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Optimally eating a stochastic cake: a recursive utility approach☆

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Abstract

In this short paper, uncertainties on resource stock and on technical progress are introduced into an intertemporal equilibrium model of optimal extraction of a non-renewable resource. The representative consumer maximizes a recursive utility function which disentangles between intertemporal elasticity of substitution and risk aversion. A closed-form solution is derived for both the optimal extraction and price paths. The value of the intertemporal elasticity of substitution relative to unity is then crucial in understanding extraction. Moreover, this model leads to a non-renewable resource price following a geometric Brownian motion.

JEL classification: Q30; D81; Q11

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1. Introduction

How much of a non-renewable resource should we consume today if there exists a lack of precise knowledge about its available stock? Since 1970s, this general problem of optimal use has received considerable attention in the literature (see Gilbert, 1978; Kemp, 1976; Loury, 1978). Only recently have models been developed that explore the effects of uncertainty by allowing information to arrive over time: in 1980, Pindyck proposed a partial

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equilibrium model in which firms continuously receive information and adapt their decisions while in previous works there was uncertainty about current reserves not future demand or reserves. In a model such as Pindyck's one, firms are supposed to be risk-neutral and finally, uncertainty has only a second-order effect on the optimal utilization of the resource.

Since then, very few models have considered the optimal extraction of a non-renewable resource under uncertainty in a general equilibrium framework: actually, only Gaudet and Khadr (1991) and Beltratti (1996) study such a problem. They stress the role of preference parameters in determining how uncertainty affects the optimal extraction of a non-renewable resource. They use standard, additive utility functions, however, so that they cannot disentangle effects of risk aversion from intertemporal substitution. It is intuitively obvious that risk aversion and intertemporal substitution should play important, but possibly distinct roles in optimal resource extraction. Such an intuition is confirmed by numerical simulations performed by Knapp and Olson (1996) who study the effect of each preference parameter in the case of a renewable resource.

This paper therefore analyzes the optimal extraction of a non-renewable resource in a stochastic general equilibrium framework using a recursive (generalized isoelastic: GIE) utility functional that permits us to distinguish between the coefficient of risk aversion and the elasticity of intertemporal substitution. The model is simple and allows a closed-form solution. We then show that the value of the intertemporal elasticity of substitution relative to unity is crucial in understanding the optimal extraction path. In particular, the direction of the effect of uncertainty on the optimal use of the resource depends crucially upon whether the intertemporal elasticity of substitution is greater or less than one. The coefficient of relative risk aversion helps determines the magnitude, but not the direction, of this effect.

Another result of our simple model is that the price of the resource (in terms of the price of the consumption good) follows a geometric Brownian motion. Even though this comes out from a model based on strong assumptions, this is an interesting result since it provides some rationale for a common assumption in the literature (see for instance Pindyck, 1981).

Since this paper focuses on how risk aversion and intertemporal substitution affect the response of resource extraction to uncertainty, we search for an analytical solution. This is why, except uncertainty modelling and the utility function, the model remains very simple. In particular, we allow for neither extraction costs (uncertainty might then also affect extraction through an irreversibility effect) nor capital accumulation. Section 2 describes first the representative consumer's recursive preferences, then the sources of uncertainty impinging upon him. A closed-form solution is derived in the third section. Results for the optimal extraction and price paths are analyzed in the fourth one. The specific role of the intertemporal elasticity of substitution is then precisely explained. Section 5 concludes.

2. The setup of the model

2.1. The recursive utility function

The model is set in continuous time. The representative agent maximizes a recursive utility function which disentangles between the risk aversion and the intertemporal elasticity of substitution. Kreps and Porteus (1978) define a recursive utility function for which utility at period t depends on current consumption as well as on the certainty equivalent of future utility. Epstein (1988), and Epstein and Zin (1989, 1991) propose some specifications in discrete time, while Duffie and Epstein (1992), and Svensson (1989) focus on specifications in continuous time. In this model, we use the same formulation of preferences[2] as Svensson (1989):

$$U(t)=\left[\frac{\varepsilon}{\varepsilon-1}c(t)^{(\varepsilon-1)/\varepsilon}+\mathrm{e}^{-\delta \mathrm{d}t}[E(U(t+\mathrm{d}t)^{(\varepsilon(1-\gamma))/(\varepsilon-1)})]^{((\varepsilon-1)/\varepsilon)(1/(1-\gamma))}\right] \quad (2.1)$$

where ε is the intertemporal elasticity of substitution, γ the relative risk aversion coefficient and δ is the time preference rate. Note that $1/\varepsilon$ may also be understood as the aversion to fluctuations. Moreover, in the special case in which the risk aversion is the inverse of the intertemporal elasticity of substitution ($\gamma = 1/\varepsilon$), this utility function reduces to the standard time-separable specification.

Recursive utility functions have now largely been used to reconsider asset pricing (Duffie and Epstein, 1992), risk sharing (Obstfeld, 1994a), growth (Smith, 1996, 1999) precautionnary savings (Weil, 1993) welfare cost of volatility (Obstfeld, 1994b; Epaulard and Pommeret, 2001a) or the equity premium puzzle (Weil, 1990; Epaulard and Pommeret, 2001b).

2.2. Technology and volatility

The consumption good is made of a non-renewable resource using a linear technology (no capital is needed);[3] per capita consumption at each time is given by (there is neither storage nor saving):

$$c(t)=\frac{A(t)R(t)}{L(t)} \quad (2.2)$$

$L(t)$ is the population size at the initial date, which we keep constant and normalized in size to unity in the rest of the paper. $R(t)$ is the level of the non-renewable resource extracted and $A(t)$ is the productivity at t. Productivity is random. We assume it follows a geometric Brownian motion:

$$\mathrm{d}A(t)=\theta A(t)\mathrm{d}t+\sigma_A A(t)\mathrm{d}z_{A_t} \quad (2.3)$$

Thus, θ is the deterministic rate of technical progress associated with the process technology converting the resource into consumption good. Random shocks perturb the productivity growth rate. $\mathrm{d}z_{A_t}$ is the increment of a standard Wiener process, z_{A_t} ($\mathrm{d}z_{A_t}=\epsilon_A(t)\sqrt{\mathrm{d}t}$, where $\epsilon_A(t)$ is a white noise). σ_A is the instantaneous standard deviation of the growth rate.

[2] After a transformation of the type $(U(t)/a)^a$ as proposed by Duffie and Epstein (1992), with $a=\varepsilon/(\varepsilon-1)$.

[3] A more realistic model would include capital in the production process and capital accumulation in the economy; Dasgupta and Heal (1974) have chosen such a framework, but as soon as uncertainty is introduced they have to impose drastic restrictions on the production function to achieve the analytical resolution of their model. As our main concern is to study analytically the effect of uncertainty on the optimal extraction of non-renewable resources, we cannot consider a more realistic production function.

$S(t)$ is the resource stock, the fluctuations of which are due to extraction ($R(t)$) and to exogenous shocks. Uncertainty is assumed to be multiplicative;[4] thus the stock of resource follows a Brownian process:

$$\mathrm{d}S(t) = -R(t)\mathrm{d}t + S(t)\sigma_S\,\mathrm{d}z_{S_t} \tag{2.4}$$

where z_{S_t} is a Wiener process. $E(\mathrm{d}z_{A_t}\,\mathrm{d}z_{S_t}) = \Omega_{AS}\,\mathrm{d}t$, where Ω_{AS} is the correlation coefficient between the processes z_{A_t} and z_{S_t}; it is the covariance per unit time between the two processes.

3. Deriving the optimal extraction path

In this section the program is solved in order to get the optimal use of the resource in terms of preferences and certainty equivalents. In the next section we will explore the comparative statics of the optimal extraction policy.

Using a recursive utility function such as (2.1), the program may be written:

$$\begin{cases} \text{Max} \quad U(t) \\ \quad = \left[\dfrac{\varepsilon}{\varepsilon-1}c(t)^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + \mathrm{e}^{-\delta \mathrm{d}t}[E(U(t+\mathrm{d}t)^{(\varepsilon/(\varepsilon-1))(1-\gamma)})]^{((\varepsilon-1)/\varepsilon)(1/(1-\gamma))}\right] \\ \text{s.c.} \\ \mathrm{d}S(t) = -R(t)\mathrm{d}t + \sigma_S S(t)\mathrm{d}z_{S_t} \\ \mathrm{d}A(t) = \theta A\,\mathrm{d}t + \sigma_A A(t)\mathrm{d}z_{A_t} \\ R(t) \geq 0,\ S(t) \geq 0 \\ A(0),\ S(0) \quad \text{given} \end{cases} \tag{3.1}$$

The Bellman equation associated with the program is:

$$V(t) = \max_{\{Rt\}}\left[\frac{\varepsilon}{\varepsilon-1}[A(t)R(t)]^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + \mathrm{e}^{-\delta \mathrm{d}t}(E_t V(t+\mathrm{d}t)^{(\varepsilon/(\varepsilon-1))(1-\gamma)})^{((\varepsilon-1)/\varepsilon)(1/(1-\gamma))}\right] \tag{3.2}$$

By analogy with the standard VNM program, an educated guess for the current value function is:

$$V(t) = D^{-(1/\varepsilon)}\frac{\varepsilon}{\varepsilon-1}[A(t)S(t)]^{(\varepsilon-1)/\varepsilon} \tag{3.3}$$

[4] It is not clear whether uncertainty on the stock of resource is multiplicative or additive. Pindyck (1980) considers an additive shock, while Beltratti (1996) uses a multiplicative one, which may be interpreted as a random rate of depreciation.

The first-order condition allows the determination of the optimal extraction of the resource $\hat{R}(t)$ and the identification of D (see Appendix A for the resolution):

$$\hat{R}(t) = DS(t) \tag{3.4}$$

$$D = \varepsilon\left[\delta - \frac{\varepsilon - 1}{\varepsilon}\underbrace{\left(\theta - \frac{1}{2}\gamma\sigma_A^2 - \frac{1}{2}\gamma\sigma_S^2 + (1-\gamma)\sigma_A\sigma_S\Omega_{AS}\right)}_{\text{CEq}}\right] \tag{3.5}$$

where CEq is the certainty equivalent of the resource "rate of return";[5]it encompasses the certainty equivalent of the rate of technical progress ($\text{CEq1} = \theta - \gamma\sigma_A^2/2$) and the certainty equivalent of the resource natural growth rate ($\text{CEq2} = -\gamma\sigma_S^2/2$) adjusted for the correlation between the two uncertainties (this last component will be neglected in comments below). One may notice that for $\gamma = 1/\varepsilon$, Eq. (3.5) reduces to the solution obtained by Beltratti (1996).

Variations in the use of the resource during time are given by:

$$\mathrm{d}\hat{R}(t) = -D\hat{R}(t)\mathrm{d}t + \hat{R}\sigma_S\,\mathrm{d}z_{S_t} \tag{3.6}$$

Thus, the resource use follows a geometric Brownian motion. The expected rate of extraction growth is:

$$E\left[\frac{\mathrm{d}\hat{R}(t)/\mathrm{d}t}{\hat{R}(t)}\right] = -D \tag{3.7}$$

The transversality condition is

$$\lim_{t\to\infty} V(t)\mathrm{e}^{-\delta t} = 0 \tag{3.8}$$

a sufficient condition for it to be satisfied is (see Appendix B):

$$\delta - \frac{\varepsilon - 1}{\varepsilon}\left[\theta - \frac{1}{2}\left(\gamma\frac{\varepsilon-1}{\varepsilon} + \frac{1}{\varepsilon^2}\right)(\sigma_A^2 + \sigma_S^2) + \frac{\varepsilon - 1}{\varepsilon}\left((1-\gamma) + \frac{1}{\varepsilon}\right)\sigma_A\sigma_S\Omega_{AS}\right] > 0 \tag{3.9}$$

The feasibility condition ($D > 0$) is verified if:

$$\delta - \frac{\varepsilon - 1}{\varepsilon}\left[\theta - \frac{1}{2}\gamma(\sigma_A^2 + \sigma_S^2) + (1-\gamma)\sigma_A\sigma_S\Omega_{AS}\right] > 0 \tag{3.10}$$

One may notice that in the case where preferences are time separable and isoelastic (here $\gamma = 1/\varepsilon$), then the transversality condition is necessary and sufficient for the feasibility condition to be satisfied (see Merton, 1990; Smith, 1996).

[5] The certainty equivalent $\hat{X}$ of X is defined as $\hat{X} = V^{-1}[E_t V(X(t+1))]$, where $V(X)$ takes the agents attitude towards risk into account; here $V(X) = X^{1-\gamma}$.

4. Analyzing the extraction and price paths

4.1. The optimal extraction path

We analyze the effect on the optimal extraction path of variations in the exogenous variables (rate of technical progress and its uncertainty, resource uncertainty), and of exogenous shocks. To get some ideas about the effect of uncertainty on the optimal extraction, we assume that the correlation between the two Wiener processes is small enough compared to the uncertainty on the technical progress and on the stock of resource to be neglected. Effects are summarized in Table 1.

As the following propositions reveal, the value of the intertemporal elasticity of substitution (ε) is crucial in understanding the extraction path.

Proposition 4.1. *An increase in the expected rate of technological change* (θ) *will decrease extraction of the resource if* $\varepsilon > 1$, *but it will increase extraction of the resource if* $\varepsilon < 1$:

$$\frac{\partial D}{\partial \theta} = \frac{1-\varepsilon}{\varepsilon}$$

If the intertemporal elasticity of substitution is high ($\varepsilon > 1$) agents are not very averse to fluctuations in their consumption stream over time. In this an increase in the expected growth rate of productivity induces them to reduce their consumption today in order to benefit from the technological progress in the future: the substitution effect is stronger than the income effect.

Proposition 4.2. *An increase in uncertainty about either productivity* (σ_A^2) *or the rate of extraction* (σ_S^2) *will increase extraction of the resource if* $\varepsilon > 1$, *but it will decrease extraction of the resource if* $\varepsilon < 1$:

$$\frac{\partial D}{\partial(\sigma_S^2 + \sigma_A^2)} = (\varepsilon - 1)\frac{\gamma}{2} \qquad (4.1)$$

Table 1
Extraction analysis

	$\varepsilon > 1$	$\varepsilon < 1$
Exogenous variables		
$\Delta\theta > 0$	$\Delta D > 0$ No effect of γ	$\Delta D < 0$ No effect of γ
$\Delta\sigma_S^2 > 0$	$\Delta D < 0$ γ strengthens the effect	$\Delta D > 0$ γ strengthens the effect
$\Delta\sigma_A^2 > 0$	$\Delta D < 0$ γ strengthens the effect	$\Delta D > 0$ γ strengthens the effect
Structural parameters		
$\Delta\gamma > 0$	$\Delta D < 0$	$\Delta D > 0$
$\Delta\varepsilon > 0$	$\Delta D > 0$ if $\delta - \theta + (1/2)\gamma\sigma_A^2 + (1/2)\gamma\sigma_S^2 - (1-\gamma)\sigma_A\sigma_S\Omega_{AS} > 0$	

As explained by Weil (1990), the higher the uncertainty, the smaller the certainty equivalent of the resource "rate of return" (noted CEq); the effect of this reduction then depends on the relative size of substitution and income effects. For a high intertemporal elasticity of substitution ($\varepsilon > 1$), the substitution effect prevails and the agent increases its current consumption. Thus, more uncertainty reduces the deterministic trend in extraction. On the opposite, less uncertainty increases the deterministic trend. Note that in the case of log intertemporal preferences, risk has no effect on resource extraction.

Using a time-separable specification for the utility function, it is not possible to distinguish the effects due to the risk aversion from those due to the intertemporal substitution: one does not know whether it is a small risk aversion or a large intertemporal elasticity of substitution that leads to an increase in the current propensity to consume when uncertainty rises.

Another way of understanding our model is to consider how positive shocks on technical progress and on the stock of resource modify the time path of extraction. As suggested by intuition, a positive shock on reserve leads to more extraction at each date. On the opposite, a positive shock on technical progress does not change the time path of extraction (as the positive shock benefits to all future generations there no reason for the extraction path to be modified).

4.2. The resource price path

The current price of the resource in terms of the consumption good price P is the ratio between the marginal utilities of these two goods:

$$P(A(t), S(t)) = \frac{P_S(A(t), S(t))}{P_c(A(t), S(t))} = \frac{\partial V(A(t), S(t))/\partial S(t)}{\partial V(A(t), S(t))/\partial c(A(t), S(t))} \tag{4.2}$$

The current price of the resource in terms of the consumption good does not depend on the stock of the resource, but only on the level of technical progress:

$$P(A(t)) = DA(t) \tag{4.3}$$

For a given level of the productivity $A(t)$, the current relative price of the resource and the current extraction both react in the same direction through D when exogenous variables or parameters vary. Moreover the current price being a linear function of a variable (the productivity) which follows a geometric Brownian motion, it follows itself a geometric Brownian motion:

$$\frac{\mathrm{d}P(A(t))}{P(A(t))} = \alpha_P\, \mathrm{d}t + \sigma_P\, \mathrm{d}z_{Pt} \tag{4.4}$$

with

$$\begin{cases} \alpha_P = \theta \\ \sigma_P = \sigma_A \\ \mathrm{d}z_{Pt} = \sigma_A\, \mathrm{d}z_{At} \end{cases} \tag{4.5}$$

If the distinction between the relative risk aversion parameter and the intertemporal elasticity of substitution is especially relevant when studying the optimal extraction of the resource, one notices here that it is of no more interest as far as the resource price path is concerned since this path does not depend on the preferences parameters at all. Neither the size of the uncertainty nor the shocks on the resource stock affect the resource price which evolves according to the same process as the productivity. Such a result may be compared with that of Young and Ryan (1996) who consider a two period model with a capital-consumption good, a risk-free bond and a resource; as is our model, technology is uncertain and the utility function is recursive. They obtain an expected return for the resource which depends on technological uncertainty. This is consistent with our findings. Nevertheless, unlike our model, the expected rate of return in their model does depend upon preferences.

It is nevertheless interesting that the model proposed in this paper leads to a non-renewable resource price following a geometric Brownian motion since this result is often assumed in the literature without theoretical foundations being suggested (see for instance Slade, 1988; Pindyck, 1981; Ekern, 1988; Brennan and Schwartz, 1985; Olsen and et Stensland, 1988; Bjerksund and Ekern, 1990; Gibson and Schwartz, 1990; Lund, 1992). Note however that such an assumption has been criticized by Lund (1993) on the basis of the existence of irreversibilities in the extraction process.

5. Conclusion

In this short paper, we have solved a model for the optimal use of a non-renewable resource under technical progress and stock uncertainties. We show that uncertainty only leads to a more conservative use of the resource if the intertemporal elasticity of substitution is less than unity, and the higher the risk aversion, the larger this effect. This confirms in a theoretical way the numerical results obtained by Knapp and Olson (1996): as far as the optimal extraction of a natural resource is concerned, each preference parameter affects the optimal extraction of a resource in a different way.

Moreover, we show that, in this simple model, the stochastic process followed by the resource price reproduces that of the technical progress. This gives a theoretical foundation to the widely used assumption of a resource price following a geometric Brownian motion. As suggested by the works of Young and Ryan (1996) and of Lund (1993), more subtle expressions for the price path may probably be obtained by introducing capital accumulation and extraction costs in our model. In such a framework, this could be done at the cost of giving up the analytical resolution.

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Appendix A

Replacing the value function by its expression in the Bellman equation, we get:

$$D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}$$
$$= \mathrm{Max}_{R(t)}[(A(t)R(t))^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + \mathrm{e}^{-\delta \mathrm{d}t}(E_t(D^{(1-\gamma)/(\varepsilon-1)}$$
$$\times (A(t+\mathrm{d}t)S(t+\mathrm{d}t))^{1-\gamma}))^{((\varepsilon-1)/\varepsilon)(1/(1-\gamma))}] \tag{A.1}$$

We define $F(t) = E_t(A(t)S(t))^{1-\gamma}$. Thus:

$$E_t(\mathrm{d}F) = E_t[(A(t+\mathrm{d}t)S(t+\mathrm{d}t))^{1-\gamma}] - (A(t)S(t))^{1-\gamma} \tag{A.2}$$

Itô's lemma gives:

$$E_t(\mathrm{d}F) = \frac{\partial F}{\partial A}E_t(\mathrm{d}A) + \frac{1}{2}\frac{\partial^2 F}{\partial A^2}E_t(\mathrm{d}A)^2 + \frac{\partial F}{\partial S}E_t(\mathrm{d}S) + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}E_t(\mathrm{d}S)^2$$
$$+\frac{\partial^2 F}{\partial S \partial A}E_t(\mathrm{d}S\,\mathrm{d}A)$$
$$= (1-\gamma)(A(t)S(t))^{1-\gamma}$$
$$\times \left[-\frac{R(t)}{S(t)} + \left(\theta - \frac{1}{2}\gamma\sigma_A^2 - \frac{1}{2}\gamma\sigma_S^2\right) + (1-\gamma)\sigma_A\sigma_S\Omega_{AS}\right]\mathrm{d}t \tag{A.3}$$

This permits us to get:

$$E_t(A(t+\mathrm{d}t)S(t+\mathrm{d}t))^{1-\gamma}$$
$$= (1-\gamma)(A(t)S(t))^{1-\gamma}\left[-\frac{R(t)}{S(t)} + \left(\theta - \frac{1}{2}\gamma\sigma_A^2 - \frac{1}{2}\gamma\sigma_S^2\right) + (1-\gamma)\sigma_A\sigma_S\Omega_{AS}\right]\mathrm{d}t$$
$$+ (A(t)S(t))^{1-\gamma} \tag{A.4}$$

Setting $\mathrm{CEq} = (\theta - (1/2)\gamma\sigma_A^2) - (1/2)\gamma\sigma_S^2 + (1-\gamma)\sigma_A\sigma_S\Omega_{AS}$, the Bellman equation becomes:

$$D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}$$
$$= \mathrm{Max}_{R(t)}\left[(A(t)R(t))^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + \mathrm{e}^{-\delta \mathrm{d}t}D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}\right.$$
$$\left.\times \left(1-\gamma\left(-\frac{R(t)}{S(t)} + \mathrm{CEq}\right)\mathrm{d}t + 1\right)^{(\varepsilon-1)/(\varepsilon(1-\gamma))}\right] \tag{A.5}$$

Using a limited expansion one gets:

$$D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}$$
$$= \mathrm{Max}_{R(t)}\left[(A(t)R(t))^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + 1 - \delta\,\mathrm{d}t D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}\right.$$
$$\left.\times \left(1 + \frac{\varepsilon-1}{\varepsilon}\left(-\frac{R(t)}{S(t)} + \mathrm{CEq}\right)\mathrm{d}t\right)\right] \tag{A.6}$$

Invoking the rules of Itô calculus, powers of dt greater than one disappear:

$$\begin{aligned}
&D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon} \\
&\quad = \mathrm{Max}_{R(t)}\Bigg[(A(t)R(t))^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon} \\
&\qquad \times \left(1+\frac{\varepsilon-1}{\varepsilon}\left(-\frac{R(t)}{S(t)}+\mathrm{CEq}\right)\mathrm{d}t\right) - \delta D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon}\mathrm{d}t\Bigg]
\end{aligned} \tag{A.7}$$

The maximization with respect to $R(t)$ gives Eq. (3.4). Replacing $R(t)$ by its optimal value, the Bellman equation may be rewritten:

$$\begin{aligned}
&D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon} \\
&\quad = (A(t)S(t))^{(\varepsilon-1)/\varepsilon}D^{(\varepsilon-1)/\varepsilon}\mathrm{d}t + D^{-(1/\varepsilon)}(A(t)S(t))^{(\varepsilon-1)/\varepsilon} \\
&\qquad \times \left(1+\frac{\varepsilon-1}{\varepsilon}(-D+\mathrm{CEq})\mathrm{d}t - \delta\,\mathrm{d}t\right)
\end{aligned} \tag{A.8}$$

After some simplifications, this reduces to Eq. (3.5).

Appendix B

A sufficient condition for the transversality condition to be satisfied is:

$$E_t\left[\frac{\mathrm{d}V(S,A)}{V}\mathrm{e}^{-\delta t}\right] < 0 \tag{B.1}$$

Using Itô's lemma, this condition may rewritten:

$$\begin{aligned}
&\frac{\varepsilon-1}{\varepsilon}A^{(\varepsilon-1)/\varepsilon}S^{-(1/\varepsilon)}E_t(\mathrm{d}S) + \frac{\varepsilon-1}{\varepsilon}S^{(\varepsilon-1)/\varepsilon}A^{-(1/\varepsilon)}E_t(\mathrm{d}A) \\
&\quad -\frac{\varepsilon-1}{2}A^{(\varepsilon-1)/\varepsilon}S^{-(1/(\varepsilon-1))}E_t(\mathrm{d}S^2) - \frac{\varepsilon-1}{2}A^{-(1/(\varepsilon-1))}S^{(\varepsilon-1)/\varepsilon}E_t(\mathrm{d}A^2) \\
&\quad +\left(\frac{\varepsilon-1}{\varepsilon}\right)^2 A^{-(1/\varepsilon)}S^{-(1/\varepsilon)}E_t(\mathrm{d}A\,\mathrm{d}S) - \delta A^{(\varepsilon-1)/\varepsilon}S^{(\varepsilon-1)/\varepsilon}\mathrm{d}t < 0
\end{aligned} \tag{B.2}$$

Replacing dA and dS by their expressions when extraction is optimal, gives Eq. (3.9).

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