Second Order Asymptotics of Aggregated Log-Elliptical Risk

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Abstract: In this paper we establish the error rate of first order asymptotic approximation for the tail probability of sums of log-elliptical risks. Our approach is motivated by extreme value theory which allows us to impose only some weak asymptotic conditions satisfied in particular by log-normal risks. Given the wide range of applications of the log-normal model in finance and insurance our result is of interest for both rare-event simulations and numerical calculations. We present numerical examples which illustrate that the second order approximation derived in this paper significantly improves over the first order approximation.

Key words: Risk aggregation; second order asymptotics; log-elliptical distribution; log-normal distribution; Gumbel max-domain of attraction.

1 Introduction

Modeling multivariate dependent risks is an important task of actuaries involved in risk management, pricing and loss reserving. The standard and most common model used in practice, both in insurance and finance, is that of dependent log-normal risks, see e.g., Mitra and Resnick (2009), Foss and Richards (2010) or Asmussen et al. (2011). Despite the tractability of multivariate log-normal distribution, the first result which derives the asymptotic tail behavior of the sum of log-normal risks appeared recently in Asmussen and Rojas-Nandayapa (2008), see also Albrecher et al. (2006). The recent contribution Asmussen et al. (2011) derives an explicit asymptotic expansion ($u \to \infty$) of

$$\mathbb{P}\{S(u) > u\}, \quad \text{with} \;\; S(u) = X_1(u) + \cdots + X_d(u),$$

where $X_i(u), i \leq d$ is a d-dimensional log-normal random vector with underlying covariance matrix depending on the threshold u. In the aforementioned paper the consideration of parametrized risks leads to the introduction of novel importance sampling estimators of $\mathbb{P}\{S(u) > u\}$.

Given the fact that Normal random vectors are a canonical example of the elliptically symmetric ones, it is natural

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to model the aggregated risk utilizing a log-elliptical framework, which has been recently discussed in Rojas-Nandayapa (2008), Kortschak and Hashorva (2013) and Hashorva (2013). The latter two papers derived (under different conditions) the following asymptotic expansion

$$\mathbb{P}\left(S(u) > u\right) \sim \sum_{i=1}^{d} \mathbb{P}\left(X_i(u) > u\right), \quad u \to \infty, \tag{1.1}$$

with $X_i(u), i \le d$ the components of some d-dimensional log-elliptical random vector indexed by u (here $a(u) \sim b(u)$ stands for the asymptotic equivalence as $u \to \infty$ of two functions $a(\cdot), b(\cdot)$).
The principal goal of this contribution is the precise quantification of the error of the approximation claimed in (1.1). Specifically for $\Upsilon(u)$ defined as

$$\Upsilon(u) := \mathbb{P}\left(S(u) > u\right) - \sum_{i=1}^{d} \mathbb{P}\left(X_i(u) > u\right), \quad u \to \infty$$

we derive in the main result (Theorem 2.2 below) the rate of convergence of $\Upsilon(u)$ to 0 as $u \to \infty$. As was already observed in Mitra and Resnick (2009) the first order approximation in (1.1) for positively correlated random variables can be rather crude even in the bivariate case $d = 2$. Now the obvious motivation of our paper is to improve this rather crude first order approximation. The numerical examples in Section 3 show that the second order approximation significantly improves over the first order one.

Two essential properties of log-normal risks are crucial for the derivation of the tail asymptotic expansion in (1.1): a) the univariate log-normal distribution belongs to the Gumbel max-domain of attraction (see below for definition), and b) log-normal risks and in particular Normal ones are asymptotically independent, see e.g., Resnick (1987).
The derivation of (1.1) for log-elliptical risks is strongly based on assumptions which agree with a) and b) above. In order to derive the asymptotics of $\Upsilon(u)$, we shall impose some additional restrictions on the probability density function of log-elliptical risks. Our result is new even when $(\log X_1(u), \dots, \log X_d(u))$ is a d-dimensional Normal random vector with mean zero and non-singular covariance matrix Σ. For this case assuming for simplicity that the off-diagonal elements of Σ are equal to $\rho \in (-1, 1)$, we obtain

$$\Upsilon(u) \quad \sim \quad \frac{d(d-1)\exp((1-\rho^2)/2)}{\sqrt{2\pi} u^{1-\rho}} \exp(-(\log u)^2/2), \quad u \to \infty. \tag{1.2}$$

The speed of convergence of $\Upsilon(u)$ to 0 is shown by (1.2) to decrease with ρ increasing, i.e., the more dependence the worse the approximation. When $\rho = 0$, then (1.2) shows that

$$\Upsilon(u) \sim d(d-1)\exp(1/2) f_*(u)$$

as $u \to \infty$ with f_* the probability density function of $X_1(u)$. As expected, (1.2) implies further that when the dimension d increases the quality of approximation also decreases.
Organisation of the rest of the paper: In Theorem 2.2 below we present our main result. Numerical comparisons are given in Section 3. Several auxiliary results and the proof of the main result are displayed in Section 4.

2 Results

Let R be a positive random variable with distribution function H being independent of $\boldsymbol{U}$ which is uniformly distributed on the unit sphere of $\mathbb{R}^d$ (with respect to the L_2-norm). For $A_u, u > 0$ a sequence of $d \times d$ non-singular matrices we shall set $\Sigma_u = A_u A_u^\top$. In the sequel we suppose that the elements of Σ_u satisfy

$$\sigma_{11}(u) = \dots = \sigma_{dd}(u) = 1, \quad \sigma_{ij}(u) \in [-1, 1], \quad i \ne j, \quad u > 0 \tag{2.3}$$

and further $\beta_i, \lambda_i, i \le d$ are positive constants such that

$$0 < \beta_d \le \cdots \le \beta_1 < \infty, \quad \lambda_1 = \max_{\beta_i=\beta_1} \lambda_i. \tag{2.4}$$

For given $\gamma_u, u > 0$ positive constants satisfying $\lim_{u\to\infty} \gamma_u = \gamma \in (0,\infty)$ we define a d-dimensional random vector

$$\boldsymbol{X}(u) := (\lambda_1 Z_1(u)^{\beta_1\gamma_u}, \cdots, \lambda_d Z_d(u)^{\beta_d\gamma_u})^\top,$$

where

$$(Z_1(u), \ldots, Z_d(u))^\top = \exp(RA_u\boldsymbol{U}), \quad u > 0.$$

The class of log-elliptical risks considered in this paper are such that R has distribution function F with infinite upper endpoint satisfying

$$\lim_{u\to\infty} \frac{1 - F(u + xb(u))}{1 - F(u)} = \exp(-x), \quad \forall x \in \mathbb{R}, \tag{2.5}$$

with some positive scaling function $b(\cdot)$. When (2.5) holds, we say that R (and alternatively also F) is in the max-domain of attraction (MDA) of the Gumbel distribution $\Lambda(x) = \exp(-\exp(-x)), x \in \mathbb{R}$ with positive scaling function $b(\cdot)$. If we assume further that

$$\lim_{u\to\infty} b(u) = 0, \tag{2.6}$$

then for $j \le d$

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(X_j(u) > u + xe_j^*(u)\right)}{\mathbb{P}\left(X_j(u) > u\right)} = \exp(-x), \quad x \in \mathbb{R}, \tag{2.7}$$

where

$$e_j^*(u) := \beta_j\gamma_u u e\left(\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\beta_j\gamma_u}}\right)\left(\frac{u}{\lambda_j}\right)^{-\frac{1}{\beta_j\gamma_u}}, \tag{2.8}$$

with

$$e(u) = ub(\log u), \quad u > 0. \tag{2.9}$$

Hereafter we shall assume that

$$\lim_{u\to\infty} e(u) = \infty. \tag{2.10}$$

The following theorem presented in Kortschak and Hashorva (2013) establishes the first order asymptotics of aggregated log-elliptical risk.

Theorem 2.1. *Suppose that* (2.6) *and* (2.10) *hold and for j with $\beta_j = \beta_1$ and every $\epsilon > 0$, $c > 0$ there exists some u_0 such that for all $u > u_0$*

$$\sigma_{ij}(u) + c\sqrt{\frac{1 - \sigma_{ij}(u)^2}{\log(u)}} \le \frac{\beta_j}{\beta_i}\frac{\log(\epsilon e_i^*(u))}{\log(u)} \tag{2.11}$$

holds for all $i \ne j$, then (1.1) *is satisfied.*

Remark: a) If $\beta_i = \beta_j$ and $\lim_{u\to\infty} \log(e_j^*(u))/\log(u) = 1$, then condition (2.11) is for example fulfilled when

$$\limsup_{u\to\infty} \frac{-\log\left(\frac{e_j^*(u)}{u}\right)}{(1-\sigma_{ij}(u))\log(u)} < 1. \tag{2.12}$$

b) If $\sigma_{ij}(u) < \kappa < \lim_{u\to\infty}(e_i^*(u))/u$ for all large u and $i \neq j$, then condition (2.11) holds.
In order to derive the asymptotics of the error term $\Upsilon(u)$ we shall impose some additional restrictions. Both conditions (2.5) and (2.6) imply that the distribution function $\widetilde{F}$ of $\widetilde{R} := \exp(R)$ is in Gumbel MDA with scaling function $e(\cdot)$ defined in (2.9). Below, that assumption will be strengthened to $\widetilde{F}$ is eventually differentiable with continuous probability density function $\widetilde{f}$ such that the following von Mises condition

$$\lim_{u\to\infty} \frac{\widetilde{f}(u + xe(u))}{\widetilde{f}(u)} = \exp(-x) \tag{2.13}$$

holds for all $x \in \mathbb{R}$. We formulate next our principal result.

Theorem 2.2. *Under the assumptions of Theorem 2.1, suppose further that the scaling function $e(\cdot)$ is ultimately monotone increasing and satisfies*

$$\lim_{\lambda\to 1} \limsup_{u\to\infty} \frac{e(\lambda u)}{e(u)} \quad = \quad 1. \tag{2.14}$$

If (2.13) *holds, and further*

$$\lim_{u\to\infty} \frac{\log(u) e_i^*(u)}{u} = c_i \in [0, \infty), \quad 1 \le i \le d, \tag{2.15}$$

then we have

$$\Upsilon(u) \quad \sim \quad \sum_{j=1}^{d} \left(\sum_{i\neq j} \frac{\lambda_i}{\beta_j \gamma} \exp\left(\frac{c_j\left(1-\sigma_{ij}^2(u)\right)}{2(\beta_j/\beta_i)^2} \right) \left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i \sigma_{ij}(u)}{\beta_j}} \right) \frac{1}{e_j^*(u)} \mathbb{P}\left(X_j(u) > u\right), \quad u \to \infty. \tag{2.16}$$

Remark 2.3. *a) For any $i \neq j, i, j \le d$ after some long calculations*

$$\mathbb{E}\left[\lambda_i Z_i(u)^{\beta_i \gamma(u)} \middle| \lambda_j Z_j(u)^{\beta_j \gamma(u)} = u\right] \sim \frac{\lambda_i}{\beta_j \gamma} \exp\left(\frac{c_j\left(1-\sigma_{ij}^2(u)\right)}{2(\beta_j/\beta_i)^2} \right) \left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i \sigma_{ij}(u)}{\beta_j}}, \quad u \to \infty$$

and thus (2.16) *is in accordance with the findings of Kortschak (2011).*
b) The von Mises condition (2.13) *is satisfied by a large class of distribution functions in the Gumbel MDA. In particular, the log-normal distribution satisfies it. Indeed, we have that with $\widetilde{f}(x) = \frac{1}{x\sqrt{2\pi}} \exp(-(\log x)^2/2), x > 0$ the scaling function can be taken to be $e(u) = u/\log u, u > 0$, and thus*

$$\frac{\widetilde{f}(u + xe(u))}{f(u)} \sim \exp(-(\log(u + xu/\log u))^2/2 + (\log u)^2/2) \sim \exp(-x), \quad \forall x \in \mathbb{R}$$

as $u \to \infty$.

Example. Given the huge interest on multivariate log-normal models for aggregated risk (see e.g., Asmussen et al. (2011)) we discuss briefly the findings of our main result when $\log X_i(u), i \le d, u > 0$ in Theorem 2.2 are Normal random variables. Since the random radius R pertaining to the stochastic representation of the Gaussian distribution and the distribution function of each $\log X_i(u)$ are in the Gumbel MDA with the same scaling function $b(u) = 1/u$, we have that $e(u) = u/\log u$, hence

$$\lim_{\lambda\to 1} \limsup_{u\to\infty} \frac{e(\lambda u)}{e(u)} = \lim_{\lambda\to 1} \limsup_{u\to\infty} \frac{\lambda u/(\log u + \log \lambda)}{u/\log u} = \lim_{\lambda\to 1} \lambda = 1.$$

Consequently, condition (2.14) is satisfied, and further (2.13) follows easily. Next, for any $j \le d$

$$e_j^*(u) = \beta_j\gamma_u u e\left(\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\beta_j\gamma_u}}\right)\left(\frac{u}{\lambda_j}\right)^{-\frac{1}{\beta_j\gamma_u}} = \frac{\beta_j\gamma_u u}{\log\left(\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\beta_j\gamma_u}}\right)} = \frac{\beta_j^2\gamma_u^2 u}{\log u - \log\lambda_j} \sim \frac{\beta_j^2\gamma^2 u}{\log u}$$

as $u \to \infty$. Further, since

$$c_i = \lim_{u\to\infty}\frac{\log(u)e_i^*(u)}{u} = \gamma^2\beta_i^2, \quad 1 \le i \le d$$

condition (2.15) holds. Therefore, (2.16) boils down to

$$\begin{aligned}\Upsilon(u) &\sim \frac{\log u}{\gamma^3 u}\sum_{j=1}^d\sum_{i\neq j}\frac{\lambda_i}{\beta_j^3}\exp\left(\frac{c_j\left(1-\sigma_{ij}^2(u)\right)}{2(\beta_j/\beta_i)^2}\right)\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\mathbb{P}\left(X_j(u) > u\right)\\ &\sim \frac{\log u}{\gamma^3 u}\sum_{j=1}^d\sum_{i\neq j}\frac{\lambda_i}{\beta_j^3}\exp\left(\frac{c_j\left(1-\sigma_{ij}^2(u)\right)}{2(\beta_j/\beta_i)^2}\right)\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\frac{\exp(-(\log(u/\lambda_j))^2/(2\beta_j^2\gamma_u^2))}{\sqrt{2\pi\beta_j^2\gamma_u^2}\log u}\\ &\sim \sum_{j=1}^d\sum_{i\neq j}\frac{\lambda_i}{(\beta_j\gamma)^4}\exp\left((\beta_i\gamma)^2(1-\sigma_{ij}^2(u))/2\right)\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\frac{\exp(-(\log(u/\lambda_j))^2/(2(\beta_j\gamma_u)^2))}{u\sqrt{2\pi}}, \quad u\to\infty.\end{aligned}$$

3 Numerical Examples

In this section we shall present some numerical examples. For comparison purposes we use the same examples as in Mitra and Resnick (2009), which means that (Y_1, Y_2) is a bivariate Normal random vector with zero mean, each component has variance 1 and correlation coefficient is $\rho(u) =: \rho \in \{-0.9, 0, 0.5, 0.9\}$, and we set $X_i = \exp(Y_i), i = 1, 2$. For this choice $e_j^*(u) = e(u)$ for any $u > 0, j = 1, 2$. For the practical implementation of the second order asymptotics we replace the term $\frac{\mathbb{P}(X_j(u)>u)}{e_j^*(u)}$ in equation (2.16) by the probability density function of $X_j(u)$. To check the accuracy of the asymptotic approximations we use Monte Carlo simulation with the estimator from Kortschak and Hashorva (2013). In tables 1–4 we present the results of the numerical study. In the first column of the tables we provide the threshold u. In the column "Asympt 1" respectively "Asympt 2" we provide the first respectively second order asymptotic approximation for the ruin probability. The results of the Monte Carlo simulation is given in the column "MC". For the Monte Carlo simulation we used so many simulations that the error of the Monte Carlo simulation is negligible compared to the error of the asymptotic approximations. The column "Ratio 1" respectively "Ratio 2" provide the ratio of the first order respectively second order asymptotic approximation and the result of the Monte Carlo simulation. The last three columns of the table provide three heuristic measures for the quality of the asymptotic approximations ϵ, u^ϵ and $\hat\rho$. For the measure ϵ which is motivated by condition (2.11) we calculate for $\theta = \log(u)/\log(u + e(u))$ the corresponding ϵ for which

$$\rho(u) + c\sqrt{1-\rho(u)^2}\sqrt{1/\theta^2 - 1} = \frac{\beta_j}{\beta_i}\frac{\log(\epsilon e_i^*(u))}{\log(u)}. \tag{3.17}$$

The measure ρ is just the $\hat\rho$ for which the inequality in (2.12) is fulfilled with equality.

For the case $\rho = 0.9$ we can observe that the second order approximation improves significantly over the first order one but is still not applicable. Indeed, in order to get a $\hat\rho = 0.9$ we have to choose $u \approx 3.4 \times 10^{15}$ and similarly if we choose $\rho = \rho(u) = 0.9$ and we want to get an $\epsilon \approx 0.1$ we need $u \approx 10^{30}$ which means that there should not be any hope that the asymptotics gives meaningful results for $\rho = \rho(u) = 0.9$ and u in a normal range (which was also found by Mitra and Resnick (2009)). Also for the other values of ρ we see from the tables that the second order

u	Asympt 1	Asympt 2	MC	Ratio 1	Ratio 2	ϵ	e^ϵ	$\hat{\rho}$
10	0.0213	0.0705	0.0522	2.45	0.74	2	7.38	0.638
30	0.000671	0.00259	0.00297	4.43	1.15	2.89	18	0.64
50	$9.15e-05$	0.000373	0.000529	5.77	1.42	3.25	25.8	0.651
75	$1.58e-05$	$6.68e-05$	0.000112	7.08	1.67	3.51	33.5	0.661
100	$4.12e-06$	$1.79e-05$	$3.36e-05$	8.16	1.88	3.68	39.7	0.668
200	$1.17e-07$	$5.31e-07$	$1.32e-06$	11.3	2.49	4.04	57.1	0.685
300	$1.17e-08$	$5.45e-08$	$1.59e-07$	13.6	2.91	4.23	68.7	0.695
500	$5.15e-10$	$2.45e-09$	$8.68e-09$	16.9	3.54	4.43	84.2	0.706
700	$5.71e-11$	$2.76e-10$	$1.11e-09$	19.4	4.02	4.55	94.7	0.713
1000	$4.92e-12$	$2.4e-11$	$1.1e-10$	22.3	4.56	4.66	106	0.72
1500	$2.61e-13$	$1.29e-12$	$6.78e-12$	26	5.27	4.77	118	0.728
2000	$2.94e-14$	$1.46e-13$	$8.52e-13$	29	5.83	4.84	127	0.733
2500	$5.12e-15$	$2.56e-14$	$1.6e-13$	31.3	6.26	4.89	133	0.737
3000	$1.18e-15$	$5.92e-15$	$3.95e-14$	33.4	6.66	4.93	138	0.74
5000	$1.63e-17$	$8.26e-17$	$6.47e-16$	39.6	7.84	5.02	151	0.748
7000	$8.47e-19$	$4.29e-18$	$3.67e-17$	43.3	8.55	5.06	158	0.754
10000	$3.25e-20$	$1.65e-19$	$1.58e-18$	48.7	9.59	5.1	165	0.759
$1e+05$	$1.14e-30$	$5.72e-30$	$9.21e-29$	81.1	16.1	5.17	176	0.788
$1e+06$	$2.05e-43$	$9.94e-43$	$2.16e-41$	105	21.7	5	148	0.81

Table 1: Results of approximation for $\rho = 0.9$

asymptotics improves significantly over the first order estimate. Further the asymptotics work better for smaller values of ρ especially for $\rho = -0.9$ the first as well as the second order asymptotics perform quite well. Finally we remark that the value of e^ϵ displayed before $\hat{\rho}$ is comparable with the value in the column in "Ratio 1" (for sufficiently large values of u).

4 Further Results and Proof of Theorem 2.2

We present first four lemmas which are of some independent interest and then proceed with the proof of our main result. In the sequel we consider some positive random variable R such that its distribution function F has an infinite upper endpoint. Under the assumption that F is in Gumbel MDA with some positive scaling function $b(\cdot)$, we have the following representation (see e.g., Resnick (1987))

$$1 - F(u) \quad = \quad c(u) \exp\Big(-\int_{x_0}^{u} \frac{g(t)}{b(t)}\, dt\Big), \tag{4.18}$$

with x_0 some constant and $c(\cdot), g(\cdot)$ two positive measurable functions such that $\lim_{u\to\infty} c(u) = \lim_{u\to\infty} g(u) = 1$. Below we assume that $e(u) = ub(\log(u))$ is a scaling function of $\widetilde{F}$, i.e., the df $\widetilde{F}$ is in the Gumbel MDA with scaling function $e(\cdot)$ (recall $\widetilde{F}$ is the distribution function of $\exp(R)$). This holds in particular when $\lim_{u\to\infty} b(u) = 0$. Next, define $e^*(u)$ by (2.8) for some λ, β positive, i.e.,

$$e^*(u) = \beta\gamma_u u e\left(\left(\frac{u}{\lambda}\right)^{\frac{1}{\beta\gamma_u}}\right)\left(\frac{u}{\lambda}\right)^{-\frac{1}{\beta\gamma_u}},$$

with γ_u such that $\lim_{u\to\infty} \gamma_u = \gamma \in (0, \infty)$.

u	Asympt 1	Asympt 2	MC	Ratio 1	Ratio 2	ϵ	e^{ϵ}	$\hat{\rho}$
10	0.0213	0.0472	0.0444	2.09	0.941	1.3	3.66	0.638
30	0.000671	0.00132	0.00165	2.46	1.25	1.2	3.33	0.64
50	$9.15e-05$	0.00017	0.000226	2.47	1.33	1.1	3	0.651
75	$1.58e-05$	$2.78e-05$	$3.79e-05$	2.4	1.36	1.01	2.74	0.661
100	$4.12e-06$	$7e-06$	$9.54e-06$	2.31	1.36	0.939	2.56	0.668
200	$1.17e-07$	$1.83e-07$	$2.41e-07$	2.06	1.32	0.779	2.18	0.685
300	$1.17e-08$	$1.75e-08$	$2.23e-08$	1.9	1.27	0.691	2	0.695
500	$5.15e-10$	$7.28e-10$	$8.86e-10$	1.72	1.22	0.589	1.8	0.706
700	$5.71e-11$	$7.82e-11$	$9.21e-11$	1.61	1.18	0.528	1.7	0.713
1000	$4.92e-12$	$6.52e-12$	$7.48e-12$	1.52	1.15	0.468	1.6	0.72
1500	$2.61e-13$	$3.34e-13$	$3.68e-13$	1.41	1.1	0.407	1.5	0.728
2000	$2.94e-14$	$3.68e-14$	$4e-14$	1.36	1.09	0.368	1.44	0.733
2500	$5.12e-15$	$6.3e-15$	$6.74e-15$	1.32	1.07	0.339	1.4	0.737
3000	$1.18e-15$	$1.44e-15$	$1.52e-15$	1.29	1.06	0.318	1.37	0.74
5000	$1.63e-17$	$1.92e-17$	$2e-17$	1.22	1.04	0.263	1.3	0.748

Table 2: Results of approximation for $\rho = 0.5$

u	Asympt 1	Asympt 2	MC	Ratio 1	Ratio 2	ϵ	e^{ϵ}	$\hat{\rho}$
10	0.0213	0.0306	0.0337	1.58	1.1	0.537	1.71	0.638
30	0.000671	0.000806	0.000864	1.29	1.07	0.28	1.32	0.64
50	$9.15e-05$	0.000104	0.000108	1.18	1.04	0.196	1.22	0.651
75	$1.58e-05$	$1.74e-05$	$1.77e-05$	1.12	1.02	0.145	1.16	0.661
100	$4.12e-06$	$4.45e-06$	$4.5e-06$	1.09	1.01	0.117	1.12	0.668
200	$1.17e-07$	$1.22e-07$	$1.23e-07$	1.05	1	0.0679	1.07	0.685
300	$1.17e-08$	$1.21e-08$	$1.21e-08$	1.03	1	0.049	1.05	0.695
500	$5.15e-10$	$5.25e-10$	$5.26e-10$	1.02	1	0.0322	1.03	0.706
700	$5.71e-11$	$5.8e-11$	$5.8e-11$	1.02	1	0.0243	1.02	0.713
1000	$4.92e-12$	$4.98e-12$	$4.98e-12$	1.01	1	0.018	1.02	0.72

Table 3: Results of approximation for $\rho = 0$

u	Asympt 1	Asympt 2	MC	Ratio 1	Ratio 2	ϵ	e^{ϵ}	$\hat{\rho}$
2	0.488	0.673	0.785	1.61	1.17	0.331	1.39	1.53
3	0.272	0.331	0.369	1.36	1.11	0.263	1.3	0.914
5	0.108	0.119	0.121	1.12	1.02	0.148	1.16	0.704
10	0.0213	0.0221	0.0221	1.04	1	0.0555	1.06	0.638
15	0.00677	0.0069	0.0069	1.02	1	0.0297	1.03	0.632
30	0.000671	0.000675	0.000675	1.01	1	0.00976	1.01	0.64
50	$9.15e-05$	$9.18e-05$	$9.18e-05$	1	1	0.00419	1	0.651
75	$1.58e-05$	$1.58e-05$	$1.58e-05$	1	1	0.00212	1	0.661
100	$4.12e-06$	$4.12e-06$	$4.12e-06$	1	1	0.0013	1	0.668

Table 4: Results of approximation for $\rho = -0.9$

The next lemma is shown in Kortschak and Hashorva (2013), whereas Lemma 4.2 follows by Berman (1992), see also Hashorva (2012). Let next $\boldsymbol{v}$ be a given vector in $\mathbb{R}^d, d \geq 2$ with L_2-norm equal 1, and define $\boldsymbol{\theta} = A_u \boldsymbol{v}$.

Lemma 4.1. *Under Assumption* (2.11), *for every j with $\beta_j = \beta_1$ and every $\epsilon > 0$ there exist some c, u_0 positive such that*

$$\theta_i \leq \theta_j \frac{\beta_j}{\beta_i} \frac{\log(\epsilon e_j^*(u))}{\log(u)}$$

holds for all $u > u_0$, provided that $\theta_j > 1 - c/\log(u)$.

Lemma 4.2. *Let R be a positive random variable, and let h be given by*

$$h(x) \quad = \quad \frac{\Gamma(d/2)}{\sqrt{\pi}\Gamma((d-1)/2)}(1-x^2)^{\frac{d-3}{2}}, \quad x \in (0,1), \tag{4.19}$$

with $\Gamma(\cdot)$ the Euler's Gamma function. If F is in the Gumbel MDA with some positive scaling function $e(\cdot)$, then for any β, λ positive

$$\mathbb{P}(X_j(u) > u) = \int_0^1 \mathbb{P}\left(\lambda e^{R\theta\beta\gamma_u} > u\right) h(\theta) d\theta \sim \frac{2^{\frac{d-3}{2}}\Gamma(d/2)}{\sqrt{\pi}} \left(\frac{e^*(u)}{u\log(u)}\right)^{\frac{d-1}{2}} \mathbb{P}\left(\lambda e^{R\beta\gamma_u} > u\right), \quad u \to \infty. \tag{4.20}$$

In the sequel for two positive functions g_1, g_2 we write $g_1 \lesssim g_2$ respectively $g_1 \gtrsim g_2$ if $\limsup_{u\to\infty} g_1(u)/g_2(u) \leq 1$ respectively $\limsup_{u\to\infty} g_1(u)/g_2(u) \geq 1$.

Lemma 4.3. *Under the assumptions of Lemma 4.2, if further*

$$\lim_{u\to\infty} \frac{e(u)\log(u)}{u} \quad = \quad c_0 \tag{4.21}$$

holds for some constant $c_0 \in [0, \infty)$, then for any $c > 1, c' > c_0$

$$\frac{\mathbb{P}(R > \log(cu))}{\mathbb{P}(R > \log(u))} \lesssim u^{-\frac{\log(c)}{c'}}$$

is valid for all u large.

Proof. By (4.21) and the representation (4.18) of the scaling function $e(\cdot)$ we obtain for $c' > c_0$ and $\varepsilon > 0, 1/u$ sufficiently small

$$\begin{aligned}
\frac{\mathbb{P}(R > \log(cu))}{\mathbb{P}(R > \log(u))} &\lesssim \exp\left(-(1-\varepsilon)\int_u^{cu} \frac{1}{e(x)} dx\right) \\
&\lesssim \exp\left(-\frac{1}{c'}\int_u^{cu} \frac{\log(x)}{x} dx\right) \\
&= \exp\left(-\frac{1}{c'}\int_1^{c} \frac{\log(ux)}{x} dx\right) \\
&\leq \exp\left(-\frac{1}{c'}\int_1^{c} \frac{\log(u)}{x} dx\right)
\end{aligned}$$

thus the claim follows. □

Next we shall consider the case that $e(\cdot)$ is O-regularly varying which means (see e.g., Bingham et al. (1987))

$$0 < \liminf_{u\to\infty} \frac{e(\lambda u)}{e(u)} \leq \limsup_{u\to\infty} \frac{e(\lambda u)}{e(u)} < \infty, \quad \forall \lambda > 1. \tag{4.22}$$

Lemma 4.4. *If $e(\cdot)$ satisfies (4.22), then there exist α, M, u_0 positive and $\varepsilon \in (0,1)$ such that*

$$\frac{\mathbb{P}(R > \log(u + xe(u)))}{\mathbb{P}(R > \log(u))} \quad \leq \quad (1+\varepsilon)e^{-(1-\varepsilon)M(1+c)^{-\alpha}x}$$

holds for any $c > 0$ and $x \in (0, \frac{cu}{e(u)})$ with $u > u_0$.

Proof. By Proposition 2.2.1 of Bingham et al. (1987) there exists M, u_0 and α such that for all $y \geq x \geq u_0$

$$\frac{e(x)}{e(y)} \geq M(x/y)^{\alpha}.$$

By the representation (4.18) of the scaling function $e(\cdot)$ and the assumptions, for $x \in (0, ce(u)/u)$ for some $\varepsilon \in (0,1)$ we may write

$$\begin{aligned}
\frac{\mathbb{P}(e^R > u + xe(u))}{\mathbb{P}(e^R > u)} &\leq (1+\varepsilon)\exp\left(-(1-\varepsilon)\int_u^{u+xe(u)} \frac{1}{e(y)}\, dy\right) \\
&= (1+\varepsilon)\exp\left(-(1-\varepsilon)\int_0^x \frac{e(u)}{e(u+ye(u))}\, dy\right) \\
&\leq (1+\varepsilon)\exp\left(-(1-\varepsilon)M\int_0^x \left(\frac{u}{u+ye(u)}\right)^{\alpha} dy\right) \\
&\leq (1+\varepsilon)\exp\left(-(1-\varepsilon)M\int_0^x (1+c)^{-\alpha}\, dy\right),
\end{aligned}$$

hence the proof follows. □

Proof of Theorem 2.2 For all u positive we have

$$\Upsilon(u) = \mathbb{P}(S(u) > u) - \sum_{j=1}^{d} \mathbb{P}\left(X_j(u) > u\right) = \sum_{j=1}^{d}\left(\mathbb{P}\left(S(u) > u, X_j(u) > \max_{j \neq i} X_j(u)\right) - \mathbb{P}\left(X_j(u) > u\right)\right).$$

For j with $\beta_j < \beta_1$ we have further

$$\mathbb{P}\left(S(u) > u, X_j(u) > \max_{j\neq i} X_i(u)\right) \leq \mathbb{P}\left(X_j(u) > u/d\right) = \mathbb{P}\left(X_1(u) > \lambda_1\left(\frac{u}{\lambda_i d}\right)^{\beta_1/\beta_j}\right) = o\left(\Upsilon(u)\right)$$

as $u \to \infty$. Hence we only need to derive the second order asymptotics of

$$\mathbb{P}\left(S(u) > u, X_j(u) > \max_{j\neq i} X_i(u)\right),$$

with $\beta_j = \beta_1$. Note that by straightforward arguments it follows from (2.14) and the monotonicity of $e(\cdot)$ that the scaling function $e(\cdot)$ is O-regularly varying. Define $\mathbf{\Theta} := A_u \boldsymbol{U}$ and write Θ_i for the ith component of $\mathbf{\Theta}_u$. Choose an index j with $\beta_1 = \beta_j$, and suppose without loss of generality that depending on j an A_u is chosen such that $\Theta_j = U_j$, with U_j the jth component of $\boldsymbol{U}$ which is uniformly distributed on the unit sphere of $\mathbb{R}^d$. Lemma 4.3 implies that we can choose a k such that for

$$a(u) = 1 - \frac{k}{\log(u)}$$

we have

$$\lim_{u\to\infty} \frac{\mathbb{P}\left(e^{R\beta_j\gamma_u} > \left(\frac{u}{\lambda_j d}\right)^{1/a(u)}\right)}{u^{-3-\frac{d-1}{2}}\mathbb{P}\left(\lambda_j e^{R\beta_j\gamma_u} > u\right)} = 0. \tag{4.23}$$

Note that it follows from (4.23) and Lemma 4.2 that for all u large

$$
\begin{aligned}
\mathbb{P}\left(S(u)>u, X_j(u)=\max_{i\neq j} X_i(u), \Theta_j\le a(u)\right) &\le \mathbb{P}\left(X_j(u)>u/d, \Theta_j\le a(u)\right)\\
&\le \mathbb{P}\left(e^{R\beta_j\gamma_u} > \left(\frac{u}{\lambda_j d}\right)^{1/a(u)}\right)\\
&= o\left(u^{-3-\frac{d-1}{2}}\mathbb{P}\left(\lambda_j e^{R\beta_j\gamma_u}>u\right)\right)\\
&= o\left(u^{-2}\mathbb{P}\left(X_j(u)>u\right)\right) = o\left(\Upsilon(u)\right).
\end{aligned}
$$

Therefore, we need to determine the second order asymptotics of $\mathbb{P}\left(S(u)>u, X_j(u)=\max_{i\neq j} X_i(u), \Theta_j > a(u)\right)$. Denote by $f_{-j}(\boldsymbol{\theta}_{-j}|\theta)$ the conditional probability density function of $\boldsymbol{\Theta}_{-j} := (\Theta_1,\dots,\Theta_{j-1},\Theta_{j+1},\dots,\Theta_d)$ given $\Theta_j=\theta$. We get by (2.11) (compare Lemma 4.1) that for sufficiently large u with h given by (4.19)

$$
\begin{aligned}
&\int_{a(u)}^1 \mathbb{P}\left(\sum_{i=1}^d \lambda_i e^{R\Theta_i\beta_i\gamma_u}>u, \lambda_j e^{R\Theta_j\beta_j\gamma_u} > \max_{k\neq j}\lambda_k e^{R\Theta_k\beta_k\gamma_u}\,\middle|\, \Theta_j=\theta_j\right) h(\theta_j)d\theta_j\\
&= \int_{a(u)}^1 \mathbb{P}\left(\sum_{i=1}^d \lambda_i e^{R\Theta_i\beta_i\gamma_u}>u\,\middle|\,\Theta_j=\theta\right)h(\theta)d\theta\\
&= \int_{a(u)}^1\int \mathbb{P}\left(\sum_{i=1}^d \lambda_i e^{R\Theta_i\beta_i\gamma_u}>u\,\middle|\,\Theta_1=\theta_1,\dots,\Theta_d=\theta_d\right)h(\theta_j)f_{\boldsymbol{\theta}_{-j}}(\boldsymbol{\theta}_{-j},\theta_j)d\boldsymbol{\theta}_{-j}d\theta_j.
\end{aligned}
$$

For sufficiently large u define the function

$$
g(u)=\left\{e^{r(u)}, \text{where } r(u) \text{ is such that } \sum_{i=1}^d \lambda_i e^{r(u)\theta_i\beta_i\gamma_u}=u\right\}.
$$

Hence, for all u large

$$
\mathbb{P}\left(\sum_{i=1}^d \lambda_i e^{R\Theta_i\beta_i\gamma_u}>u\,\middle|\,\Theta_1=\theta_1,\dots,\Theta_d=\theta_d\right)=\mathbb{P}\left(e^R>g(u)\right)=\mathbb{P}\left(e^R>g_0(u)+g_1(u)\right),
$$

where

$$
g_0(u)=\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\theta_j\beta_j\gamma_u}},\quad g_1(u)=g(u)+g_0(u),\quad u>0,
$$

with $g_1(u)<0$ for all u positive. From Lemma 4.1 we get that uniformly in $\boldsymbol{\theta}=(\theta_1,\dots,\theta_d)$ with $\theta_j>a(u)$ $\lim_{u\to\infty} g_1(u)/g_0(u)=0$ and hence for some $|\xi|\le g_1(u)$ the following equalities are equivalent

$$
\begin{aligned}
u &= \sum_{i=1}^d \lambda_i (g_0(u)+g_1(u))^{\theta_i\beta_i\gamma_u}\\
u &= \lambda_i (g_0(u))^{\theta_j\beta_j\gamma_u} + \theta_j\beta_j\gamma_u\lambda_i g_1(u)(g_0(u)+\xi)^{\theta_j\beta_j\gamma_u-1}+\sum_{i\neq j}^d \lambda_i(g_0(u)+g_1(u))^{\theta_i\beta_i\gamma_u}\\
-\theta_j g_1(u) &= \left(\frac{g_0(u)+g_1(u)}{g_0(u)+\xi}\right)^{\theta_j\beta_j\gamma_u-1}\sum_{i\neq j}^d \frac{\lambda_i}{\beta_j\gamma\lambda_j}(g_0(u)+g_1(u))^{\gamma_u(\theta_i\beta_i-\theta_i\beta_i)+1}.
\end{aligned}
$$

It follows that for some $c>0$ and uniformly in $\boldsymbol{\theta}$ with $\theta_j>a(u)$

$$
g_1(u)\sim -\sum_{i\neq j}\frac{\lambda_i}{\beta_j\gamma\lambda_j}g_0(u)^{\gamma_u(\theta_i\beta_i-\theta_j\beta_j)+1}
$$

$$\gtrsim -g_0(u)^{-\gamma_u\theta_j\beta_j+1}\sum_{i\neq j}\frac{\lambda_i}{\beta_j\gamma\lambda_j}g_0(u)^{\gamma_u\beta_j\theta_j\frac{\log(\epsilon e_j^*(u))}{\log(u)}}\gtrsim -cg_0(u)^{-\gamma_u\theta_j\beta_j+1}\epsilon e_j^*(u).$$

Since the scaling function $e(\cdot)$ is O-regularly varying we get for some $c_1>0$ that $|g_1(u)|\lesssim c_1\epsilon e(g_0(u))$ for any $\epsilon>0$ and uniformly in $\boldsymbol{\theta}$ with $\theta_j>a(u)$. Taylor expansion implies for a $g_0(u)+g_1(u)\leq\xi_u\leq g_0(u)$

$$\begin{aligned}\mathbb{P}\left(e^R>g(u)\right)&=\mathbb{P}\left(e^R>\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\Theta_j\beta_j\gamma_u}}\right)-g_1(u)\widetilde{f}(\xi_u)\\&=\mathbb{P}\left(e^R>\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\Theta_j\beta_j\gamma_u}}\right)-(1+o(1))g_1(u)\frac{1}{e(\xi_u)}\mathbb{P}(R>\log(\xi_u)).\end{aligned}$$

Next, note that with (4.23) and Lemma 4.2 we get for u large enough such that $\mathbb{P}(X_j(u)>u,\theta_j\leq u)=0$ that

$$\begin{aligned}&\int_{a(u)}^1\int\mathbb{P}\left(e^R>\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\Theta_j\beta_j\gamma_u}}\middle|\Theta_1=\theta_1,\dots,\Theta_d=\theta_d\right)h(\theta_j)f_{\boldsymbol{\theta}_{-j}}(\boldsymbol{\theta}_{-j},\theta_j)d\boldsymbol{\theta}_{-j}d\theta_j\\&=\mathbb{P}\left(X_j(u)>u\right)-\int_0^{a(u)}\mathbb{P}\left(\lambda_je^{R\beta_j\gamma_u}>\lambda_j^{\frac{1}{\Theta_j}+1}u^{\frac{1}{\Theta_j}}\middle|\Theta_j=\theta_j\right)h(\theta_j)d\theta_j\\&=\mathbb{P}\left(X_j(u)>u\right)+o\left(u^{-3-\frac{d-1}{2}}\mathbb{P}\left(\lambda_je^{R\beta_j\gamma_u}>u\right)\right)\\&=\mathbb{P}\left(X_j(u)>u\right)+o\left(u^{-2}\mathbb{P}\left(X_j(u)>u\right)\right).\end{aligned}$$

We are left with finding the asymptotics of

$$\begin{aligned}&-\int_{a(u)}^1\int g_1(u)\frac{1}{e(\xi_u)}\mathbb{P}(R>\log(\xi_u))h(\theta_j)f_{\boldsymbol{\theta}_{-j}}(\boldsymbol{\theta}_{-j},\theta_j)d\boldsymbol{\theta}_{-j}d\theta_j\\&\sim\sum_{i\neq j}\int_{a(u)}^1\int\frac{\lambda_i}{\beta_j\gamma\lambda_j}g_0(u)^{\gamma_u(\theta_i\beta_i-\theta_j\beta_j)+1}\frac{1}{e(\xi_u)}\mathbb{P}(R>\log(\xi_u))h(\theta_j)f_{\boldsymbol{\theta}_{-j}}(\boldsymbol{\theta}_{-j},\theta_j)d\boldsymbol{\theta}_{-j}d\theta_j.\end{aligned}$$

Since any scaling function, and therefore $e(\cdot)$ is self-neglecting (see Bingham et al. (1987) for the main properties), i.e.,

$$\frac{e(u+xe(u))}{e(u)}\to 1,\quad u\to\infty$$

locally uniformly for $x\in\mathbb{R}$ we get that

$$\begin{aligned}g_1(u)\frac{1}{e(\xi_u)}\mathbb{P}(e^R>\xi_u)&\sim g_1(u)\frac{1}{e(g_0(u))}\mathbb{P}(e^R>g_0(u))\\&=g_1(u)\frac{1}{e(g_0(u))}\mathbb{P}\left(\lambda_je^{\beta_j\gamma_u\Theta_j}>u\right).\end{aligned}$$

It follows that we need further to calculate the asymptotics of

$$\delta(u):=\int_{a(u)}^1\int_{-1}^1\frac{\lambda_i}{\beta_j\gamma\lambda_j}g_0(u)^{\gamma_u(\theta_i\beta_i-\theta_j\beta_j)+1}\frac{1}{e(g_0(u))}\mathbb{P}(R>\log(g_0(u)))h(\theta_j)f_{ij}(\theta_i|\theta_j)d\boldsymbol{\theta}_{-j}d\theta_j$$

where $f_{ij}(\theta_i|\theta)$ is the probability density function of $\Theta_i|\Theta_j=\theta$. To evaluate $\delta(u)$ we choose for a specific index i the matrix A_u in such a way that $a_{jj}=1$, $a_{ji}=\sigma_{ij}(u)$ and $a_{ii}=\sqrt{1-\sigma_{ij}(u)^2}$. If U_i and U_j are two components of a random vector that is uniformly distributed on the d–dimensional unit sphere and V_j and V_i are independent and have the same distribution as the marginal distribution of random vector that is uniformly distributed on the d respectively $(d-1)$–dimensional unit sphere, then Cambanis et al. (1981) Lemma 2 shows that (U_i,U_j) can be represented as

$$(U_j,U_i)\stackrel{d}{=}\left(V_j,V_i\sqrt{1-V_j^2}\right).$$

It follows that we can assume that $\Theta_j = V_j$ and $\Theta_i = \sigma_{ij}(u)V_j + \sqrt{1-\sigma_{ij}(u)^2}\sqrt{1-V_j^2}V_i$. Define next the function $\widehat{e}_j(u,v)$ by

$$\widehat{e}_j(u,v) := u\beta_j\gamma_v e\left(\left(\frac{u}{\lambda_j}\right)^{\frac{1}{\beta_j\gamma_v}}\right)\left(\frac{u}{\lambda_j}\right)^{-\frac{1}{\beta_j\gamma_v}}, \quad u,v>0$$

and note that $\widehat{e}_j(u,v) = e_j^*(u)$. We have

$$\frac{g_0(u)^{-\gamma_u\theta_j\beta_j+1}}{e(g_0(u))} = \frac{\lambda_j g_0(u)}{ue(g_0(u))} = \lambda_j\left(\frac{u}{\lambda_j}\right)^{1-\theta_j}\frac{1}{\widehat{e}_j\left(u^{1/\theta_j}(\lambda_j)^{1-1/\theta_j},u\right)}.$$

It follows that

$$\begin{aligned}\delta(u) = {} & \frac{\Gamma(d/2)}{\pi\Gamma((d-2)/2)}\frac{\lambda_i}{\beta_j\gamma}\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\int_{a(u)}^1\int_{-1}^1\left(\frac{u}{\lambda_j}\right)^{1-v_j}\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i v_i\sqrt{1-\sigma_{ij}(u)^2}\sqrt{1-v_j^2}}{v_j\beta_j}}\\ & \times\frac{\mathbb{P}\left(\lambda_j e^{\gamma_u v_j\beta_j R}>u\right)}{\widehat{e}_j\left(u^{1/v_j}(\lambda_j)^{1-1/v_j},u\right)}(1-v_j^2)^{\frac{d-3}{2}}(1-v_i^2)^{\frac{d-4}{2}}dv_i dv_j.\end{aligned}$$

Substituting $v_j = \frac{\log(u)}{\log(u)+\log(1+xe_j^*(u)/u)} = \frac{\log(u)}{\log(u+xe_j^*(u))}$ we obtain (set next $\eta_j(u) := 1+xe_j^*(u)/u$)

$$\begin{aligned}\delta(u) = {} & \frac{\Gamma(d/2)}{\pi\Gamma((d-2)/2)}\frac{\lambda_i}{\beta_j\gamma}\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\\ & \times\int_0^{\frac{e^{\log(u)/a(u)}-u}{e_j^*(u)}}\int_{-1}^1\frac{\log(u)}{(\log(u)+\log(\eta_j(u)))^2}\frac{e_j^*(u)/u}{\eta_j(u)}\\ & \times\left(\frac{u}{\lambda_j}\right)^{\frac{\log(\eta_j(u))}{\log(u)+\log(\eta_j(u))}}\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i v_i\sqrt{1-\sigma_{ij}(u)^2}\sqrt{2\log(u)\log(\eta_j(u))+\log(\eta_j(u))^2}}{\beta_j\log(u)}}\\ & \times\frac{\mathbb{P}\left(\lambda_j e^{\gamma_u\beta_j R}>u+xe_j^*(u)\right)}{\widehat{e}_j\left(\left(u+xe_j^*(u)\right)(\lambda_j)^{-\frac{\log(\eta_j(u))}{\log(u)}},u\right)}\left(\frac{2\log(u)\log(\eta_j(u))+\log(\eta_j(u))^2}{(\log(u)+\log(\eta_j(u)))^2}\right)^{\frac{d-3}{2}}(1-v_i^2)^{\frac{d-4}{2}}dv_i dx.\end{aligned}$$

We remark that by (2.14)

$$\lim_{u\to\infty}\frac{\widehat{e}_j(u,u)}{\widehat{e}_j\left(\left(u+xe_j^*(u)\right)(\lambda_j)^{-\frac{\log(\eta_j(u))}{\log(u)}},u\right)} = 1$$

and using Lemma 4.4 to get an integrable upper bound (note that $x \lesssim e^k u/e_j^*(u)$), by the bounded convergence theorem, we obtain

$$\begin{aligned}\delta(u) \sim {} & \frac{2\Gamma(d/2)}{\pi\Gamma((d-2)/2)}\frac{\lambda_i}{\beta_j\gamma}\left(\frac{u}{\lambda_j}\right)^{\frac{\beta_i\sigma_{ij}(u)}{\beta_j}}\left(\frac{2e_j^*(u)}{u\log(u)}\right)^{\frac{d-1}{2}}\frac{\mathbb{P}\left(\lambda_j e^{\gamma_u\beta_j R}>u\right)}{e_j^*(u)}\\ & \times\int_0^\infty\int_{-1}^1 e^{\sqrt{x}v\frac{\beta_i\sqrt{2c_0\left(1-\sigma_{ij}^2\right)}}{\beta_j}}e^{-x}x^{\frac{d-3}{2}}(1-v^2)^{\frac{d-4}{2}}dvdx, \quad u\to\infty.\end{aligned}$$

Using Euler's duplication formula

$$\Gamma(s)\Gamma(s+1/2) = \sqrt{\pi}2^{1-2s}\Gamma(2s)$$

for $n\geq 0$ we have

$$\Gamma(n+1/2)\Gamma(n+1) = \sqrt{\pi}2^{1-2(n+1/2)}\Gamma(2n+1) = \sqrt{\pi}4^{-n}\Gamma(2n+1),$$

hence for the last integral above we may further write (below $I_{\{\cdot\}}$ stands for the indicator function and $q := \beta_i\sqrt{2c_j\left(1-\sigma_{ij}^2\right)}/\beta_j$)

$$\begin{aligned}
&\int_0^\infty \int_{-1}^1 e^{\sqrt{x}vq} e^{-x} x^{\frac{d-3}{2}} (1-v^2)^{\frac{d-4}{2}} dv dx \\
&= \sum_{n=0}^\infty \frac{q^n}{n!} \int_0^\infty \int_{-1}^1 e^{-x} x^{\frac{d+n-3}{2}} v^n (1-v^2)^{\frac{d-4}{2}} dv dx \\
&= \sum_{n=0}^\infty \frac{q^n}{n!} \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{d-2}{2}\right)}{\Gamma\left(\frac{d+n-1}{2}\right)} \Gamma\left(\frac{d+n-1}{2}\right) I_{\{n=0 \mod 2\}} \\
&= \Gamma\left(\frac{d-2}{2}\right) \sum_{n=0}^\infty q^{2n} \frac{\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(2n+1)} \\
&= \sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right) \sum_{n=0}^\infty (q^2/4)^n \frac{1}{\Gamma(n+1)} = \sqrt{\pi}\Gamma\left(\frac{d-2}{2}\right) \exp\left(q^2/4\right).
\end{aligned}$$

Consequently, the claim follows by Lemma 4.2. □

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