THE HARMONIC MEAN FORMULA FOR RANDOM PROCESSES

KRZYSZTOF BISEWSKI, ENKELEJD HASHORVA, AND GEORGIY SHEVCHENKO

ABSTRACT. Motivated by the harmonic mean formula in [1], we investigate the relation between the sojourn time and supremum of a random process $X(t), t \in \mathbb{R}^d$ and extend the harmonic mean formula for general stochastically continuous X. We discuss two applications concerning the continuity of distribution of supremum of X and representations of classical Pickands constants.

1. INTRODUCTION

Let $X(t), t \in \mathbb{R}^d$ be a measurable and separable real-valued random process. Given a measurable function $f : \mathbb{R} \mapsto \mathbb{R}$, define the f-sojourn time (or f-occupation time) of X in the set $\mathscr{A} \subset \mathbb{R}^d$ by

$$\mathcal{L}(\mathscr{A}, f(X)) = \int_{\mathscr{A}} f(X(t))\lambda(dt),$$

where λ is the Lebesgue measure on $\mathbb{R}^d$ and further A is λ-measurable. When $\mathscr{A}$ is countable, then we simply take λ to be the counting measure on $\mathscr{A}$. An example of f, which is particularly interesting in application is

$$f(t) = \mathcal{J}_z(t) := \mathbb{I}(t > z), \ z \in \mathbb{R},$$

where $\mathbb{I}(\cdot)$ is the indicator function. Hereafter we set $\mathcal{M}(\mathscr{A}, X) = \sup_{t \in \mathscr{A}} X(t)$. Note in passing that when $\lambda(A)$ is finite, both $\mathcal{M}(\mathscr{A}, X)$ and $\mathcal{L}(\mathscr{A}, \mathcal{J}_z(X))$ afe determined by finite-dimensional distributions (fidi's) of X (see [2, Lem 10.4.2]). Consider next the simpler case that $\mathscr{A}$ is a countable subset of $\mathbb{R}^d$. Clearly, for all $z \in \mathbb{R}$ satisfying $\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} > 0$ we have

$$(1.1) \qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z, \mathcal{L}(\mathscr{A}, \mathcal{J}_z(X)) = 0\} = 0.$$

An implication of (1.1) is an elegant and important result of Aldous, the so called harmonic mean formula, which is stated in (1.2) below.

Lemma 1.1 ([1, Lem 1.5, Eq. (1.7)]). *Let* $\mathscr{A} = \{t_1, \ldots, t_n\} \subset \mathbb{R}^d, n > 1$ *be given and let* $X(t), t \in \mathbb{R}^d$ *be a real-valued random process. If* $z \in \mathbb{R}$ *is such that* $\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} > 0$, *then*

$$(1.2) \qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} = \sum_{i=1}^{n} \mathbb{P}\{X(t_i) > z\}\mathbb{E}\left\{\frac{1}{\mathcal{L}(\mathscr{A}, \mathcal{J}_z(X))}\Big| X(t_i) > z\right\}.$$

For general $\mathscr{A}$ and X the claim in (1.1) does not hold in general. For instance, considering a deterministic process $X(t) = 0, t \in [0, 1] \setminus \{1/2\}$ and $X(1/2) = 1$ we clearly have that

$$\sup_{t \in [0,1]} X(t) = \sup_{t \in (0,1)} X(t) = 1 > 0$$

almost surely but $\mathbb{P}\{\int_0^1 X(t)\lambda(dt) = 0\} = 1$.

Two natural questions which arise here are:

  Q1) Under what conditions on fidi's of X does (1.1) hold?

  Q2) Can the mean harmonic formula be extended to more general f and uncountable $\mathscr{A}$?

Lemma 1.1 is of interest in several applications. In the present form or with some adjustments, it has been utilised extensively for rare events simulations (by importance sampling technique), see, e.g., [3, 4]. We shall show in the next section that a natural assumption under which Item Q1) has a positive answer is that of stochastically continuous X, which is satisfied if, for instance, X is separable, jointly measurable and has stationary increments, see Remark 2.3, Item (vi) below. Moreover, for stochastically continuous X we shall derive the harmonic mean formula in Theorem 2.1.

The manuscript is organized as follows. Our main findings are presented in Section 2, where we also discuss two applications concerning the continuity of the distribution of $\mathcal{M}(\mathscr{A}, X)$ and derive new representations for the classical Pickands constants. All the proofs are relegated to Section 3.

2. Main Result

In this section we shall consider $X(t), t \in \mathbb{R}^d$ stochastically continuous, i.e. for all $t \in \mathbb{R}^d$ we have the convergence in probability $X(s) \xrightarrow{p} X(t)$ as $s \to t$. In view of [5, Thm 1, p. 171] this guarantees that there exists a jointly measurable and separable version. Therefore in the following, whenever X is stochastically continuous we shall further assume that X is jointly measurable and separable, see [6, Thm 1], [7, Thm 9.4.2] for equivalent conditions that guarantee measurability of a random process. Below we shall consider open $\mathscr{A} \subset \mathbb{R}^d$ answering both questions raised in the Introduction.

Theorem 2.1. *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be a measurable function and let $z \in \mathbb{R}, \kappa \in [-\infty, z]$ be given. Suppose that $\mathscr{A} \subset \mathbb{R}^d$ is open and $X(t), t \in \mathbb{R}^d$ is a stochastically continuous real-valued process. If $f(x) > 0$ for all $x > \kappa$ and $\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} > 0$, then*

$$(2.1) \qquad\qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z, \mathcal{L}(\mathscr{A}, f_\kappa(X)) = 0\} = 0,$$

where $f_\kappa(x) = f(x)\mathbb{I}(x > \kappa)$, $x \in \mathbb{R}$. If further $\mathcal{L}(\mathscr{A}, f_\kappa(X))$ is almost surely finite, then

$$(2.2) \qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} \;\; = \;\; \int_{\mathscr{A}} \mathbb{E}\left\{ \frac{f_\kappa(X(t))\mathbb{I}(\mathcal{M}(\mathscr{A}, X) > z)}{\mathcal{L}(\mathscr{A}, f_\kappa(X))} \right\} \lambda(dt).$$

A simple application of Theorem 2.1 concerns the question of continuity of $\mathcal{M}(\mathscr{A}, X)$, which has been investigated in various generalities in numerous contributions, see e.g., [8–11] and the excellent contribution [9], where the methodology is explained in details.

Corollary 2.2. *Let $X(t), t \in \mathbb{R}^d$ be as in Theorem 2.1, let $z \in \mathbb{R}$ be given and suppose the open set $\mathscr{A} \subset \mathbb{R}^d$ is bounded. If $\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) \geq z\} > 0$, $\mathbb{P}\{X(t) = z\} = 0$ for all $t \in \mathscr{A}$, and*

$$(2.3) \qquad\qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) \geq z, \mathcal{L}(\mathscr{A}, I_z) = 0\} = 0,$$

where $I_z(t) := \mathbb{I}(t \geq z)$, then the distribution of $\mathcal{M}(\mathscr{A}, X)$ is continuous at z.

Remark 2.3.   (i) Theorem 2.1 is also true for countable $\mathscr{A}$, with λ being the counting measure. Moreover, the assumption that $\mathscr{A}$ is open can be relaxed to $\mathscr{A}$ is a Borel subset of $\mathbb{R}^d$ with non-empty interior $\mathscr{A}^o$ satisfying $\mathbb{P}\{\sup_{t \in \mathscr{A}} X(t) = \sup_{t \in \mathscr{A}^\circ} X(t)\} = 1$;

(ii) If $\kappa = z$, then the indicator function in (2.2) can be dropped;

(iii) For $X(t), t \in \mathbb{R}^d$ measurable and separable with separant D, the assumption of stochastic continuity can be relaxed to the following: For all $\varepsilon > 0$ positive and given $t_i \in D$

$$(2.4) \qquad \mathbb{P}\{X(t) \leq z, X(t_i) > z + \varepsilon\} \to 0, \ t \to t_i.$$

To see that, note that in the proof of Theorem 2.1 below the crucial argument of the proof (along with stochastic continuity) is (3.2). In particular, (2.4) holds if for all $\varepsilon > 0$

$$\mathbb{P}\{X(t) < X(t_i) - \varepsilon\} \to 0, \ t \to t_i,$$

or X is stochastically continuous from the right when $d = 1$. For $d > 1$ the latter assumption can be formulated in terms of quadrant stochastic continuity;

(iv) If X is Gaussian with continuous covariance function on $\mathbb{R}^d$, then (2.4) is satisfied;

(v) Two common choices for f in the results above are $f(x) = e^{bx}$ and $f(x) = |x - z|^b, b \geq 0$, for which we have that $f(x) > 0$ for all $x > z$. In particular, from (1.1) for all $b \geq 0$ we have

$$(2.5) \qquad \mathbb{P}\left\{\sup_{t \in \mathscr{A}} |X(t)| > 0, \int_{\mathscr{A}} |X(t)|^b \lambda(dt) = 0\right\} = 0.$$

(vi) When X is stationary with $p_\kappa = \mathbb{P}\{X(0) > \kappa\} > 0$, then $p_\kappa = \mathbb{P}\{X(t) > \kappa\}, \forall t \in \mathbb{R}^d$, and thus, under the assumptions of Theorem 2.1, by the shift-invariance of Lebesgue measure we can rewrite (2.2) as

$$(2.6) \qquad \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} \ = \ p_\kappa \int_{\mathscr{A}} \mathbb{E}\left\{\frac{f(X(0))\mathbb{I}(\max_{s \in \mathscr{A}} X(s-t) > z)}{\int_{\mathscr{A}} f(X(s-t))\mathbb{I}(X(s-t) > \kappa)\lambda(ds)}\Big| X(0) > \kappa\right\}\lambda(dt).$$

Moreover, if we assume that X is separable and jointly measurable, then by [12, Prop 3.1] (see [2][Thm 1.3.3] for the more general case of processes with stationary increments) we have that X is stochastically continuous. If $\mathscr{A} \subset \delta\mathbb{Z}^d, \delta > 0$ and λ is the counting measure on $\delta\mathbb{Z}^d$ we have that (2.6) still holds;

(vii) If we take $X(t) = 2\mathbb{I}(U < 0) + \sin(t - U)\mathbb{I}(U \geq 0), t \in [-1, 1]$, with U uniformly distributed on $[-1, 1]$, it follows that for any open $\mathscr{A} \subset [0, 1]$

$$\mathbb{P}\left\{\mathcal{M}(\mathscr{A}, X) \geq 1, \int_0^1 \mathbb{I}(X(t) \geq 1)dt > 0\right\} = 1/2,$$

hence condition (2.3) in Corollary 2.2 cannot be removed. This example was kindly suggested by Mikhail Lifshits (personal communication).

Next, we present another application of the harmonic mean formula. The idea, which appears for instance in [13, 14] is to find some positive functions $v(x), q(x), x > 0$ such that

$$Y_\kappa(t) = v(\kappa)[X(q(\kappa)t) - \kappa], \ t \in \mathbb{R}^d,$$

conditioned on $Y_\kappa(0) > 0$ converges weakly to some random process $Y(t), t \in \mathbb{R}^d$, as $\kappa \to \infty$.

In the sequel, let $X(t), t \geq 0$ be a centered stationary Gaussian process with unit variance and correlation function r satisfying the so-called Pickands condition see [15], i.e.,

$$(2.7) \qquad r(t) < 1, \forall t \geq 0, \ 1 - r(t) = C |t|^\alpha (1 + o(1)), \ t \to 0$$

for some $C > 0, \alpha \in (0, 2]$. In view of Lemma 3.1 below for this case we can take

$$v(\kappa) = \kappa, \ q(\kappa) = \kappa^{-2/\alpha}$$

and the limit process Y is given with B being a standard fractional Brownian motion with Hurst index $\alpha/2$ by

$$(2.8) \qquad Y(t) = W_1(C^{1/\alpha}t) + \eta, \quad W_b(s) = b[\sqrt{2}B(t) - |t|^\alpha], \quad b \geq 0, t \in \mathbb{R},$$

where the unit exponential rv η is independent of B.

In the following, for $\delta > 0$ let $\delta\mathbb{Z}$ denote the infinite grid of uniformly spaced points $\delta\mathbb{Z} := \{\ldots, -2\delta, -\delta, 0, \delta, 2\delta, \ldots\}$ and set $0\mathbb{Z} = \mathbb{R}$.

Theorem 2.4. *If $X(t), t \geq 0$ is a centered stationary Gaussian process with unit variance, continuous sample paths and correlation function satisfying the Pickands condition for some $C > 0, \alpha \in (0, 2]$, and $T_z > 0$ is such that $\lim_{z\to\infty} T_z z^{2/\alpha} = \infty$, then for all any $\delta \geq 0$*

$$(2.9) \qquad \mathbb{P}\left\{ \sup_{t\in[0,T_z]\cap\delta\mathbb{Z}} X(t) > z \right\} \sim C^{1/\alpha} \mathcal{H}_\alpha^\delta T_z z^{2/\alpha} \mathbb{P}\{X(0) > z\}, \quad z \to \infty,$$

provided that the left-hand side above converges to 0 as $z \to \infty$, with

$$(2.10) \qquad \mathcal{H}_\alpha^\delta = e^\theta \mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t\in\delta\mathbb{Z}} W_1(t) + \eta > \theta)}{\delta \sum_{t\in\delta\mathbb{Z}} e^{W_b(t)} \mathbb{I}(W_1(t) + \eta > 0)} \right\},$$

for all $\delta > 0$, and

$$(2.11) \qquad \mathcal{H}_\alpha := \mathcal{H}_\alpha^0 = \lim_{\delta\to 0} \mathcal{H}_\alpha^\delta = e^\theta \mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t\in\mathbb{R}} W_1(t) + \eta > \theta)}{\int_\mathbb{R} e^{W_b(t)} \mathbb{I}(W_1(t) + \eta > 0)\lambda(dt)} \right\},$$

where $b, \theta \in [0, \infty)$ are arbitrary.

Remark 2.5.   (i) The new results above are the expressions for Pickands constant H_α, which are known for the case $a = \theta = 0$, see [16, 3.6] and [17, Thm 1.1], see also [18, J20a, J20b];

(ii) The discrete Pickands constant is defined for X as in Theorem 2.4 by

$$(2.12) \qquad \mathcal{H}_\alpha^\delta = \lim_{T\to\infty} T^{-1}\mathbb{E}\left\{ \sup_{t\in[0,T]\cap\delta\mathbb{Z}} e^{W_1(t)} \right\} = \lim_{T\to\infty}\lim_{z\to\infty} \frac{\mathbb{P}\{\sup_{t\in[0,T]\cap\delta\mathbb{Z}} X(q(z)t) > z\}}{T\mathbb{P}\{X(0) > z\}},$$

with $q(z) \sim z^{-2/\alpha}$ as $z \to \infty$. See e.g. [19, p. 1605] for the first formula and [20, p. 164] for the second one. Furthermore, we have

$$(2.13) \qquad \delta\mathcal{H}_\alpha^\delta = e^\theta \mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t\in\delta\mathbb{Z}} W_1(t) + \eta > \theta)}{\sum_{t\in\delta\mathbb{Z}} e^{W_b(t)} \mathbb{I}(W_1(t) + \eta > 0)} \right\} = \mathbb{E}\left\{ \frac{\sup_{t\in\delta\mathbb{Z}} e^{W_1(t)}}{\sum_{t\in\delta\mathbb{Z}} e^{W_1(t)}} \right\}$$

$$(2.14) \qquad = \mathbb{P}\left\{ \sup_{t\in\delta\mathbb{N}} W_1(t) + \eta \leq 0 \right\} > 0$$

for all $\delta > 0$. The first formula for $\theta = 0$ and both two other ones for $\mathcal{H}_\alpha^\delta$ in (2.13) and (2.14) can be derived also as in [21] utilising previous findings of [22], see also [23–27]. The expression in (2.14) was derived in [28] and appeared also latter in [29, 30];

(iii) Explicit formulas for $\mathcal{H}_1^\delta, \delta > 0$ are known, see [31, Lem 5.16, Rem 5.17]. Such a formula appeared also in other connections, see definition of $\nu(x)$ function in [32, Eq. (2.1)]. An alternative expression is given in [33].

3. Proofs

Proof of Theorem 2.1 We show first that

$$(3.1) \qquad \mathbb{P}\left\{\mathcal{M}(\mathscr{A}, X) > 0, \int_{\mathscr{A}} \mathbb{I}(X(t) > 0)\,\lambda(dt) = 0\right\} = 0.$$

Suppose for simplicity that $d = 1$ and consider without loss of generality X to be further measurable and separable (we use definition in [5] for (joint) measurability). Let $D \subset \mathbb{R}$ be a countable dense set that is a separant for X and set $D \cap \mathscr{A} = \{t_i, i \in \mathbb{N}\}$. For fixed arbitrary $\varepsilon > 0$, by definition of separability we have

$$\mathcal{M}(\mathscr{A}, X) = \sup_{t \in \mathscr{A} \cup D} X(t) = \sup_{i \in \mathbb{N}} X(t_i)$$

almost surely. By the stochastic continuity of X, for each positive integer i and given $\varepsilon > 0$

$$\mathbb{P}\{|X(t) - X(t_i)| > \varepsilon\} \to 0, \quad t \to t_i.$$

Therefore, since $\mathscr{A}$ is open, there exists some open interval $\Delta_i \subset \mathscr{A}$ containing t_i such that

$$(3.2) \qquad \mathbb{P}\{X(t) \le 0, X(t_i) > \varepsilon\} \le \varepsilon 2^{-i}, \quad \forall t \in \Delta_i.$$

Consequently, for all $i \in \mathbb{N}$, the Fubini-Tonelli theorem yields

$$\mathbb{E}\left\{\int_{\Delta_i} \mathbb{I}(X(t) \le 0)\lambda(dt)\mathbb{I}(X(t_i) > \varepsilon)\right\} = \int_{\Delta_i} \mathbb{P}\{X(t) \le 0, X(t_i) > \varepsilon\}\lambda(dt) \le \varepsilon 2^{-i}\lambda(\Delta_i).$$

Hence, by the Markov inequality

$$\begin{aligned}
\mathbb{P}\left\{\int_{\Delta_i} \mathbb{I}(X(t) > 0)\,dt = 0, X(t_i) > \varepsilon\right\} &= \mathbb{P}\left\{\int_{\Delta_i} \mathbb{I}(X(t) \le 0)\lambda(dt) = \lambda(\Delta_i), X(t_i) > \varepsilon\right\} \\
&\le \mathbb{P}\left\{\int_{\Delta_i} \mathbb{I}(X(t) \le 0)\lambda(dt) \ge \lambda(\Delta_i), X(t_i) > \varepsilon\right\} \\
&\le \frac{1}{\lambda(\Delta_i)}\mathbb{E}\left\{\int_{\Delta_i} \mathbb{I}(X(t) \le 0)\lambda(dt)\mathbb{I}(X(t_i) > \varepsilon)\right\} \\
&\le \varepsilon 2^{-i}.
\end{aligned}$$

Since further

$$\begin{aligned}
\mathbb{P}\left\{\mathcal{M}(\mathscr{A}, X) > \varepsilon, \int_{\mathscr{A}} \mathbb{I}(X(t) > 0)\lambda(dt) = 0\right\} &= \mathbb{P}\left\{\sup_{i \ge 1} X(t_i) > \varepsilon, \int_{\mathscr{A}} \mathbb{I}(X(t) > 0)\lambda(dt) = 0\right\} \\
&\le \sum_{i \ge 1} \mathbb{P}\left\{X(t_i) > \varepsilon, \int_{\Delta_i} \mathbb{I}(X(t) > 0)\,dt = 0\right\} \\
&\le \varepsilon \sum_{i \ge 1} 2^{-i} = \varepsilon
\end{aligned}$$

the claim in (3.1) follows by letting $\varepsilon \to 0$.

Next, by our assumption, $f_\kappa(X(t)) = f(X(t))\mathbb{I}(X(t) > \kappa) > 0$ whenever $X(t) > \kappa$, thus the equality

$$\mathcal{L}(\mathscr{A}, f_\kappa(X)) = \int_{\mathscr{A}} f(X(t))\mathbb{I}(X(t) > \kappa)\lambda(dt) = 0$$

implies

$$\lambda(\{t \in \mathscr{A} : X(t) > \kappa\}) = 0.$$

Therefore, applying (3.1) to the process $X(t) - \kappa, t \in \mathbb{R}^d$ we obtain

$$
\begin{aligned}
\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z, \mathcal{L}(\mathscr{A}, f_\kappa(X)) = 0\} & \\
\leq \quad & \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z, \lambda(\{t \in \mathscr{A} : X(t) > \kappa\}) = 0\} \\
\leq \quad & \mathbb{P}\left\{\mathcal{M}(\mathscr{A}, X) > z, \int_\mathscr{A} \mathbb{I}(X(t) > \kappa)\lambda(dt) = 0\right\} \\
\leq \quad & \mathbb{P}\left\{\mathcal{M}(\mathscr{A}, X) > z, \int_\mathscr{A} \mathbb{I}(X(t) > z)\lambda(dt) = 0\right\} \\
= \quad & 0,
\end{aligned}
$$

which establishes the first claim.

Next, by the assumption,

$$
\mathcal{L}(\mathscr{A}, f_\kappa(X)) = \int_\mathscr{A} f(X(t))\mathbb{I}(X(t) > \kappa)\lambda(dt)
$$

is almost surely finite and non-zero, interpreting $\frac{0}{0}$ as 0, by (1.1) we find that

$$
\mathbb{I}(\mathcal{M}(\mathscr{A}, X) > z) = \frac{\mathcal{L}(\mathscr{A}, f_\kappa(X))}{\mathcal{L}(\mathscr{A}, f_\kappa(X))}\mathbb{I}(\mathcal{M}(\mathscr{A}, X) > z)
$$

holds almost surely. Hence by the Fubini-Tonelli theorem and the assumption $\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} > 0$

$$
\begin{aligned}
\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\} & = \mathbb{E}\left\{\frac{\mathcal{L}(\mathscr{A}, f_\kappa(X))}{\mathcal{L}(\mathscr{A}, f_\kappa(X))}\mathbb{I}(\mathcal{M}(\mathscr{A}, X) > z)\right\} \\
& = \int_\mathscr{A} \mathbb{E}\left\{\frac{f(X(t))\mathbb{I}(\mathcal{M}(\mathscr{A}, X) > z)\mathbb{I}(X(t) > \kappa)}{\mathcal{L}(\mathscr{A}, f_\kappa(X))}\right\}\lambda(dt) > 0
\end{aligned}
$$

establishing the proof. □

PROOF OF COROLLARY 2.2. Recall that $\mathcal{J}_z(s) = \mathbb{I}(s > z)$ and we set $I_z(s) = \mathbb{I}(s \geq z)$. Since we suppose that

$$
(3.3) \qquad\qquad \mathbb{P}\{\mathbb{I}(X(t) = z) = 0\} = 1, \quad \forall t \in \mathscr{A}
$$

and $\mathscr{A}$ is bounded, then

$$
\infty > \mathcal{L}(\mathscr{A}, I_z) = \int_\mathscr{A} \mathbb{I}(X(t) \geq z)\lambda(dt) \geq \mathcal{L}(\mathscr{A}, \mathcal{J}_z) = \int_\mathscr{A} \mathbb{I}(X(t) > z)\lambda(dt)
$$

almost surely and hence

$$
\mathbb{E}\{\mathcal{L}(\mathscr{A}, I_z) - \mathcal{L}(\mathscr{A}, \mathcal{J}_z)\} = \int_\mathscr{A} \mathbb{E}\{\mathbb{I}(X(t) = z)\}\lambda(dt) = 0
$$

implying

$$
\mathbb{P}\{\mathcal{L}(\mathscr{A}, I_z) = \mathcal{L}(\mathscr{A}, \mathcal{J}_z)\} = 1.
$$

Since, by the assumption, $\mathcal{M}(\mathscr{A}, X) \geq z$ implies almost surely $\mathcal{L}(\mathscr{A}, \mathcal{J}_z) \in (0, \infty)$, as in the proof of Theorem 2.1, using further (3.3) we have

$$
\mathbb{P}\{\mathcal{M}(\mathscr{A}, X) \geq z\} = \int_\mathscr{A} \mathbb{E}\left\{\frac{\mathbb{I}(X(t) \geq z)}{\mathcal{L}(\mathscr{A}, I_z)}\right\}\lambda(dt) = \int_\mathscr{A} \mathbb{E}\left\{\frac{\mathbb{I}(X(t) > z)}{\mathcal{L}(\mathscr{A}, \mathcal{J}_z)}\right\}\lambda(dt) = \mathbb{P}\{\mathcal{M}(\mathscr{A}, X) > z\},
$$

where the last equality follows from Theorem 2.1 and Remark 2.3, Item (ii), This concludes the proof. □

Next we state a lemma which is needed in the proof of Theorem 2.4. Denote by $C([0, T])$ the space of real-valued continuous functions on $[0, T]$ equipped with a metric which turns it into a Polish space and let $\mathcal{C}$ be the corresponding

Borel σ-field. We write $\overset{d}{\to}$ for the weak convergence of fidi's. Hereafter $W_b(t)$ and $Y(t) = W_1(t) + \eta$ are as in (2.8), with η a unit exponential rv independent of W_1. Set n the following

$$(3.4) \qquad \mathcal{H}_\alpha^\delta(T) := \lim_{z \to \infty} \frac{\mathbb{P}\{\sup_{t \in \delta\mathbb{Z} \cap [0,T]} X(tz^{-2/\alpha}) > z\}}{\mathbb{P}\{X(0) > z\}}$$

for some $T > 0$.

Lemma 3.1. *Let the stationary process $X(t), t \in [-T,T], T > 0$ have almost surely sample paths in $C([-T,T])$. Suppose further that X is Gaussian with mean zero and unit variance function satisfying (2.7) for some $C > 0$ and $\alpha \in (0,2]$ and let $q(z)$ be positive such that $q(z) \sim z^{-2/\alpha}$ as $z \to \infty$.*

(i) As $z \to \infty$ we have the weak convergence $z[X(q(z)t) - z] \mid (X(0) > z) \overset{d}{\to} Y(t)$;

(ii) For all $b, \theta \geq 0$ and all δ, T positive

$$(3.5) \qquad \mathcal{H}_\alpha^\delta(T) = e^\theta \sum_{\tau \in \delta\mathbb{Z} \cap [0,T]} \mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t \in \delta\mathbb{Z} \cap [0,T]} W_1(t-\tau) + \eta > \theta)}{\sum_{t \in \delta\mathbb{Z} \cap [0,T]} e^{W_b(t-\tau)} \mathbb{I}(W_1(t-\tau) + \eta > 0)} \right\}.$$

PROOF OF LEMMA 3.1 Item (i): The convergence of fidi's is well-known, see for instance [30, Lem 2].

Item (ii): For any fixed $\theta > 0$ set $\kappa(z) := z - \theta/z, z > 0$. In the following, for brevity we write $\kappa = \kappa(z)$. Take

$$(3.6) \qquad f_\kappa(x) = e^{b\kappa x} \mathbb{I}(x > \kappa), \quad q_\theta(x) := q(\tfrac{x + \sqrt{x^2 + 4\theta}}{2})$$

such that $q_\theta(\kappa(z)) = q(z)$ and $Y_\kappa(t) = \kappa(z)[X(q_\theta(\kappa(z)t) - \kappa(z)]$. Using Theorem 2.1, with f_κ defined in (3.6) we obtain

$$\mathcal{H}_\alpha^\delta(T) = \lim_{z \to \infty} \frac{\sum_{\tau \in \delta\mathbb{Z} \cap [0,T]} \mathbb{E}\{U_z(\tau, T)\}}{\mathbb{P}\{X(0) > z\}}, \quad U_z(\tau, T) := \frac{f_\kappa(X(q(z)\tau))}{\sum_{t \in K} f_\kappa(X(q(z)t))} \mathbb{I}\left(\sup_{s \in K} X(q(z)s) > z\right).$$

For any $\tau \in \delta\mathbb{Z} \cap [0,T]$ we have

$$U_z(\tau, T) = \frac{e^{b\kappa X(q_\theta(\kappa)\tau)} \mathbb{I}(X(q_\theta(\kappa)\tau) > \kappa)}{\sum_{t \in \delta\mathbb{Z} \cap [0,T]} e^{b\kappa X(q_\theta(\kappa)t)} \mathbb{I}(X(q_\theta(\kappa)t) > \kappa)} \mathbb{I}\left(\sup_{t \in \delta\mathbb{Z} \cap [0,T]} \kappa(X(q_\theta(\kappa)t) - \kappa) > \kappa(z - \kappa)\right)$$

$$= \frac{e^{b\kappa(X(q_\theta(\kappa)\tau) - \kappa)} \mathbb{I}(\kappa(X(q_\theta(\kappa)\tau) - \kappa) > 0)}{\sum_{t \in \delta\mathbb{Z} \cap [0,T]} e^{b\kappa(X(q_\theta(\kappa)t) - \kappa)} \mathbb{I}(\kappa(X(q_\theta(\kappa)t) - \kappa) > 0)} \mathbb{I}\left(\sup_{t \in \delta\mathbb{Z} \cap [0,T]} \kappa(X(q_\theta(\kappa)t) - \kappa) > \theta\kappa/z\right).$$

Further by the the stationarity of X

$$U_z(\tau, T) \overset{d}{=} \frac{e^{bY_\kappa(0)} \mathbb{I}(Y_\kappa(0) > 0)}{\sum_{t \in \delta\mathbb{Z} \cap [0,T]} e^{bY_\kappa(t-\tau)} \mathbb{I}(Y_\kappa(t-\tau) > 0)} \mathbb{I}\left(\sup_{t \in \delta\mathbb{Z} \cap [0,T]} Y_\kappa(t-\tau) > \theta\right) := U_z^*(\tau, T).$$

Hence applying Item (i) and the continuous mapping theorem we obtain the convergence in distribution

$$(3.7) \qquad (U_z^*(\tau, T) \mid Y_\kappa(0) > 0) \overset{d}{\to} \frac{\mathbb{I}(\sup_{t \in \delta\mathbb{Z} \cap [0,T]} W_1(t-\tau) + \eta > \theta)}{\sum_{t \in \delta\mathbb{Z} \cap [0,T]} e^{W_b(t-\tau)} \mathbb{I}(W_1(t-\tau) + \eta > 0)} := U(\tau, T), \; z \to \infty.$$

Moreover, since $U_z^*(\tau) \leq 1$, then by the dominated convergence theorem $\lim z\mathbb{E}\{U_z(\tau, T)\} = \mathbb{E}\{U(\tau, T)\}$. Consequently, using further

$$\mathbb{P}\{Y_\kappa(0) > 0\} = \mathbb{P}\{X(0) > z - \theta/z\} \sim e^\theta \mathbb{P}\{X(0) > z\}, \quad z \to \infty$$

implies

$$\mathcal{H}_\alpha^\delta(T) = \lim_{z \to \infty} \frac{\mathbb{P}\{Y_\kappa(0) > z\}}{\mathbb{P}\{X(0) > z\}} \sum_{\tau \in \delta\mathbb{Z} \cap [0,T]} \mathbb{E}\{U_z^*(\tau) \mid Y_\kappa(0) > 0\} = e^\theta \sum_{\tau \in \delta\mathbb{Z} \cap [0,T]} \mathbb{E}\{U(\tau, T)\}$$

establishing proof. $\qquad \square$

PROOF OF THEOREM 2.4 . The claim in (2.9) is shown in [34, Lem 9.15], [35, Thm 2.1] for $\delta = 0$. The proof for $\delta > 0$ follows with similar arguments and is omitted. Alternatively, it can be derived utilising the approach in [30, p. 339] and applying Lemma 3.1. In view of [17, Thm 1.1] and [16, Eq. 5.2] we have the following expressions

$$\mathcal{H}_\alpha = \mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t\in\mathbb{R}} W_1(t) + \eta > 0)}{\int_\mathbb{R} \mathbb{I}(W_1(t) + \eta > 0)\lambda(dt)} \right\} = \lim_{\delta\downarrow 0} \mathcal{H}_\alpha^\delta = \lim_{\delta\downarrow 0} \frac{1}{\delta}\mathbb{E}\left\{ \frac{\mathbb{I}(\sup_{t\in\delta\mathbb{Z}} W_1(t) + \eta > 0)}{\sum_{t\in\delta\mathbb{Z}} \mathbb{I}(W_1(t) + \eta > 0)} \right\} > 0.$$

Note that the positivity of the constants follows from the fact that almost surely (see e.g., [36, Thm 6.1, Prop 6.1])

$$(3.8) \qquad\qquad\qquad\qquad W_1(t) \to -\infty, \ t \to \infty.$$

Further, $\mathcal{H}_\alpha = \lim_{\delta\downarrow 0} \mathcal{H}_\alpha^\delta$ is well-known, see e.g., [37]. Consequently, (2.11) follows for $b = 0$ and all $\delta, \theta \geq 0$. Next, in view of [38][Thm 2.8] or [39, Eq. (5.4)] (see also [25] for case $d = 1$) for all $b, \theta \in [0, \infty), x > 0$ we have (write $B^h f(\cdot) = f(\cdot - h), h \in \mathbb{R}$)

$$(3.9) \qquad \mathbb{E}\{F(xe^{\eta+W_1})\mathbb{I}(W_1(h) + \eta > -\ln x)\} = x\mathbb{E}\{F(e^{\eta+B^h W_1})\mathbb{I}(W_1(-h) + \eta > \ln x)\}, \forall h \in \mathbb{R}^d,$$

valid for all measurable maps $F : C([0, \infty)) \mapsto [0, \infty]$; we interpret $0/0$ and $0 \cdot \infty$ as 0. For some countable dense set $D \subset \mathbb{R}$ set

$$L(t) = e^{W_1(t)+\eta}, \ M = \sup_{t\in\mathbb{R}\cap D} L(t), \ S = \int_\mathbb{R} L(t)\lambda(dt), \quad \chi = e^\theta.$$

Observe that

$$L(s)/M \in (0, 1], \ M \in (0, \infty), \ S \in (0, \infty), \ \int_\mathbb{R} [L(t)]^b\mathbb{I}(L(t) > 1)dt \in (0, \infty)$$

almost surely (the last claim follows from (3.8), the previous last follows from [36, Thm 6.1, Prop 6.1]). Hence, using (3.9), we obtain from the Fubini-Tonelli theorem that

$$\mathbb{E}\left\{ \frac{e^\theta\mathbb{I}(\sup_{t\in\mathbb{R}} W_1(t) + \eta > \theta)}{\int_\mathbb{R} e^{W_b(t)}\mathbb{I}(W_1(t) + \eta > 0)\lambda(dt)} \right\}$$
$$= \chi\mathbb{E}\left\{ \frac{S}{S} \frac{[L(0)]^b\mathbb{I}(\sup_{t\in\mathbb{R}\cap D} L(t) > \chi)}{\int_\mathbb{R} [L(t)]^b\mathbb{I}(L(t) > 1)\lambda(dt)} \right\}$$
$$= \chi\int_\mathbb{R} \mathbb{E}\left\{ \frac{[L(0)]^b L(s)\mathbb{I}(M > \chi)}{S\int_\mathbb{R} [L(t)]^b\mathbb{I}(L(t) > 1)\lambda(dt)} \right\}\lambda(ds)$$
$$= \chi\int_0^\infty \int_\mathbb{R} \mathbb{E}\left\{ \frac{[L(0)]^b\mathbb{I}(rL(s) > 1)\mathbb{I}(M > \chi)}{S\int_\mathbb{R} [L(t)]^b\mathbb{I}(L(t) > 1)\lambda(dt)} \right\}r^{-2}\lambda(ds)dr.$$

Write $M^* = rM$ and $S^* = rS$, $L^* = rL$. Then the expectation above can be written as

$$\mathbb{E}\left\{ \frac{r[L^*(0)]^b\mathbb{I}(rL(s) > 1)\mathbb{I}(M^* > r\chi)}{S^*\int_\mathbb{R} [L^*(t)]^b\mathbb{I}(L^*(t) > r)\lambda(dt)} \right\} = r\mathbb{E}\{G(rL)\mathbb{I}(rL(s) > 1)\},$$

where

$$G(f) := \frac{f^b(0)\mathbb{I}(\sup_{t\in\mathbb{R}\cap D} f(t) > r\chi)}{\int_\mathbb{R} f(t)dt \int_\mathbb{R} f^b(t)\mathbb{I}(f(t) > r)\lambda(dt)}.$$

After applying (3.9) for any $r > 0$ we obtain

$$r\mathbb{E}\{G(rL)\mathbb{I}(rL(s) > 1)\} = r^2\mathbb{E}\{G(B^s L)\mathbb{I}(L(-s) > r)\},$$

which, after applying substitution $z := r\chi/M$, further gives us

$$
\chi \int_0^\infty \int_\mathbb{R} \mathbb{E}\{G(B^s L)1(L(-s) > r)\}\lambda(ds)dr
$$
$$
= \chi \int_\mathbb{R} \int_0^\infty \mathbb{E}\left\{\frac{[L(-s)]^b \mathbb{I}(L(-s) > r)\mathbb{I}(M > r\chi)}{S \int_\mathbb{R} [L(t)]^b \mathbb{I}(L(t) > r)\lambda(dt)}\right\}\lambda(ds)dr
$$
$$
= \int_0^\infty \int_\mathbb{R} \mathbb{E}\left\{\frac{M[L(-s)]^b \mathbb{I}(\chi L(-s)/M > z, 1 > z)}{S \int_\mathbb{R} [L(t)]^b \mathbb{I}(\chi L(t)/M > z)\lambda(dt)}\right\}\lambda(ds)dz
$$
$$
= \int_0^1 \mathbb{E}\left\{\frac{M}{S} \cdot \frac{\int_\mathbb{R} [L(-s)]^b \mathbb{I}(\chi L(-s)/M > z)ds}{\int_\mathbb{R} [L(t)]^b \mathbb{I}(\chi L(t)/M > z)\lambda(dt)}\right\}dz
$$
$$
= \mathbb{E}\left\{\frac{\sup_{t\in\mathbb{R}} e^{W_1(t)}}{\int_\mathbb{R} e^{W_1(t)}\lambda(dt)}\right\} = \mathcal{H}_\alpha.
$$

In the last equality we used that $z < 1$ and $L(s)/M < 1$ almost surely implies $z < L(s)/M\chi$ almost surely for all $s \in \mathbb{R}$ since $\chi \geq 1$, hence the proof for the case $\delta = 0$ in complete. If $\delta > 0$ the above calculations can be repeated. An alternative proof follows by passing to the limit in the expression given in (3.5). $\qquad\square$

ACKNOWLEDGMENTS

REFERENCES

[1] D. Aldous, "The harmonic mean formula for probabilities of unions: applications to sparse random graphs," *Discrete Math.*, vol. 76, no. 3, pp. 167–176, 1989.

[2] G. Samorodnitsky, *Stochastic processes and long range dependence*. Springer Series in Operations Research and Financial Engineering, Springer, Cham, 2016.

[3] R. J. Adler, J. H. Blanchet, and J. Liu, "Efficient Monte Carlo for high excursions of Gaussian random fields," *The Annals of Applied Probability*, vol. 22, no. 3, pp. 1167 – 1214, 2012.

[4] K. Bisewski, D. Crommelin, and M. Mandjes, "Controlling the time discretization bias for the supremum of Brownian motion," *ACM Trans. Model. Comput. Simul.*, vol. 28, no. 3, pp. Art. 24, 25, 2018.

[5] I. I. Gīhman and A. V. Skorohod, *The theory of stochastic processes. I*, vol. 210 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin-New York, english ed., 1980. Translated from the Russian by Samuel Kotz.

[6] S. Cambanis, "The measurability of a stochastic process of second order and its linear space," *Proc. Amer. Math. Soc.*, vol. 47, pp. 467–475, 1975.

[7] G. Samorodnitsky and M. S. Taqqu, *Stable non-Gaussian random processes*. Stochastic Modeling, Chapman & Hall, New York, 1994. Stochastic models with infinite variance.

[8] V. Tsirel'Son, "The density of the distribution of the maximum of a gaussian process," *Theory of Probability & Its Applications*, vol. 20, no. 4, pp. 847–856, 1976.

[9] Y. A. Davydov, M. A. Lifshits, and N. V. Smorodina, *Local properties of distributions of stochastic functionals*, vol. 173 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1998. Translated from the 1995 Russian original by V. E. Nazaĭkinskiĭ and M. A. Shishkova.

[10] M. Lifshits, *Lectures on Gaussian processes*. Springer Briefs in Mathematics, Springer, Heidelberg, 2012.

[11] J.-M. Azaïs and M. Chassan, "Discretization error for the maximum of a Gaussian field," *Stochastic Process. Appl.*, vol. 130, no. 2, pp. 545–559, 2020.

[12] P. Roy, "Nonsingular group actions and stationary $S\alpha S$ random fields," *Proc. Amer. Math. Soc.*, vol. 138, no. 6, pp. 2195–2202, 2010.

[13] S. Berman, "Sojourns and extremes of stationary processes," *Ann. Probab.*, vol. 10, no. 1, pp. 1–46, 1982.

[14] S. Berman, *Sojourns and Extremes of Stochastic Processes*. The Wadsworth & Brooks/Cole Statistics/Probability Series, Pacific Grove, CA: Wadsworth & Brooks/Cole Advanced Books & Software, 1992.

[15] V. I. Piterbarg, *Asymptotic methods in the theory of Gaussian processes and fields*, vol. 148 of *Translations of Mathematical Monographs*. Providence, RI: American Mathematical Society, 1996. Translated from the Russian by V.V. Piterbarg, revised by the author.

[16] E. Hashorva, "On extremal index of max-stable random fields," *arXiv:2003.00727, Lithuanian Math J, in press*, 2021.

[17] K. Dębicki, Z. Michna, and X. Peng, "Approximation of sojourn times of Gaussian processes," *Methodol. Comput. Appl. Probab.*, vol. 21, no. 4, pp. 1183–1213, 2019.

[18] D. Aldous, *Probability approximations via the Poisson clumping heuristic*, vol. 77 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1989.

[19] A. B. Dieker and B. Yakir, "On asymptotic constants in the theory of extremes for Gaussian processes," *Bernoulli*, vol. 20, no. 3, pp. 1600–1619, 2014.

[20] V. I. Piterbarg, "Discrete and continuous time extremes of Gaussian processes," *Extremes*, vol. 7, no. 2, pp. 161–177 (2005), 2004.

[21] K. Dębicki and E. Hashorva, "On extremal index of max-stable stationary processes," *Probab. Math. Statist.*, vol. 37, no. 2, pp. 299–317, 2017.

[22] B. Basrak and J. Segers, "Regularly varying multivariate time series," *Stochastic Process. Appl.*, vol. 119, no. 4, pp. 1055–1080, 2009.

[23] H. Planinić and P. Soulier, "The tail process revisited," *Extremes*, vol. 21, no. 4, pp. 551–579, 2018.

[24] B. Basrak and H. Planinić, "Compound Poisson approximation for random fields with application to sequence alignment," *Bernoulli*, vol. 27, no. 2, p. 1371–1408, 2021.

[25] P. Soulier, "The tail process and tail measure of continuous time regularly varying stochastic processes," *arXiv:2004.00325*, 2020.

[26] R. Kulik and P. Soulier, *Heavy tailed time series*. Cham: Springer, 2020.

[27] H. Planinić, "Palm theory for extremes of stationary regularly varying time series and random fields," *arXiv:2104.03810*, 2021.

[28] J. M. P. Albin, "On extremal theory for stationary processes," *Ann. Probab.*, vol. 18, no. 1, pp. 92–128, 1990.

[29] J. Hüsler, "Extremes of a Gaussian process and the constant H_α," *Extremes*, vol. 2, no. 1, pp. 59–70, 1999.

[30] J. Albin and H. Choi, "A new proof of an old result by Pickands," *Electron. Commun. Probab.*, vol. 15, pp. 339–345, 2010.

[31] Z. Kabluchko and Y. Wang, "Limiting distribution for the maximal standardized increment of a random walk," *Stochastic Process. Appl.*, vol. 124, no. 9, pp. 2824–2867, 2014.

[32] D. Siegmund and E. S. Venkatraman, "Using the generalized likelihood ratio statistic for sequential detection of a change-point," *Ann. Statist.*, vol. 23, no. 1, pp. 255–271, 1995.

[33] G. Buriticá, M. Nicolas, T. Mikosch, and O. Wintenberger, "Some variations on the extremal index," 2021.

[34] S. G. Kobelkov and V. I. Piterbarg, "On maximum of Gaussian random field having unique maximum point of its variance," *Extremes*, vol. 22, no. 3, pp. 413–432, 2019.

[35] K. Dębicki, E. Hashorva, and L. Wang, "Extremes of vector-valued Gaussian processes," *Stochastic Process. Appl.*, vol. 130, no. 9, pp. 5802–5837, 2020.

[36] Y. Wang and S. A. Stoev, "On the structure and representations of max-stable processes," *Adv. in Appl. Probab.*, vol. 42, no. 3, pp. 855–877, 2010.

[37] K. Dębicki, E. Hashorva, and Z. Michna, "On continuity of Pickands constants," *J. Appl. Prob., in press*, 2020.

[38] E. Hashorva, "Representations of max-stable processes via exponential tilting," *Stochastic Process. Appl.*, vol. 128, no. 9, pp. 2952–2978, 2018.

[39] M. Bladt, E. Hashorva, and G. Shevchenko, "Tail measures and regular variation," *arXiv:2103.04396v3*, 2021.

DEPARTMENT OF ACTUARIAL SCIENCE, UNIVERSITY OF LAUSANNE, SWITZERLAND

Email address: `Krzysztof.Bisewski@unil.ch`

Email address: `Enkelejd.Hashorva@unil.ch`

KYIV SCHOOL OF ECONOMICS, UKRAINE

Email address: `gshevchenko@kse.org.ua`