Higher-order expansions of distributions of maxima in a Hüsler-Reiss model

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Abstract: The max-stable Hüsler-Reiss distribution which arises as the limit distribution of maxima of bivariate Gaussian triangular arrays has been shown to be useful in various extreme value models. For such triangular arrays, this paper establishes higher-order asymptotic expansions of the joint distribution of maxima under refined Hüsler-Reiss conditions. In particular, the rate of convergence of normalized maxima to the Hüsler-Reiss distribution is explicitly calculated.

Key Words: Hüsler-Reiss max-stable distribution; higher-order asymptotic expansion; triangular arrays; Gaussian random vector.

1 Introduction

The fact that the componentwise maxima of bivariate Gaussian random vectors possess asymptotic independent components (see e.g., [7]) has been seen as a drawback in extreme value theory since for modeling asymptotically dependent risks the classical and tractable Gaussian framework is inadequate. In the seminal paper [16] this drawback was removed by considering triangular arrays where the dependence may increase with n. Specifically, let $\{(X_{nk}, Y_{nk}), 1 \le k \le n, n \ge 1\}$ be a triangular array of independent standard (mean-zero and unit variance) bivariate Gaussian random vectors with correlations $\{\rho_n, n \ge 1\}$ and joint distribution function F_{ρ_n}. The principal finding of [16] is

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{R}} \left| F^n_{\rho_n}(x/b_n + b_n,\ y/b_n + b_n) - H_\lambda(x,y) \right| = 0, \tag{1.1}$$

provided that the so-called Hüsler-Reiss condition

$$\lim_{n\to\infty} \frac{1}{2} b_n^2 (1-\rho_n) = \lambda^2 \quad \text{with } \lambda \in [0,\infty] \tag{1.2}$$

holds with b_n given by

$$n(1-\Phi(b_n)) = 1 \tag{1.3}$$

or $nb_n^{-1}\varphi(b_n) = 1$, where Φ denotes the $N(0,1)$ distribution function and $\varphi(x) = \Phi'(x)$; see [16] for more details. The max-stable Hüsler-Reiss distribution H_λ is given by

$$H_\lambda(x,y) = \exp\left(-\Phi\left(\lambda + \frac{y-x}{2\lambda}\right)e^{-x} - \Phi\left(\lambda + \frac{x-y}{2\lambda}\right)e^{-y}\right), \quad x,y\in\mathbb{R},$$

with $H_0(x,y) = \exp(-e^{-\min(x,y)})$ and $H_\infty(x,y) = \Lambda(x)\Lambda(y)$, where $\Lambda(x) = \exp(-e^{-x}), x\in\mathbb{R}$ is the Gumbel distribution.

In fact, the bivariate Hüsler-Reiss distribution appeared in another context in [1], see for recent contribution in this direction [4, 17, 20, 2]. Related results for more general triangular arrays can be found in [9, 5, 6, 8, 11, 13, 14, 15,

10, 3, 12]; an interesting statistical applications related to the Hüsler-Reiss distribution is presented in [6].
For both applications and various theoretical investigations, it is of interest to know how good the Hüsler-Reiss distribution approximates the distribution of the bivariate maxima. So, a natural goal of this paper is to investigate the rate of convergence in (1.1), i.e., the speed of convergence to 0 as $n \to \infty$ of the following difference

$$\Delta(F^n_{\rho_n}, H_\lambda; x, y) := F^n_{\rho_n}(u_n(x), u_n(y)) - H_\lambda(x, y),$$

where $u_n(s) = s/b_n + b_n$ with norming constant b_n given by (1.3). In the literature the only available results concern the univariate problem, namely in [19] it has been shown that

$$\lim_{n\to\infty} b_n^2 \Big[b_n^2 \Big(\Phi^n(u_n(x)) - \Lambda(x) \Big) - s(x)\Lambda(x) \Big] = \Big(t(x) + \frac{1}{2} s^2(x) \Big) \Lambda(x), \tag{1.4}$$

with b_n given by (1.3) and $s(x), t(x)$ defined as

$$s(x) = 2^{-1}(x^2 + 2x)e^{-x} \quad \text{and} \quad t(x) = -8^{-1}(x^4 + 4x^3 + 8x^2 + 16x)e^{-x}. \tag{1.5}$$

In order to derive the rate of convergence of $\Delta(F^n_{\rho_n}, H_\lambda; x, y)$ to 0 we shall introduce a refinement of the Hüsler-Reiss condition (1.2), namely we shall suppose that

$$\lim_{n\to\infty} b_n^2(\lambda - \lambda_n) = \alpha \in \mathbb{R} \tag{1.6}$$

holds with $\lambda_n = (b_n^2(1 - \rho_n)/2)^{1/2}$ and $\lambda \in (0, \infty)$. By assuming further that $\delta_n = b_n^2(\lambda - \lambda_n) - \alpha$ also converges to 0 with a speed determined again by b_n^2, we are able to refine the second-order approximation significantly. The analysis of the two extreme cases $\lambda = 0$ and $\lambda = \infty$ are more complicated and more information related to ρ_n is needed. Two special cases $\rho_n = 1$ and $\rho_n \in [-1, 0]$ for all large n are explicitly solved.
The rest of the paper is organized as follows. In Section 2 we present the main results . All the proofs are relegated to Section 3.

2 Main Results

In the following we shall denote throughout by b_n the constants defined in (1.3) and further λ shall always be defined with respect to the Hüsler-Reiss condition (1.2). Next, we derive the second-order expansions of bivariate extremes under the second-order Hüsler-Reiss condition (1.6).

Theorem 2.1 *If* (1.6) *holds with* $\lambda_n = (b_n^2(1 - \rho_n)/2)^{1/2}$ *and* $\lambda \in (0, \infty)$*, then for all* $x, y \in \mathbb{R}$ *we have*

$$\lim_{n\to\infty} b_n^2 \Delta(F^n_{\rho_n}, H_\lambda; x, y) \quad = \quad \kappa(\alpha, \lambda, x, y) H_\lambda(x, y), \tag{2.1}$$

where

$$\kappa(\alpha, \lambda, x, y) \quad = \quad s(x)\Phi\left(\lambda + \frac{y - x}{2\lambda}\right) + s(y)\Phi\left(\lambda + \frac{x - y}{2\lambda}\right) + (2\alpha - \lambda(\lambda^2 + x + y + 2))e^{-x}\varphi\left(\lambda + \frac{y - x}{2\lambda}\right),$$

where $s(z), z \in \mathbb{R}$ *is defined by* (1.5).

If the second-order Hüsler-Reiss condition is further refined to a third-order one, a finer result than that stated in (2.1) can be obtained. Indeed, this can be achieved by introducing a restriction on the difference $\delta_n := b_n^2(\lambda - \lambda_n) - \alpha$, namely

$$\lim_{n\to\infty} b_n^2 \delta_n = \beta \in \mathbb{R}. \tag{2.2}$$

Utilising further condition (2.2) we derive below a third-order expansion of the joint distribution of extremes. For simplicity we shall omit the expression of the function τ below, it is specified in (3.17).

Theorem 2.2 *If* (2.2) *holds with* $\lambda \in (0,\infty)$*, then for all* $x, y \in \mathbb{R}$ *we have*

$$\lim_{n\to\infty} b_n^2\left(b_n^2\Delta(F_{\rho_n}^n, H_\lambda; x,y) - \kappa(\alpha,\lambda,x,y)H_\lambda(x,y)\right) = \left(\tau(\alpha,\beta,\lambda,x,y) + \frac{1}{2}\kappa^2(\alpha,\lambda,x,y)\right)H_\lambda(x,y). \tag{2.3}$$

For the two extreme cases $\lambda = 0$ and $\lambda = \infty$ we first consider two special cases satisfied for all large n, namely $\rho_n \in [-1,0]$ and $\rho_n = 1$ including components of each Gaussian vector with independence ($\rho_n = 0$), complete negative dependence ($\rho_n = -1$) and complete positive dependence ($\rho_n = 1$), respectively.

Theorem 2.3 *Let* $s(z)$ *and* $t(z)$ *be those defined as in* (1.5) *and set* $u_n(z) = b_n + z/b_n$, $z \in \mathbb{R}$.

(i). For $\rho_n \in [-1,0], n \ge 1$ *and any* $x, y \in \mathbb{R}$ *we have*

$$\lim_{n\to\infty} b_n^2\left[b_n^2\Delta(F_{\rho_n}^n, H_\infty; x,y) - (s(x)+s(y))H_\infty(x,y)\right] = \left(t(x)+t(y)+\frac{1}{2}(s(x)+s(y))^2\right)H_\infty(x,y). \tag{2.4}$$

(ii). If $\rho_n = 1, n \ge 1$*, then for any* $x, y \in \mathbb{R}$ *we have*

$$\lim_{n\to\infty} b_n^2\left[b_n^2\Delta(F_1^n, H_0; x,y) - s(\min(x,y))H_0(x,y)\right] = \left(t(\min(x,y)) + \frac{1}{2}(s(\min(x,y)))^2\right)H_0(x,y). \tag{2.5}$$

We consider next the other cases of $\rho_n \in (0,1)$ such that $\lambda_n \to 0$ or $\lambda_n \to \infty$. With more information on the asymptotic behavior of ρ_n we obtain below upper bounds for the convergence rates of $F_{\rho_n}^n$ to H_0 or H_∞.

Corollary 2.1 *For some* $\mathbb{C} > 0$ *and* $R(x,y) = \mathbb{C}(\exp(2|x|) + \exp(2|y|))$ *we have:*
(i). Suppose that $\rho_n \in (0,1), n \ge 1$ *and* (1.2) *holds with* $\lambda = \infty$*. If further* $\frac{1}{2}((1-\rho_n)\ln n - (2+\rho_n)\ln\ln n) \to \gamma \in (-\infty,\infty]$ *as* $n \to \infty$*, then for all* $x, y \in \mathbb{R}$

$$\limsup_{n\to\infty} b_n^2\left|\Delta(F_{\rho_n}^n, H_\infty; x,y)\right| \le (|s(x)| + |s(y)|)H_\infty(x,y) + e^{-\gamma}R(x,y).$$

(ii). If $(1-\rho_n)(\ln n)^3 \to \tau^2 \in [0,\infty)$ *as* $n \to \infty$*, then for all* $x, y \in \mathbb{R}$

$$\limsup_{n\to\infty} b_n^2\left|\Delta(F_{\rho_n}^n, H_0; x,y)\right| \le |s(\min(x,y))|H_0(x,y) + \tau R(x,y).$$

Remark 2.1 *For the Hüsler-Reiss model the rates of convergence of* $F_{\rho_n}^n(u_n(x), u_n(y))$ *to its ultimate max-stable distribution* $H_\lambda(x,y)$ *is proportional to* $O(1/\ln n)$ *for all cases studied in this paper.*

3 Proofs

Recall that we set $u_n(x) = b_n + x/b_n, x \in \mathbb{R}$ with b_n satisfying equation (1.3). Define further below

$$\overline{\Phi}(x) = 1 - \Phi(x), \quad \overline{\Phi}_n(s) = n\overline{\Phi}(u_n(s))$$

and

$$I_k := \int_y^\infty \varphi\left(\lambda + \frac{x-z}{2\lambda}\right)e^{-z}z^k \mathrm{d}z, \quad k = 0,\cdots,3.$$

The following formulas obtained by partial integration will be used in the proofs below:

$$I_0 = 2\lambda e^{-x}\overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right), \tag{3.1}$$

$$I_1 = (2\lambda x - 4\lambda^3)e^{-x}\overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right) + 4\lambda^2 e^{-x}\varphi\left(\lambda + \frac{y-x}{2\lambda}\right), \tag{3.2}$$

$$I_2 = (8\lambda^5 - 8\lambda^3 x + 8\lambda^3 + 2\lambda x^2)e^{-x}\overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right) + (-8\lambda^4 + 4\lambda^2 x + 4\lambda^2 y)e^{-x}\varphi\left(\lambda + \frac{y-x}{2\lambda}\right) \tag{3.3}$$

$$\begin{aligned} I_3 = {} & (24\lambda^5 x - 12\lambda^3 x^2 + 24\lambda^3 x + 2\lambda x^3 - 16\lambda^7 - 48\lambda^5)e^{-x}\overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda}\right) \\ & + (16\lambda^6 - 16\lambda^4 x - 8\lambda^4 y + 32\lambda^4 + 4\lambda^2 x^2 + 4\lambda^2 xy + 4\lambda^2 y^2)e^{-x}\varphi\left(\lambda + \frac{y-x}{2\lambda}\right). \end{aligned} \tag{3.4}$$

Lemma 3.1 *If (X,Y) is a bivariate normal vector with correlation $\rho\in(-1,1)$, then*

$$\begin{aligned}
&n\mathbb{P}\left(X>u_n(x),Y>u_n(y)\right)\\
=\;& \overline{\Phi}_n(y)-\int_y^\infty \Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)e^{-z}\left[1+\left(1-\frac{z^2}{2}\right)\frac{1}{b_n^2}+\left(\frac{z^4}{8}-\frac{z^2}{2}-2\right)\frac{1}{b_n^4}\right]\mathrm{d}z+O(b_n^{-6}) \quad (3.5)\\
=\;& \overline{\Phi}_n(y)-\int_y^\infty \Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)e^{-z}\left[1+\left(1-\frac{z^2}{2}\right)\frac{1}{b_n^2}\right]\mathrm{d}z+O(b_n^{-4}). \quad (3.6)
\end{aligned}$$

Proof of Lemma 3.1 First note that

$$\left|e^{-x}-\left(1-x+\frac{x^2}{2}\right)\right|<\frac{x^3}{6}+\frac{x^4}{24}$$

for $x>0$, which implies

$$\int_{u_n(y)}^\infty \Phi\left(\frac{u_n(x)-\rho z}{\sqrt{1-\rho^2}}\right)\varphi(z)\mathrm{d}z \;=\; b_n^{-1}\varphi(b_n)\left[\int_y^\infty \Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}+\frac{z^4}{8b_n^4}\right)\mathrm{d}z+O(b_n^{-6})\right]$$

for large n. Hence

$$\begin{aligned}
&\mathbb{P}\left(X>u_n(x),Y>u_n(y)\right)\\
&= \int_{u_n(y)}^\infty \overline{\Phi}\left(\frac{u_n(x)-\rho z}{\sqrt{1-\rho^2}}\right)\varphi(z)\mathrm{d}z\\
&= \overline{\Phi}(u_n(y))-b_n^{-1}\varphi(b_n)\int_y^\infty \Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}+\frac{z^4}{8b_n^4}\right)\mathrm{d}z+O(b_n^{-7}\varphi(b_n))\\
&= \overline{\Phi}(u_n(y))-b_n^{-1}\varphi(b_n)\int_y^\infty \Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)e^{-z}\left(1-\frac{z^2}{2b_n^2}\right)\mathrm{d}z+O(b_n^{-5}\varphi(b_n)).
\end{aligned}$$

According to the definition of b_n we have

$$n^{-1}=\overline{\Phi}(b_n)=b_n^{-1}\varphi(b_n)(1-b_n^{-2}+3b_n^{-4}+O(b_n^{-6}))$$

for large n thus the claim follows. □

For notational simplicity hereafter we set

$$A_{1n}=b_n^2\left(\lambda-\lambda_n\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}}\right),\qquad A_{2n}=\frac{1}{2}b_n^2\left(\frac{1}{\lambda}-\frac{1}{\lambda_n}\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}}\right)$$

and

$$A_{3n}=\lambda_n\left(1-\frac{\lambda_n^2}{b_n^2}\right)^{-\frac{1}{2}}.$$

Lemma 3.2 *Under the conditions of Theorem 2.1, we have*

$$\lim_{n\to\infty} b_n^2\int_y^\infty\left(\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)-\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}\mathrm{d}z=\kappa_1(\alpha,\lambda,x,y,),$$

where

$$\kappa_1(\alpha,\lambda,x,y)=(2\lambda^4-2\lambda^2x)e^{-x}\overline{\Phi}\left(\lambda+\frac{y-x}{2\lambda}\right)+(2\alpha-3\lambda^3)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right).$$

Proof of Lemma 3.2 Using the assumption (1.6) we have

$$\lim_{n\to\infty}A_{1n} \;=\; \lim_{n\to\infty}b_n^2\left(\lambda-\lambda_n-\frac{1}{2b_n^2}\lambda_n^3+O(b_n^{-4})\right)=\alpha-\frac{1}{2}\lambda^3,$$

$$\begin{aligned}
\lim_{n\to\infty} A_{2n} &= \lim_{n\to\infty} \frac{1}{2} b_n^2 \left(\frac{\lambda_n - \lambda}{\lambda \lambda_n} - \frac{1}{2b_n^2} \lambda_n + O(b_n^{-4}) \right) = -\frac{1}{2} \alpha \lambda^{-2} - \frac{1}{4} \lambda, \\
\lim_{n\to\infty} A_{3n} &= \lim_{n\to\infty} \lambda_n (1 + O(b_n^{-2})) = \lambda.
\end{aligned}$$

Hence since

$$\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} = \left(\lambda_n + \frac{x-z}{2\lambda_n} + \frac{\lambda_n z}{b_n^2} \right) \left(1 - \frac{\lambda_n^2}{b_n^2} \right)^{-\frac{1}{2}} \to \lambda + \frac{x-z}{2\lambda}, \quad n \to \infty, \tag{3.7}$$

then we obtain

$$\begin{aligned}
& b_n^2 \int_y^\infty \left(\lambda + \frac{x-z}{2\lambda} - \frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) \varphi\left(\lambda + \frac{x-z}{2\lambda} \right) e^{-z} \mathrm{d}z \\
=\ & (A_{1n} + A_{2n} x) I_0 - (A_{2n} + A_{3n}) I_1 \\
\to\ & \left(\alpha - \frac{1}{2}\lambda^3 - \frac{1}{2}\alpha\lambda^{-2} x - \frac{1}{4}\lambda x \right) I_0 - \left(\frac{3}{4}\lambda - \frac{1}{2}\alpha\lambda^{-2} \right) I_1 \\
=\ & (2\lambda^4 - 2\lambda^2 x) e^{-x} \overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda} \right) + (2\alpha - 3\lambda^3) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda} \right)
\end{aligned} \tag{3.8}$$

as $n \to \infty$. Using Taylor's expansion with Lagrange remainder term, we have

$$\begin{aligned}
\Phi\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) &= \Phi\left(\lambda + \frac{x-z}{2\lambda} \right) + \varphi\left(\lambda + \frac{x-z}{2\lambda} \right) v_n(x, y, \lambda) \\
&\quad - \frac{1}{2} \xi_n(x,z) \varphi(\xi_n(x,z)) v_n^2(x, z, \lambda)
\end{aligned} \tag{3.9}$$

with $v_n(x,z,\lambda) := \left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}} - \lambda - \frac{x-z}{2\lambda} \right)$ and some $\xi_n(x,z)$ between $\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}$ and $\lambda + \frac{x-z}{2\lambda}$. Moreover, by arguments similar to (3.8), combining with (3.7) we have

$$\begin{aligned}
& b_n^2 \int_y^\infty v_n^2(x,z,\lambda) \xi_n(x,z) \varphi(\xi_n(x,z)) e^{-z} \mathrm{d}z \\
=\ & b_n^{-2} \int_y^\infty (A_{1n} + A_{2n} x - (A_{2n} + A_{3n}) z)^2 \, \xi_n(x,z) \varphi(\xi_n(x,z)) e^{-z} \mathrm{d}z \\
=\ & O(b_n^{-2}),
\end{aligned}$$

which together with (3.8) and (3.9) established the proof. □

Lemma 3.3 *Under the conditions of Theorem 2.2, we have*

$$\lim_{n\to\infty} b_n^2 \left[b_n^2 \int_y^\infty \left(\Phi\left(\lambda + \frac{x-z}{2\lambda} \right) - \Phi\left(\frac{u_n(x) - \rho_n u_n(z)}{\sqrt{1-\rho_n^2}} \right) \right) e^{-z} \mathrm{d}z - \kappa_1(\alpha, \lambda, x, y) \right] = \tau_1(\alpha, \beta, \lambda, x, y),$$

where $\kappa_1(\alpha,\lambda,x,y)$ is defined in Lemma 3.2 and

$$\begin{aligned}
\tau_1(\alpha,\beta,\lambda,x,y) &= (2\lambda^8 + 8\lambda^6 - 4\lambda^6 x + 2\lambda^4 x^2 - 4\lambda^4 x - 8\alpha\lambda^3 + 4\alpha\lambda x) e^{-x} \overline{\Phi}\left(\lambda + \frac{y-x}{2\lambda} \right) \\
&+ \Big(2\beta + 9\alpha\lambda^2 - \frac{23}{4}\lambda^5 - \frac{3}{8}\lambda^3 xy - \alpha\lambda^2 x + \frac{3}{4}\alpha y^2 - \frac{1}{4}\alpha^2\lambda^{-3} y^2 - \frac{1}{4}\alpha^2\lambda^{-3} x^2 - \alpha\lambda^2 y - \frac{1}{4}\alpha x^2 - \frac{7}{4}\lambda^7 \\
&+ \frac{7}{2}\lambda^5 x - \frac{1}{16}\lambda^3 x^2 - \alpha\lambda^4 + \alpha^2\lambda + \frac{3}{2}\lambda^5 y - \frac{9}{16}\lambda^3 y^2 - \frac{1}{2}\alpha xy + \frac{1}{2}\alpha^2\lambda^{-3} xy \Big) e^{-x} \varphi\left(\lambda + \frac{y-x}{2\lambda} \right).
\end{aligned}$$

Proof of Lemma 3.3 By assumption (2.2) we have

$$\lim_{n\to\infty} b_n^2 \left(A_{1n} - \alpha + \frac{1}{2}\lambda^3 \right) = \beta + \frac{3}{2}\alpha\lambda^2 - \frac{3}{8}\lambda^5,$$

$$\lim_{n\to\infty} b_n^2\left(A_{2n}+\frac{1}{2}\alpha\lambda^{-2}+\frac{1}{4}\lambda\right) = -\frac{1}{2}\beta\lambda^{-2}-\frac{1}{2}\alpha^2\lambda^{-3}+\frac{1}{4}\alpha-\frac{3}{16}\lambda^3$$

$$\lim_{n\to\infty} b_n^2(A_{3n}-\lambda) = \lim_{n\to\infty} b_n^2\left[\lambda_n+\frac{\lambda^3}{2b_n^2}+O(b_n^{-4})-\lambda\right]=-\alpha+\frac{1}{2}\lambda^3.$$

Hence, using further (3.1), (3.2) and (3.7) we obtain

$$\begin{aligned}
& b_n^2\left[b_n^2\int_y^\infty\left(\lambda+\frac{x-z}{2\lambda}-\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\mathrm{d}z-\kappa_1(x,y,\lambda,\alpha)\right]\\
= & \; b_n^2\left[A_{1n}+A_{2n}x-\left(\alpha-\frac{1}{2}\lambda^3-\frac{1}{2}\alpha\lambda^{-2}x-\frac{1}{4}\lambda x\right)\right]I_0\\
& -b_n^2\left[A_{2n}+A_{3n}-\left(\frac{3}{4}\lambda-\frac{1}{2}\alpha\lambda^{-2}\right)\right]I_1\\
\to & \left(\frac{1}{2}\lambda^6+2\alpha\lambda x-\lambda^4x-2\alpha^2\right)e^{-x}\left(1-\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)\right)\\
& +\left(2\beta+2\alpha^2\lambda^{-1}+3\alpha\lambda^2-\frac{5}{4}\lambda^5\right)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right) \qquad (3.10)
\end{aligned}$$

as $n\to\infty$. Consequently, using (3.3), (3.4), (3.7) and combining with the limits of $A_{in}, i=1,2,3$ we have

$$\begin{aligned}
& \frac{1}{2}b_n^4\int_y^\infty v_n^2(x,z,\lambda)\left(\lambda+\frac{x-z}{2\lambda}\right)\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\mathrm{d}z\\
= & \;(A_{1n}+A_{2n}x)^2\left(\frac{\lambda}{2}+\frac{x}{4\lambda}\right)I_0\\
& -\left[(A_{1n}+A_{2n}x)^2\frac{1}{4\lambda}+(A_{1n}+A_{2n}x)(A_{2n}+A_{3n})\left(\lambda+\frac{x}{2\lambda}\right)\right]I_1\\
& +\left[(A_{1n}+A_{2n}x)(A_{2n}+A_{3n})\frac{1}{2\lambda}+(A_{2n}+A_{3n})^2\left(\frac{\lambda}{2}+\frac{x}{4\lambda}\right)\right]I_2\\
& -(A_{2n}+A_{3n})^2\frac{1}{4\lambda}I_3\\
\to & \left(2\lambda^8+\frac{15}{2}\lambda^6-8\alpha\lambda^3+2\alpha^2-3\lambda^4x-4\lambda^6x+2\alpha\lambda x+2\lambda^4x^2\right)e^{-x}\overline{\Phi}\left(\lambda+\frac{y-x}{2\lambda}\right)\\
& +\left(-\frac{3}{8}\lambda^3xy-\alpha\lambda^2x+\frac{3}{4}\alpha y^2-\frac{\alpha^2y^2}{4\lambda^3}-\frac{\alpha^2x^2}{4\lambda^3}+\frac{\alpha^2xy}{2\lambda^3}-\alpha\lambda^2y-\frac{1}{4}\alpha x^2-\frac{7}{4}\lambda^7+\frac{7}{2}\lambda^5x-\frac{1}{16}\lambda^3x^2-\frac{9}{2}\lambda^5\right.\\
& \left.-\alpha\lambda^4+\alpha^2\lambda+\frac{3}{2}\lambda^5y-\frac{9}{16}\lambda^3y^2+6\alpha\lambda^2-\frac{2\alpha^2}{\lambda}-\frac{1}{2}\alpha xy\right)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right), n\to\infty \qquad (3.11)
\end{aligned}$$

where $v_n(x,z,\lambda)$ is as in the previous lemma. Using Taylor expansion with Lagrange remainder term, we have

$$\begin{aligned}
\Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right) = & \;\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)+\varphi\left(\lambda+\frac{x-z}{2\lambda}\right)v_n(x,y,\lambda)\Big[1-\frac{1}{2}\left(\lambda+\frac{x-z}{2\lambda}\right)v_n(x,y,\lambda)\Big]\\
& +\frac{1}{6}\varphi(\xi_n(x,z))(\xi_n^2(x,z)-1)v_n^3(x,y,\lambda) \qquad (3.12)
\end{aligned}$$

for some $\xi_n(x,z)$ between $\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}$ and $\lambda+\frac{x-z}{2\lambda}$. Since further

$$b_n^4\int_y^\infty\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}-\lambda-\frac{x-z}{2\lambda}\right)^3(\xi_n^2(x,z)-1)\varphi(\xi_n(x,z))e^{-z}\mathrm{d}z=O(b_n^{-2}) \qquad (3.13)$$

the desired result follows by (3.10)-(3.13). ☐

Proof of Theorem 2.1 Define

$$h_n(x,y,\lambda)=n\ln F_{\rho_n}(u_n(x),u_n(y))+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}.$$

In view of (1.4), (3.6) and Lemma 3.2 we have

$$
\begin{aligned}
b_n^2 h_n(x,y,\lambda) &= b_n^2 \Big[-n(1-F_{\rho_n}(u_n(x),u_n(y))) - \frac{n}{2}(1-F_{\rho_n}(u_n(x),u_n(y)))^2(1+o(1)) \\
&\quad +\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y} + \Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\Big] \\
&\to \left(\frac{1}{2}x^2+x\right)e^{-x} + \kappa_1(x,y,\lambda,\alpha) - \int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\left(1-\frac{z^2}{2}\right)\mathrm{d}z
\end{aligned}
$$

as $n\to\infty$. By partial integration we have

$$
\begin{aligned}
&\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\left(1-\frac{z^2}{2}\right)\mathrm{d}z \\
&= -\left(\frac{1}{2}y^2+y\right)e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right) + \left(2\lambda^4-2\lambda^2x+x+\frac{1}{2}x^2\right)e^{-x}\overline{\Phi}\left(\lambda+\frac{y-x}{2\lambda}\right) \\
&\quad +(-2\lambda^3+\lambda x+\lambda y+2\lambda)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right).
\end{aligned}
$$

Further as $n\to\infty$

$$
h_n(x,y,\lambda)\to 0 \quad \text{and} \quad \left|\sum_{i=2}^\infty \frac{h_n^{i-2}(x,y,\lambda)}{i!}\right| < \exp(h_n(x,y,\lambda)) \to 1. \tag{3.14}
$$

Hence,

$$
\begin{aligned}
b_n^2\Big(F_{\rho_n}^n(u_n(x),u_n(y)) - H_\lambda(x,y)\Big) &= b_n^2\Big(\exp(h_n(x,y,\lambda))-1\Big)H_\lambda(x,y) \\
&= b_n^2 h_n(x,y,\lambda)\left(1+h_n(x,y,\lambda)\sum_{i=2}^\infty \frac{h_n^{i-2}(x,y,\lambda)}{i!}\right)H_\lambda(x,y) \\
&\to \kappa(\alpha,\lambda,x,y)H_\lambda(x,y)
\end{aligned}
$$

as $n\to\infty$, where

$$
\begin{aligned}
\kappa(\alpha,\lambda,x,y) &= 2^{-1}(x^2+2x)e^{-x}\Phi\left(\lambda+\frac{y-x}{2\lambda}\right) + 2^{-1}(y^2+2y)e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right) \\
&\quad +(2\alpha-\lambda^3-\lambda x-\lambda y-2\lambda)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right).
\end{aligned}
$$

The proof is complete. □

Proof of Theorem 2.2 By arguments similar to that of Lemma 3.2, we have

$$
\begin{aligned}
&b_n^2\int_y^\infty \left(\Phi\left(\lambda+\frac{x-z}{2\lambda}\right) - \Phi\left(\frac{u_n(x)-\rho_n u_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}\left(1-\frac{z^2}{2}\right)\mathrm{d}z \\
&= (A_{1n}+A_{2n}x)\left(I_0-\frac{1}{2}I_2\right) - (A_{2n}+A_{3n})\left(I_1-\frac{1}{2}I_3\right) + O(b_n^{-2}) \\
&\to \left(\alpha-\frac{\lambda^3}{2}-\frac{\alpha x}{2\lambda^2}-\frac{\lambda x}{4}\right)\left(I_0-\frac{1}{2}I_2\right) - \left(\frac{3}{4}\lambda-\frac{\alpha}{2\lambda^2}\right)\left(I_1-\frac{1}{2}I_3\right) \\
&= (2\lambda^4-4\alpha\lambda x-2\lambda^2x+8\lambda^6x-5\lambda^4x^2+10\lambda^4x+\lambda^2x^3+8\alpha\lambda^3-4\lambda^8-16\lambda^6)e^{-x}\overline{\Phi}\left(\lambda+\frac{y-x}{2\lambda}\right) \\
&\quad +\left(2\alpha+4\lambda^7+12\lambda^5-3\lambda^3-6\lambda^5x+2\lambda^3x^2-\alpha y^2+2\lambda^3xy-2\lambda^5y+\frac{3}{2}\lambda^3y^2-8\alpha\lambda^2\right)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right) \\
&= \tau_2(\alpha,\lambda,x,y)
\end{aligned}
\tag{3.15}
$$

as $n\to\infty$. By partial integration we get

$$
\int_y^\infty \Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\left(\frac{z^4}{8}-\frac{z^2}{2}-2\right)\mathrm{d}z
$$

$$
\begin{aligned}
&= 8^{-1}(y^4+4y^3+8y^2+16y)e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)-\frac{1}{16\lambda}\int_y^\infty \varphi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}(z^4+4z^3+8z^2+16z)\mathrm{d}z\\
&= 8^{-1}(y^4+4y^3+8y^2+16y)e^{-y}\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)\\
&\quad+\Big(4\lambda^6x-3\lambda^4x^2+\lambda^2x^3-2\lambda^8-8\lambda^6-\frac{1}{8}x^4-2\lambda^2x-\frac{1}{2}x^3+2\lambda^4+6\lambda^4x-x^2-2x\Big)e^{-x}\overline{\Phi}\left(\lambda+\frac{y-x}{2\lambda}\right)\\
&\quad+\Big(2\lambda^7-\frac{1}{4}\lambda x^3-\lambda^5y+\frac{1}{2}\lambda^3y^2+\lambda^3xy-3\lambda^5x+\frac{3}{2}\lambda^3x^2-\frac{1}{4}\lambda y^3-\frac{1}{4}\lambda y^2x-\frac{1}{4}\lambda yx^2-\lambda^3y-\lambda y^2-\lambda xy-\lambda^3x\\
&\quad-\lambda x^2-4\lambda^3+6\lambda^5-2\lambda x-2\lambda y-4\lambda\Big)e^{-x}\varphi\left(\lambda+\frac{y-x}{2\lambda}\right)\\
&= \tau_3(\lambda,x,y). \qquad (3.16)
\end{aligned}
$$

Hence, using (1.4), (3.5) and Lemma 3.3 we have

$$
\begin{aligned}
& b_n^2\Big[b_n^2h_n(\lambda,x,y)-\kappa(\alpha,\lambda,x,y)\Big]\\
&= b_n^2\Bigg(b_n^2\Bigg(-n(1-F_{\rho_n}(u_n(x),u_n(y)))-\frac{n}{2}(1-F_{\rho_n}(u_n(x),u_n(y)))^2(1+o(1))+\Phi\left(\lambda+\frac{x-y}{2\lambda}\right)e^{-y}\\
&\quad+\Phi\left(\lambda+\frac{y-x}{2\lambda}\right)e^{-x}\Bigg)-\kappa(\alpha,\lambda,x,y)\Bigg)\\
&= b_n^2\Big[b_n^2\left[-\overline{\Phi}_n(x)+e^{-x}\right]-\left(\frac{x^2}{2}+x\right)e^{-x}\Big]\\
&\quad+b_n^2\left[b_n^2\int_y^\infty\left(\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)-\Phi\left(\frac{u_n(x)-\rho u_n(z)}{\sqrt{1-\rho^2}}\right)\right)e^{-z}\mathrm{d}z-\kappa_1(x,y,\lambda,\alpha)\right]\\
&\quad+b_n^2\int_y^\infty\left(\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)-\Phi\left(\frac{u_n(x)-\rho_nu_n(z)}{\sqrt{1-\rho_n^2}}\right)\right)e^{-z}\left(1-\frac{z^2}{2}\right)\mathrm{d}z\\
&\quad-\int_y^\infty\Phi\left(\lambda+\frac{x-z}{2\lambda}\right)e^{-z}\left(\frac{z^4}{8}-\frac{z^2}{2}-2\right)\mathrm{d}z+O(b_n^{-2})-\frac{1}{2}b_n^4n(1-F_{\rho_n}(u_n(x),u_n(y)))^2(1+o(1))\\
&\to -8^{-1}(x^4+4x^3+8x^2+16x)e^{-x}+\tau_1(\alpha,\beta,\lambda,x,y)+\tau_2(\alpha,\lambda,x,y)-\tau_3(\lambda,x,y)\\
&= \tau(\alpha,\beta,\lambda,x,y),\quad n\to\infty, \qquad (3.17)
\end{aligned}
$$

where $\tau_i, i\le 3$ are given by Lemma 3.3, (3.15) and (3.16), respectively. Hence, (3.14) entails

$$
\begin{aligned}
& b_n^2\Big[b_n^2\Big(F_{\rho_n}^n(u_n(x),u_n(y))-H_\lambda(x,y)\Big)-\kappa(\alpha,\lambda,x,y)H_\lambda(x,y)\Big]\\
&= b_n^2\Big[b_n^2\Big(\exp(h_n(x,y,\lambda))-1\Big)-\kappa(\alpha,\lambda,x,y)\Big]H_\lambda(x,y)\\
&= \left[b_n^2\Big[b_n^2h_n(x,y,\lambda)-\kappa(\alpha,\lambda,x,y)\Big]+b_n^4h_n^2(x,y,\lambda)\left(\frac{1}{2}+h_n(x,y,\lambda)\sum_{i=3}^\infty\frac{h_n^{i-3}(x,y,\lambda)}{i!}\right)\right]H_\lambda(x,y)\\
&\to \left(\tau(\alpha,\beta,\lambda,x,y)+\frac{1}{2}\kappa^2(\alpha,\lambda,x,y)\right)H_\lambda(x,y)
\end{aligned}
$$

as $n\to\infty$ establishing the proof. □

Proof of Theorem 2.3 (*i*) For the case of $\rho_n\in[-1,0]$, we first consider that the bivariate Gaussian are either complete independent ($\rho_n=0$) or complete negative dependent ($\rho_n=-1$). Both imply $\lambda=\infty$. Let $\hat{h}_n(x,y)=n\ln F_0(u_n(x),u_n(y))+e^{-x}+e^{-y}$ and $\tilde{h}_n(x,y)=n\ln F_{-1}(u_n(x),u_n(y))+e^{-x}+e^{-y}$. In view of Lemma 2.1 in [19]

$$
\begin{aligned}
b_n^2\hat{h}_n(x,y) &= b_n^2[-\overline{\Phi}_n(x)+e^{-x}]+b_n^2[-\overline{\Phi}_n(y)+e^{-y}]\\
&\quad+\frac{b_n^2}{n}\overline{\Phi}_n(x)\overline{\Phi}_n(y)-\frac{1}{2}b_n^2n(1-F_0(u_n(x),u_n(y)))^2(1+o(1))\\
&\to s(x)+s(y)
\end{aligned}
$$

and

$$\begin{aligned}
b_n^2\tilde{h}_n(x,y) &= b_n^2[-\overline{\Phi}_n(x)+e^{-x}]+b_n^2[-\overline{\Phi}_n(y)+e^{-y}] \\
&\quad +b_n^2 n\mathbb{P}\left(u_n(x)<X<-u_n(y)\right)-\frac{1}{2}b_n^2 n(1-F_{-1}(u_n(x),u_n(y)))^2(1+o(1)) \\
&\to s(x)+s(y)
\end{aligned}$$

as $n\to\infty$, where X is a standard normal variable. By Lemma 2.1 in [19] once again we have

$$\lim_{n\to\infty} b_n^2\left[b_n^2\hat{h}_n(x,y)-(s(x)+s(y))\right]=t(x)+t(y)$$

and

$$\lim_{n\to\infty} b_n^2\left[b_n^2\tilde{h}_n(x,y)-(s(x)+s(y))\right]=t(x)+t(y).$$

Consequently,

$$\lim_{n\to\infty} b_n^2\left[b_n^2\Delta(F_{\rho_n}^n,H_\infty;x,y)-(s(x)+s(y))H_\infty(x,y)\right]=\left(t(x)+t(y)+\frac{1}{2}(s(x)+s(y))^2\right)H_\infty(x,y) \tag{3.18}$$

holds with $\rho_n=-1$ and $\rho_n=0$ (for all n large), respectively. Consequently, by using Slepian's Lemma and (3.18), the claimed result (2.4) holds for $\rho_n\in[-1,0]$.

(ii) For the complete positive dependence case, without loss of generality, assume that $x<y$. Hence

$$F_1(u_n(x),u_n(y))=\Phi(u_n(x)),\quad H_0(x,y)=\Lambda(x)$$

(1.4) follows and thus the proof is complete. □

Proof of Corollary 2.1 *(i)* Obviously, $n(1-\Phi(b_n))=1$ implies

$$b_n\sim(2\ln n)^{1/2},\quad e^{-\frac{b_n^2}{2}}\sim\sqrt{2\pi}\frac{b_n}{n}$$

for large n. For the case of $\lambda=\infty$, according to Berman's inequality (see e.g., [21]), with some positive constant which may change from line to line $\mathbb{C}$ and all n large we have

$$\begin{aligned}
&b_n^2\left|F_{\rho_n}^n(u_n(x),u_n(y))-F_0^n(u_n(x),u_n(y))\right| \\
&\le \mathbb{C}n(\ln n)\rho_n\exp\left(-\frac{u_n^2(x)+u_n^2(y)}{2(1+\rho_n)}\right) \\
&\le \mathbb{C}n(\ln n)\left(\exp\left(-\frac{u_n^2(x)}{(1+\rho_n)}\right)+\exp\left(-\frac{u_n^2(y)}{(1+\rho_n)}\right)\right) \\
&\le \mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big)n^{-\frac{1-\rho_n}{1+\rho_n}}(\ln n)^{1+\frac{1}{1+\rho_n}} \\
&\le \mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big)\exp\left(-\frac{1}{2}((1-\rho_n)\ln n-(2+\rho_n)\ln\ln n)\right) \\
&\to \mathbb{C}e^{-\gamma}\Big(\exp(2|x|)+\exp(2|y|)\Big),\quad n\to\infty
\end{aligned}$$

since by the assumption $\lim_{n\to\infty}\frac{1}{2}((1-\rho_n)\ln n-(2+\rho_n)\ln\ln n)=\gamma$. By Theorem 2.3 we have

$$\begin{aligned}
\limsup_{n\to\infty} b_n^2\Delta(F_{\rho_n}^n,H_\infty;x,y) &\le \lim_{n\to\infty} b_n^2\Delta(F_0^n,H_\infty;x,y)+\lim_{n\to\infty} b_n^2\left|F_{\rho_n}^n(u_n(x),u_n(y))-F_0^n(u_n(x),u_n(y))\right| \\
&\le (|s(x)|+|s(y)|)H_\infty(x,y)+\mathbb{C}e^{-\gamma}\Big(\exp(2|x|)+\exp(2|y|)\Big).
\end{aligned}$$

(ii) The condition $\lim_{n\to\infty}(1-\rho_n)(\ln n)^3=\tau^2\in[0,\infty)$ implies $\lambda=0$ and $\lim_{n\to\infty}\rho_n=1$. By using Berman's inequality in [18] we have

$$b_n^2\left|F_{\rho_n}^n(u_n(x),u_n(y))-F_1^n(u_n(x),u_n(y))\right| \le \mathbb{C}n(\ln n)\left(\frac{\pi}{2}-\arcsin(\rho_n)\right)\exp\left(-\frac{u_n^2(x)+u_n^2(y)}{4}\right)$$

$$
\begin{aligned}
&\leq \mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big)\Big(\frac{\pi}{2}-\arcsin(\rho_n)\Big)(\ln n)^{\frac{3}{2}}\\
&\leq \mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big)(1-\rho_n)^{\frac{1}{2}}(\ln n)^{\frac{3}{2}}\\
&\to \tau\mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big), \quad n\to\infty
\end{aligned}
$$

since $\lim_{n\to\infty}(1-\rho_n)(\ln n)^3=\tau^2$ which also implies $\lim_{n\to\infty}\frac{\frac{\pi}{2}-\arcsin(\rho_n)}{(1-\rho_n)^{1/2}}=\sqrt{2}$. Hence Theorem 2.3 yields

$$
\begin{aligned}
\limsup_{n\to\infty} b_n^2\Delta(F_{\rho_n}^n,H_0;x,y) &\leq \lim_{n\to\infty} b_n^2\Delta(F_1^n,H_0;x,y)+\lim_{n\to\infty} b_n^2\Big|F_{\rho_n}^n(u_n(x),u_n(y))-F_1^n(u_n(x),u_n(y))\Big|\\
&= |s(\min(x,y))|H_0(x,y)+\tau\mathbb{C}\Big(\exp(2|x|)+\exp(2|y|)\Big)
\end{aligned}
$$

establishing the claim. □

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