On the Distribution of Dividend Payments and the Discounted Penalty Function in a Risk Model with Linear Dividend Barrier

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Abstract

In the framework of classical risk theory we investigate a model that allows for dividend payments according to a time-dependent linear barrier strategy. Partial integro-differential equations for Gerber and Shiu's discounted penalty function and for the moment generating function of the discounted sum of dividend payments are derived, which generalizes several recent results. Explicit expressions for the nth moment of the discounted sum of dividend payments and for the joint Laplace transform of the time to ruin and the surplus prior to ruin are derived for exponentially distributed claim amounts.

Keywords: Classical Risk Theory; Collective Model; Linear Dividend Barrier; Time of Ruin; Deficit at Ruin; Surplus prior to Ruin

1 Introduction

The classical collective risk model describing the surplus process of an insurance portfolio assumes independent and identically distributed claims $X_i, i = 1, 2, \ldots,$ (with distribution function F), which occur according to a homogeneous Poisson process N_t with intensity λ. The premium income is modelled by a constant premium density c and the net profit condition is then $c > \lambda\mu$, where the expected value of the individual claim amounts $\mu = \mathbb{E}[X_i]$ is assumed to be finite (cf. [5]).

Let us consider the following extension of the classical model: Whenever the surplus R_t reaches a time-dependent barrier of type

$$b_t = b_0 + at, \qquad (0 \le a < c),$$

dividends are paid out to the shareholders with intensity $c - a$ and the surplus remains on the barrier until the next claim occurs. The dynamics of R_t are thus given by

$$\begin{aligned} dR_t &= cdt - dS_t \quad \text{if } R_t < b_0 + at \\ dR_t &= adt - dS_t \quad \text{if } R_t = b_0 + at, \end{aligned} \tag{1}$$

where $S_t = \sum_{i=1}^{N(t)} X_i$ (cf. Figure 1).

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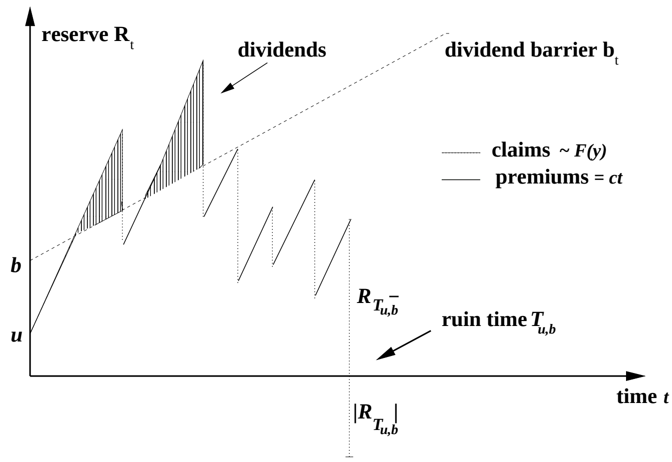

Figure 1: A sample path of the surplus process R_t

Dividend barrier models allow for a certain type of profit participation for the shareholders of an insurance company and have a long history within risk theory going back to de Finetti [7]. The linear barrier model (1) was introduced by Gerber [10] to overcome the deficiency of horizontal barrier models that they lead to ruin with probability 1. More general barrier models were recently investigated in [2, 3]. Some optimality results of dividend payment schemes can e.g. be found in [9, 15] (for related results with a surplus process modelled by a Brownian motion, see [6, 14, 16]). For a general overview of dividend models, we refer to [8].

For $0 \leq u \leq b$, define $T_{u,b} = \inf\{t : R_t < 0 \,|\, R_0 = u,\ b_0 = b\}$ to be the time of ruin of the surplus process R_t and let $D_{u,b}$ denote the present value of the discounted sum of dividend payments until ruin occurs (discount factor $\delta \geq 0$). Among the crucial quantities in risk theory are the probability of ruin $\psi(u, b) = \mathbb{P}(T_{u,b} < \infty)$ and the expected value of discounted dividend payments $V_1(u, b) = \mathbb{E}[D_{u,b}]$. In [12], Gerber derived exact formulae for $\psi(u, b)$ and $V_1(u, b)$ for the model (1) in the case of exponentially distributed claim amounts. This solution algorithm was generalized to the case of Erlang distributed claims by Siegl and Tichy [19] and Albrecher and Tichy [4].

Although the quantities $\psi(u, b)$ and $V_1(u, b)$ provide some rough insight into the risk and the effectiveness of a dividend barrier strategy, there is a need for more refined measures of the inherent risk such as the time and the severity of ruin given ruin occurs, or higher-order moments of the discounted dividend payments until ruin. This paper focuses on some of these extensions. For a horizontal dividend barrier, Dickson and Waters [9] have recently derived an equation for the nth moment $V_n(u, b) = \mathbb{E}[D^n_{u,b}]$. In Section 2 we will derive a partial integro-differential equation for the moment generating function of $D_{u,b}$ in the linear dividend barrier model, thereby simplifying and generalizing the corresponding result in [9]. The used technique is related to Gerber and Shiu [14]. For the case of exponential claim amounts we provide an explicit solution for $V_n(u, b)$ in terms of an infinite series and prove its convergence.

The discounted penalty function introduced by Gerber and Shiu [13] allows to study properties of the time of ruin, the severity of ruin and the surplus prior to ruin at the same time (for a detailed discussion in the classical risk model without barriers, see [17] and the references therein). In a model with a horizontal dividend barrier, this penalty function was recently studied in detail by Lin et al. [18]. In Section 3, we derive a partial integro-differential equation for this penalty function for the linear barrier model and use it to derive explicit formulae for the time of ruin, the surplus prior to ruin and the discounted nth moment of the deficit at ruin for exponential claim amounts, which generalizes corresponding results for horizontal barriers in [9]. Numerical illustrations of these explicit solutions in comparison with values obtained by stochastic simulation are given in Section 4.

2 The moments of the discounted dividends

2.1 Dividend payments continued after ruin

For later use, let us first consider a slightly modified situation, where the dividend payments according to the linear barrier strategy are not stopped at the event of ruin (this model may be of independent interest, since ruin is only a technical term useful for risk management and does not necessarily imply bankruptcy). Clearly, in this model the discounted sum of dividend payments $\tilde{D}_{u,b}$ depends on u and b only through the difference $x = b - u$ (i.e. the difference between the initial value $b_0 = b$ of the barrier and the initial capital $R_0 = u$, see also [19]). This observation simplifies the analysis considerably. Let $Z_n(x) := \mathbb{E}\left[\tilde{D}^n_{u,b} \,|\, b - u = x\right]$ and denote by $Z(x, y)$ the moment-generating function

$$Z(x,y) := \mathbb{E}\left[e^{y\tilde{D}_{u,b}}\middle|\; b-u=x\right] \tag{2}$$

for suitable values of y. One can now derive an integro-differential equation for $Z(x, y)$ by conditioning on the occurrence of a claim. For $x > 0$, we have

$$\begin{aligned} Z(x,y) = (1-\lambda dt)\, Z(x-(c-a)dt, y\, e^{-\delta dt}) \\ + \lambda dt \int_0^\infty Z(x-(c-a)dt+v, ye^{-\delta dt})\, \mathrm{d}F(v) + o(dt). \end{aligned}$$

Taylor expansion and collecting all terms of order dt yields

$$\lambda\, Z(x,y) + (c-a)\frac{\partial Z}{\partial x}(x,y) + \delta\, y \frac{\partial Z}{\partial y}(x,y) - \lambda \int_0^\infty Z(x+v,y)\, \mathrm{d}F(v) = 0. \tag{3}$$

One obvious boundary condition is given by

$$\lim_{x\to\infty} Z(x,y) = 1. \tag{4}$$

A second boundary condition follows from $x = 0$:

$$Z(0,y) = (1-\lambda dt)e^{y(c-a)dt} Z(0, ye^{-\delta dt}) + \lambda\, dt\; e^{y(c-a)dt} \int_0^\infty Z(v, ye^{-\delta dt})\, \mathrm{d}F(v) + o(dt),$$

which leads to

$$(\lambda - y(c-a))Z(0,y) + \delta\, y\, \frac{\partial Z}{\partial y}\bigg|_{x=0} - \lambda \int_0^\infty Z(v,y)\,\mathrm{d}F(v) = 0. \tag{5}$$

Now, assuming continuity at $x = 0$, a comparison of (3) and (5) yields

$$\frac{\partial Z}{\partial x}\bigg|_{x=0} = -yZ(0,y). \tag{6}$$

Using the representation

$$Z(x,y) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{n!}\, Z_n(x), \tag{7}$$

and equating the coefficients of y^n in (3), one obtains the integro-differential equations

$$(c-a)Z_n'(x) + (\lambda + n\delta)Z_n(x) - \lambda \int_0^\infty Z_n(x+v)dF(v) = 0 \tag{8}$$

with boundary conditions

$$Z_n'(0) = -n\, Z_{n-1}(0) \quad \text{and} \quad \lim_{x\to\infty} Z_n(x) = 0. \tag{9}$$

Note that (8) and (9) generalize equations (7) and (8) of [19], where only $Z_1(x)$ had been considered.

Example 2.1. For exponential claim amounts ($F(v) = 1 - e^{-\alpha v}$), (8) together with (9) can be solved in a straight-forward way: differentiating (8) yields the linear homogeneous second-order differential equation

$$(c-a)Z_n''(x) + (\lambda + n\delta - \alpha(c-a))Z_n'(x) - \alpha n\delta Z_n(x) = 0,$$

the solution of which is of the form

$$Z_n(x) = A_{n,1}e^{r_{n,1}x} + A_{n,2}e^{r_{n,2}x},$$

where $r_{n,1}, r_{n,2}$ denote the solutions in R of the equation

$$R^2 + \left(\frac{\lambda + n\delta}{c-a} - \alpha\right) R - \frac{\alpha n\delta}{c-a} = 0. \tag{10}$$

Since $n \geq 1, \alpha, \delta > 0$ and $c > a$, (10) has exactly one positive and one negative solution. The boundary condition $\lim_{x\to\infty} Z_n(x) = 0$ then implies that the coefficient corresponding to the positive solution equals zero. Let $-\rho_n$ denote the negative solution of (10) and A_n the corresponding non-zero coefficient, i.e.

$$Z_n(x) = A_n e^{-\rho_n x}.$$

The condition $Z_n'(0) = -n\, Z_{n-1}(0)$ then imposes

$$A_n = \frac{n\, A_{n-1}}{\rho_n}$$

and together with $A_1 = 1/\rho_1$ (which follows from $Z_1'(0) = -1$), we thus obtain the solution

$$Z_n(x) = \frac{n!}{\rho_1 \rho_2 \cdots \rho_n} e^{-\rho_n x}. \tag{11}$$

Along the same line of arguments, one can determine the solution of (8) together with (9) for $\Gamma(\beta, \alpha)$-distributed claim sizes with density $f(v) = \alpha^\beta v^{\beta-1} e^{-\alpha v}/\Gamma(\beta)$ ($\beta \in \mathbb{N}$). It is again of the form (11), but now $-\rho_n$ is the unique negative real solution in R of

$$(c-a)R(1-R/\alpha)^\beta + (\lambda + n\delta)(1-R/\alpha)^\beta - \lambda = 0.$$

From (11) and (7) it follows that the moment-generating function of $\tilde{D}_{u,b}$ for $\Gamma(\beta, \alpha)$-distributed claim sizes ($\beta \in \mathbb{N}$) is given by

$$\mathbb{E}\left[e^{y\tilde{D}_{u,b}}\right] = 1 + \sum_{n=1}^{\infty} \frac{y^n}{\prod_{j=1}^n \rho_j}\, e^{-\rho_n(b-u)}.$$

2.2 Dividend payments stopped at ruin

Let us now turn to the original problem of deriving a partial integro-differential equation for the moments $V_n(u, b)$ of the sum of the discounted dividend payments until ruin for arbitrary $n \in \mathbb{N}$. More generally, let us consider the moment-generating function

$$M(u, y, b) = \mathbb{E}\left[e^{yD} \middle| R_0 = u, b_0 = b\right],$$

for those values of y where it exists. Again, we condition on the occurrence of a claim. For $0 < u < b$, we have

$$\begin{aligned} M(u, y, b) &= (1 - \lambda dt) M(u + cdt, ye^{-\delta dt}, b + adt) \\ &+ \lambda dt \int_0^{u+cdt} M(u + cdt - v, ye^{-\delta dt}, b + adt)\, \mathrm{d}F(v) + \lambda dt \int_{u+cdt}^{\infty} \mathrm{d}F(v) + o(dt), \end{aligned}$$

which, by Taylor expansion and collection of terms of order dt, yields

$$\begin{aligned} c \frac{\partial M}{\partial u}(u, y, b) + a \frac{\partial M}{\partial b}(u, y, b) - \lambda\, M(u, y, b) - \delta y \frac{\partial M}{\partial y}(u, y, b) \\ + \lambda \int_0^u M(u - v, y, b)\, \mathrm{d}F(v) + \lambda(1 - F(u)) = 0. \end{aligned} \tag{12}$$

For this equation, we have the two boundary conditions

$$\lim_{b\to\infty} M(u, y, b) = 1 \tag{13}$$

and

$$\lim_{u\to\infty} M(u,y,u+x) = Z(x,y), \tag{14}$$

where $Z(x,y)$ is defined by (2). A third boundary condition can be found at $u=b$:

$$\begin{aligned} M(b,y,b) = (1-\lambda dt)\, e^{y(c-a)dt}\, M(b+adt, ye^{-\delta dt}, b+adt) \\ + \lambda dt \int_0^{b+adt} e^{y(c-a)dt} M(b+adt-v, ye^{-\delta dt}, b+adt)\,\mathrm{d}F(v) \\ + \lambda dt\; e^{y(c-a)dt} \int_{b+adt}^{\infty} \mathrm{d}F(v) + o(dt), \end{aligned}$$

which implies

$$\begin{aligned} a\frac{\partial M}{\partial u}\Big|_{u=b} + a\frac{\partial M}{\partial b}\Big|_{u=b} + (y(c-a)-\lambda)M(b,y,b) - \delta y \frac{\partial M}{\partial y}\Big|_{u=b} \\ + \lambda \int_0^b M(b-v,y,b)\,\mathrm{d}F(v) + \lambda(1-F(b)) = 0. \end{aligned} \tag{15}$$

Setting $u=b$ in (12), we obtain by continuity

$$\frac{\partial M}{\partial u}\Big|_{u=b} = y\, M(b,y,b). \tag{16}$$

Using the representation

$$M(u,y,b) = 1 + \sum_{n=1}^{\infty} \frac{y^n}{n!}\, V_n(u,b)$$

and equating the coefficients of y^n in (12) leads to the following partial integro-differential equation for $V_n(u,b)$:

$$c\frac{\partial V_n}{\partial u}(u,b) + a\frac{\partial V_n}{\partial b}(u,b) - (\lambda + n\delta)V_n(u,b) + \lambda \int_0^u V_n(u-v,b)dF(v) = 0, \tag{17}$$

with boundary conditions

$$\frac{\partial V_n}{\partial u}\Big|_{u=b} = nV_{n-1}(b,b), \tag{18}$$

$$\lim_{b\to\infty} V_n(u,b) = 0, \tag{19}$$

and

$$\lim_{u\to\infty} V_n(u,u+x) = Z_n(x). \tag{20}$$

Remark 2.1. Note that this at the same time simplifies and generalizes a corresponding derivation for horizontal dividend barriers by Dickson and Waters [9]. For $n=1$, due to $V_0(b,b)=1$, (17) together with boundary conditions (18)-(20) reduce to the well-known integro-differential equation for the expected value of discounted dividend payments in the presence of a linear barrier (cf. [12, 19]).

Remark 2.2. One might wonder, whether there is an intuitive reason for the fact that V_n depends on lower moments only through the $(n-1)$th moment. This can actually be understood by a direct derivation of $V_n(u,b)$ using the differential argument and the binomial formula (see Albrecher [1], where this aspect was worked out in detail for a horizontal barrier).

2.3 An explicit formula for $V_n(u,b)$ for exponential claim amounts

Let us now assume $F(v) = 1 - e^{-\alpha v}$. In this case, (17) can be rewritten as a linear, homogeneous partial differential equation of second order

$$c\,\frac{\partial^2 V_n}{\partial u^2} + a\,\frac{\partial^2 V_n}{\partial u \partial b} + a\alpha\,\frac{\partial V_n}{\partial b} + (c\alpha - \lambda - n\delta)\,\frac{\partial V_n}{\partial u} - n\alpha\delta\, V_n = 0 \tag{21}$$

with boundary conditions

$$c\frac{\partial V_n}{\partial u}\Big|_{u=0} + a\frac{\partial V_n}{\partial b}\Big|_{u=0} - (\lambda + n\delta)V_n(0,b) = 0, \tag{22}$$

and (18)-(20). It is immediately clear that a function of the form

$$(C_1 e^{r_1 u} + C_2 e^{r_2 u})\, e^{sb}, \qquad (C_1, C_2 \ldots \text{constants}) \tag{23}$$

satisfies (21), if r_1 and r_2 are the two solutions in R of the equation

$$cR^2 + (as + c\alpha - \lambda - n\delta)R + \alpha(as - n\delta) = 0. \tag{24}$$

For $s < 0$ we have $\alpha(as - n\delta)/c < 0$ and thus (24) has exactly one positive and one negative solution (the positive solution will always be denoted by r_1). Substitution of (23) into (22) shows that condition (22) is automatically fulfilled, if

$$(\alpha + r_2)C_1 = -(\alpha + r_1)C_2,$$

which we will assume from now on. Thus the challenge is to find a combination of functions of type

$$C(e^{r_1 u} - \frac{\alpha + r_2}{\alpha + r_1}\, e^{r_2 u})\, e^{sb}, \qquad (C \ldots \text{constant})$$

that satisfies the boundary conditions (18)-(20).

Remark 2.3. By a renewal argument similar to the one used in [2] and [12], one can identify the solution of (18)-(22) as the fixed point of a contracting integral operator in the Banach space of bounded functions, which ensures the existence and uniqueness of a bounded solution of (18)-(22) (and at the same time gives rise to efficient number-theoretic simulation techniques as developed in Albrecher and Kainhofer [2]). It is not a priori clear that the solution is of the above form, but if such a solution can be found, we have solved (18)-(22) by uniqueness.

In [12], Gerber solved the case $n = 1$, the expected value of discounted dividend payments, in terms of an infinite series:

$$V_1(u,b) = \sum_{k=0}^{\infty} C_k e^{s_k\, b} \left(e^{r_{1,k}\, u} - \frac{\alpha + r_{2,k}}{\alpha + r_{1,k}}\, e^{r_{2,k}\, u} \right) \tag{25}$$

with coefficients

$$C_0 = \frac{1}{\rho_1}, \quad s_0 = -\rho_1, \quad r_{1,0} = \rho_1, \quad r_{2,0} = -\alpha \frac{a\rho_1 + \delta}{c\rho_1}, \tag{26}$$

$$r_{2,k} + s_k = r_{1,k+1} + s_{k+1}, \quad k \geq 0 \tag{27}$$

and

$$C_k = C_{k-1} \frac{r_{2,k-1}}{r_{1,k}} \frac{\alpha + r_{2,k-1}}{\alpha + r_{1,k-1}}, \quad k \geq 1. \tag{28}$$

In the sequel we will show how this approach can be generalized to obtain an expression for the nth moment of the discounted dividend payments of the following form:

$$V_n(u,b) = \sum_{k_1=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} C_{k_1,\ldots,k_n} e^{s_{k_1,\cdots,k_n}\, b} \left(e^{r_{1,k_1,\ldots,k_n}\, u} - \frac{\alpha + r_{2,k_1,\ldots,k_n}}{\alpha + r_{1,k_1,\ldots,k_n}}\, e^{r_{2,k_1,\ldots,k_n}\, u} \right). \tag{29}$$

For convenience, we restrict ourselves to the case $n = 2$ (leading to a double series). The calculation of the nth moment ($n \geq 3$) is possible along the same lines.

In accordance with (20) we choose the starting parameters of the double series for $V_2(u,b)$ in the following way:

$$C_{0,0} = \frac{2}{\rho_1\rho_2}, \qquad s_{0,0} = -\rho_2, \quad \text{and} \quad r_{1,0,0} = \rho_2 \tag{30}$$

(note that $Z_2(x) = \frac{2}{\rho_1\rho_2} e^{-\rho_2 u}$). By Vieta's rule and (24) we then get

$$r_{2,0,0} = -\frac{\alpha a}{c} - \frac{2\alpha\delta}{c\rho_2} < 0. \tag{31}$$

Thus, (20) will be fulfilled, if all terms with higher index vanish in the limit $\lim_{u\to\infty} V_2(u+x, x)$, i.e. for $k_1 + k_2 \geq 1$ we have the condition

$$r_{1,k_1,k_2} + s_{k_1,k_2} < 0 \ \text{ and } \ r_{2,k_1,k_2} + s_{k_1,k_2} < 0. \tag{32}$$

Furthermore, condition (19) will be satisfied, if for all $k_1 + k_2 \geq 0$

$$s_{k_1,k_2} < 0. \tag{33}$$

Now, let us turn to (18), which due to (25) and (27) we can rewrite to

$$\frac{\partial V_n}{\partial u}\Big|_{u=b} = 2\, V_1(b,b) = \sum_{k=0}^{\infty} D_k e^{z_k\, b} \tag{34}$$

with

$$D_k = 2\, C_k \left(1 - \frac{r_{1,k}}{r_{2,k-1}}\right) \quad \text{and} \quad z_k = s_k + r_{1,k}.$$

From condition (20) for $V_1(u,b)$ we have $z_k < 0$ for $k \geq 1$.
In order to satisfy condition (18), we now choose $C_{k_1,0}$, $s_{k_1,0}$ and $r_{1,k_1,0}$ such that

$$s_{k_1,0} + r_{1,k_1,0} = z_{k_1}, \quad C_{k_1,0}\, r_{1,k_1,0} = D_{k_1}. \tag{35}$$

By combining (24) and (35) we can fix $r_{1,k_1,0}$ as the unique positive solution of the equation

$$R^2 + \frac{az_{k_1} + (c-a)\alpha - (\lambda + 2\delta)}{c-a} R + \frac{\alpha}{c-a}(az_{k_1} - 2\delta) = 0. \tag{36}$$

Then, we construct for every $k_1 \geq 0$ a series in such a way that all but the term $C_{k_1,0}\ r_{1,k_1,0}\ e^{z_{k_1} b}$ of the derivative of (29) evaluated at b cancel out. This can be achieved, if $\forall\ k_1 \geq 0$ we set

$$C_{k_1,k_2} := C_{k_1,k_2-1} \frac{r_{2,k_1,k_2-1}}{r_{1,k_1,k_2}} \frac{\alpha + r_{2,k_1,k_2-1}}{\alpha + r_{1,k_1,k_2-1}} \quad (k_2 \geq 1) \tag{37}$$

and

$$r_{2,k_1,k_2} + s_{k_1,k_2} = r_{1,k_1,k_2+1} + s_{k_1,k_2+1} := z_{k_1,k_2} \quad (k_2 \geq 0), \tag{38}$$

where r_{1,k_1,k_2+1} is the unique positive solution of

$$R^2 + \frac{az_{k_1,k_2} + (c-a)\alpha - (\lambda + 2\delta)}{c-a} R + \frac{\alpha}{c-a}(az_{k_1,k_2} - 2\delta) = 0$$

and r_{2,k_1,k_2} is the corresponding negative solution from (24).

From $z_{k_1} < 0$ and (35) we obtain $s_{k_1,0} < 0$ and furthermore

$$s_{k_1,k_2+1} = r_{2,k_1,k_2} + s_{k_1,k_2} - r_{1,k_1,k_2+1} < 0$$

so that indeed (33) holds $\forall\ k_1, k_2 \geq 0$. Also, from $s_{k_1,k_2+1} < 0$, $r_{2,k_1,k_2} < 0$ and (38) we immediately see that (32) is fulfilled $\forall\ k_1 + k_2 \geq 1$.

One can now derive a recursion for the values r_{1,k_1,k_2} and r_{2,k_1,k_2}:

$$r_{1,k_1,k_2+1} = \frac{\alpha a(r_{2,k_1,k_2} + s_{k_1,k_2}) - \alpha 2\delta}{(c-a) r_{2,k_1,k_2}} = \frac{\alpha a}{(c-a)} + \frac{c}{c-a} r_{1,k_1,k_2}, \tag{39}$$

$$r_{2,k_1,k_2+1} = \frac{\alpha(as_{k_1,k_2+1} - 2\delta)}{cr_{1,k_1,k_2+1}} = \frac{-\alpha a}{c} + \frac{(c-a)}{c} r_{2,k_1,k_2}, \tag{40}$$

and we have

$$s_{k_1,k_2+1} = \frac{c}{\alpha a} r_{1,k_1,k_2+1} r_{2,k_1,k_2+1} + \frac{2\delta}{a} \,. \tag{41}$$

These recursions can be solved explicitly in a straight-forward way:

$$r_{1,k_1,k_2} = \left(\frac{c}{c-a}\right)^{k_2} r_{1,k_1,0} + \alpha\left(\left(\frac{c}{c-a}\right)^{k_2} - 1\right) \tag{42}$$

and

$$r_{2,k_1,k_2} = -\left(\frac{c-a}{c}\right)^{k_2} \frac{2\alpha\delta}{c\, r_{1,k_1,0}} - \alpha\left(1 - \left(\frac{c-a}{c}\right)^{k_2+1}\right) . \tag{43}$$

The second moment of the sum of discounted dividends in a linear barrier model can thus be calculated as a double series of the form (29) with the coefficients given by (26)-(28), (30), (31), (37) and (41)-(43). In Appendix A, it is shown that this double series is absolutely convergent, justifying this solution approach. The series turns out to converge quite fast. Numerical illustrations are given in Section 4.

Remark 2.4. The same idea can in principle be used to derive an explicit expression for $V_n(u, b)$ for arbitrary hyper-exponential and Erlang distributed claim sizes (using the technique of [4, 19]), which then consists of several sums of the form (29).

3 The discounted penalty function

For the Cramér-Lundberg model, Gerber and Shiu [13] introduced the by now classical discounted penalty function at ruin, which adapted to our dividend model is given by

$$m(u, b) := \mathbb{E}\Big(w(R_{T_{u,b}^-}, |R_{T_{u,b}}|)e^{-\delta T_{u,b}} 1_{\{T_{u,b}<\infty\}}\Big), \tag{44}$$

where $T_{u,b}$ denotes the time of ruin, $R_{T_{u,b}^-}$ is the surplus immediately before ruin, $|R_{T_{u,b}}|$ is the deficit at ruin and the penalty $w(x_1, x_2)$ is an arbitrary non-negative function on $[0, \infty) \times [0, \infty)$. $\delta \geq 0$ may be interpreted as a force of interest, but (44) may also be considered as a Laplace transform with δ as its argument.

The function $m(u, b)$ contains a lot of useful information about the ruin process. For example, if $w \equiv 1$, then $m(u, b)$ is the Laplace transform of the time to ruin given it occurs, and if furthermore $\delta = 0$, then $m(u, b)$ is simply the ruin probability. For $\delta = 0$ and $w(x_1, x_2) = 1_{\{x_1 \leq x\}} 1_{\{x_2 \leq y\}}$, $m(u, b)$ represents the joint distribution of the surplus before ruin and the deficit at ruin. Properties of $m(u, b)$ for a horizontal dividend barrier were recently studied in detail by Lin et al. [18] using a renewal theory approach. We will now derive a partial integro-differential equation for $m(u, b)$ in the presence of a linear dividend barrier.

3.1 A partial integro-differential equation for $m(u,b)$

Conditioning on the occurrence of a claim, we obtain

$$\begin{aligned} m(u,b) = (1-\lambda dt)e^{-\delta dt} m(u+cdt, b+a\,dt) \\ + e^{-\delta dt}\lambda dt \int_0^{u+cdt} m(u+cdt-v, b+a\,dt)dF(v) \\ + \lambda dt\, e^{-\delta dt} \int_{u+c\,dt}^{\infty} w(u+c\,dt, v-u-c\,dt)dF(v) + o(dt), \end{aligned}$$

from which it follows along the same line of arguments as in Section 2 that

$$c\frac{\partial m}{\partial u} + a\frac{\partial m}{\partial b} - (\lambda+\delta)m + \lambda \int_0^u m(u-v,b)dF(v) + \lambda \int_u^\infty w(u,v-u)dF(v) = 0. \quad (45)$$

A boundary condition can be obtained from

$$\begin{aligned} m(b,b) = (1-\lambda dt)e^{-\delta dt} m(b+a\,dt, b+a\,dt) \\ + e^{-\delta dt}\lambda dt \int_0^{b+a\,dt} m(b+a\,dt-v, b+a\,dt)dF(v) \\ + \lambda dt\, e^{-\delta dt} \int_{b+a\,dt}^{\infty} w(b+a\,dt, v-b-a\,dt)dF(v) + o(dt), \end{aligned}$$

and thus by continuity we have to have

$$\left.\frac{\partial m}{\partial u}\right|_{u=b} = 0. \quad (46)$$

Furthermore,

$$\lim_{b\to\infty} m(u,b) = m(u), \quad (47)$$

where $m(u)$ is the discounted penalty function without barrier.

Remark 3.1. Again, it is straight-forward to identify $m(u,b)$ as the fixed point of a contracting integral operator implying existence and uniqueness of a bounded solution of (45)-(47), cf. Remark 2.3.

Solving (45) together with (46) and (47) is a difficult problem in general. However, for exponential claim sizes and specific choices of the penalty function w, it is possible to derive explicit solutions as shown in the sequel:

3.2 Exponential claim amounts

Let $F(v) = 1 - e^{-\alpha v}$, then multiplying (45) with $e^{\alpha u}$ and differentiating w.r.t. u, we obtain the second-order partial differential equation

$$c\frac{\partial^2 m}{\partial u^2} + a\frac{\partial^2 m}{\partial u\,\partial b} + (c\alpha - \delta - \lambda)\frac{\partial m}{\partial u} + a\alpha\frac{\partial m}{\partial b} - \alpha\delta m + e^{-\alpha u}\lambda\xi'(u) = 0 \quad (48)$$

with boundary conditions (46),(47) and

$$c\frac{\partial m}{\partial u}\Big|_{u=0} + a\frac{\partial m}{\partial b}\Big|_{u=0} - (\lambda+\delta)m\Big|_{u=0} + \lambda\,\xi(0) = 0, \tag{49}$$

where $\xi(u) := \alpha\int_0^\infty w(u,x)e^{-\alpha x}dx$.

3.2.1 The distribution of the time to ruin

We will first solve this equation for the case, when the penalty function only depends on its second argument ($w(x_1,x_2) = w(x_2)$). Then $\xi(u)$ is constant, so that $\xi'(u) = 0$ (two particular examples are the choice $w \equiv 1$ which leads to the Laplace transform of the time of ruin, and $w = x_2^n$ leading to the (discounted) nth moment of the deficit at ruin, cf. Examples 3.1 and 3.2).

Suppose that $\delta > 0$ and $\xi(u) \equiv \eta$ for some positive constant η. Mimicking the procedure of Section 2, we try to find a solution of (48) of the form

$$m(u,b) = m(u) + \sum_{k=0}^{\infty} e^{s_k b}\,(C_{1,k}e^{r_{1,k}\,u} + C_{2,k}\,e^{r_{2,k}\,u}) \tag{50}$$

where $r_{1,k} > 0$, $r_{2,k} < 0$ are the solutions in R of the equation

$$cR^2 + (as_k + \alpha c - \lambda - \delta)R + \alpha(as_k - \delta) = 0 \tag{51}$$

and $s_k < 0$ for all $k \geq 0$. If such a solution exists, then (47) is automatically satisfied. (50) will now fulfill (48), if the function $m(u)$ (which does not depend on b) is a solution of

$$c\frac{\partial^2 m}{\partial u^2} + (c\alpha - \delta - \lambda)\frac{\partial m}{\partial u} - \alpha\delta m = 0, \tag{52}$$

from which it follows that

$$m(u) = A_1 e^{\rho_1 u} + A_2 e^{-\rho_2 u},$$

where ρ_1 and $-\rho_2$ denote the positive and negative solution of the equation

$$cR^2 + (\alpha c - (\lambda+\delta))R - \delta\alpha = 0. \tag{53}$$

Since $\lim_{u\to\infty} m(u) = 0$, $A_1 = 0$ so that $m(u) = A_2\,e^{-\rho_2 u}$.
By substituting (50) in (49) and comparing the coefficients of the terms without factor $e^{s_k b}$, we obtain

$$c\,m'(0) - (\lambda+\delta)m(0) + \lambda\,\eta = 0,$$

which determines the constant A_2. Using Vieta's rule we obtain

$$m(u) = \eta\frac{\lambda}{c}\,\frac{e^{-\rho_2 u}}{\alpha+\rho_1}, \tag{54}$$

which is the discounted penalty function without barrier. Note that ρ_1 and $-\rho_2$ are a function of δ. Comparing coefficients of the terms with factor $e^{s_k b}$, we get the condition

$$\sum_{k=0}^{\infty} \left(c\left(C_{1,k} r_{1,k} + C_{2,k} r_{2,k}\right) + a s_k \left(C_{1,k} + C_{2,k}\right) - (\lambda + \delta)\left(C_{k_1} + C_{k,2}\right)\right) = 0.$$

This condition is in particular fulfilled if every summand equals zero, i.e.

$$C_{1,k}(\alpha + r_{2,k}) = -C_{2,k}(\alpha + r_{1,k}). \tag{55}$$

Now it only remains to satisfy (46) by suitably combining the choices of constants in (50) for $k = 0, 1, \ldots$ (this part works in the same way as developed by Gerber [12] for the calculation of the ruin probability, which is the special case $\delta = 0, w \equiv 1$). Setting

$$r_{1,0} + s_0 = -\rho_2 = z_0, \qquad C_{1,0} = \frac{1}{r_{1,0}} \frac{\rho_2}{\alpha + \rho_1} \frac{\lambda}{c} \eta, \qquad r_{2,0} = \frac{\alpha a s_0 - \alpha\delta}{c r_{1,0}} \tag{56}$$

and for $k \geq 0$,

$$r_{1,k+1} + s_{k+1} = r_{2,k} + s_k = z_{k+1} \text{ and } C_{1,k+1} = \frac{r_{2,k}(\alpha + r_{2,k})}{r_{1,k}(\alpha + r_{1,k})} C_{1,k}, \tag{57}$$

(46) is fulfilled. All constants $r_{1,k}, r_{2,k}$ and s_k $(k \geq 0)$ are uniquely determined by the above relations and can be worked out iteratively. Again, we get the recurrences

$$r_{1,k} = \frac{\alpha a}{c - a} + \frac{c}{c - a}\, r_{1,k-1} \quad (k > 0)$$

and

$$r_{2,k} = \frac{-\alpha a}{c} + \frac{(c - a)}{c}\, r_{2,k-1} \quad (k > 0),$$

or explicitly

$$r_{1,k} = \left(\frac{c}{c - a}\right)^k r_{1,0} + \alpha\left(\left(\frac{c}{c - a}\right)^k - 1\right) \quad (k > 0) \tag{58}$$

and

$$r_{2,k} = \left(\frac{c - a}{c}\right)^k r_{2,0} - \alpha\left(1 - \left(\frac{c - a}{c}\right)^k\right) \quad (k > 0), \tag{59}$$

from which by similar arguments as in the previous section the absolute convergence of the series (50) can easily be shown.

Remark 3.2. This solution algorithm can be generalized to hyper-exponential and Erlang distributed claim sizes in a straight-forward way using the techniques developed in Siegl and Tichy [19].

Example 3.1. For the choice $\xi(u) \equiv \eta = 1$, we can identify (50) as the Laplace transform of the time of ruin

$$\mathbb{E}[e^{-\delta T_{u,b}} I_{\{T_{u,b}<\infty\}}] \tag{60}$$

in a ruin model with linear dividend barrier and exponential claims, where the coefficients are determined by (56),(57), (58) and (59). The moments $\mathbb{E}[T_{u,b}^n\, I_{\{T_{u,b}<\infty\}}]$ may easily be found by evaluating the nth derivative of (60) at $\delta = 0$ (a numerical example is given in Section 4). Setting $\delta = 0$ in (60) gives back the well-known expression for the ruin probability derived in [12].

Example 3.2. For the choice $w = x_2^n$ (implying $\xi(u) \equiv n!/\alpha^n$), $m(u, b)$ denotes the discounted nth moment of the deficit at ruin given ruin occurs (see e.g. [13]). Due to the lack-of-memory property of the exponential distribution, the deficit at ruin is exponentially distributed, independent of the time of ruin. From this it follows that

$$\mathbb{E}\Big[|R_{T_{u,b}}|^n\, e^{-\delta T_{u,b}}\, 1_{\{T_{u,b}<\infty\}}\Big] = \frac{n!}{\alpha^n}\, \mathbb{E}\Big[e^{-\delta T_{u,b}}\, 1_{\{T_{u,b}<\infty\}}\Big].$$

A numerical example is given in Section 4.

Remark 3.3. For the special case of a horizontal dividend barrier b ($a = 0$), equations (48) and (49) considerably simplify and for the determination of the Laplace transform of $T_{u,b}$ it suffices to look for a solution of (48) of the form $A_1(b)e^{\rho_1 u} + A_2(b)e^{-\rho_2 u}$. $A_1(b)$ and $A_2(b)$ can then be calculated from the boundary conditions (46), (47), (49) and (54) to give

$$\mathbb{E}[e^{-\delta T_{u,b}} I_{\{T_{u,b}<\infty\}}] = \frac{\lambda}{c}\, \frac{\rho_1\, e^{\rho_1 b-\rho_2 u} + \rho_2\, e^{\rho_1 u-\rho_2 b}}{\rho_1(\alpha+\rho_1)\, e^{\rho_1 b} + \rho_2(\alpha-\rho_2)\, e^{-\rho_2 b}}, \tag{61}$$

see Dickson and Waters [9]. This formula was originally derived by martingale arguments in Gerber [11]; for yet another derivation, see Lin et al. [17].
Of course, formula (61) can also be identified as the limiting case $a = 0$ of our above approach: from (24) it follows that for $a = 0$ we have $r_{1,k} = \rho_1$ and $r_{2,k} = -\rho_2$ for all $k \geq 0$. But then (56) and (57) imply $s_k = -(k+1)(\rho_1+\rho_2)$, $C_{1,0} = \lambda\rho_2/(c\,\rho_1(\alpha+\rho_1))$ and $C_{1,k+1} = -C_{1,k}\rho_2(\alpha-\rho_2)/(\rho_1(\alpha+\rho_1))$ for all $k \geq 0$. From (50) we obtain

$$\mathbb{E}[e^{-\delta T_{u,b}} I_{\{T_{u,b}<\infty\}}] = \frac{\lambda}{c}\left(\frac{e^{-\rho_2 u}}{\alpha+\rho_1} + \Big(e^{\rho_1 u} - \frac{\alpha-\rho_2}{\alpha+\rho_1}\, e^{-\rho_2 u}\Big)\sum_{k=0}^{\infty} C_{1,k}\, e^{-(k+1)(\rho_1+\rho_2)b}\right),$$

which after a little algebra yields (61).

3.2.2 More general penalty functions

The technique of Section 3.2.1 to derive exact solutions can be generalized to non-constant functions $\xi(u)$ in the following way: Let $m(u)$ be the solution of the ordinary differential equation

$$c\frac{\partial^2 m}{\partial u^2} + (c\alpha - \delta - \lambda)\frac{\partial m}{\partial u} - \alpha\delta m + e^{-\alpha u}\lambda\xi'(u) = 0 \tag{62}$$

with boundary conditions

$$c\frac{\partial m}{\partial u}\Big|_{u=0} - (\lambda+\delta)m\Big|_{u=0} + \lambda\,\xi(0) = 0 \tag{63}$$

and

$$\lim_{u\to\infty} m(u) = 0. \tag{64}$$

Then, the solution of (46)-(49) is of the form

$$m(u,b) = m(u) + \sum_{l=1}^{L}\sum_{k=0}^{\infty} e^{s_k^{(l)}\,b}\, C_{1,k}^{(l)}\left(e^{r_{1,k}^{(l)}\,u} - \frac{\alpha+r_{2,k}^{(l)}}{\alpha+r_{1,k}^{(l)}}\, e^{r_{2,k}^{(l)}\,u}\right), \tag{65}$$

if $m(u)$ is a linear combination of L exponentials in u, since then each of the L series can control one summand of $m(u)$ with respect to condition (47) in just the same way as in (50). By construction, the function $m(u)$ is then again the discounted penalty function in the classical risk model without dividend barrier.
The solution of (62) is given by

$$m(u) = A_1 e^{\rho_1 u} + A_2 e^{-\rho_2 u} + m_p(u), \tag{66}$$

with the particular solution

$$m_p(u) = \frac{\lambda}{c(\rho_1+\rho_2)}\left(e^{-\rho_2 u}\int^u e^{-(\alpha-\rho_2)t}\xi'(t)dt - e^{\rho_1 u}\int^u e^{-(\alpha+\rho_1)t}\xi'(t)dt\right),$$

where ρ_1 and $-\rho_2$ are again the positive and negative solution of (53). So whenever the penalty function $w(x_1,x_2)$ (through $\xi'(u)$) allows for a solution of (66) together with (63) and (64) of exponential type, an explicit expression for the discounted penalty function of the form (65) can be obtained. Condition (47) ensures that $z_0^{(l)} < 0,\ l = 1,\dots,L$, from which the absolute convergence of (65) again follows.

As an illustration, we will utilize this approach to derive the joint Laplace transform of the surplus prior to ruin and the time of ruin $m(u,b) := \mathbb{E}\Big(e^{-\nu R(T_{u,b}^-)}\, e^{-\delta T_{u,b}} 1_{\{T_{u,b}<\infty\}}\Big)$. Thus we set $w(x_1,x_2) = e^{-\nu x_1}$ or equivalently $\xi(u) = e^{-\nu u}$ from which the solution of (62)-(64) follows easily:

$$m(u) = \frac{\lambda\,\nu\, e^{-(\nu+\alpha)u} + \lambda\,(\alpha-\rho_2)\, e^{-\rho_2 u}}{c(\nu+\alpha+\rho_1)(\nu+\alpha-\rho_2)}, \tag{67}$$

(this formula could also have been obtained by calculating the Laplace transform of the generalized Dickson formulae (6.5) and (6.6) of Gerber and Shiu [13] for the (discounted) defective density function of the surplus prior to ruin in the classical risk model without barrier (note that their function $\psi(u)$, which is the joint Laplace transform of the deficit at ruin and the time of ruin, is in our case, using the results of Section 3.2.1, given by $\psi(u) = \frac{\lambda\,\alpha\, e^{-\rho_2 u}}{c\,(\alpha+\rho_1)^2}$)).

Indeed, $m(u)$ in (67) is a sum of two exponentials and we can solve (48) by (65) with $L = 2$, where each of the two series corrects the deviation from condition (46) for one of the two summands of $m(u)$. This can be achieved by starting the recursion for the coefficients with

$$\begin{aligned} r_{1,0}^{(1)} + s_0^{(1)} &= -\rho_2 = z_0^{(1)}, & C_{1,0}^{(1)} &= \tfrac{\lambda\,\rho_2(\alpha-\rho_2)}{c\,r_{1,0}^{(1)}(\nu+\alpha+\rho_1)(\nu+\alpha-\rho_2)}, \\ r_{1,0}^{(2)} + s_0^{(2)} &= -(\nu+\alpha) = z_0^{(2)}, & C_{1,0}^{(2)} &= \tfrac{\lambda\,\nu\,(\nu+\alpha)}{c\,r_{1,0}^{(2)}(\nu+\alpha+\rho_1)(\nu+\alpha-\rho_2)} \end{aligned}$$

and (57). The resulting analytic expression for the joint Laplace transform can then for instance be used to determine the (discounted) moments of the surplus prior to ruin by calculating the derivatives at $\nu = 0$ (for a numerical example see Section 4).

Remark 3.4. For the special case of a horizontal dividend barrier b ($a = 0$), we obtain, analogously to Remark 3.3, $r_{1,k}^{(l)} = \rho_1$ and $r_{2,k}^{(l)} = -\rho_2$ for all $k \geq 0$ and $l = 1, 2$. Moreover, $s_k^{(1)} = -(k+1)(\rho_1+\rho_2)$ and $s_k^{(2)} = -(\nu+\alpha+\rho_1) - k\,(\rho_1+\rho_2)$, from which we obtain

$$\mathbb{E}\Big(e^{-\nu R(T_{u,b}^-)}\, e^{-\delta T_{u,b}} 1_{\{T_{u,b}<\infty\}}\Big) = \frac{\lambda}{c\,(\alpha+\nu+\rho_1)(\alpha+\nu-\rho_2)} \cdot \\ \Bigg(\nu e^{-(\alpha+\nu)u} + (\alpha-\rho_2)e^{-\rho_2 u} + \big((\alpha+\rho_1)\,e^{\rho_1 u} - (\alpha-\rho_2)\,e^{-\rho_2 u}\big) \cdot \\ \Big(\frac{e^{-(\alpha+\nu-\rho_2)b}\nu(\alpha+\nu)}{\rho_1(\alpha+\rho_1)e^{(\rho_1+\rho_2)b} - \rho_2(\alpha-\rho_2)} + \frac{\rho_2(\alpha-\rho_2)}{\rho_1(\alpha+\rho_1)e^{(\rho_1+\rho_2)b} + \rho_2(\alpha-\rho_2)}\Big)\Bigg).$$

For $\delta = 0$ (which implies $\rho_1 = 0$ and $\rho_2 = \alpha - \lambda/c$), the inverse Laplace transform of the above expression leads to the density of the surplus prior to ruin in a model with horizontal dividend barrier

$$f(x) = \begin{cases} \frac{\alpha\lambda(e^{-\lambda x/c} - e^{-\alpha x})}{c\alpha-\lambda} & 0 < x \leq u, \\ \frac{\lambda\,c\,\alpha\,e^{-\lambda x/c} - \lambda^2\,e^{-\alpha u+\lambda(u-y)/c)}}{c(c\alpha-\lambda)} & u < x \leq b, \end{cases}$$

and

$$\mathbb{P}(R_{T_{u,b}} = b) = \frac{c\alpha\,e^{-\lambda b/c} - \lambda\,e^{-\alpha\,u+\lambda(u-b)/c}}{c\alpha-\lambda},$$

a result that was recently derived by other techniques in Lin et al. [18, p.562].

The solution approach presented in this section can again be extended to hyper-exponential and Erlang claim sizes. In the latter cases, the distribution of the deficit at ruin can also be considered (which for exponential claim sizes is trivially exponential, independent of the surplus prior to ruin and the time of ruin). For instance, the joint Laplace transform of $R_{T_{u,b}^-}$, $|R_{T_{u,b}}|$ and $T_{u,b}$ can be obtained in the above way, since the penalty function $w(x_1, x_2) = e^{-\nu_1 x_1 - \nu_2 x_2}$ also leads to an exponential expression for $m(u)$.

4 Numerical illustrations

Finally, we briefly illustrate the exact solutions derived in this paper for the following set of parameters: $X_i \sim \text{Exp}(1), \lambda = 1, c = 1.5, a = 1.1$ (and $\delta = 0.1$, if applicable). In each example, the infinite series representation of the explicit solution is approximated by an appropriate truncation of the series and these results are then compared with values from stochastic simulation (see e.g. [3]). In order to achieve a comparable degree of accuracy, each simulation estimate is based on 1 million sample paths. Note that each entry for the exact values in the tables below can be calculated on a normal PC in less than 0.01 seconds, whereas each simulation estimate takes several minutes.

Table 1 shows the results for the expected sum of discounted dividends $V_1(u, b)$ as given by (25). Table 2 contains the results for the corresponding standard deviation of $D_{u,b}$, i.e. $\sqrt{V_1(u, b)^2 - V_2(u, b)}$, where $V_2(u, b)$ is given by the double series (29) together with (26), (30), (31), (37), (41), (42) and (43) (cf. Section 2). Note that from Tables 1 and 2 one can observe that the standard deviation of $D_{u,b}$ is of about the same size as its expected value, which indicates that $V_1(u, b)$ can be a highly insufficient measure of the effectiveness of the dividend strategy and the consideration of higher order moments of $D_{u,b}$ is important.

Tables 3, 4 and 5 give the results for the expected value of the time to ruin $T_{u,b}$, the expected value of the surplus prior to ruin and the discounted expectation of the deficit at ruin, respectively, where the algorithms of Section 3 are used to determine the exact values.

Appendix A

Lemma A.1. *The double series representation of $V_2(u, b)$ derived in Section 2.3 is absolutely convergent.*

Proof: In the setting of Section 2.3, define

$$f_{z_{k_1}}(R) = R^2 + \frac{az_{k_1} + (c-a)\alpha - (\lambda + 2\delta)}{c-a} R + \frac{\alpha}{c-a}(az_{k_1} - 2\delta) \tag{68}$$

and recall that $r_{1,k_1,0}$ is the positive solution of $f_{z_{k_1}}(R) = 0$. Observe that due to $z_{k_1} \le 0$ for $k_1 \ge 0$ we have $f_{z_{k_1}}(2\delta/(c-a)) < 0$ and $\lim_{R\to\infty} f_{z_{k_1}}(R) = +\infty$. Thus

$$r_{1,k_1,0} > 2\delta/(c-a) \quad \forall\, k_1 \ge 0. \tag{69}$$

The sequence $(z_k)_{k\ge 0}$ is strictly decreasing. Set $z_{k+1} = z_k - h$ $(h > 0)$ to see that $f_{z_{k_1+1}}(r_{1,k_1,0}) = -ah(r_{1,k_1,0} + \alpha)/(c-a) < 0$. Thus, for every $k_1 \ge 0$,

$$r_{1,k_1+1,0} > r_{1,k_1,0}. \tag{70}$$

From (69) and (42) we get $\forall\, k_1, k_2 \geq 0$ a lower bound for r_{1,k_1,k_2}

$$r_{1,k_1,k_2} > \left(\frac{c}{c-a}\right)^{k_2} \frac{2\delta}{c-a}, \tag{71}$$

and (43) gives $\forall\, k_1, k_2 \geq 0$ a bound for r_{2,k_1,k_2}

$$-\alpha < r_{2,k_1,k_2} < -\alpha + \alpha \left(\frac{c-a}{c}\right)^{k_2+1}. \tag{72}$$

By (37), we now have for every $i \geq 0$ and $j \geq 1$

$$|C_{i,j}| \;=\; |C_{i,j-1}| \left| \frac{r_{2,i,j-1}}{r_{1,i,j}} \frac{(\alpha + r_{2,i,j-1})}{(\alpha + r_{1,i,j-1})} \right| \;\leq\; |C_{i,j-1}| \left| \frac{\alpha^2 (c-a)^2}{4\delta^2} \left(\frac{c-a}{c}\right)^{3j} \right|. \tag{73}$$

Let L be the smallest number such that

$$q_L := \left| \frac{\alpha^2 (c-a)^2}{4\delta^2} \left(\frac{c-a}{c}\right)^{3L} \right| < 1.$$

Then we have for every $j \geq L$

$$|C_{i,j}| \leq |C_{i,0}| \left(\frac{\alpha^2 (c-a)^2}{4\delta^2} \right)^{L-1} q_L^{j-L},$$

so that $\forall\, k_1 \geq 0$

$$\sum_{k_2=0}^{\infty} |C_{k_1,k_2}| \leq c_1\, |C_{k_1,0}|, \tag{74}$$

where c_1 is a constant which is independent of k_1. Furthermore, from (28), (34) and estimates from [12] it follows that for sufficiently large k_1

$$|C_{k_1+1,0}| \leq |C_{k_1,0}| \,\frac{\alpha c}{2\delta} \left(\frac{c-a}{c}\right)^{2k_1+1}.$$

Hence

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} |C_{k_1,k_2}| < c_1 \sum_{k_1=0}^{\infty} |C_{k_1,0}| < \infty.$$

Observe now that due to (32) and (72), for all $0 \leq u \leq b$ and all $k_1, k_2 \geq 0$

$$\left| \; C_{k_1,k_2} e^{s_{k_1,k_2} b} \left(e^{r_{1,k_1,k_2} u} - \frac{\alpha + r_{2,k_1,k_2}}{\alpha + r_{1,k_1,k_2}}\, e^{r_{2,k_1,k_2} u} \right) \right| < |C_{k_1,k_2}|\,,$$

which finally shows that the series (29) for $n = 2$ is absolutely convergent. $\square$

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$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.0	0.485										
0.1	0.403	0.495									
0.2	0.334	0.412	0.504								
0.3	0.277	0.341	0.418	0.510							
0.4	0.230	0.283	0.347	0.423	0.515						
0.5	0.190	0.234	0.287	0.351	0.427	0.518					
0.6	0.157	0.194	0.238	0.290	0.354	0.430	0.521				
0.7	0.130	0.161	0.197	0.241	0.293	0.356	0.432	0.523			
0.8	0.108	0.133	0.163	0.199	0.243	0.295	0.358	0.434	0.525		
0.9	0.090	0.110	0.135	0.165	0.201	0.244	0.296	0.359	0.435	0.526	
1.0	0.074	0.091	0.112	0.137	0.166	0.202	0.246	0.298	0.360	0.436	0.528

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.486										
0.1	0.404	0.496									
0.2	0.334	0.415	0.505								
0.3	0.279	0.340	0.418	0.510							
0.4	0.231	0.284	0.346	0.422	0.513						
0.5	0.191	0.236	0.287	0.350	0.427	0.519					
0.6	0.157	0.194	0.238	0.292	0.353	0.430	0.522				
0.7	0.132	0.160	0.198	0.242	0.294	0.356	0.432	0.524			
0.8	0.108	0.134	0.164	0.199	0.243	0.295	0.359	0.437	0.526		
0.9	0.090	0.109	0.135	0.166	0.201	0.245	0.298	0.358	0.436	0.528	
1	0.074	0.092	0.111	0.136	0.166	0.203	0.245	0.298	0.360	0.437	0.527

Table 1: Exact (above) and simulated (below) values of $V_1(u,b)$.

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.0	0.447										
0.1	0.438	0.447									
0.2	0.416	0.436	0.447								
0.3	0.390	0.417	0.438	0.446							
0.4	0.361	0.391	0.417	0.437	0.445						
0.5	0.333	0.363	0.392	0.417	0.437	0.444					
0.6	0.304	0.334	0.364	0.392	0.417	0.436	0.444				
0.7	0.278	0.306	0.335	0.365	0.392	0.417	0.436	0.443			
0.8	0.252	0.279	0.307	0.336	0.364	0.393	0.417	0.436	0.443		
0.9	0.229	0.254	0.281	0.308	0.337	0.365	0.393	0.417	0.435	0.443	
1.0	0.206	0.230	0.255	0.281	0.309	0.337	0.365	0.393	0.417	0.435	0.442

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.446										
0.1	0.437	0.447									
0.2	0.416	0.441	0.448								
0.3	0.389	0.415	0.436	0.447							
0.4	0.361	0.391	0.417	0.437	0.445						
0.5	0.335	0.362	0.389	0.417	0.436	0.446					
0.6	0.305	0.332	0.363	0.394	0.416	0.437	0.445				
0.7	0.279	0.307	0.336	0.366	0.393	0.417	0.434	0.443			
0.8	0.252	0.280	0.309	0.336	0.365	0.392	0.419	0.438	0.443		
0.9	0.229	0.251	0.281	0.308	0.337	0.365	0.393	0.415	0.436	0.444	
1	0.207	0.232	0.253	0.282	0.306	0.336	0.363	0.392	0.415	0.437	0.442

Table 2: Exact (above) and simulated (below) values of the standard deviation $\sqrt{V_1(u,b)^2 - V_2(u,b)}$.

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.372										
0.1	1.412	1.427									
0.2	1.429	1.463	1.476								
0.3	1.430	1.475	1.508	1.521							
0.4	1.424	1.476	1.518	1.548	1.560						
0.5	1.415	1.468	1.516	1.556	1.584	1.595					
0.6	1.404	1.458	1.508	1.552	1.589	1.615	1.625				
0.7	1.393	1.447	1.497	1.543	1.584	1.618	1.642	1.652			
0.8	1.383	1.436	1.485	1.531	1.574	1.612	1.646	1.666	1.675		
0.9	1.374	1.425	1.473	1.519	1.562	1.601	1.636	1.665	1.686	1.694	
1.0	1.366	1.416	1.463	1.508	1.549	1.588	1.624	1.657	1.684	1.703	1.710

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	1.371										
0.1	1.406	1.429									
0.2	1.429	1.460	1.482								
0.3	1.430	1.482	1.511	1.523							
0.4	1.423	1.482	1.518	1.540	1.552						
0.5	1.419	1.469	1.514	1.558	1.590	1.594					
0.6	1.406	1.462	1.511	1.546	1.585	1.609	1.624				
0.7	1.395	1.442	1.500	1.544	1.586	1.624	1.65	1.654			
0.8	1.380	1.437	1.486	1.534	1.573	1.623	1.647	1.668	1.680		
0.9	1.373	1.427	1.475	1.521	1.563	1.601	1.636	1.660	1.687	1.694	
1.0	1.369	1.420	1.469	1.515	1.551	1.587	1.627	1.658	1.680	1.713	1.714

Table 3: Exact (above) and simulated (below) values of the mean ruin time $\mathbb{E}[T_{u,b} I_{\{T_{u,b}<\infty\}}]$.

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.488										
0.1	0.518	0.534									
0.2	0.527	0.557	0.569								
0.3	0.526	0.562	0.588	0.598							
0.4	0.522	0.560	0.591	0.613	0.622						
0.5	0.517	0.554	0.587	0.613	0.632	0.639					
0.6	0.512	0.550	0.581	0.608	0.630	0.646	0.652				
0.7	0.508	0.545	0.576	0.602	0.624	0.642	0.655	0.660			
0.8	0.505	0.541	0.571	0.596	0.618	0.636	0.650	0.660	0.665		
0.9	0.503	0.538	0.567	0.591	0.612	0.629	0.643	0.654	0.662	0.666	
1	0.502	0.536	0.565	0.588	0.607	0.623	0.636	0.647	0.655	0.662	0.664

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.487										
0.1	0.520	0.533									
0.2	0.526	0.557	0.570								
0.3	0.526	0.562	0.589	0.599							
0.4	0.522	0.559	0.591	0.613	0.622						
0.5	0.517	0.555	0.587	0.614	0.632	0.640					
0.6	0.512	0.550	0.581	0.609	0.629	0.646	0.649				
0.7	0.509	0.546	0.575	0.601	0.625	0.643	0.655	0.659			
0.8	0.507	0.540	0.572	0.597	0.618	0.636	0.648	0.660	0.665		
0.9	0.506	0.539	0.567	0.592	0.613	0.628	0.644	0.656	0.662	0.667	
1	0.503	0.535	0.565	0.587	0.607	0.623	0.636	0.646	0.655	0.662	0.664

Table 4: Exact (above) and simulated (below) values of the expected surplus before ruin $\mathbb{E}[R_{T^-_{u,b}} I_{\{T_{u,b}<\infty\}}]$

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.646										
0.1	0.624	0.621									
0.2	0.609	0.599	0.595								
0.3	0.599	0.584	0.575	0.571							
0.4	0.592	0.574	0.560	0.551	0.548						
0.5	0.589	0.567	0.551	0.537	0.528	0.525					
0.6	0.583	0.562	0.544	0.578	0.515	0.506	0.503				
0.7	0.581	0.559	0.539	0.521	0.506	0.494	0.486	0.482			
0.8	0.580	0.557	0.536	0.531	0.500	0.485	0.473	0.465	0.462		
0.9	0.578	0.555	0.534	0.514	0.496	0.479	0.465	0.454	0.446	0.443	
1.0	0.578	0.554	0.533	0.512	0.493	0.475	0.459	0.446	0.435	0.428	0.425

$b \setminus u$	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.0	0.646										
0.1	0.624	0.621									
0.2	0.610	0.600	0.595								
0.3	0.600	0.584	0.575	0.571							
0.4	0.592	0.574	0.560	0.551	0.548						
0.5	0.587	0.568	0.550	0.537	0.528	0.5255					
0.6	0.583	0.563	0.544	0.528	0.516	0.506	0.504				
0.7	0.581	0.558	0.540	0.521	0.506	0.493	0.485	0.483			
0.8	0.579	0.558	0.536	0.517	0.500	0.485	0.473	0.465	0.462		
0.9	0.579	0.556	0.533	0.514	0.495	0.479	0.465	0.453	0.446	0.444	
1.0	0.578	0.555	0.533	0.512	0.493	0.475	0.460	0.446	0.434	0.428	0.425

Table 5: Exact (above) and simulated (below) values for the discounted expectation of the deficit at ruin $\mathbb{E}[e^{-\delta T_{u,b}}|R(T_{u,b})|]$.