# On Piterbarg's Max-discretisation Theorem for Multivariate Stationary Gaussian Processes

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**Abstract:** Let $\{X(t), t \geq 0\}$ be a stationary Gaussian process with zero-mean and unit variance. A deep result derived in Piterbarg (2004), which we refer to as Piterbarg's max-discretisation theorem gives the joint asymptotic behaviour ($T \to \infty$) of the continuous time maximum $M(T) = \max_{t \in [0,T]} X(t)$, and the maximum $M^\delta(T) = \max_{t \in \mathfrak{R}(\delta)} X(t)$, with $\mathfrak{R}(\delta) \subset [0,T]$ a uniform grid of points of distance $\delta = \delta(T)$. Under some asymptotic restrictions on the correlation function Piterbarg's max-discretisation theorem shows that for the limit result it is important to know the speed $\delta(T)$ approaches 0 as $T \to \infty$. The present contribution derives the aforementioned theorem for multivariate stationary Gaussian processes.

**Key Words**: Berman condition; strong dependence; time discretisation; Piterbarg's max-discretisation theorem; limit theorems; multivariate stationary Gaussian processes.

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## 1 Introduction

Let $\{X(t), t \geq 0\}$ be a standard (zero-mean, unit-variance) stationary Gaussian process with correlation function $r(\cdot)$ and continuous sample paths. A tractable and very large class of correlation functions satisfy

$$r(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0 \tag{1}$$

for some positive constant $C$ and $\alpha \in (0, 2]$, see e.g., Piterbarg (1996). If further, the Berman condition (see Berman (1964) or Berman (1992))

$$\lim_{T \to \infty} r(T) \ln T = 0 \tag{2}$$

holds, then it is well-known, see e.g., Leadbetter et al. (1983) that the maximum $M(T) = \max_{t \in [0,T]} X(t)$ obeys the Gumbel law as $T \to \infty$, namely

$$\lim_{T \to \infty} \sup_{x \in \mathbb{R}} \Big| P\{a_T(M(T) - b_T) \leq x\} - \Lambda(x) \Big| = 0 \tag{3}$$

is valid with $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$ the cumulative distribution function of a Gumbel random variable and normalising constants defined for all large $T$ by

$$a_T = \sqrt{2 \ln T}, \quad b_T = a_T + a_T^{-1} \ln\Big((2\pi)^{-1/2} C^{1/\alpha} H_\alpha a_T^{-1+2/\alpha}\Big). \tag{4}$$

Here $H_\alpha$ denotes the well-known Pickands constant given by the limit relation

$$H_\alpha = \lim_{S \to \infty} S^{-1} \mathbb{E}\Big\{\exp\Big(\max_{t \in [0,S]} \Big(\sqrt{2} B_{\alpha/2}(t) - t^\alpha\Big)\Big)\Big\} \in (0, \infty),$$

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with $B_\alpha$ a standard fractional Brownian motion with Hurst index $\alpha$, see e.g., Mishura and Valkeila (2011) for recent characterisations of $B_\alpha$. For the main properties of Pickands and related constants, see for example Adler (1990), Piterbarg (1996), Dębicki (2002), Dębicki et al. (2003), Wu (2007), Dębicki and Kisowski (2009), Dębicki and Tabiś (2011) and Hashorva et al. (2013a). We note in passing that the first correct proof of Pickands theorem where $H_\alpha$ appears (see Pickands (1969)) is derived in Piterbarg (1972).

We say that $X$ is weakly dependent if its correlation function satisfies the Berman condition (2). A natural generalisation of (2) is the following assumption

$$\lim_{T\to\infty} r(T)\ln T = r \in (0,\infty) \tag{5}$$

in which case we say that $X$ is a strongly dependent Gaussian process. Mittal and Ylvisaker (1975) prove the limit theorem for the normalised maximum of strongly dependent stationary Gaussian processes showing that the limiting distribution is a mixture of Gumbel and Gaussian distribution. In fact, a similar result is shown therein also for the extreme case that (5) holds with $r=\infty$ with the limiting distribution being Gaussian. For other related results on extremes of strongly dependent Gaussian sequences and processes, we refer to McCormick and Qi (2000), James et al. (2007), Tan and Wang (2012), Hashorva and Weng (2013), Hashorva et al. (2013b) and the references therein.

In this paper the random variable $M(T)=\sup_{0\le t\le T} X(t), T>0$ denotes the continuous-time maximum and $M^\delta(T)=\max_{t\in\delta\mathbb{N}\cap[0,T]} X(t)$ stands for the maximum over the uniform grid $\delta\mathbb{N}\cap[0,T]$. Under the assumption (1) we need to distinguish between three types of grids: A uniform grid of points $\mathfrak{R}(\delta)=\delta\mathbb{N}$ is called sparse if $\delta=\delta(T)$ is such that

$$\lim_{T\to\infty} \delta(T)(2\ln T)^{1/\alpha} = D, \tag{6}$$

with $D=\infty$. When (6) holds for some $D\in(0,\infty)$, then the grid is referred to as Pickands grid, whereas when (6) holds with $D=0$, it is called a dense grid. Throughout this paper we assume that $\alpha\in(0,2]$.

Piterbarg (2004) derived the joint asymptotic behaviour of $M^\delta(T)$ and $M(T)$ for weakly dependent stationary Gaussian processes. As shown therein, after a suitable normalisation (as in (3)) $M^\delta(T)$ and $M(T)$ are asymptotically independent, dependent or totally dependent if the grid is a sparse, a Pickands or a dense grid, respectively. We shall refer to that result as Piterbarg's max-discretisation theorem.

For a large class of locally stationary Gaussian processes Hüsler (2004) proved a similar result to Piterbarg (2004) considering only sparse and dense grids. In another investigation concerning the storage process with fractional Brownian motion as input, it was shown in Hüsler and Piterbarg (2004) that the continuous time maximum and the discrete time maximum over the dense grid are asymptotically completely dependent. Tan and Tang (2012) and Tan and Wang (2013) recently proved Piterbarg's max-discretisation theorem for strongly dependent stationary Gaussian processes, whereas Tan and Hashorva (2012) derives similar results for sparse and dense grids for standardised maximum of stationary Gaussian processes. Novel and deep results concerning stationary non-Gaussian processes are derived in Turkman (2012).

As noted in Piterbarg (2004) derivation of the joint asymptotic behaviour of $M^\delta(T)$ and $M(T)$ is important for theoretical problems and at the same time is crucial for applications, see Piterbarg (2004), Hüsler (2004) and Tan and Hashorva (2012) for more details.

The main contribution of this paper is the derivation of Piterbarg's max-discretisation theorem for multivariate stationary Gaussian processes. Our results show that, despite the high technical difficulties, it is possible to state Piterbarg's result in multidimensional settings allowing for asymptotic conditions and the two maxima are no longer asymptotically independent.

Brief organisation of the paper: In Section 2 we present the principal results. Section 3 presents some auxiliary results followed by Section 4 which is dedicated to the proofs of the our main theorems. Several technical lemmas and the proof of Lemma 3.1 are displayed in Appendix.

## 2 Main Results

Consider $(X_1(t), \cdots, X_p(t)), p \in \mathbb{N}$ a $p$-dimensional centered Gaussian vector process with covariance functions $r_{kk}(\tau) = Cov(X_k(t), X_k(t+\tau)), k \le p$. Hereafter we shall assume that the components have continuous sample paths and further $Cov(X_k(t), X_l(t+\tau))$ does not dependent on $t$ so we shall write

$$r_{kl}(\tau) = Cov(X_k(t), X_l(t+\tau))$$

for the cross-covariance function. Further we shall suppose that each component $X_i$ has a unit variance function; in short we shall refer to such vector processes as standard stationary Gaussian vector process. Similarly to (1) we suppose that for all indices $k \le p$

$$r_{kk}(t) = 1 - C|t|^\alpha + o(|t|^\alpha) \quad \text{as} \quad t \to 0, \tag{7}$$

with some positive constants $C$, and further

$$\lim_{T\to\infty} r_{kl}(T) \ln T = r_{kl} \in (0, \infty) \tag{8}$$

holds for $1 \le k, l \le p$. In order to exclude the possibility that $X_k(t) = \pm X_l(t + t_0)$ for some $k \ne l$, and some choice of $t_0$ and $+$ or $-$, we assume that

$$\max_{k\ne l} \sup_{t\in[0,\infty)} |r_{kl}(t)| < 1. \tag{9}$$

For any $k \le p$ and a given uniform grid of points $\mathfrak{R}(\delta)$ we define the componentwise maximum (in continuous and discrete time) by

$$(M_k(T), M_k^\delta(T)) := \Big(\max_{t\in[0,T]} X_k(t), \max_{t\in\mathfrak{R}(\delta)\cap[0,T]} X_k(t)\Big).$$

Let $\boldsymbol{Z} = (Z_1, \ldots, Z_p)$ be a $p$-dimensional centered Gaussian random vector with covariances

$$Cov(Z_k, Z_l) = \frac{r_{kl}}{\sqrt{r_{kk} r_{ll}}}.$$

Further, let $\Psi$ denote the survival function of a $N(0,1)$ random variable and put $\boldsymbol{x} := (x_1, \ldots, x_p), \boldsymbol{y} := (y_1, \ldots, y_p)$. In our theorem below we consider the case of sparse grids, followed then by two results on Pickands and dense grids.

**Theorem 2.1**. *Let $(X_1(t), \cdots, X_p(t))$ be standard stationary Gaussian vector process with covariance functions satisfying (7), (8) and (9). If further $\boldsymbol{Z}$ has a positive-definite covariance matrix, then for any sparse grid $\mathfrak{R}(\delta)$*

$$\lim_{T\to\infty} \sup_{\boldsymbol{x}\in\mathbb{R}^p, \boldsymbol{y}\in\mathbb{R}^p} \left| P\left\{a_T(M_k(T) - b_T) \le x_k, a_T(M_k^\delta(T) - b_T^\delta) \le y_k, k = 1, \cdots, p\right\} - \mathbb{E}\Big\{\exp\left(-f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z})\right)\Big\} \right| = 0, \tag{10}$$

*where $a_T$ is defined in (4),*

$$f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{Z}) = \sum_{k=1}^{p} \left(e^{-x_k - r_{kk} + \sqrt{2r_{kk}} Z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}} Z_k}\right)$$

*and*

$$b_T^\delta = a_T + a_T^{-1}\ln\Big((2\pi)^{-1/2}\delta^{-1}a_T^{-1}\Big). \tag{11}$$

**Corollary 2.1**. *Under the assumptions of Theorem 2.1 we have*

$$\lim_{T\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^p}\left|P\left\{a_T(M_k(T)-b_T)\le x_k, k=1,\cdots,p\right\}-\mathbb{E}\Big\{\exp\left(-h(\boldsymbol{x},\boldsymbol{x},\boldsymbol{Z})\right)\Big\}\right|=0, \tag{12}$$

*where*

$$h(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})=\sum_{k=1}^{p}e^{-\min(x_k,y_k)-r_{kk}+\sqrt{2r_{kk}}Z_k} \tag{13}$$

*and $a_T, b_T$ are defined in* (4).

Before presenting the result for Pickands grids, we introduce the following constants which can be found in Leadbetter et al. (1983). For any $d>0, \lambda>0$, $k\in\mathbb{Z}$ and $x,y\in\mathbb{R}$ define

$$H_{d,\alpha}(\lambda)=\mathbb{E}\Big\{\exp\left(\max_{kd\in[0,\lambda]}\left(\sqrt{2}B_{\alpha/2}(kd)-(kd)^\alpha\right)\right)\Big\}$$

and

$$H_{d,\alpha}^{x,y}(\lambda)=\lim_{\lambda\to\infty}\frac{1}{\lambda}\int_{s\in\mathbb{R}}e^sP\left(\max_{t\in[0,\lambda]}\left(\sqrt{2}B_{\alpha/2}(t)-t^\alpha\right)>s+x,\ \max_{k:kd\in[0,\lambda]}\left(\sqrt{2}B_{\alpha/2}(kd)-(kd)^\alpha\right)>s+y\right)ds.$$

In view of Leadbetter et al. (1983) both constants $H_{d,\alpha}$ and $H_{d,\alpha}^{x,y}$ defined as

$$H_{d,\alpha}=\lim_{\lambda\to\infty}\frac{H_{d,\alpha}(\lambda)}{\lambda}\quad\text{ and }\quad H_{d,\alpha}^{x,y}=\lim_{\lambda\to\infty}\frac{H_{d,\alpha}^{x,y}(\lambda)}{\lambda}$$

are finite and positive.

**Theorem 2.2**. *Let $(X_1(t),\cdots,X_p(t))$ satisfy the assumptions of Theorem 2.1, and let $a_T$ be as in* (4). *For any Pickands grid $\mathfrak{R}(da_T^{-2/\alpha})$ with $d>0$ we have*

$$\lim_{T\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^p,\boldsymbol{y}\in\mathbb{R}^p}\left|P\left\{a_T(M_k(T)-b_T)\le x_k, a_T(M_k^\delta(T)-b_{d,T})\le y_k, k=1,\cdots,p\right\}-\mathbb{E}\Big\{\exp\left(-g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})\right)\Big\}\right|=0, \tag{14}$$

*where*

$$g(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})=\sum_{k=1}^{p}\left(e^{-x_k-r_{kk}+\sqrt{2r_{kk}}Z_k}+e^{-y_k-r_{kk}+\sqrt{2r_{kk}}Z_k}-H_{d,\alpha}^{\ln H_\alpha+x_k,\ln H_{d,\alpha}+y_k}e^{-r_{kk}+\sqrt{2r_{kk}}Z_k}\right),$$

*with*

$$b_{d,T}=a_T+a_T\ln\Big((2\pi)^{-1/2}C^{1/\alpha}H_{d,\alpha}a_T^{-1+2/\alpha}\Big). \tag{15}$$

**Theorem 2.3**. *Under the assumptions of Theorem 2.1 for any dense grid $\mathfrak{R}(\delta)$*

$$\lim_{T\to\infty}\sup_{\boldsymbol{x}\in\mathbb{R}^p,\boldsymbol{y}\in\mathbb{R}^p}\left|P\left\{a_T(M_k(T)-b_T)\le x_k, a_T(M_k^\delta(T)-b_T)\le y_k, k=1,\cdots,p\right\}-\mathbb{E}\Big\{\exp\left(-h(\boldsymbol{x},\boldsymbol{y},\boldsymbol{Z})\right)\Big\}\right|=0, \tag{16}$$

*with $a_T, b_T$ as defined in* (4) *and $h$ defined in* (13).

**Remark 2.1**. a) In condition (7) we can use different $C's$ and $\alpha's$, i.e., condition (7) can be replaced by

$$r_{kk}(t) = 1 - C_k|t|^{\alpha_k} + o(|t|^{\alpha_k}) \quad \text{as} \quad t \to 0.$$

In that case, the above results still hold with some obvious modifications of $b_T$, $b_T^\delta, b_{d,T}$ and the grid $\mathfrak{R}(\delta)$.
b) If $\boldsymbol{Z}$ has a singular covariance matrix, then we still can derive the above results. To see that consider the simple case $Z_p = c_1 Z_1 + \cdots + c_{p-1}Z_{p-1}$ with $c_i, i \le p-1$ some constants, and $\boldsymbol{Z}^* = (Z_1, \ldots, Z_{p-1})$ has a non-singular covariance matrix. In the proofs below we need to condition on $Z_1 = z_1, \ldots, Z_{p-1} = z_{p-1}$, and then put $c_1z_1 + \cdots + c_{p-1}z_{p-1}$ instead of $z_p$ therein.

# 3 Auxiliary Results

In this section we present several lemmas needed for the proof of the main results, where Lemma 3.1 plays a crucial role. In order to establish Piterbarg's max-discretisation theorem for standard stationary vector Gaussian processes we need to closely follow the steps of the proofs in Piterbarg (2004), and of course to strongly rely on the deep ideas and techniques presented in Piterbarg (1996). First, for $1 \le k, l \le p$ define

$$\rho_{kl}(T) = r_{kl}/\ln T.$$

Following the former reference, we divide the interval $[0, T]$ onto intervals of length $S$ alternating with shorter intervals of length $R$. Let $a > b$ be constants which will be determined in the proof of Lemma 3.1. We shall denote throughout in the sequel

$$S = T^a, \quad R = T^b, \quad T > 0.$$

Denote the long intervals by $\mathcal{S}_l$, $l = 1, 2, \cdots, n_T = [T/(S+R)]$, and the short intervals by $\mathcal{R}_l$, $l = 1, 2, \cdots, n_T$. It will be seen from the proofs, that a possible remaining interval with length different than $S$ or $R$ plays no role in our asymptotic considerations; we call also this interval a short interval. Define further $\mathbf{S} = \cup_{l=1}^{n_T}\mathcal{S}_l, \mathbf{R} = \cup_{l=1}^{n_T}\mathcal{R}_l$ and thus $[0, T] = \mathbf{S} \cup \mathbf{R}$.
Our proofs also rely on the main ideas of Mittal and Ylvisaker (1975) by constructing new Gaussian processes to approximate the original Gaussian processes. For each index $k \le p$ we define a new Gaussian process $\eta_k$ by taking $\{Y_k^{(j)}(t), t \ge 0\}$, $j = 1, 2, \cdots, n_T$ independent copies of $\{X_k(t), t \ge 0\}$ and setting $\eta_k(t) = Y_k^{(j)}(t)$ for $t \in \mathcal{R}_j \cup \mathcal{S}_j = [(j-1)(S+R), j(S+R))$. We construct the processes so that $\eta_k, k = 1, \cdots, p$ are independent by taking $Y_k^{(j)}$ to be independent for any $j$ and $k$ two possible indices. The independence of two different processes $\eta_k$ and $\eta_l$ implies

$$\gamma_{kl}(s,t) := \mathbb{E}\Big\{\eta_k(s)\eta_l(t)\Big\} = 0, \quad k \ne l,$$

whereas for any fixed $k$

$$\gamma_{kk}(s,t) := \mathbb{E}\Big\{\eta_k(s)\eta_k(t)\Big\} = \begin{cases} \mathbb{E}\Big\{Y_k^{(i)}(t), Y_k^{(i)}(s)\Big\} = r_{kk}(s,t), & \text{if } t, s \in \mathcal{R}_i \cup \mathcal{S}_i, \text{ for some } i \le n_T; \\ \mathbb{E}\Big\{Y_k^{(i)}(t), Y_k^{(j)}(s)\Big\} = 0, & \text{if } t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, \text{ for some } i \ne j \le n_T. \end{cases}$$

For $k = 1, 2, \cdots, p$ define

$$\xi_k^T(t) = \big(1 - \rho_{kk}(T)\big)^{1/2}\eta_k(t) + \rho_{kk}^{1/2}(T)Z_k, \quad 0 \le t \le T,$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_p)$ is a $p$-dimensional centered Gaussian random vector defined in Section 2, which is independent of $\{\eta_k(t), t \ge 0\}$, $k = 1, 2, \cdots, p$. Denote by $\{\varrho_{kl}(s,t), 1 \le k, l \le p\}$ the covariance functions of $\{\xi_k^T(t), 0 \le t \le T, k =$

$1, 2, \cdots, p\}$. We have

$$\varrho_{kl}(s,t) = \mathbb{E}\Big\{\xi_k^T(s)\xi_l^T(t)\Big\} = \rho_{kl}(T), \quad k \neq l$$

and

$$\varrho_{kk}(s,t) = \begin{cases} r_{kk}(s,t) + (1 - r_{kk}(s,t))\rho_{kk}(T), & t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, i = j; \\ \rho_{kk}(T), & t \in \mathcal{R}_i \cup \mathcal{S}_i, s \in \mathcal{R}_j \cup \mathcal{S}_j, i \neq j. \end{cases}$$

For notational simplicity we write for any $x_k, y_k \in \mathbb{R}$

$$u_k^{(1)} = b_T + x_k/a_T, \quad u_k^{(2)} = b_T^{'} + y_k/a_T,$$

where $b_T^{'} = b_T^{\delta}$ for a sparse grid and $b_T^{'} = b_{d,T}$ for Pickands grid. Further, for any $\varepsilon > 0$ set

$$q = \frac{\varepsilon}{(\ln T)^{1/\alpha}}.$$

We give first a crucial result which shows that the maximums of the original Gaussian processes $(X_1(t), \cdots, X_p(t))$ can be approximated by that of the Gaussian processes $\{\xi_k^T(t), 0 \leq t \leq T, k = 1, 2, \cdots, p\}$. The proof of the next Lemma, due to its complications, is relegated to the Appendix.

**Lemma 3.1**. *Suppose that the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid. For any $B > 0$ for all $x_k, y_k \in [-B, B], k \leq p$ we have*

$$\lim_{T\to\infty} \Bigg| P\left\{ \max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} X_k(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \\ - P\left\{ \max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} \xi_k^T(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} \xi_k^T(t) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \Bigg| = 0.$$

In order to deal with our multivariate framework in Lemmas 3.2 and 3.3 below we present the multivariate versions of Lemmas 6 and 4 in Piterbarg (2004), respectively. Lemma 3.4 is a new result.

**Lemma 3.2**. *Suppose that the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid. For any $B > 0$ there exits a positive constant $K$ such that for all $x_k, y_k \in [-B, B]$ we have*

$$\Bigg| P\left\{ M_k(T) \leq u_k^{(1)}, M_k^{\delta}(T) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \\ - P\left\{ \max_{t\in\mathbf{S}} X_k(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \Bigg| \leq K(\ln T)^{1/\alpha-1/2} T^{b-a},$$

*with $0 < b < a < 1$ given constants and all $T$ large.*

**Proof:** In order to obtain the upper bound, we shall use the following inequality

$$\Bigg| P\left\{ M_k(T) \leq u_k^{(1)}, M_k^{\delta}(T) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \\ - P\left\{ \max_{t\in\mathbf{S}} X_k(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \leq u_k^{(2)}, k = 1, \cdots, p \right\} \Bigg| \\ \leq \sum_{k=1}^{p} P\left\{ \max_{t\in\mathbf{R}} X_k(t) > b_T + x_k/a_T \right\} + \sum_{k=1}^{p} P\left\{ \max_{t\in\mathbf{R}\cap\mathfrak{R}(\delta)} X_k(t) > b_T^{'} + y_k/a_T \right\} \tag{17}$$

valid for any $x_i \in \mathbb{R}, y_i \in \mathbb{R}, i \leq k$. By Pickands theorem

$$P\left\{ \max_{t\in\mathbf{R}} X_k(t) > b_T + x_k/a_T \right\} = O(mes(\mathbf{R})(b_T + x_k/a_T)^{2/\alpha})\Psi(b_T + x_k/a_T)$$

as $T \to \infty$, where $mes(\mathbf{R})$ denotes the Lebesgue measure of $\mathbf{R}$. In the light of (11) and (16) of Piterbarg (2004) for a sparse grid and Pickands grid, respectively, we get the same order for the second probability in the right-hand side of (17), hence the proof is complete. $\square$

**Lemma 3.3**. *Suppose that the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid. For any $B > 0$ for all $x_k, y_k \in [-B, B], k \le p$ and for the Pickands grid $\mathfrak{R}(q) = \mathfrak{R}(\varepsilon/(\ln T)^{1/\alpha})$ we have*

$$\Bigg| P\left\{\max_{t\in\mathbf{S}} X_k(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \le u_k^{(2)}, k = 1, \cdots, p\right\}$$
$$-P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} X_k(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \le u_k^{(2)}, k = 1, \cdots, p\right\}\Bigg| \to 0$$

*as $\varepsilon \downarrow 0$.*

**Proof:** In view of Lemma 4 of Piterbarg (2004)

$$\begin{aligned}
&\Bigg| P\left\{\max_{t\in\mathbf{S}} X_k(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \le u_k^{(2)}, k = 1, \cdots, p\right\} \\
&\qquad -P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} X_k(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \le u_k^{(2)}, k = 1, \cdots, p\right\}\Bigg| \\
&\le \sum_{k=1}^{p}\left[P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} X_k(t) \le u_k^{(1)}\right\} - P\left\{\max_{t\in\mathbf{S}} X_k(t) \le u_k^{(1)}\right\}\right] \\
&\le g(\varepsilon)\sum_{k=1}^{p} n_T T^a (u_k^{(1)})^{2/\alpha}\Psi(u_k^{(1)}) \le K g(\varepsilon),
\end{aligned}$$

where $g(\varepsilon) \to 0$ as $\varepsilon \downarrow 0$, hence the claim follows. $\square$

**Lemma 3.4**. *Suppose that the grid $\mathfrak{R}(\delta)$ is a sparse grid or a Pickands grid. For any $B > 0$ for all $x_k, y_k \in [-B, B], k \le p$ and for the Pickands grid $\mathfrak{R}(q) = \mathfrak{R}(\varepsilon/(\ln T)^{1/\alpha})$ we have*

$$\Bigg| P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} \xi_k^T(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} \xi_k^T(t) \le u_k^{(2)}, k = 1, \cdots, p\right\}$$
$$-\int_{\mathbf{z}\in\mathbb{R}^p}\prod_{i=1}^{n_T} P\left\{\max_{t\in\mathcal{S}_i}\eta_k(t) \le u_k^*, \max_{t\in\mathfrak{R}(\delta)\cap\mathcal{S}_i}\eta_k(t) \le u_k^{*'}, k = 1, \cdots, p\right\} d\Phi_p(\boldsymbol{z})\Bigg| \to 0$$

*as $\varepsilon \downarrow 0$, where $\Phi_p$ is the distribution function of $\boldsymbol{Z}$*

$$u_k^* := \frac{b_T + x_k/a_T - \rho_{kk}^{1/2}(T)z_k}{(1-\rho_{kk}(T))^{1/2}} = \frac{x_k + r_{kk} - \sqrt{2r_{kk}}z_k}{a_T} + b_T + o(a_T^{-1}), \tag{18}$$

$$u_k^{*'} := \frac{b_T' + y_k/a_T - \rho_{kk}^{1/2}(T)z_k}{(1-\rho_{kk}(T))^{1/2}} = \frac{y_k + r_{kk} - \sqrt{2r_{kk}}z_k}{a_T} + b_T' + o(a_T^{-1}) \tag{19}$$

*and $b_T' = b_T^\delta$ for a sparse grid and $b_T' = b_{a,T}$ for a Pickands grid.*

**Proof:** First, by the definition of $\xi_k^T$ and $\eta_k$ we have

$$\begin{aligned}
&P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} \xi_k^T(t) \le u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} \xi_k^T(t) \le u_k^{(2)}, k = 1, \cdots, p\right\} \\
&= \int_{\mathbf{z}\in\mathbb{R}^p} P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}}\eta_k(t) \le u_k^*, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}}\eta_k(t) \le u_k^{*'}, k = 1, \cdots, p\right\} d\Phi_p(\boldsymbol{z}) \\
&= \int_{\mathbf{z}\in\mathbb{R}^p}\prod_{i=1}^{n_T} P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathcal{S}_i}\eta_k(t) \le u_k^*, \max_{t\in\mathfrak{R}(\delta)\cap\mathcal{S}_i}\eta_k(t) \le u_k^{*'}, k = 1, \cdots, p\right\} d\Phi_p(\boldsymbol{z}).
\end{aligned} \tag{20}$$

As for the discrete case, see p. 137 on Leadbetter et al. (1983) direct calculations lead to (18) and (19). Next, similarly to the proof of Lemma 3.3, for all large $T$

$$\begin{aligned}
&\Bigg|\prod_{i=1}^{n_T} P\left\{\max_{t\in\Re(q)\cap\mathcal{S}_i}\eta_k(t)\le u_k^*,\ \max_{t\in\Re(\delta)\cap\mathcal{S}_i}\eta_k(t)\le u_k^{*'},k=1,\cdots,p\right\}\\
&\qquad-\prod_{i=1}^{n_T} P\left\{\max_{t\in\mathcal{S}_i}\eta_k(t)\le u_k^*,\ \max_{t\in\Re(\delta)\cap\mathcal{S}_i}\eta_k(t)\le u_k^{*'},k=1,\cdots,p\right\}\Bigg|\\
&\le\sum_{i=1}^{n_T}\left[P\left\{\max_{t\in\Re(q)\cap\mathcal{S}_i}\eta_k(t)\le u_k^*,k=1,\cdots,p\right\}-P\left\{\max_{t\in\mathcal{S}_i}\eta_k(t)\le u_k^{*'},k=1,\cdots,p\right\}\right]\\
&\le g(\varepsilon)T^a\sum_{i=1}^{n_T}(u_k^*)^{2/\alpha}\Psi(u_k^*)\le Kg(\varepsilon)
\end{aligned}$$

holds for some constant $K$, thus the claim follows by applying the dominated convergence theorem and letting $\varepsilon\downarrow 0$.
□

# 4 Proofs

**Proof of Theorem 2.1**. From Lemmas 3.1-3.4 and the dominated convergence theorem, we known that in order to prove Theorem 2.1, it suffice to show that

$$\begin{aligned}
\lim_{T\to\infty}&\Bigg|\prod_{i=1}^{n_T} P\left\{\max_{t\in\mathcal{S}_i}\eta_k(t)\le u_k^*,\ \max_{t\in\Re(\delta)\cap\mathcal{S}_i}\eta_k(t)\le u_k^{*'},k=1,\cdots,p\right\}\\
&\qquad-\exp\left(-\sum_{k=1}^{p}(e^{-x_k-r_{kk}+\sqrt{2r_{kk}}z_k}+e^{-y_k-r_{kk}+\sqrt{2r_{kk}}z_k})\right)\Bigg|=0. \qquad (21)
\end{aligned}$$

Define next the events

$$\mathcal{A}_i=\left\{\max_{t\in[0,S]}\eta_i(t)>u_i^*\right\},\quad \mathcal{A}_{p+i}=\left\{\max_{t\in\Re(\delta)\cap[0,S]}\eta_i(t)>u_i^{*'}\right\},\quad i=1,\cdots,p.$$

Using the stationarity of $\{\eta_k(t),k=1,2,\cdots,p\}$ (we write $\mathcal{A}_i^c$ for the complimentary event of $\mathcal{A}_i$)

$$\begin{aligned}
\prod_{i=1}^{n_T} P\left\{\max_{t\in\mathcal{S}_i}\eta_k(t)\le u_k^*,\ \max_{t\in\Re(\delta)\cap\mathcal{S}_i}\eta_k(t)\le u_k^{*'},k=1,\cdots,p\right\} &= (P\{\cap_{i=1}^{2p}\mathcal{A}_i^c\})^{n_T}\\
&= \exp\big(n_T\ln(P\{\cap_{i=1}^{2p}\mathcal{A}_i^c\})\big)\\
&= \exp\big(-n_TP\{\cup_{i=1}^{2p}\mathcal{A}_i\}+W_{n_T}\big).
\end{aligned}$$

Since $\lim_{T\to\infty}P\{\cap_{i=1}^{2p}\mathcal{A}_i\}=1$ we get that the remainder $W_{n_T}$ satisfies

$$W_{n_T}=o(n_TP\{\cup_{i=1}^{2p}\mathcal{A}_i\}),\quad T\to\infty.$$

Next, by Bonferroni inequality

$$\begin{aligned}
\sum_{i=1}^{2p}P\{\mathcal{A}_i\} &\ge P\{\cup_{i=1}^{2p}\mathcal{A}_i\}\ge\sum_{i=1}^{2p}P\{\mathcal{A}_i\}-\sum_{1\le i<j\le 2p}P\{\mathcal{A}_i,\mathcal{A}_j\}\\
&= \sum_{i=1}^{2p}P\{\mathcal{A}_i\}-\sum_{1\le k<l\le p}P\{\mathcal{A}_k,\mathcal{A}_l\}-\sum_{1\le k<l\le p}P\{\mathcal{A}_{p+k},\mathcal{A}_{p+l}\}-2\sum_{1\le k<l\le p}P\{\mathcal{A}_k,\mathcal{A}_{p+l}\}
\end{aligned}$$

$$-\sum_{i=1}^{p} P\{\mathcal{A}_i, \mathcal{A}_{p+i}\} =: A_1 - A_2 - A_3 - 2A_4 - A_5. \tag{22}$$

Further, Lemma 2 of Piterbarg (2004) and (18), (19) imply

$$\begin{aligned} A_1 &\sim \sum_{k=1}^{p} ST^{-1}(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + + \sqrt{2r_{kk}}z_k}) \\ &= \sum_{k=1}^{p} \frac{T^a}{T}(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}}z_k}), \quad T \to \infty. \end{aligned}$$

For $A_2$, by the independence of $\eta_k(t)$ and $\eta_l(t)$, $k \neq l$, Lemma 2 of Piterbarg (2004) and (18), (19), we have

$$\begin{aligned} A_2 &= \sum_{1 \le k < l \le p} P\left\{\max_{t \in [0,S]} \eta_k(t) > u_k^*, \max_{t \in [0,S]} \eta_l(t) > u_l^*\right\} \\ &= \sum_{1 \le k < l \le p} P\left\{\max_{t \in [0,S]} \eta_k(t) > u_k^*\right\} P\left\{\max_{t \in [0,S]} \eta_l(t) > u_l^*\right\} \\ &\sim \sum_{1 \le k < l \le p} ST^{-1}e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} ST^{-1}e^{-x_l - r_{ll} + \sqrt{2r_{ll}}z_l} = o(A_1). \end{aligned}$$

Note that $\mathfrak{R}(\delta)$ is a sparse grid, similar arguments as for $A_2$ imply

$$A_k = o(A_1), \quad k = 2, 3.$$

Further, Lemma 2 of Piterbarg (2004) implies $A_5 = o(A_1)$. Consequently, as $T \to \infty$

$$n_T P\{\cup_{i=1}^{2p} \mathcal{A}_i\} \sim \sum_{k=1}^{p} \left(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}}z_k}\right),$$

which completes the proof of (21). $\square$

**Proof of Theorem 2.2**. In view of Lemmas 3.1-3.4 and the dominated convergence theorem in order to establish the proof we need to show

$$\begin{aligned} &\Bigg| \prod_{i=1}^{n_T} P\left\{\max_{t \in \mathcal{S}_i} \eta_k(t) \le u_k^*, \max_{t \in \mathfrak{R}(\delta) \cap \mathcal{S}_i} \eta_k(t) \le u_k^{*'}, k = 1, \cdots, p\right\} \\ &\quad - \exp\left(-\sum_{k=1}^{p} \left(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}}z_k} - H_{d,\alpha}^{\ln H_\alpha + x_k, \ln H_{d,\alpha} + y_k} e^{-r_{kk} + \sqrt{2r_{kk}}z_k}\right)\right) \Bigg| \to 0 \end{aligned} \tag{23}$$

as $T \to \infty$. We proceed as for the proof of (21) using the lower bound (22); we have thus

$$\begin{aligned} P\{\cup_{i=1}^{2p} \mathcal{A}_i\} &= \sum_{i=1}^{2p} P\{\mathcal{A}_i\} - \sum_{1 \le k < l \le p} P\{\mathcal{A}_k, \mathcal{A}_l\} - \sum_{1 \le k < l \le p} P\{\mathcal{A}_{p+k}, \mathcal{A}_{p+l}\} - 2\sum_{1 \le k < l \le p} P\{\mathcal{A}_k, \mathcal{A}_{p+l}\} \\ &\quad - \sum_{i=1}^{p} P\{\mathcal{A}_i, \mathcal{A}_{p+i}\} + \sum_{1 \le k < \cdots < l \le p} =: A_1 - A_2 - A_3 - 2A_4 - A_5 + A_6. \end{aligned} \tag{24}$$

By Lemma 3 of Piterbarg (2004) and (18), (19) as $T \to \infty$

$$\begin{aligned} A_1 &\sim \sum_{k=1}^{p} ST^{-1}(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}}z_k}) \\ &= \sum_{k=1}^{p} \frac{T^a}{T}(e^{-x_k - r_{kk} + \sqrt{2r_{kk}}z_k} + e^{-y_k - r_{kk} + \sqrt{2r_{kk}}z_k}). \end{aligned}$$

With similar arguments as for $A_2, A_3, A_4$ in the proof of Theorem 2.1, we conclude that

$$A_k = o(A_1), \quad k = 2, 3, 4.$$

For the sum $A_6$, it is easy to see that each term in $A_6$ can be bounded by $A_3$ or $A_4$, and thus $A_6 = o(A_1)$. The claim follows now easily borrowing the arguments in p. 176 of Piterbarg (2004). $\square$

**Proof of Theorem 2.3**. First, recall that $\mathfrak{R}(\delta)$ is a dense grid in this case. By Lemma 5 of Piterbarg (2004) we have

$$\begin{aligned}
&\Big| P\left\{a_T(M_k(T) - b_T) \le x_k, a_T(M_k^\delta(T) - b_T) \le y_k, k = 1, \cdots, p\right\} \\
&\qquad - P\left\{a_T(M_k(T) - b_T) \le x_k, a_T(M_k(T) - b_T) \le y_k, k = 1, \cdots, p\right\} \Big| \\
&\le \sum_{k=1}^{p} \Big| P\left\{a_T(M_k^\delta(T) - b_T) \le y_k\right\} - P\left\{a_T(M_k(T) - b_T) \le y_k\right\} \Big| \to 0, \quad T \to \infty.
\end{aligned}$$

Since

$$\begin{aligned}
&P\left\{a_T(M_k(T) - b_T) \le x_k, a_T(M_k(T) - b_T) \le y_k, k = 1, \cdots, p\right\} \\
&= P\left\{a_T(M_k(T) - b_T) \le \min(x_k, y_k), k = 1, \cdots, p\right\}
\end{aligned}$$

in order to complete the proof, we only need to show that

$$\lim_{T\to\infty} P\left\{a_T(M_k(T) - b_T) \le \min(x_k, y_k), k = 1, \cdots, p\right\} = \int_{\mathbf{z}\in\mathbb{R}^p} \exp\left(-h(\mathbf{x}, \mathbf{y}, \mathbf{z})\right) d\Phi_p(\boldsymbol{z}),$$

which follows from Corollary 2.1. $\square$

# 5 Appendix

In this section, we give the detailed proof of Lemma 3.1 which is based on the results of six lemmas given below. Let in the following $\mathcal{C}$ be constant whose value will change from place to place. Define further $r_{kl}^{(h)}(t, s) = hr_{kl}(t, s) + (1 - h)\varrho_{kl}(t, s)$ for $h \in [0, 1]$ and $1 \le k, l \le p$ and let $\vartheta_{kk}(t) = \sup_{t<|nq-mq|\le T}\{\varpi_{kk}(nq, mq)\}$, where $\varpi_{kk}(nq, mq) = \max\{r_{kk}(nq, mq), \varrho_{kk}(nq, mq)\}$. Assumption (7) implies that $\vartheta_{kk}(\epsilon) < 1$ for all $T$ and any $\epsilon \in (0, 2^{-1/\alpha})$. Consequently, we may choose some positive constant $\beta_{kk}$ such that

$$\beta_{kk} < \frac{1 - \vartheta_{kk}(\epsilon)}{1 + \vartheta_{kk}(\epsilon)} \tag{25}$$

for all sufficiently large $T$. In the following we choose

$$0 < a < b < \min_{k\in\{1,2,\cdots,p\}} \beta_{kk}$$

and we set $\Delta_{kl}(ns, mt) := |r_{kl}(ns, mt) - \varrho_{kl}(ns, mt)|$ for all possible indices $k, l$.

**Lemma 5.1**. *Under conditions of Lemma 3.1, we have*

$$\sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T}} \Delta_{kk}(nq, mq) \int_0^1 \frac{1}{\sqrt{1 - r_{kk}^{(h)}(nq, mq)}} \exp\left(-\frac{(u_k^{(1)})^2}{1 + r_{kk}^{(h)}(nq, mq)}\right) dh \to 0$$

*as* $T \to \infty$.

**Proof**: First, we consider the case t $nq$ and $mq$ are in the same interval $\mathcal{S}$ which implies $\varrho_{kk}(nq, mq) = r_{kk}(nq, mq) + (1 - r_{kk}(nq, mq))\rho_{kk}(T) \sim r_{kk}(nq, mq)$ as $T \to \infty$. Split the sum in the lemma into two parts as

$$\sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|<\epsilon}} (\cdots) + \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|\geq\epsilon}} (\cdots) =: J_{T,1} + J_{T,2}.$$

For the term $J_{T,1}$ note that Assumption (7) implies for all $|s-t| \leq \epsilon < 2^{-1/\alpha}$

$$\frac{1}{2}|s-t|^{\alpha} \leq 1 - r_{kk}(s,t) \leq 2|s-t|^{\alpha}.$$

By the definition of $u_k^{(1)}$

$$(u_k^{(1)})^2 = 2\ln T - \ln\ln T + \frac{2}{\alpha}\ln\ln T + O(1), \quad T \to \infty. \tag{26}$$

Consequently, since further $q = \varepsilon(\ln T)^{-1/\alpha}$

$$\begin{aligned}
J_{T,1} &\leq \mathcal{C} \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|<\epsilon}} \Delta_{kk}(nq,mq)\frac{1}{\sqrt{1-r_{kk}(nq,mq)}}\exp\left(-\frac{(u_k^{(1)})^2}{1+r_{kk}(nq,mq)}\right) \\
&\leq \mathcal{C}\rho_{kk}(T) \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|<\epsilon}} \frac{1-r_{kk}(nq,mq)}{\sqrt{1-r_{kk}(nq,mq)}}\exp\left(-\frac{(u_k^{(1)})^2}{2}\right)\exp\left(-\frac{(1-r_{kk}(nq,mq))(u_k^{(1)})^2}{2(1+r_{kk}(nq,mq))}\right) \\
&\leq \mathcal{C}\rho_{kk}(T)T^{-1}(\ln T)^{1/2-1/\alpha} \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|<\epsilon}} \sqrt{1-r_{kk}(nq,mq)}\exp\left(-\frac{(1-r_{kk}(nq,mq))(u_k^{(1)})^2}{2(1+r_{kk}(nq,mq))}\right) \\
&\leq \mathcal{C}\rho_{kk}(T)T^{-1}(\ln T)^{1/2-1/\alpha} \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|<\epsilon}} |nq-mq|^{\alpha/2}\exp\left(-\frac{1}{8}|nq-mq|^{\alpha}(u_k^{(1)})^2\right) \\
&\leq \mathcal{C}(\ln T)^{-1/2}\sum_{0<nq<\epsilon}(nq)^{\alpha/2}\exp\left(-\frac{1}{8}(nq)^{\alpha}(u_k^{(1)})^2\right) \\
&\leq \mathcal{C}(\ln T)^{-1/2}\sum_{0<nq<\epsilon}\exp\left(-\frac{1}{8}(nq)^{\alpha}\ln T\right) \\
&\leq \mathcal{C}(\ln T)^{-1/2}\sum_{n=1}^{\infty}\exp\left(-\frac{1}{8}(\varepsilon n)^{\alpha}\right) \\
&\leq \mathcal{C}(\ln T)^{-1/2}
\end{aligned} \tag{27}$$

implying thus $\lim_{T\to\infty} J_{T,1} = 0$. Using the fact that $u_k^{(1)} \sim (2\log T)^{1/2}$ as $T \to \infty$ we obtain

$$\begin{aligned}
J_{T,2} &\leq \mathcal{C} \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|\geq\epsilon}} \Delta_{kk}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2}{1+\varpi_{kk}(nq,mq)}\right) \\
&\leq \mathcal{C} \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|\geq\epsilon}} \Delta_{kk}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(\epsilon)}\right) \\
&\leq \mathcal{C}\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(\epsilon)}\right) \sum_{\substack{nq,mq\in\mathcal{S}_i \\ i=1,2\cdots,n_T,|nq-mq|\geq\epsilon}} 1 \\
&\leq \mathcal{C}\frac{T}{q}T^{-\frac{2}{1+\vartheta_{kk}(\epsilon)}}\sum_{\epsilon<nq\leq T^a} 1
\end{aligned}$$

$$\leq \quad \mathcal{C}T^{a-\frac{1-\vartheta_{kk}(\epsilon)}{1+\vartheta_{kk}(\epsilon)}}(\ln T)^{2/\alpha}. \tag{28}$$

Since $a < \min_{k\in\{1,2,\cdots,p\}} \beta_{kk} < \min_{k\in\{1,2,\cdots,p\}} \frac{1-\vartheta_{kk}(\epsilon)}{1+\vartheta_{kk}(\epsilon)}$ we have $\lim_{T\to\infty} J_{T,2} = 0$. Second, we deal with the case that $nq \in \mathcal{S}_i$ and $mq \in \mathcal{S}_j$, $i \neq j$. Note that in this case, the distance between any two intervals $\mathcal{S}_i$ and $\mathcal{S}_j$ is large than $T^b$. Split the sum into two parts as

$$\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j,i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|<T^{\beta_{kk}}}} (\cdots) + \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\geq T^{\beta_{kk}}}} (\cdots) =: J_{T,3} + J_{T,4}.$$

Similarly to the derivation of (28), for large enough $T$ we have

$$\begin{aligned}
J_{T,3} &\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|<T^{\beta_{kk}}}} \Delta_{kk}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2}{1+\varpi_{kk}(nq,mq)}\right) \\
&\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|<T^{\beta_{kk}}}} \Delta_{kk}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(T^b)}\right) \\
&\leq \mathcal{C}\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(\epsilon)}\right) \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|<T^{\beta_{kk}}}} 1 \\
&\leq \mathcal{C}\frac{T}{q}T^{-\frac{2}{1+\vartheta_{kk}(\epsilon)}} \sum_{T^b<nq\leq T^{\beta_{kk}}} 1 \\
&\leq \mathcal{C}T^{\beta_{kk}-\frac{1-\vartheta_{kk}(\epsilon)}{1+\vartheta_{kk}(\epsilon)}}(\ln T)^{2/\alpha}.
\end{aligned} \tag{29}$$

Consequently, by (25) $J_{T,3} \to 0$ as $T \to \infty$. By the Assumption (8) we have $\vartheta_{kk}(t)\ln t \leq K$ for all sufficiently large $t$ and some constant $K$. Thus, $\varpi_{kk}(nq,mq) \leq \vartheta_{kk}(T^{\beta_{kk}}) \leq K/\ln T^{\beta_{kk}}$ for $|nq-mq| > T^{\beta_{kk}}$. Now using (26) again for all large $T$ we obtain

$$\begin{aligned}
\frac{T^2}{q^2\ln T}\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(T^{\beta_{kk}})}\right) &\leq \frac{T^2}{q^2\ln T}\exp\left(-\frac{(u_k^{(1)})^2}{1+K/\ln T^{\beta_{kk}}}\right) \\
&= \frac{T^2}{q^2\ln T}\left(T^{-2}\ln T(\ln T)^{-2/\alpha}\right)^{\frac{1}{1+K/\ln T^{\beta_{kk}}}}(1+o(1)) \\
&= \varepsilon^{-2}T^{(2K/\ln T^{\beta_{kk}})/(1+K/\ln T^{\beta_k})}(\ln T)^{((2/\alpha_{kk}-1)K/\ln T^{\beta_{kk}})/(1+K/\ln T^{\beta_{kk}})} \\
&= O(1).
\end{aligned} \tag{30}$$

Further, we have

$$\begin{aligned}
J_{T,4} &\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\geq T^{\beta_{kk}}}} \Delta_{kk}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2}{1+\varpi_{kk}(nq,mq)}\right) \\
&\leq \mathcal{C}\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(T^{\beta_{kk}})}\right) \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\geq T^{\beta_{kk}}}} \Delta_{kk}(nq,mq) \\
&= \mathcal{C}\frac{T^2}{q^2\ln T}\exp\left(-\frac{(u_k^{(1)})^2}{1+\vartheta_{kk}(T^{\beta_{kk}})}\right)\cdot\frac{q^2\ln T}{T^2} \sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\geq T^{\beta_{kk}}}} \Delta_{kk}(nq,mq)
\end{aligned}$$

$$
\begin{aligned}
&\le \mathcal{C}\frac{q^2\ln T}{T^2}\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\ge T^{\beta_{kk}}}}\left|r_{kk}(nq,mq)-\frac{r_{kk}}{\ln T}\right|\\
&\le \mathcal{C}\frac{q^2}{\beta^{kk}T^2}\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\ge T^{\beta_{kk}}}}|r_{kk}(nq,mq)\ln(|nq-mq|)-r_{kk}|\\
&\quad+\mathcal{C}r_{kk}\frac{q^2}{T^2}\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-mq|\ge T^{\beta_{kk}}}}\left|1-\frac{\ln T}{\ln(|nq-mq|)}\right|\\
&\le \mathcal{C}\frac{q}{\beta^{kk}T}\sum_{T^{\beta_{kk}}\le nq\le T}|r_{kk}(nq)\ln(nq)-r_{kk}|+\mathcal{C}r_{kk}\frac{q}{T}\sum_{T^{\beta_{kk}}\le nq\le T}\left|1-\frac{\ln T}{\ln(nq)}\right|. \qquad (31)
\end{aligned}
$$

By the Assumption (8) the first term on the right-hand-side of (31) tends to 0 as $T\to\infty$. Furthermore, the second term of the right-hand-side of (31) also tends to 0 by an integral estimate as in the proof of Lemma 6.4.1 of Leadbetter et al. (1983). Now from (27)-(31), we get that the sum in the claim of the lemma tends to 0 as $T\to\infty$. $\square$

**Lemma 5.2**. *Under conditions of Lemma 3.1, we have*

$$
\sum_{\substack{n\delta\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kk}(n\delta,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kk}^{(h)}(n\delta,m\delta)}}\exp\left(-\frac{(u_k^{(2)})^2}{1+r_{kk}^{(h)}(n\delta,m\delta)}\right)dh\to 0
$$

*as* $T\to\infty$.

**Proof:** The claim is established by following very closely the proof of Lemma 5.1. $\square$

**Lemma 5.3**. *Under conditions of Lemma 3.1, we have*

$$
\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j,nq\neq m\delta\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(nq,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kk}^{(h)}(nq,m\delta)}}\exp\left(-\frac{(u_k^{(1)})^2+(u_k^{(2)})^2}{2(1+r_{kk}^{(h)}(nq,m\delta))}\right)dh\to 0
$$

*as* $T\to\infty$.

**Proof:** We only prove the case that $\mathfrak{R}(\delta)$ is a sparse grid, since the proof of the Pickands grid is the same. First, we consider the case that $nq, m\delta$ in the same interval $\mathcal{S}$. Note that in this case, $\varrho_{kk}(nq,m\delta)=r_{kk}(nq,m\delta)+(1-r_{kk}(nq,m\delta))\rho_{kk}(T)\sim r_{kk}(nq,m\delta)$ for sufficiently large $T$. Split the sum into two parts as

$$
\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}(\cdots)+\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|\ge\epsilon}}(\cdots)=:S_{T,1}+S_{T,2}.
$$

The definition of $u_k^{(2)}$ implies thus

$$
(u_k^{(2)})^2=2\ln T-\ln\ln T+2\ln\delta^{-1}+O(1) \qquad (32)
$$

for sparse grids. Consequently, since further $q=\varepsilon(\ln T)^{-1/\alpha}$ and the definition of $\delta$ we obtain (set $u_{k,12}:=(u_k^{(1)})^2+(u_k^{(2)})^2$)

$$
\begin{aligned}
S_{T,1}&\le \mathcal{C}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}\Delta_{kl}(nq,m\delta)\frac{1}{\sqrt{1-r_{kk}(nq,m\delta)}}\exp\left(-\frac{u_{k,12}}{2(1+r_{kk}(nq,m\delta)}\right)\\
&\le \mathcal{C}\rho_{kk}(T)\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}\frac{1-r_{kk}(nq,m\delta)}{\sqrt{1-r_{kk}(nq,m\delta)}}\exp\left(-\frac{u_{k,12}}{4}\right)\exp\left(-\frac{(1-r_{kk}(nq,m\delta))u_{k,12}}{4(1+r_{kk}(nq,m\delta))}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \mathcal{C}\rho_{kk}(T)T^{-1}(\ln T)^{1/2-1/2\alpha}\delta^{1/2}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}\sqrt{1-r_{kk}(nq,m\delta)}\exp\left(-\frac{(1-r_{kk}(nq,m\delta))u_{k,12}}{4(1+r_{kk}(nq,m\delta))}\right)\\
&\leq \mathcal{C}\rho_{kk}(T)T^{-1}(\ln T)^{1/2-1/2\alpha}\delta^{1/2}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}|nq-m\delta|^{\alpha/2}\exp\left(-\frac{1}{16}|nq-m\delta|^{\alpha}u_{k,12}\right)\\
&\leq \mathcal{C}T^{-1}(\ln T)^{-1/2-1/2\alpha}\delta^{1/2}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}\exp\left(-\frac{1}{16}|nq-m\delta|^{\alpha}\ln T\right). \qquad (33)
\end{aligned}
$$

Since $\mathfrak{R}(\delta)$ is a sparse grid, $\delta(\ln T)^{1/\alpha}\to\infty$. A simple calculation shows that

$$
\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|<\epsilon}}\exp\left(-\frac{1}{16}|nq-m\delta|^{\alpha}\ln T\right)\leq \mathcal{C}T\delta^{-1}\sum_{0<nq<\epsilon}\exp\left(-\frac{1}{16}(nq)^{\alpha}\ln T\right)\leq \mathcal{C}T\delta^{-1}.
$$

Hence the assumption $\delta(\ln T)^{1/\alpha}\to\infty$ implies $S_{T,1}\leq\mathcal{C}(\ln T)^{-1/2-1/2\alpha}\delta^{-1/2}=o((\ln T)^{-1/2})$. Since $u_k^{(i)}\sim(2\log T)^{1/2}, i=1,2$

$$
\begin{aligned}
S_{T,2} &\leq \mathcal{C}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|\geq\epsilon}}\Delta_{kl}(nq,m\delta)\exp\left(-\frac{u_{k,12}}{2(1+\varpi_{kk}(nq,m\delta))}\right)\\
&\leq \mathcal{C}\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|\geq\epsilon}}\Delta_{kl}(nq,m\delta)\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(\epsilon))}\right)\\
&\leq \mathcal{C}\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(\epsilon))}\right)\sum_{\substack{nq,m\delta\in\mathcal{S}_i\\ i=1,2\cdots,n_T,|nq-m\delta|\geq\epsilon}}1\\
&\leq \mathcal{C}\frac{T}{q}T^{-\frac{2}{1+\vartheta_{kk}(\epsilon)}}\sum_{0<m\delta\leq T^{a}}1\\
&\leq \mathcal{C}T^{a-\frac{1-\vartheta_{kk}(\epsilon)}{1+\vartheta_{kk}(\epsilon)}}\delta^{-1}(\ln T)^{1/\alpha}. \qquad (34)
\end{aligned}
$$

In view of (25) and the fact that $\mathfrak{R}(\delta)$ is a sparse grid, $\lim_{T\to\infty}S_{T,2}=0$.
Second, we deal with the case that $nq\in\mathcal{S}_i$ and $mq\in\mathcal{S}_j$, $i\neq j$. Again we split the sum into two parts as

$$
\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j,i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|<T^{\beta_{kk}}}}(\cdots)+\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}(\cdots)=:S_{T,3}+S_{T,4}.
$$

Similarly to the derivation of (34) for large enough $T$ we have

$$
\begin{aligned}
S_{T,3} &\leq \mathcal{C}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|<T^{\beta_{kk}}}}\Delta_{kl}(nq,m\delta)\exp\left(-\frac{u_{k,12}}{2(1+\varpi_{kk}(nq,m\delta))}\right)\\
&\leq \mathcal{C}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|<T^{\beta_{kk}}}}\Delta_{kl}(nq,m\delta)\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(T^{b}))}\right)\\
&\leq \mathcal{C}\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(\epsilon))}\right)\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|<T^{\beta_{kk}}}}1\\
&\leq \mathcal{C}\frac{T}{q}T^{-\frac{2}{1+\vartheta_{kk}(\epsilon)}}\sum_{0<n\delta\leq T^{\beta_{kk}}}1
\end{aligned}
$$

$$\leq \quad \mathcal{C}T^{\beta_{kk}-\frac{1-\vartheta_{kk}(\epsilon)}{1+\vartheta_{kk}(\epsilon)}}\delta^{-1}(\ln T)^{1/\alpha}. \tag{35}$$

Consequently, by (25) $\lim_{T\to\infty} S_{T,3}=0$. By the Assumption (8) we have $\vartheta_{kk}(t)\ln t\leq K$ for all sufficiently large $t$ and some constant $K$. Thus, $\varpi_{kk}(nq,m\delta)\leq \vartheta_{kk}(T^{\beta_{kk}})\leq K/\ln T^{\beta_{kk}}$ for $|nq-m\delta|>T^{\beta_{kk}}$. Now using (26) and (32) again for all large $T$ and $|nq-m\delta|>T^{\beta_{kk}}$ we obtain

$$\begin{aligned}
\frac{T^2}{q\delta\ln T}\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(T^{\beta_{kk}}))}\right) &\leq \frac{T^2}{q\delta\ln T}\exp\left(-\frac{u_{k,12}}{2(1+K/\ln T^{\beta_{kk}})}\right)\\
&= \frac{T^2}{q\delta\ln T}\left(T^{-2}\ln T(\ln T)^{-1/\alpha}\delta\right)^{\frac{1}{1+K/\ln T^{\beta_{kk}}}}(1+o(1))\\
&= O(1).
\end{aligned} \tag{36}$$

Now, with similar arguments as in the proof of (31) we obtain

$$\begin{aligned}
S_{T,4} &\leq \mathcal{C}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}\Delta_{kl}(nq,m\delta)\exp\left(-\frac{u_{k,12}}{2(1+\varpi_{kk}(nq,m\delta))}\right)\\
&\leq \mathcal{C}\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(T^{\beta_{kk}}))}\right)\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}\Delta_{kl}(nq,m\delta)\\
&= \mathcal{C}\frac{T^2}{q\delta\ln T}\exp\left(-\frac{u_{k,12}}{2(1+\vartheta_{kk}(T^{\beta_{kk}}))}\right)\frac{q\delta\ln T}{T^2}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}\Delta_{kl}(nq,m\delta)\\
&\leq \mathcal{C}\frac{q\delta\ln T}{T^2}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}\left|r_{kk}(nq,m\delta)-\frac{r_{kk}}{\ln T}\right|\\
&\leq \mathcal{C}\frac{q\delta}{\beta^{kk}T^2}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}|r_{kk}(nq,m\delta)\ln(|nq-m\delta|)-r_{kk}|\\
&\quad +\mathcal{C}r_{kk}\frac{q\delta}{T^2}\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\, i\neq j\\ i,j=1,2\cdots,n_T,|nq-m\delta|\geq T^{\beta_{kk}}}}\left|1-\frac{\ln T}{\ln(|nq-m\delta|)}\right|.
\end{aligned} \tag{37}$$

By the Assumption (8) the first term on the right-hand-side of (37) tends to 0 as $T\to\infty$. Furthermore, the second term of the right-hand-side of (37) also tends to 0 by an integral estimate as in the proof of Lemma 6.4.1 of Leadbetter et al. (1983). Now the claim follows from (33)-(37). □

**Lemma 5.4**. *Under conditions of Lemma 3.1, we have for $k<l$*

$$\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(nq,mq)\int_0^1\frac{1}{\sqrt{1-r_{kl}^{(h)}(nq,mq)}}\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+r_{kl}^{(h)}(nq,mq))}\right)dh\to 0$$

*as $T\to\infty$.*

**Proof:** Let $\vartheta_{kl}(t)=\sup_{|nq-mq|\geq t}\{\varpi_{kl}(nq,mq)\}$, where $\varpi_{kl}(nq,mq)=\max\{r_{kl}(nq,mq),\varrho_{kl}(nq,mq)\}$. From Assumption (9) and the definition of $\varrho_{kl}(nq,mq)$, we have $\vartheta_{kl}(0)<1$ for all $T$. Consequently, we may choose some positive constant $\beta_{kl}$ such that $\beta_{kl}<\frac{1-\vartheta_{kl}(0)}{1+\vartheta_{kl}(0)}$ for all sufficiently large $T$. Split the sum into two parts as

$$\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T,|nq-mq|<T^{\beta_{kl}}}}(\cdots)+\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T,|nq-mq|\geq T^{\beta_{kl}}}}(\cdots)=:R_{T,1}+R_{T,2}.$$

Similarly to the derivation of (28), for large enough $T$ we have

$$\begin{aligned}
R_{T,1} &\le \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|<T^{\beta_{kl}}}} \Delta_{kl}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\varpi_{kl}(nq,mq))}\right) \\
&\le \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|<T^{\beta_{kl}}}} \Delta_{kl}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\vartheta_{kl}(0))}\right) \\
&\le \mathcal{C}\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\vartheta_{kl}(0))}\right) \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|<T^{\beta_{kl}}}} 1 \\
&\le \mathcal{C}\frac{T}{q}T^{-\frac{2}{1+\vartheta_{kl}(0)}} \sum_{0<nq\le T^{\beta_{kl}}} 1 \\
&\le \mathcal{C}T^{\beta_{kl}-\frac{1-\vartheta_{kl}(0)}{1+\vartheta_{kl}(0)}}(\ln T)^{2/\alpha}.
\end{aligned}$$

Consequently, $R_{T,1}\to 0$ as $T\to\infty$ which follows by the fact that $\beta_{kl} < \frac{1-\vartheta_{kl}(0)}{1+\vartheta_{kl}(0)}$.
By the Assumption (8) we have $\vartheta_{kl}(t)\ln t \le K$ for all sufficiently large $t$. Thus, $\varpi_{kl}(nq,mq) \le \vartheta_{kl}(T^{\beta_{kl}}) \le K/\ln T^{\beta_{kl}}$ for $|nq-mq| > T^{\beta_{kl}}$. Now with the similar arguments as for (30) we obtain

$$\frac{T^2}{q^2\ln T}\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right) = O(1).$$

Thus, for $R_{T,2}$ we have

$$\begin{aligned}
R_{T,2} &\le \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|\ge T^{\beta_{kl}}}} \Delta_{kl}(nq,mq)\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\varpi_{kl}(nq,mq))}\right) \\
&\le \mathcal{C}\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right) \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|\ge T^{\beta_{kl}}}} \Delta_{kl}(nq,mq) \\
&\le \mathcal{C}\frac{T^2}{q^2\ln T}\exp\left(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right)\cdot\frac{q^2\ln T}{T^2} \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|\ge T^{\beta_{kl}}}} \Delta_{kl}(nq,mq) \\
&\le \mathcal{C}\frac{q^2\ln T}{T^2} \sum_{\substack{nq\in\mathcal{S}_i, mq\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T, |nq-mq|\ge T^{\beta_{kl}}}} \left| r_{kl}(nq,mq) - \frac{r_{kl}}{\ln T}\right|.
\end{aligned}$$

By the same arguments as those in Lemma 5.1, we have $\lim_{T\to\infty} R_{T,2} = 0$ and thus the claim follows. □

**Lemma 5.5**. *Under conditions of Lemma 3.1, we have for $k<l$*

$$\sum_{\substack{n\delta\in\mathcal{S}_i, m\delta\in\mathcal{S}_j \\ i,j=1,2\cdots,n_T}} \Delta_{kl}(n\delta,m\delta)\int_0^1 \frac{1}{\sqrt{1-r_{kl}^{(h)}(n\delta,m\delta)}}\exp\Big(-\frac{(u_k^{(2)})^2+(u_l^{(2)})^2}{2(1+r_{kl}^{(h)}(n\delta,m\delta))}\Big)dh \to 0$$

*as $T\to\infty$.*

**Proof:** The claim follows with the same arguments as in the proof of Lemma 5.3. □

**Lemma 5.6**. *Under conditions of Lemma 3.1, we have for $k<l$*

$$\sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j, nq\ne m\delta \\ i,j=1,2\cdots,n_T}} \Delta_{kl}(nq,m\delta)\int_0^1 \frac{1}{\sqrt{1-r_{kl}^{(h)}(nq,m\delta)}}\exp\Big(-\frac{(u_k^{(1)})^2+(u_l^{(2)})^2}{2(1+r_{kl}^{(h)}(nq,m\delta))}\Big)dh \to 0$$

as $T \to \infty$.

**Proof:** As in Lemma 5.2, we also only prove the case that $\mathfrak{R}(\delta)$ is a sparse grid. Split the sum into two parts as (with the same definition $T^{\beta_{kl}}$ as in Lemma 5.3.)

$$\sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|<T^{\beta_{kl}}}} (\cdots) + \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|\geq T^{\beta_{kl}}}} (\cdots) =: M_{T,1} + M_{T,2}.$$

Similarly to the derivation of (28), for large enough $T$ we have

$$\begin{aligned}
M_{T,1} &\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|<T^{\beta_{kl}}}} \Delta_{kl}(nq, m\delta) \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\varpi_{kl}(nq, m\delta))}\right)\\
&\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|<T^{\beta_{kl}}}} \Delta_{kl}(nq, m\delta) \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\vartheta_{kl}(0))}\right)\\
&\leq \mathcal{C} \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\vartheta_{kl}(0))}\right) \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|<T^{\beta_{kl}}}} 1\\
&\leq \mathcal{C}\frac{T}{q} T^{-\frac{2}{1+\vartheta_{kl}(0)}} \sum_{0<m\delta\leq T^{\beta_{kl}}} 1\\
&\leq \mathcal{C} T^{\beta_{kl}-\frac{1-\vartheta_{kl}(0)}{1+\vartheta_{kl}(0)}} (\ln T)^{1/\alpha}\delta^{-1}.
\end{aligned}$$

Thus, $M_{T,1} \to 0$ as $T \to \infty$ from the facts that $\beta_{kl} < \frac{1-\vartheta_{kl}(0)}{1+\vartheta_{kl}(0)}$ and $\delta(\ln T)^{1/\alpha} \to \infty$. As for (36) we have

$$\frac{T^2}{q\delta \ln T} \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right) = O(1).$$

Consequently

$$\begin{aligned}
M_{T,2} &\leq \mathcal{C} \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|\geq T^{\beta_{kl}}}} \Delta_{kl}(nq, m\delta) \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\varpi_{kl}(nq, m\delta))}\right)\\
&\leq \mathcal{C} \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right) \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|\geq T^{\beta_{kl}}}} \Delta_{kl}(nq, m\delta)\\
&\leq \mathcal{C}\frac{T^2}{q\delta\ln T} \exp\left(-\frac{(u_k^{(1)})^2 + (u_l^{(2)})^2}{2(1+\vartheta_{kl}(T^{\beta_{kl}}))}\right) \cdot \frac{q\delta\ln T}{T^2} \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|\geq T^{\beta_{kl}}}} \Delta_{kl}(nq, m\delta)\\
&\leq \mathcal{C}\frac{q\delta\ln T}{T^2} \sum_{\substack{nq\in\mathcal{S}_i, m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T, |nq-m\delta|\geq T^{\beta_{kl}}}} \left| r_{kl}(nq, m\delta) - \frac{r_{kl}}{\ln T}\right|.
\end{aligned}$$

By the same arguments as those in Lemma 5.3 we obtain $\lim_{T\to\infty} M_{T,2} = 0$, and thus the claim follows. $\square$

**Proof of Lemma 3.1**: Using Berman's inequality (see for example Piterbarg (1996, Theorem 1.2)) we have

$$\begin{aligned}
&\Bigg| P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} X_k(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} X_k(t) \leq u_k^{(2)}, k=1,\cdots,p\right\}\\
&\quad - P\left\{\max_{t\in\mathfrak{R}(q)\cap\mathbf{S}} \xi_k^T(t) \leq u_k^{(1)}, \max_{t\in\mathfrak{R}(\delta)\cap\mathbf{S}} \xi_k^T(t) \leq u_k^{(2)}, k=1,\cdots,p\right\} \Bigg|
\end{aligned}$$

$$\leq \mathcal{C}\sum_{k=1}^{p}\Big[\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kk}(nq,mq)\int_0^1\frac{1}{\sqrt{1-r_{kk}^{(h)}(nq,mq)}}\exp\Big(-\frac{(u_k^{(1)})^2}{1+r_{kk}^{(h)}(nq,mq)}\Big)dh$$

$$+\sum_{\substack{n\delta\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kk}(n\delta,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kk}^{(h)}(n\delta,m\delta)}}\exp\Big(-\frac{(u_k^{(2)})^2}{1+r_{kk}^{(h)}(n\delta,m\delta)}\Big)dh$$

$$+\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j,nq\neq m\delta\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(nq,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kk}^{(h)}(nq,m\delta)}}\exp\Big(-\frac{(u_k^{(1)})^2+(u_k^{(2)})^2}{2(1+r_{kk}^{(h)}(nq,m\delta))}\Big)dh\Big]$$

$$+2\mathcal{C}\sum_{1\leq k<l\leq p}\Big[\sum_{\substack{nq\in\mathcal{S}_i,mq\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(nq,mq)\int_0^1\frac{1}{\sqrt{1-r_{kl}^{(h)}(nq,mq)}}\exp\Big(-\frac{(u_k^{(1)})^2+(u_l^{(1)})^2}{2(1+r_{kl}^{(h)}(nq,mq))}\Big)dh$$

$$+\sum_{\substack{n\delta\in\mathcal{S}_i,m\delta\in\mathcal{S}_j\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(n\delta,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kl}^{(h)}(n\delta,m\delta)}}\exp\Big(-\frac{(u_k^{(2)})^2+(u_l^{(2)})^2}{2(1+r_{kl}^{(h)}(n\delta,m\delta))}\Big)dh$$

$$+\sum_{\substack{nq\in\mathcal{S}_i,m\delta\in\mathcal{S}_j,nq\neq m\delta\\ i,j=1,2\cdots,n_T}}\Delta_{kl}(nq,m\delta)\int_0^1\frac{1}{\sqrt{1-r_{kl}^{(h)}(nq,m\delta)}}\exp\Big(-\frac{(u_k^{(1)})^2+(u_l^{(2)})^2}{2(1+r_{kl}^{(h)}(nq,m\delta))}\Big)dh\Big].$$

Now, the claim of Lemma 3.1 follows from Lemma 5.5-5.6. □

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