S}} \tag{46}$$

where $\rho_i(c_s^i) \equiv -u_{cc}^i(c_s^i,s)c_s^i/u_c^i(c_s^i,s)$ denotes agent i's relative risk aversion on his optimal consumption path.

Proof of Proposition 5: Rewriting (25) as

$$\frac{d\ln(\hat{P}(S))}{dS} = \frac{d\ln(P(S))}{dS} + \epsilon(S) \tag{47}$$

yields, for an arbitrary reference point $\underline{S}$,

$$\ln\left(\frac{\hat{P}(S)}{\hat{P}(\underline{S})}\right) = \ln\left(\frac{P(S)}{P(\underline{S})}\right) + \int_{\underline{S}}^{S}\epsilon(z)dz \tag{48}$$

or

$$\frac{\hat{P}(S)}{P(S)} = \frac{\hat{P}(\underline{S})}{P(\underline{S})}\exp\left(\int_{\underline{S}}^{S}\epsilon(z)dz\right) = \gamma\exp\left(\int_{\underline{S}}^{S}\epsilon(z)dz\right) \tag{49}$$

with $\gamma \equiv \hat{P}(\underline{S})/P(\underline{S})$ a constant that ensures that $P(S)$ integrates to 1.

B Computation of P and Q using Pan's (2002) Model

Pan (2002) adopts the Bates (2000) stochastic volatility and jumps model to characterize the index return and volatility dynamics. Letting r_t denote the riskless interest rate (assumed to follow a square root process with long-run mean $\bar{r}$, mean-reversion rate κ_r, and volatility coefficient σ_r), q_t the dividend yield (with long-run mean $\bar{q}$, mean-reversion rate κ_q, and volatility coefficient σ_q), the index price S_t and volatility V_t under the physical measure P are assumed to follow:

$$dS_t = (r_t - q_t + \eta^s V_t + \lambda V_t(\mu - \mu^*))\, S_t dt + \sqrt{V_t} S_t dB_t^1 + dZ_t - \mu S_t \lambda V_t dt \quad (50)$$

$$dV_t = \kappa_v(\bar{v} - V_t)dt + \sigma_v \sqrt{V_t} \left(\rho dB_t^1 + \sqrt{1-\rho^2} dB_t^2\right) \quad (51)$$

where dB_t^1 and dB_t^2 are independent standard Brownian motions and Z is a pure jump process. Jumps are assumed to arrive with a stochastic intensity λV_t. Conditional on a jump occurring, the logarithm of the relative jump size is assumed to be normally distributed with mean $\mu_J = \ln(1+\mu) - \sigma_J^2/2$ and variance σ_J^2. Hence, the last term in (50), $\mu S_t \lambda V_t dt$, compensates for the change in instantaneous expected index returns introduced by Z. The volatility process has long-run mean $\bar{v}$, mean-reversion rate κ_v, and volatility coefficient σ_v, and the Brownian shocks to price S and volatility V are correlated with constant coefficient ρ.

Under the risk-neutral measure Q, the dynamics of r_t and q_t are assumed to be the same as under P, but the dynamics of S_t and V_t follow:

$$dS_t = (r_t - q_t) S_t dt + \sqrt{V_t} S_t dB_t^1(Q) + dZ_t^Q - \mu^* S_t \lambda V_t dt \quad (52)$$

$$dV_t = (\kappa_v(\bar{v} - V_t) + \eta^v V_t)dt + \sigma_v \sqrt{V_t} \left(\rho dB_t^1(Q) + \sqrt{1-\rho^2} dB_t^2(Q)\right) \quad (53)$$

where $dB_t^1(Q)$ and $dB_t^2(Q)$ are independent standard Brownian motions under Q and Z^Q is a pure jump process under Q. Jumps are assumed to arrive with a stochastic intensity λV_t that is the same as under P; in other words, there is no risk premium for jump timing uncertainty. Conditional on a jump occurring, the logarithm of the relative jump size under Q is assumed to be normally distributed with mean $\mu_J^* = \ln(1+\mu^*) - \sigma_J^2/2$ and variance σ_J^2. The volatility process has mean-reversion rate $\kappa_v^* = \kappa_v - \eta_v$, long-run mean $\bar{v}^* = \kappa_v \bar{v}/\kappa_v^*$, and volatility coefficient σ_v.

Thus, the model allows for three risk premia: the risk premium for "Brownian" return risk, $\eta^s V_t$, that for jump return risk, $\lambda V_t(\mu - \mu^*)$, and that for volatility risk, $\eta^v V_t$.

In this setting, from Appendices B and D in Pan (2002), given initial values of the interest rate, r, the dividend yield, q, and volatility, v, and a time horizon $\tau = s - t$, the time-t conditional transform of $\ln(S_s)$ under P is given by:

$$\psi(c; v, r, q, \tau) = \exp(\alpha_r(c) + \alpha_q(c) + \alpha_v(c) + \beta_r(c)r + \beta_q(c)q + \beta_v(c)v) \tag{54}$$

where, letting $\gamma_r = \sqrt{\kappa_r^2 + 2(1-c)\sigma_r^2}$, $\gamma_q = \sqrt{\kappa_q^2 + 2c\sigma_q^2}$,

$$a = c(1-c) - 2\lambda\left(\exp\left(c\mu_J + c^2\frac{\sigma_J^2}{2}\right) - 1 - c\mu^*\right) - 2c\eta^s \tag{55}$$

$b = \sigma_v \rho c - \kappa_v$ and $\gamma_v = \sqrt{b^2 + a\sigma_v^2}$, one has [21]

$$\alpha_r = -\frac{\kappa_r \bar{r}}{\sigma_r^2}\left((\gamma_r - \kappa_r)\tau + 2\ln\left(1 - (\gamma_r - \kappa_r)\frac{1-\exp(-\gamma_r\tau)}{2\gamma_r}\right)\right) \tag{56}$$

$$\alpha_q = -\frac{\kappa_q \bar{q}}{\sigma_q^2}\left((\gamma_q - \kappa_q)\tau + 2\ln\left(1 - (\gamma_q - \kappa_q)\frac{1-\exp(-\gamma_q\tau)}{2\gamma_q}\right)\right) \tag{57}$$

$$\alpha_v = -\frac{\kappa_v \bar{v}}{\sigma_v^2}\left((\gamma_v + b)\tau + 2\ln\left(1 - (\gamma_v + b)\frac{1-\exp(-\gamma_v\tau)}{2\gamma_v}\right)\right) \tag{58}$$

$$\beta_r = -\frac{2(1-c)(1-\exp(-\gamma_r\tau))}{2\gamma_r - (\gamma_r - \kappa_r)(1-\exp(-\gamma_r\tau))} \tag{59}$$

$$\beta_q = -\frac{2c(1-\exp(-\gamma_q\tau))}{2\gamma_q - (\gamma_q - \kappa_q)(1-\exp(-\gamma_q\tau))} \tag{60}$$

$$\beta_v = -\frac{a(1-\exp(-\gamma_v\tau))}{2\gamma_v - (\gamma_v + b)(1-\exp(-\gamma_v\tau))} \tag{61}$$

Given the values of the parameters $\kappa_r = 0.20$, $\bar{r} = 0.058$, $\sigma_r = 0.0415$, $\kappa_q = 0.24$, $\bar{q} = 0.025$, $\sigma_q = 0.0269$ reported in Table 6 of Pan (2002) and the values $\kappa_v = 6.4$, $\bar{v} = 0.0153$, $\sigma_v = 0.30$, $\rho = -0.53$, $\eta^s = 3.6$, $\lambda = 12.3$, $\mu = -0.008$, $\mu^* = -0.192$, and $\sigma_J = 0.0387$ reported in Table 3, and selecting initial values for r, q and v and the time horizon τ considered, the density of S_s under P can be obtained by numerical integration:

$$P(S; v, r, q, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi(iz; v, r, q, \tau) \exp(-izS) dz \tag{62}$$

An analogous procedure is used to obtain the density under Q. From Pan's (2002) Appendix B, the transform under Q has a similar form as under P, except that some of the parameters used are the risk-neutral counterparts of those under P. Specifically, one has

$$\psi^*(c; v, r, q, \tau) = \exp\left(\alpha_r(c) + \alpha_q(c) + \alpha_v^*(c) + \beta_r(c)r + \beta_q(c)q + \beta_v^*(c)v\right) \tag{63}$$

where α_r, α_q, β_r and β_q are defined as under P and, letting $\kappa_v^* = \kappa_v - \eta^v$, $\bar{v}^* = \kappa_v \bar{v}/\kappa_v^*$, $a^* = c(1-c) - 2\lambda\left(\exp\left(c\mu_J^* + c^2\sigma_J^2/2\right) - 1 - c\mu^*\right)$, $b^* = \sigma_v \rho c - \kappa_v^*$ and $\gamma_v^* = \sqrt{(b^*)^2 + a^*\sigma_v^2}$, one has

$$\alpha_v^* = -\frac{\kappa_v^* \bar{v}^*}{\sigma_v^2}\left((\gamma_v^* + b^*)\tau + 2\ln\left(1 - (\gamma_v^* + b^*)\frac{1 - \exp(-\gamma_v^*\tau)}{2\gamma_v^*}\right)\right) \tag{64}$$

$$\beta_v^* = -\frac{a^*(1 - \exp(-\gamma_v^*\tau))}{2\gamma_v^* - (\gamma_v^* + b^*)(1 - \exp(-\gamma_v^*\tau))} \tag{65}$$

Using the parameter values above and the additional value $\eta^v = 3.1$ reported in Table 3 of Pan (2002), the density of S_s under Q for initial values r, q, v and a time horizon τ can again be obtained by numerical integration:

$$Q(S; v, r, q, \tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi^*(iz, v, r, q, \tau) \exp(-izS) dz \tag{66}$$

C Joint Distribution of c_t and the ζ_t^is

The computations in this section follow the approach used in Buraschi and Jiltsov (2005), who perform analogous computations in a slightly different setting.

The distribution of c_s conditional on c_t as perceived by agent i can be computed by noting that agent i perceives c_t to have dynamics

$$\frac{dc_t}{c_t} = \mu_t^i dt + \sigma dB_t^i \tag{67}$$

where dB_t^i is a Brownian innovation. Therefore,

$$d\ln(c_t) = \left(\mu_t^i - \frac{\sigma^2}{2}\right) dt + \sigma dB_t^i \tag{68}$$

Since

$$d\mu_t^i = \frac{V}{\sigma} dB_t^i \tag{69}$$

one has

$$\mu_s^i = \mu_t^i + \int_t^s \frac{V}{\sigma} dB_s^i = \mu_t^i + \frac{V}{\sigma}(B_s^i - B_t^i) \tag{70}$$

Hence,

$$\begin{aligned} \ln(c_s) &= \ln(c_t) + \int_t^s \left(\mu_u^i - \frac{\sigma^2}{2}\right) du + \sigma \int_t^s dB_u^i \\ &= \ln(c_t) + \int_t^s \left(\mu_t^i + \frac{V}{\sigma}(B_u^i - B_t^i) - \frac{\sigma^2}{2}\right) du + \sigma \int_t^s dB_u^i \\ &= \ln(c_t) + \left(\mu_t^i - \frac{\sigma^2}{2}\right)(s-t) + \frac{V}{\sigma}\int_t^s (B_u^i - B_t^i) du + \sigma(B_s^i - B_t^i) \end{aligned} \tag{71}$$

Thus, conditional on $\ln(c_t)$, agent i perceives $\ln(c_s)$ to be normally distributed with expectation

$$E_t^i(\ln(c_s)) = \ln(c_t) + \left(\mu_t^i - \frac{\sigma^2}{2}\right)(s-t) \equiv \mu_c^i \tag{72}$$

and variance

$$\begin{aligned} \mathrm{Var}_t^i(\ln(c_s)) &= \sigma^2(s-t) + \left(\frac{V}{\sigma}\right)^2 E^i\left(\left(\int_t^s (B_u^i - B_t^i) du\right)^2\right) \\ &= \sigma^2(s-t) + \left(\frac{V}{\sigma}\right)^2 \frac{(s-t)^2}{2} \equiv \sigma_c^2 \end{aligned} \tag{73}$$

In order to determine the distribution of the density processes $\{\zeta_s^i\}$ from agent 1's perspective, observe that since

$$dB_t^i = \frac{1}{\sigma}\left(\frac{dc_t}{c_t} - \mu_t^i dt\right) \tag{74}$$

the Brownian innovations of group i and the reference group 1 are related by

$$dB_t^i = dB_t^1 + \frac{\mu_t^1 - \mu_t^i}{\sigma} dt = dB_t^1 + \overline{\mu}_t^i dt \tag{75}$$

where $\overline{\mu}_t^i \equiv (\mu_t^1 - \mu_t^i)/\sigma$ denotes the normalized divergence in beliefs about μ_t between both groups. Since $\overline{\mu}_t^i$ has dynamics

$$d\overline{\mu}_t^i = \frac{d\mu_t^1 - d\mu_t^i}{\sigma} = \frac{V}{\sigma^2}\frac{\mu_t^i - \mu_t^1}{\sigma}dt = -\frac{V}{\sigma^2}\overline{\mu}_t^i dt \tag{76}$$

it is a deterministic function of time given by

$$\overline{\mu}_s^i = \overline{\mu}_t^i \exp\left(-\frac{V}{\sigma^2}(s-t)\right) \tag{77}$$

By Girsanov's theorem, the density process for group i, ζ_t^i, follows

$$\frac{d\zeta_t^i}{\zeta_t^i} = -\overline{\mu}_t^i dB_t^1 \tag{78}$$

Hence,

$$\begin{aligned}
\ln(\zeta_s^i) &= \ln(\zeta_t^i) - \int_t^s \overline{\mu}_u^i dB_u^1 - \frac{1}{2}\int_t^s (\overline{\mu}_u^i)^2 du \qquad (79)\\
&= \ln(\zeta_t^i) - \overline{\mu}_t^i \int_t^s \exp\left(-\frac{V}{\sigma^2}(u-t)\right) dB_u^1 - \frac{1}{2}(\overline{\mu}_t^i)^2 \int_t^s \exp\left(-\frac{2V}{\sigma^2}(u-t)\right) du
\end{aligned}$$

Thus, conditional on $\ln(\zeta_t^i)$, agent 1 perceives $\ln(\zeta_s^i)$ to be normally distributed with mean

$$\begin{aligned}
E_t^1(\ln(\zeta_s^i)) &= \ln(\zeta_t^i) - \frac{1}{2}(\overline{\mu}_t^i)^2 \int_t^s \exp\left(-\frac{2V}{\sigma^2}(u-t)\right) du \qquad (80)\\
&= \ln(\zeta_t^i) - \frac{1}{2}(\overline{\mu}_t^i)^2 \frac{\sigma^2}{2V}\left(1 - \exp\left(-\frac{2V}{\sigma^2}(s-t)\right)\right) \equiv \ln(\zeta_t^i) + \mu_{\zeta^i}
\end{aligned}$$

and variance

$$\begin{aligned}
\mathrm{Var}_t^1(\ln(\zeta_s^i)) &= (\overline{\mu}_t^i)^2 \int_t^s \exp\left(-\frac{2V}{\sigma^2}(u-t)\right) du\\
&= (\overline{\mu}_t^i)^2 \frac{\sigma^2}{2V}\left(1 - \exp\left(-\frac{2V}{\sigma^2}(s-t)\right)\right) \equiv \sigma_{\zeta^i}^2 \qquad (81)
\end{aligned}$$

The covariance between $\ln(\zeta_s^i)$ and $\ln(\zeta_t^j)$ as perceived by agent 1 at time t is given by

$$\begin{aligned}
\mathrm{Cov}_t^1(\ln(\zeta_s^i), \ln(\zeta_s^j)) &= \int_t^s \overline{\mu}_u^i \overline{\mu}_u^j du = \overline{\mu}_t^i \overline{\mu}_t^j \int_t^s \exp\left(-\frac{2V}{\sigma^2}(u-t)\right) du\\
&= \overline{\mu}_t^i \overline{\mu}_t^j \frac{\sigma^2}{2V}\left(1 - \exp\left(-\frac{2V}{\sigma^2}(s-t)\right)\right) \qquad (82)
\end{aligned}$$

with the consequence that the correlation between $\ln(\zeta_s^i)$ and $\ln(\zeta_s^j)$ is perfect but can be either positive or negative, depending on whether $\overline{\mu}_t^i$ and $\overline{\mu}_t^j$ have similar signs or not, i.e., $\rho_{\zeta^i\zeta^j} = \text{sign}(\overline{\mu}_t^i\overline{\mu}_t^j)$.

Finally, the covariance between $\ln(c_s)$ and $\ln(\zeta_s^i)$ as perceived by agent 1 at time t is given by

$$\begin{aligned}\text{Cov}_t^1(\ln(c_s), \ln(\zeta_s^i)) &= -\int_t^s \sigma\overline{\mu}_u^i du = -\sigma\overline{\mu}_t^i \int_t^s \exp\left(-\frac{V}{\sigma^2}(u-t)\right) du \\ &= -\overline{\mu}_t^i \frac{\sigma^3}{V}\left(1 - \exp\left(-\frac{V}{\sigma^2}(s-t)\right)\right) \equiv \sigma_{c\zeta^i} \qquad (83)\end{aligned}$$

References

Aït-Sahalia, Y., and A. W. Lo, 2000, "Nonparametric Risk Management and Implied Risk Aversion," *Journal of Econometrics*, 94, 9–51.

Aït-Sahalia, Y., Y. Wang, and F. Yared, 2001, "Do Option Markets Correctly Price the Probabilities of Movement of the Underlying Asset?" *Journal of Econometrics*, 102, 67–110.

Aït-Sahalia, Y., 2002, "Telling from Discrete Data whether the Underlying Continuous-Time Model is a Diffusion," *Journal of Finance*, 57, 2075–2112.

Aït-Sahalia, Y., and J. Duarte, 2003, "Nonparametric Option Pricing under Shape Restrictions," *Journal of Econometrics*, 116, 9–47.

Andersen, T. G., L. Benzoni, and J. Lund, 2002, "An Empirical Investigation of Continuous-Time Equity Return Models," *Journal of Finance*, 57, 1239–1284.

Anderson, E. W., E. Ghysels, and J. L. Juergens, 2005, "Do Heterogeneous Beliefs Matter for Asset Pricing?" *Review of Financial Studies*, 18, 875–924.

Arrow, K. J., 1970, *Essays in the Theory of Risk-Bearing*, North-Holland, Amsterdam.

Ball, C. A., and W. N. Torous, 1983, "A Simplified Jump Process for Common Stock Returns," *Journal of Financial and Quantitative Analysis*, 18, 53–65.

Ball, C. A., and W. N. Torous, 1985, "On Jumps in Common Stock Prices and Their Impact on Call Option Pricing," *Journal of Finance*, 40, 155–173.

Basak, S., 2005, "Asset Pricing with Heterogeneous Beliefs," *Journal of Banking and Finance*, 29, 2849–2881.

Bates, D. S., 2000, "Post-'87 Crash Fears in the S&P 500 Futures Option Market," *Journal of Econometrics*, 94, 181–238.

Bates, D. S., 2001, "The Market for Crash Risk," Working Paper 8557, NBER.

Benninga, S., and J. Mayshar, 2000, "Heterogeneity and Option Pricing," *Review of Derivatives Research*, 4, 7–27.

Bliss, R. R., and N. Panigirtzoglou, 2000, "Testing the Stability of Implied Probability Density Functions," Working Paper 114, Bank of England.

Brown, D. P., and J. C. Jackwerth, 2004, "The Pricing Kernel Puzzle: Reconciling Index Option Data and Economic Theory," working paper, University of Wisconsin at Madison and University of Konstanz.

Buraschi, A., and A. Jiltsov, 2005, "Model Uncertainty and Option Markets with Heterogeneous Agents," forthcoming in *Journal of Finance*.

Chernov, M., et al., 2003, "Alternative Models for Stock Price Dynamics," *Journal of Econometrics*, 116, 225–257.

Constantinides, G. M., and D. Duffie, 1996, "Asset Pricing with Heterogeneous Consumers," *Journal of Political Economy*, 104, 219–240.

Detemple, J. B., and S. Murthy, 1994, "Intertemporal Asset Pricing with Heterogeneous Beliefs," *Journal of Economic Dynamics and Control*, 62, 294–320.

Drèze, J. H., 1981, "Inferring Risk Tolerance from Deductibles in Insurance Contracts," *Geneva Papers on Risk and Insurance*, 20, 48–52.

Eraker, B., 2004, "Do Stock Prices and Volatility Jump? Reconciling Evidence from Spot and Option Prices," *Journal of Finance*, 59, 1367–1403.

Eraker, B., M. Johannes, and N. Polson, 2003, "The Impact of Jumps in Volatility and Returns," *Journal of Finance*, 58, 1269–1300.

Friend, I., and M. E. Blume, 1975, "The Demand for Risky Assets," *American Economic Review*, 65, 900–922.

Jackwerth, J. C., 2000, "Recovering Risk Aversion from Option Prices and Realized Returns," *Review of Financial Studies*, 13, 433–451.

Jackwerth, J. C., and M. Rubinstein, 2004, "Recovering Probabilities and Risk Aversion from Option Prices and Realized Returns," in B. Lehmann (ed.), *The Legacy of Fisher Black*, Oxford University Press, Oxford.

Jondeau, E., and M. Rockinger, 2000, "Reading the Smile: The Message Conveyed by Methods which Infer Risk Neutral Densities," *Journal of International Money and Finance*, 19, 885–915.

Lochstoer, L., 2004, "Expected Returns and the Business Cycle: Heterogeneous Agents and Heterogeneous Goods," working paper, University of California, Berkeley.

Pan, J., 2002, "The Jump-Risk Premia Implicit in Options: Evidence from an Integrated Time-Series Study," *Journal of Financial Economics*, 63, 3–50.

Platen, E., and W. J. Runggaldier, 2005, "A Benchmark Approach to Filtering in Finance," *Asia Pacific Financial Markets*, 11, 79–105.

Poon, S. H., and R. Stapleton, 2005, *Asset Pricing in Discrete Time*, Oxford University Press, Oxford.

Rosenberg, J. V., and R. F. Engle, 2002, "Empirical Pricing Kernels," *Journal of Financial Economics*, 64, 341–372.

Rubinstein, M., 1985, "Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active COE Option Classes from August 23, 1976 through August 31, 1978," *Journal of Finance*, 40, 455–480.

Rubinstein, M., 1994, "Implied Binomial Trees," *Journal of Finance*, 49, 771–818.

Siegel, S., 2004, "Consumption-Based Asset Pricing: Durable Goods, Adjustment Costs, and Aggregation," working paper, Columbia University.

Wilson, R., 1968, "The Theory of Syndicates," *Econometrica*, 36, 119–132.

Footnotes

1. Constantinides and Duffie (1996) resolve the empirical difficulties encountered by representative agent consumption-based models using uninsurable labor income shocks. Lochstoer (2004) investigates the role of heterogeneous agents and heterogeneous goods, while Siegel (2004) analyzes the impact of frictions in a setting where adjusting durable good consumption is costly.

2. Although market incompleteness may cause implied risk aversion to smile, it need not do so. For instance, Poon and Stapleton (2005) show that with CRRA preferences, the introduction of background risk causes the pricing kernel to exhibit declining elasticity, i.e., implied relative risk aversion to be strictly decreasing.

3. Although the implied risk aversion literature uses option-based state-price density estimates, Aït-Sahalia, Wang and Yared (2001) show that the state-price density can also be estimated using historical index returns and index futures prices.

4. In order to guarantee that financial markets with a finite number of securities remain complete even in the presence of jumps, the jump process must satisfy $J_t = \int_0^t \phi_u dN_u$, where ϕ is predictable and $\sigma(B, N)$-measurable. In this case, the jump size distribution ν_t is the distribution of ϕ. I am grateful to Julien Hugonnier for pointing this out.

5. See, for example, Detemple and Murthy (1994) for a general description and Basak (2005) for a parametric example in the Gaussian case.

6. A detailed exposition of filtering in a setting which is quite similar to that used here can be found in Platen and Runggaldier (2005).

7. Although Aït-Sahalia and Lo (2000) estimate a *relative* risk aversion function, it is easy to check from their Figure 4 that their representative agent also exhibits increasing absolute risk aversion by noting that the slope of rays drawn through

the origin of their diagram and the points on their implied risk aversion function is increasing for index values between 415 and 440, between 465 and 475 and above 500. Rosenberg and Engle (2002) do not report risk aversion patterns directly, but the implied risk aversion pattern implicit in their pricing kernel estimates is similar to Jackwerth's.

8. Although Jackwerth (2000) tries to address this problem by varying the length of the historical sample he uses from 2 to 10 years and shows that his results do not change significantly, the basic issue of using historical returns to estimate beliefs remains.

9. The evidence on the need to account for jumps in volatility is mixed. Eraker, Johannes and Polson (2003) and Eraker (2004) consider models involving jumps in both returns and volatility and find evidence that both types of jumps are present in the data. However, in their analysis of the suitability of a number of alternative models for stock price dynamics, Chernov et al. (2003) find that the improvement in statistical fit achieved by going from a model with jumps in returns only to a model with jumps in both returns and volatility is too small to consider the latter to be the ideal model.

10. A Monte Carlo analysis in which the initial volatility parameter is drawn from a normal distribution with mean and standard deviation equal to the estimated long-term mean of volatility and the standard error of the estimate, respectively, was also performed. The average implied risk aversion and its 95% confidence bounds obtained in this simulation are virtually identical to the estimates corresponding to the long-term mean and the mean plus or minus two standard deviations shown in Figure 1.

11. Further computations were performed for other time horizons between 1 month and 1 year. The implied risk aversion patterns were similar.

12. Bates (2001) comes to a similar conclusion in the context of a general equilibrium model with jumps. He demonstrates that such a model is unable to generate negative implied risk aversion estimates. However, he also notes that the empirical risk aversion patterns could arise if there is a disparity between the risk-neutral distribution and the *estimate* of the objective distribution.

13. The value of 4 is arbitrary but not out of line with existing empirical evidence. Based on an analysis of the demand for risky assets, Friend and Blume (1975) find that the average coefficient of relative risk aversion is probably well in excess of one and perhaps in excess of two. Using an analysis of deductibles in insurance contracts, Drèze (1981) finds somewhat higher values. When fitting CRRA preferences to their data, Aït-Sahalia and Lo (2000) find a value of 12.7.

14. Rubinstein (1994) shows that beliefs inferred from the state-price density and some set of (assumed) investor preferences are not sensitive to the particular functional form of the market-wide risk aversion function chosen.

15. This specification of agents' beliefs is exact under the assumptions that (i) the index price has dynamics $dS_t = \mu S_t dt + \sigma S_t dB_t + k S_t dq_t$, where q is a Poisson counter, (ii) the jump size k is lognormally distributed, and (iii) there is at most one jump over the estimation horizon considered [see Ball and Torous (1983) and (1985), and Jondeau and Rockinger (2000)]. Since we are concerned with a Peso problem, where jumps are infrequent, this latter assumption is reasonable.

16. Note that, as one would expect, the standard deviation of returns conditional on a jump occurring over the period, σ_2, is much higher than that conditional on no jump occurring, σ_1.

17. As in previous sections of the paper, this analysis assumes that the stock index is a good proxy for the aggregate endowment, $c_s = S$. In this section, in order to avoid going back and forth between notation in S and in c_s, we formulate all results

in terms of c_s.

18. Although the initial weights of the different groups are given by $\{\kappa_i\}$, their effective weights at time t are $\{\kappa_i \zeta_t^i\}$. It is therefore appropriate to weight beliefs using $\kappa_i \zeta_t^i$ rather than κ_i.

19. An alternative way to derive (42) is to set $\beta = 1$ in (34) and compute the state-price density based on the expression $q_{t,s}(c_s) = \frac{c_t}{c_s} p_{t,s}^1(c_s) E_t^1 \left(\frac{1+\sum_{i=2}^I \kappa_i \zeta_s^i}{1+\sum_{i=2}^I \kappa_i \zeta_t^i} | c_s \right)$ using the fact that $p_{t,s}^1(c_s)$ is a lognormal density and that, conditional on c_s, $\ln(\zeta_s^i)$ is normally distributed with mean $\ln(\zeta_t^i) + \mu_{\zeta^i} + \frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1)$ and variance $\sigma_{\zeta^i}^2(1 - \rho_{c\zeta^i}^2)$, so that $E_t(\zeta_s^i | c_s) = \zeta_t^i \exp\left(\mu_{\zeta^i} + \frac{\sigma_{c\zeta^i}}{\sigma_c^2}(\ln(c_s) - \mu_c^1) + \left(1 - \rho_{c\zeta^i}^2\right) \frac{\sigma_{\zeta^i}^2}{2} \right)$.

20. These results suggest a probability of 39% of jumps of an average magnitude of -12% and a probability of 21% of jumps of an average magnitude of over -27%. Thus, mixing three densities instead of two does not change the conclusion from Section 2.2.3 that the crash probabilities required to generate the empirical state-price density are implausibly high – in this case, the probability of no crash is less than 40%.

21. The expression for β_v in Pan (2002) contains an additional term,
$\lambda_0 \left(\exp\left(c\mu_J + c^2\sigma_j^2/2\right) - 1 - c\mu^*\right)$, capturing the effect of a jump arrival intensity of the form $\lambda_0 + \lambda_1 V_t$. The calibrated model, however, uses the intensity specification λV_t. The transforms reproduced here are those that apply in this special case.

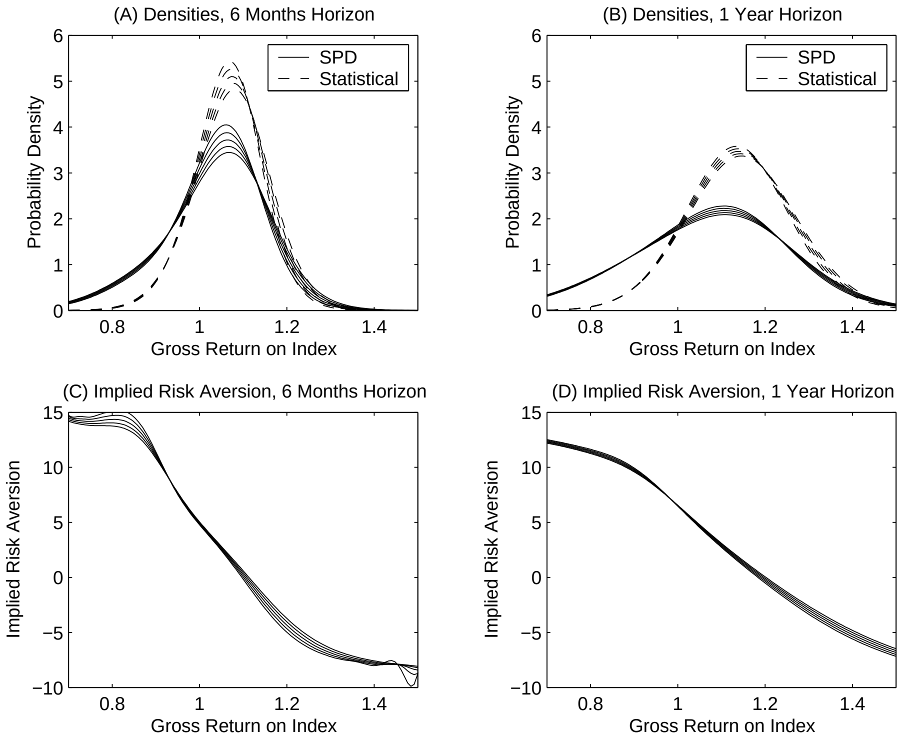

Figure 1: State-price density, statistical density and implied risk aversion with stochastic volatility and jumps

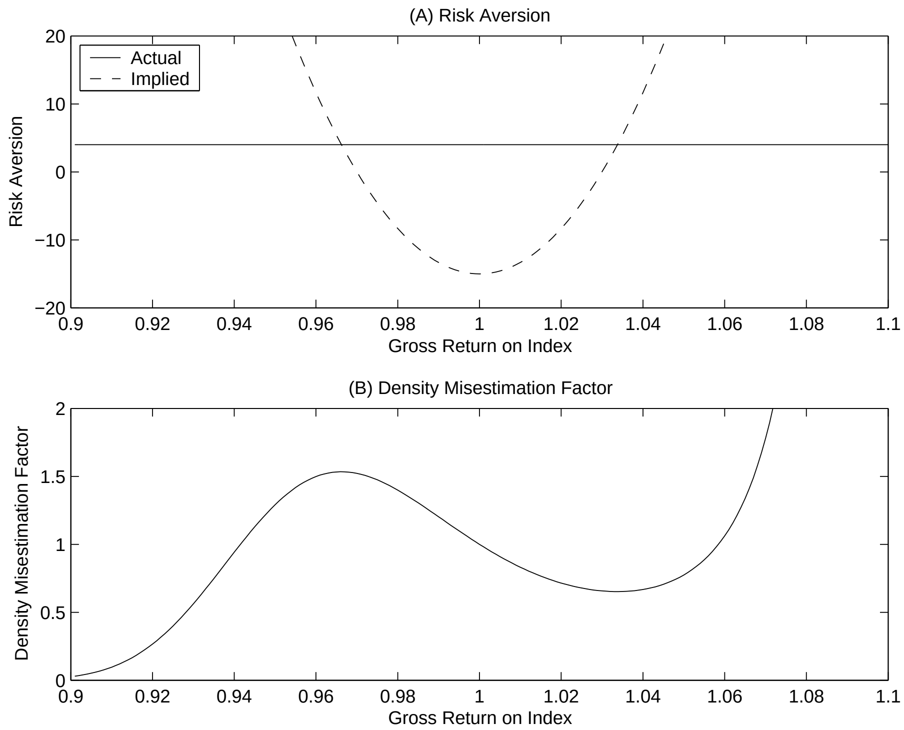

Figure 2: Link between beliefs estimation errors and risk aversion estimation errors

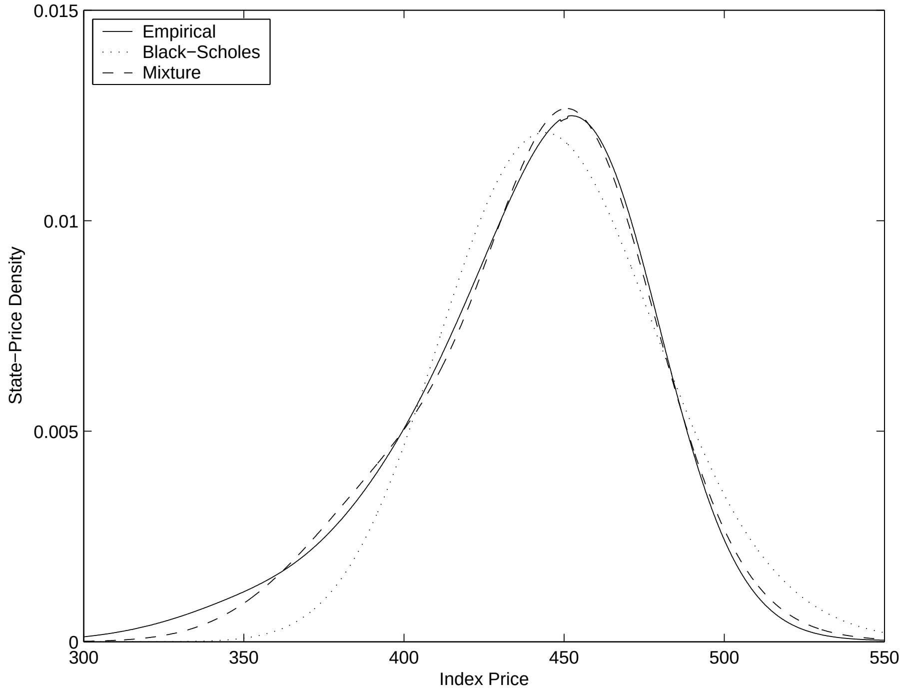

Figure 3: A Peso problem: CRRA state-price density with a mixture of lognormal distributions

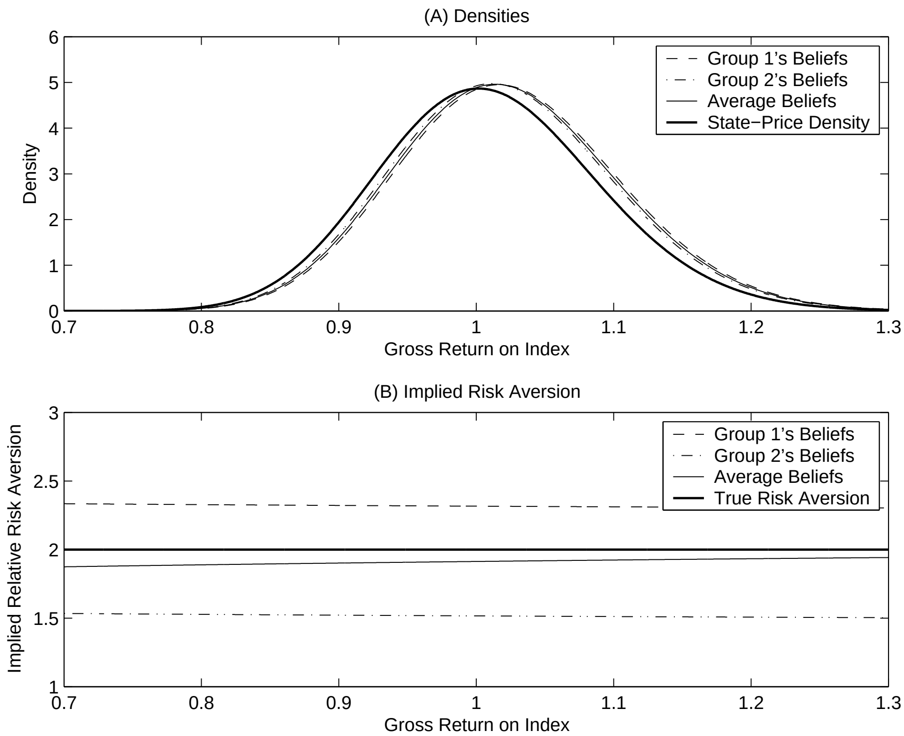

Figure 4: Heterogeneous beliefs and implied risk aversion – small heterogeneity

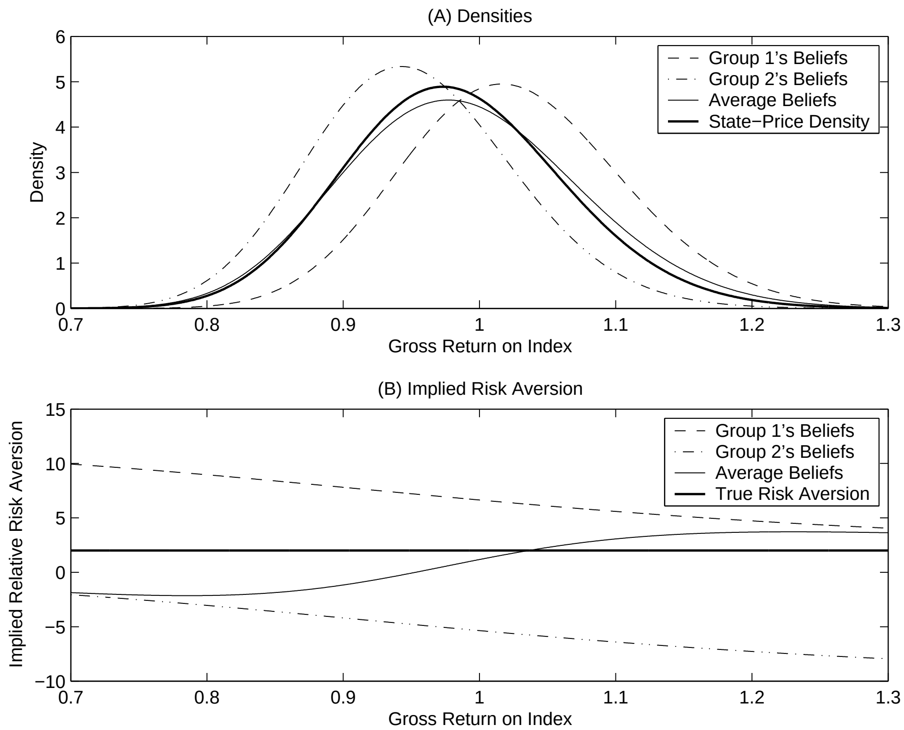

Figure 5: Heterogeneous beliefs and implied risk aversion – large heterogeneity

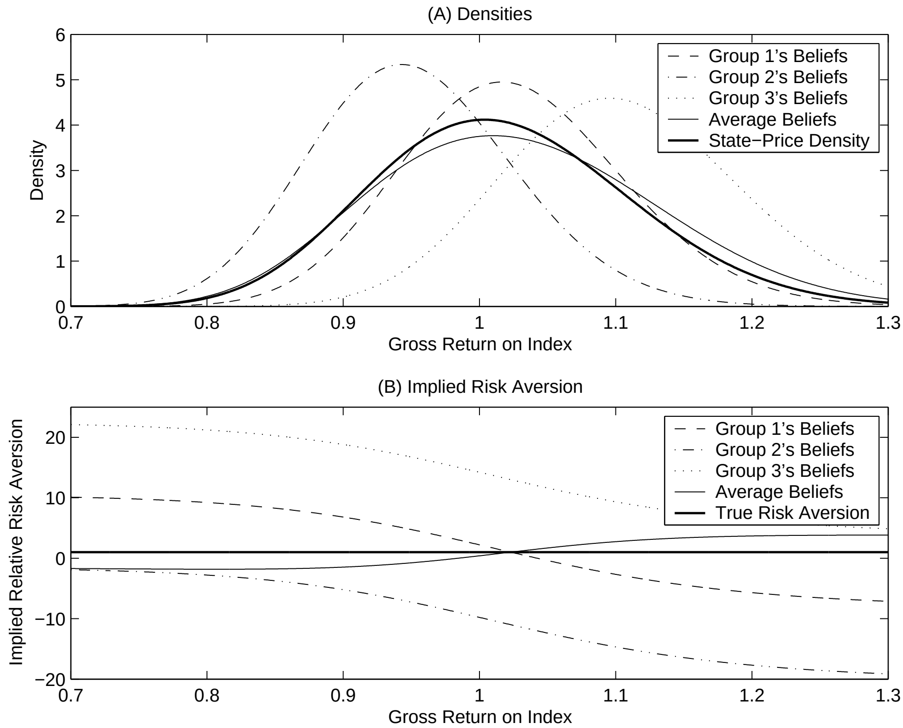

Figure 6: Heterogeneous beliefs and implied risk aversion – logarithmic utility, three groups

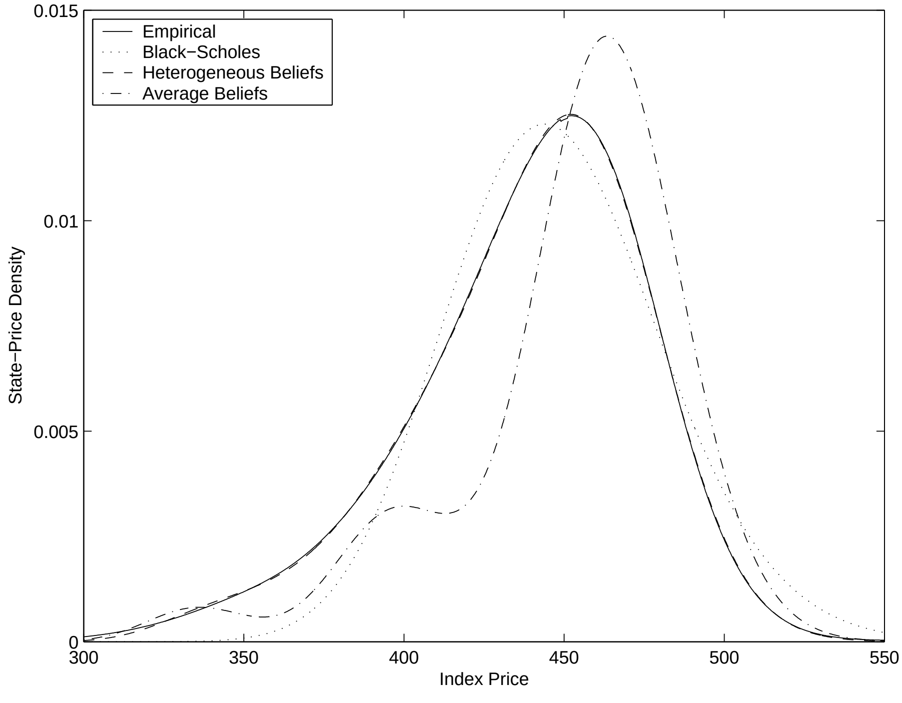

Figure 7: CRRA state-price density with heterogeneous beliefs

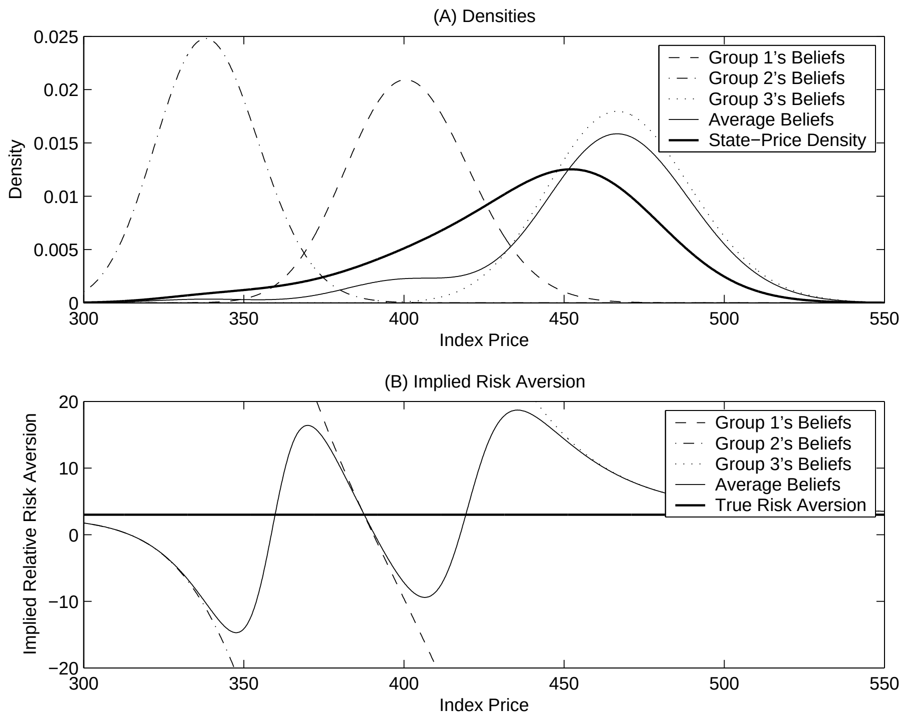

Figure 8: CRRA state-price density with heterogeneous beliefs: density and implied risk aversion estimates