OPTIMAL DIVIDEND STRATEGIES FOR A RISK PROCESS UNDER FORCE OF INTEREST*

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Abstract

In the classical Cramér-Lundberg model in risk theory the problem of maximizing the expected cumulated discounted dividend payments until ruin is a widely discussed topic. In the most general case within that framework it is proved (Gerber (1969), Azcue & Muler (2005), Schmidli (2007)) that the optimal dividend strategy is of band type. In the present paper we discuss this maximization problem in a generalized setting including a constant force of interest in the risk model. The value function is identified in the set of viscosity solutions of the associated Hamilton-Jacobi-Bellman equation and the optimal dividend strategy in this risk model with interest is derived, which in the general case is again of band type and for exponential claim sizes collapses to a barrier strategy. Finally, an example is constructed for Erlang(2)-claim sizes, in which the bands for the optimal strategy are explicitly calculated.

1 Introduction

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ be a filtered probability space on which all random processes and variables introduced in the sequel are defined. Consider the following stochastic model for the risk reserve process $R = \{R_t\}_{t\geq 0}$ of an insurance portfolio

$$R_t = x + ct - \sum_{k=1}^{N_t} Y_k + i\int_0^t R_s ds. \tag{1}$$

The number of claims $N = \{N_t\}_{t\geq 0}$ is modelled as a homogeneous Poisson process with parameter λ which has the càdlàg property ($N_{t+} = N_t$). The incoming premiums are assumed to be collected continuously over time at a constant rate c. The claim amounts $\{Y_k\}_{k\in\mathbb{N}}$ are an iid sequence of positive random variables with continuous distribution function F_Y. The integral term represents the additional income resulting from the constant force of interest $i > 0$ on the free surplus (see for instance Paulsen [9], where the existence of such a process R is proved). A similar model was dealt with in Albrecher et al. [2] and Paulsen & Gjessing [10, 11]. In this paper we are interested in identifying the optimal strategy to pay out dividends from process (1) to shareholders during the period of solvency.

Let L_t denote the accumulated paid dividends up to time t. We call a dividend strategy $L = \{L_t\}_{t\geq 0}$ admissible if it is an adapted càglàd (previsible, $L_{t-} = L_t$) and non-decreasing process. Further we require $L_{t+} - L_t \leq R_t^L$ such that paying dividends can not cause ruin, where the controlled process is defined via

$$R_t^L = x + ct - \sum_{k=1}^{N_t} Y_k + i\int_0^t R_s^L ds - L_t.$$

The càdlàg property of the reserve process and the càglàd property of the dividends process imply that $R_{t-}^L \neq R_t^L$ is always due to a claim and $R_{t+}^L \neq R_t^L$ is due to some singular dividend payment. Although not standard in the literature, this càglàd assumption for the dividends will simplify the analysis (and the previsibility of the control is then also ensured by the càglàd property).

The performance of an admissible strategy L is measured by the function

$$V_L(x) = \mathbb{E}\left(\int_0^{\tau^L} e^{-\delta s}\, dL_s \,\middle|\, R_0^L = x\right), \tag{2}$$

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i.e. the expectation of the discounted dividend payments until the time of ruin $\tau^L = \inf\{t | R_t^L < 0\}$ of the controlled process. Here $\delta > 0$ denotes the discount factor, which can also be interpreted as a measure of the preference of shareholders to receive payments earlier rather than later during the lifetime of the risk process. The value function of the maximization problem is then given through

$$V(x) = \sup_{L \in \Pi} V_L(x), \tag{3}$$

where the supremum is taken over the set Π of all admissible strategies.

Optimization problems of the form (3) are a classical topic in stochastic control theory (see for instance Schmidli [13] for a recent survey). Under the assumption that the underlying risk process R is modelled by a Cramér-Lundberg process (i.e. (1) with $i = 0$), it was first shown in Gerber [7] by a discrete approximation and then a limiting argument that the optimal dividend strategy according to the criterion (2) is of so-called band type. This result was recently rederived by means of viscosity theory in Azcue and Muler [3]. It is a natural question to ask for an analogous result in the presence of an interest force $i > 0$ on the free surplus, not the least because, from a practical perspective, the use of a discount factor $\delta > 0$ for the dividends in the objective function acknowledging the time value of money should be complemented by such an effect for the underlying risk process, too. It is intuitively not surprising that the dividend maximization problem is only well formulated for $i < \delta$ (for exponential claims we will also demonstrate this fact explicitly in Section 4).

As in the classical Cramér-Lundberg case, one can not expect the value function (3) to be a classical solution of the associated Hamilton-Jacobi-Bellman (HJB) equation. Like Azcue & Muler [3] in the case $i = 0$, we therefore use the methodology of viscosity solutions to identify the optimal strategy for $i > 0$.

The outline of the paper is as follows. After establishing some basic properties of the value function (3), the corresponding HJB equation is derived and the value function is identified as a viscosity solution of this HJB equation (Proposition 2.3). Typically, dividend maximization problems in the Cramér-Lundberg setting lack an initial condition (cf. Azcue & Muler [3], Gerber [7], Schmidli [13]; Mnif & Sulem [8] circumvent this problem by considering a slightly different risk model that does provide an initial value for the maximization problem). Therefore we first prove uniqueness of the viscosity solution of the HJB equation for a given initial condition via a comparison principle (Proposition 2.4) and in a second step we show that every viscosity supersolution dominates the value function (Proposition 2.6). In that way we can characterize the value function as the viscosity supersolution with the smallest initial value fulfilling the same growth conditions.
The construction of the optimal strategy of *band* type needs some care concerning the behaviour of the value function at points where differentiability may not be fulfilled (Propositions 2.11 and 2.12, which also indicate already how to construct the optimal solution along the arguments of Schmidli [13]).
In Section 3 the existence and uniqueness of the solution to the integro-differential part of the HJB equation in the respective regions are established and properties of the crucial sets needed for the definition of the optimal strategy are derived. Eventually the approriate *band* strategy is formulated and its optimality is proved (Proposition 3.3).
In Section 4.1 the case of exponentially distributed claim sizes is investigated in more detail and it is shown that in this case the optimal band strategy collapses to a *barrier* strategy, including a study of conditions on parameter values under which the optimal barrier is in fact in 0 (this complements results of Paulsen & Gjessing [10], who investigated optimal barrier values for the risk process (1) within the class of barrier strategies).
Finally, in Section 4.2 an example for Erlang(2)-distributed claims is identified for which the optimal band strategy can be explicitly calculated.

2 Value function and viscosity solutions

2.1 Basic properties of the value function

Let us first derive some bounds for the value function and its first derivative.

Proposition 2.1. *For $i < \delta$ we have*

$$x + \frac{c}{\delta + \lambda} \leq V(x) \leq \frac{\delta x + c}{\delta - i}.$$

Proof. The controlled process

$$R_t^L = x + ct - \sum_{k=1}^{N_t} Y_k + i \int_0^t R_s^L ds - L_t$$

is clearly upper-bounded by

$$R_t^L \leq e^{it} \left(x + c \int_0^t e^{-is} ds \right)$$

and the growth rate in t of the right hand side is $e^{it}(ix + c)$. Because payments due to an admissible strategy L can not cause ruin, the cumulated dividends up to time t are bounded by the maximal possible position of the reserve at that time,

$$L_t \leq x + \int_0^t e^{is} (ix + c) \, ds,$$

and since dividend payments stop at the time of ruin, by partial integration we arrive at

$$\begin{aligned} V_L(x) &= \mathbb{E} \left(\int_0^\infty e^{-\delta s} \, dL_s \right) = \mathbb{E} \left(\int_0^\infty \delta \, e^{-\delta s} L_s \, ds \right) \\ &\leq x + \int_0^\infty \delta \, e^{-\delta s} \left(\int_0^s e^{iu} (ix + c) \, du \right) ds \\ &= \frac{\delta x + c}{\delta - i}. \end{aligned}$$

On the other hand, we get a lower bound for $V(x)$ when we pay the initial surplus x and all incoming premia immediately as dividends and the first claim that occurs (after an exponential time τ_1) causes ruin:

$$V(x) \geq V_{L_0}(x) = x + c \, \mathbb{E} \left(\int_0^{\tau_1} e^{-\delta t} dt \right) = x + \frac{c}{\delta + \lambda}.$$

□

Proposition 2.2. *For $0 \leq x < y$ we have the following inequalities*

$$y - x \leq V(y) - V(x) \leq V(x) \left(\left(\frac{iy + c}{ix + c} \right)^{\frac{\delta + \lambda}{i}} - 1 \right)$$

Proof. For $\epsilon > 0$ let L_ϵ be an ϵ-optimal strategy for initial capital x (i.e. $V_{L\epsilon}(x) \geq V(x) - \epsilon$). For $y > x$ define $\overline{L}$ such that an amount $y - x$ is paid as dividend immediately followed by using the strategy L_ϵ. We have

$$V(y) \geq y - x + V_{L_\epsilon}(x) \geq y - x + V(x) - \epsilon.$$

Because this holds for all $\epsilon > 0$ we get

$$V(y) - V(x) \geq y - x.$$

For the other direction let $0 \leq x < y$ and $\epsilon > 0$. Define $\hat{L}$ for initial capital x as follows. Nothing is done as long as the reserve stays below y and then an ϵ-optimal strategy L_ϵ for initial capital y is applied. The reserve reaches y not before time $t_0 = \frac{1}{i} \log \left(\frac{iy+c}{ix+c} \right)$ and it is further assumed that there is no payment at all if a claim occurs before t_0. Hence

$$V(x) \geq V_{\hat{L}}(x) \geq e^{-(\delta + \lambda) t_0} V_{L_\epsilon}(y) \geq e^{-(\delta + \lambda) t_0} \left(V(y) - \epsilon \right).$$

Finally we arrive at

$$V(y) - V(x) \leq V(x) \left(\left(\frac{iy + c}{ix + c} \right)^{\frac{\delta+\lambda}{i}} - 1 \right).$$

□

From the above and [15], we get that $V(x)$ is increasing and locally Lipschitz on $[0, \infty)$ (apply a Taylor expansion to the upper bound around x to see this) which by Rademacher's Theorem ensures the existence of the derivative almost everywhere and then $1 \leq V'(x) \leq \frac{\delta+\lambda}{ix+c} V(x)$. Furthermore $V(x)$ is Lipschitz on compact sets which implies that it is absolutely continuous.

2.2 Representation as a viscosity solution

The value function $V(x)$ fulfills the dynamic programming principle for any stopping time γ,

$$V(x) = \sup_{L \in \Pi} \mathbb{E} \left(\int_0^{\tau \wedge \gamma} e^{-\delta s} dL_s + e^{-\delta(\tau \wedge \gamma)} V(R^L_{\tau \wedge \gamma}) \right), \tag{4}$$

which can be shown analogously to the proof of Proposition 3.1 of [3] (with $x_{\max}$ replaced by $e^{i\gamma} \left(x + c \int_0^\gamma e^{-is} ds \right)$). Now let us define the operator

$$\mathcal{L}_u(x) = (c + ix)u'(x) - (\delta + \lambda)u(x) + \lambda \int_0^x u(x - y) dF_Y(y).$$

Standard arguments from stochastic control (see [6]) imply the HJB equation

$$\max \{1 - u'(x), \mathcal{L}_u(x)\} = 0. \tag{5}$$

But, as mentioned in the introduction, we can not expect the value function to be a classical solution to (5). Therefore we need another concept of solutions for this type of equation. We choose the concept of viscosity solutions which is introduced in the following.

Definition 2.1. *A function $\underline{u} : [0, \infty) \to \mathbb{R}$ is called a viscosity subsolution of (5) at $x \in (0, \infty)$ if any continuously differentiable function $\psi(x) : (0, \infty) \to \mathbb{R}$ with $\psi(x) = \underline{u}(x)$ such that $\underline{u} - \psi$ reaches a maximum at x satisfies*

$$\max \{1 - \psi'(x), \mathcal{L}_\psi(x)\} \geq 0.$$

We say that a function $\overline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (5) at $x \in (0, \infty)$ if any continuously differentiable function $\phi(x) : (0, \infty) \to \mathbb{R}$ with $\phi(x) = \overline{u}(x)$ such that $\overline{u} - \phi$ reaches a minimum at x satisfies

$$\max \{1 - \phi'(x), \mathcal{L}_\phi(x)\} \leq 0.$$

A function $u(x) : [0, \infty) \to \mathbb{R}$ is a viscosity solution if it is both a viscosity sub- and supersolution.

Remark 2.1. At some points later on will also make use of a different but equivalent (Sayah [12], Benth et al. [4]) definition of a viscosity sub- and supersolution: Define the modified operator

$$\mathcal{L}^*_{u,v}(x) = (c + ix)v'(x) - (\delta + \lambda)u(x) + \lambda \int_0^x u(x - y) dF_Y(y).$$

A function $\underline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity subsolution of (5) at $x \in (0, \infty)$ if any continuously differentiable function $\psi(x) : (0, \infty) \to \mathbb{R}$ with $\psi(x) = \underline{u}(x)$ such that $\underline{u} - \psi$ reaches a maximum at x satisfies

$$\max \left\{ 1 - \psi'(x), \mathcal{L}^*_{\underline{u},\psi}(x) \right\} \geq 0.$$

A function $\overline{u} : [0, \infty) \to \mathbb{R}$ is a viscosity supersolution of (5) at $x \in (0, \infty)$ if any continuously differentiable function $\phi(x) : (0, \infty) \to \mathbb{R}$ with $\phi(x) = \overline{u}(x)$ such that $\overline{u} - \phi$ reaches a minimum at x satisfies

$$\max \left\{ 1 - \phi'(x), \mathcal{L}^*_{\overline{u},\phi}(x) \right\} \leq 0. \tag{6}$$

Later on we will need the following two properties of the derivatives of some test functions.

Remark 2.2. A continuously differentiable function $\psi : (0,\infty) \to \mathbb{R}$ such that $\underline{u} - \psi$ reaches a maximum at $y > 0$ with $\psi'(y) = q$ exists if and only if

$$\liminf_{x\uparrow y} \frac{\underline{u}(y) - \underline{u}(x)}{y - x} \geq q \geq \limsup_{x\downarrow y} \frac{\underline{u}(y) - \underline{u}(x)}{y - x}.$$

A continuously differentiable function $\phi : (0,\infty) \to \mathbb{R}$ such that $\overline{u} - \phi$ reaches a minimum at $y > 0$ with $\phi'(y) = q$ exists if and only if

$$\liminf_{x\downarrow y} \frac{\overline{u}(y) - \overline{u}(x)}{y - x} \geq q \geq \limsup_{x\uparrow y} \frac{\overline{u}(y) - \overline{u}(x)}{y - x}. \tag{7}$$

Remark 2.3. Note that for a continuously differentiable test function ψ (as required in the definition of viscosity solutions) the operators $\mathcal{L}_\psi$ and $\mathcal{L}^*_{u,\psi}$ are continuous functions for $x \geq 0$, so that we do not have to work with the upper semi-continuity as in Mnif and Sulem [8].

The next proposition characterizes the value function as a viscosity solution. The supersolution proof is in the spirit of [3], whereas the subsolution proof is related to the approach in [8].

Proposition 2.3. *The value function V is a viscosity solution of the HJB equation (5).*

Proof. We start with showing that V is a viscosity supersolution. Fix $l \geq 0$ and let $h > 0$ be small enough such that $e^{ih}\left(x + (c-l)\int_0^h e^{-is}ds\right) \geq 0$. Let τ_1 denote the time of the first claim occurrence. From the dynamic programming principle we derive

$$\begin{aligned}
V(x) = &\sup_{L\in\Pi} \mathbb{E}\left(\int_0^{\tau_1\wedge h} e^{-\delta s}\, dL_s + e^{-\delta(\tau_1\wedge h)}\, V\left(R^L_{\tau_1\wedge h}\right)\right)\\
\geq &e^{-\lambda h}\int_0^h e^{-\delta s} l\, ds + e^{-(\delta+\lambda)h}\, V\left(e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right)\right)\\
&+ \int_0^h \lambda e^{-\lambda t}\left[\int_0^t e^{-\delta s} l\, ds + e^{-\delta t}\int_0^{e^{it}\left(x+(c-l)\int_0^t e^{-is}\, ds\right)} V\left(e^{it}\left(x + (c-l)\int_0^t e^{-is}\, ds\right) - y\right) dF_Y(y)\right] dt.
\end{aligned}$$

This further leads to

$$\begin{aligned}
0 \geq &\frac{1 - e^{-(\delta+\lambda)h}}{h(\delta+\lambda)} l + \frac{V\left(e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right)\right) - V(x)}{h} - \frac{1 - e^{-(\delta+\lambda)h}}{h} V\left(e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right)\right)\\
&+ \frac{\lambda}{h}\int_0^h e^{-(\delta+\lambda)t}\int_0^{e^{it}\left(x+(c-l)\int_0^t e^{-is}\, ds\right)} V\left(e^{it}\left(x + (c-l)\int_0^t e^{-is}\, ds\right) - y\right) dF_Y(y)\, dt.
\end{aligned}$$

Now let ϕ be a continuously differentiable test function with $V(x) = \phi(x)$ and $V - \phi$ attaining a minimum in x. We get

$$\begin{aligned}
0 \geq &\frac{1 - e^{-(\delta+\lambda)h}}{h(\delta+\lambda)} l + \frac{\phi\left(e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right)\right) - \phi(x)}{h} - \frac{1 - e^{-(\delta+\lambda)h}}{h} V\left(e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right)\right)\\
&+ \frac{\lambda}{h}\int_0^h e^{-(\delta+\lambda)t}\int_0^{e^{it}\left(x+(c-l)\int_0^t e^{-is}\, ds\right)} V\left(e^{it}\left(x + (c-l)\int_0^t e^{-is}\, ds\right) - y\right) dF_Y(y)\, dt.
\end{aligned}$$

Using Taylor expansion w.r.t. h at $h = 0$ and neglecting second order terms,

$$e^{ih}\left(x + (c-l)\int_0^h e^{-is}\, ds\right) \approx x + h(ix + (c-l)),$$

we get for $h \to 0$ and using continuity of V and differentiability of ϕ

$$0 \geq l(1-\phi'(x)) + (ix+c)\phi'(x) - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)dF_Y(y). \tag{8}$$

Inequality (8) holds for an arbitrary $l \geq 0$ (using a strategy $L_t = tl$). This gives $1-\phi'(x) \leq 0$ and for $l = 0$ we get $\mathcal{L}^*_{V,\phi}(x) \leq 0$. Therefore we have that V is a viscosity supersolution of (5).
Next we will identify the viscosity subsolution property using Definition 2.1. For some function $\psi \in C^1(0,\infty)$ fulfilling

$$0 = V(x_0) - \psi(x_0) > V(x) - \psi(x) \quad \forall x \neq x_0,\ x \in (0,\infty),$$

for some $x_0 \in (0,\infty)$, we have to show

$$\max\{1-\psi'(x_0), \mathcal{L}_\psi(x_0)\} \geq 0.$$

Assume the contrary. Because ψ, ψ' and V are continuous, the operator $\mathcal{L}_\psi$ is continuous, too. Therefore some $r > 0$ and $\xi > 0$ exist with

$$\max\{1-\psi'(x), \mathcal{L}_\psi(x)\} < -\delta\xi, \quad \forall x \in (x_0 - r, x_0 + r) = B,$$

and such that for $x' = x_0 \pm r$ we have

$$V(x') \leq \psi(x') - \xi.$$

Further choose r such that $B \subset (0,\infty)$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence with $x_n \to x_0$ and without loss of generality assume $x_n \in B$ for all $n \in \mathbb{N}$. Because of the continuity of ψ and V we have $\mid V(x_n) - \psi(x_n) \mid \to 0$. From now on we look at the reserve with initial capital x_n which is controlled by an arbitrary admissible strategy $L \in \Pi$, $R^{L,x_n} = \{R^{L,x_n}_t\}_{t\geq 0}$. Define

$$\tau_n = \inf\{t > 0 \mid R^{L,x_n}_t \notin B\}$$

and denote by $\tau^* = \tau_n \wedge T$ for some $T > 0$. Look now at the set $\{\tau^* = \tau_n\}$ first, leaving B before time T. We have, from the construction of the process, that either $x_0 + r$ is reached which implies $R^{L,x_n}_{\tau^*-} = R^{L,x_n}_{\tau^*} = x_0 + r$, or a jump happens leading to $R^{L,x_n}_{\tau^*-} \geq R^{L,x_n}_{\tau^*}$ and $R^{L,x_n}_{\tau^*} \leq x_0 - r$. Since V is increasing and also ψ is increasing on B, we get from $\psi' > 1$,

$$V(R^{L,x_n}_{\tau^*}) \leq V(x') \leq \psi(x') - \xi \leq \psi(R^{L,x_n}_{\tau^*-}) - \xi.$$

On the set $\{\tau^* = T\}$, $R^{L,x_n}_{\tau^*} \leq R^{L,x_n}_{\tau^*-}$ gives

$$V(R^{L,x_n}_{\tau^*}) \leq \psi(R^{L,x_n}_{\tau^*-}).$$

Altogether

$$e^{-\delta\tau^*} V(R^{L,x_n}_{\tau^*}) \leq e^{-\delta\tau^*-}\psi(R^{L,x_n}_{\tau^*}) - e^{-\delta\tau^*}\xi I_{\{\tau_n = \tau^*\}}.$$

Apply the Itô formula to $e^{-\delta\tau^*}\ \psi\left(R^{L,x_n}_{\tau^*-}\right)$:

$$\begin{aligned} e^{-\delta\tau^*}\ \psi\left(R^{L,x_n}_{\tau^*-}\right) - \psi(x) &= \int_0^{\tau^*} e^{-\delta s}(c + iR^{L,x_n}_{s-})\psi'(R^{L,x_n}_{s-})ds - \delta\int_0^{\tau^*} \psi(R^{L,x_n}_{s-})e^{-\delta s}ds \\ &\quad - \int_0^{\tau^*} \psi'(R^{L,x_n}_{s-})e^{-\delta s}\, dL^c_s + \sum_{R^{L,x_n}_{s-} \neq R^{L,x_n}_s \wedge s \leq \tau^*} \left(\psi(R^{L,x_n}_s) - \psi(R^{L,x_n}_{s-})\right) e^{-\delta s} \\ &\quad + \sum_{R^{L,x_n}_{s+} \neq R^{L,x_n}_s \wedge s < \tau^*} \left(\psi(R^{L,x_n}_{s+}) - \psi(R^{L,x_n}_s)\right) e^{-\delta s}. \end{aligned} \tag{9}$$

Note that $R^{L,x_n}_{s+} - R^{L,x_n}_s = -(L_{s+} - L_s)$ and therefore $\sum_{R^{L,x_n}_{s+} \neq R^{L,x_n}_s \wedge s<\tau^*} \left(\psi(R^{L,x_n}_{s+}) - \psi(R^{L,x_n}_s)\right) e^{-\delta s} = -\sum_{L_{s+}\neq L_s \wedge s<\tau^*} e^{-\delta s} \left(\int_0^{L_{s+}-L_s} \psi'(R^{L,x_n}_s - u)du\right)$. Because $\psi'(x) > 1$ for $x \in B$ we get

$$-\left(\int_0^{\tau^*} \psi'(R^{L,x_n}_{s-})e^{-\delta s}\, dL^c_s + \sum_{L_{s+}\neq L_s \wedge s<\tau^*} e^{-\delta s} \left(\int_0^{L_{s+}-L_s} \psi'(R^{L,x_n}_s - u)du\right)\right) \leq$$
$$-\left(\int_0^{\tau^*} e^{-\delta s}\, dL^c_s + \sum_{L_{s+}\neq L_s \wedge s<\tau^*} e^{-\delta s}\,(L_{s+} - L_s)\right) = -\int_0^{\tau^*} e^{-\delta s} dL_s.$$

The last equality holds because the dividends process is left-continuous. Plugging this into (9) we obtain the inequality

$$\begin{aligned} e^{-\delta\tau^*}\psi(R^{L,x_n}_{\tau^*-}) \leq& \psi(x_n) - \int_0^{\tau^*} e^{-\delta s}\, dL_s + \int_0^{\tau^*} e^{-\delta s}\left((c + iR^{L,x_n}_{s-})\psi'(R^{L,x_n}_{s-}) - \delta\psi(R^{L,x_n}_{s-})\right) ds \\ &+ \sum_{R^{L,x_n}_{s-}\neq R^{L,x_n}_s \wedge s\leq\tau^*} \left(\psi(R^{L,x_n}_s) - \psi(R^{L,x_n}_{s-})\right) e^{-\delta s}. \end{aligned}$$

Further we know (see e.g. [5]) that

$$\sum_{R^{L,x_n}_{s-}\neq R^{L,x_n}_s \wedge s\leq\tau^*} \left(\psi(R^{L,x_n}_s) - \psi(R^{L,x_n}_{s-})\right) e^{-\delta s} - \int_0^{\tau^*} \lambda e^{-\delta s}\left(\int_0^{R^{L,x_n}_{s-}} \psi(R^{L,x_n}_{s-} - y)dF_Y(y) - \psi(R^{L,x_n}_{s-})\right) ds$$

is a martingale. Therefore taking expectations on both sides yields

$$\mathbb{E}\left(e^{-\delta\tau^*}\psi(R^{L,x_n}_{\tau^*-}) + \int_0^{\tau^*} e^{-\delta s}\, dL_s\right) \leq \psi(x_n) + \mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\mathcal{L}_\psi(R^{L,x_n}_{s-})ds\right).$$

Because of $R^{L,x_n}_{s-} \in B$ for $s \in [0,\tau^*)$ we have $\mathcal{L}_\psi(R^{L,x_n}_{s-}) < -\delta\xi$. We can use this to derive

$$\mathbb{E}\left(e^{-\delta\tau^*}V(R^{L,x_n}_{\tau^*}) + \int_0^{\tau^*} e^{-\delta s}\, dL_s\right) + \mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right) \leq V(x_n) + \gamma_n,$$

where $\gamma_n = \psi(x_n) - V(x_n)$ converges to zero. Therefore choose n large enough such that $\gamma_n \leq \frac{1}{2}\mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right)$. For arbitrary L we arrive at

$$\mathbb{E}\left(e^{-\delta\tau^*}V(R^{L,x_n}_{\tau^*}) + \int_0^{\tau^*} e^{-\delta s}\, dL_s\right) + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right) \leq V(x_n).$$

This leads to the following contradiction to the dynamic programming principle:

$$\begin{aligned} &V(x_n) + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right) = \\ &\quad \sup_{L\in\Pi}\mathbb{E}\left(e^{-\delta\tau^*}V(R^{L,x_n}_{\tau^*}) + \int_0^{\tau^*} e^{-\delta s}\, dL_s\right) + \frac{1}{2}\mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right) \leq V(x_n). \end{aligned}$$

If there is a positive probability for the event $\tau^* = 0$ which is only possible if $\tau^* = \tau_n$, then for the second term above $e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}} > 0$. Therefore $\mathbb{E}\left(\int_0^{\tau^*} e^{-\delta s}\delta\xi\, ds + e^{-\delta\tau^*}\xi I_{\{\tau_n=\tau^*\}}\right) > 0$ holds and leads indeed to a contradiction. □

2.3 Uniqueness

The following comparison principle allows us to decide whether a viscosity supersolution dominates anonther viscosity subsolution by looking at their initial value. Since every viscosity solution is both a sub- and supersolution, this will imply uniqueness for a given initial value. Actually in our situation we have to modify the proof presented by Azcue and Muler [3]. Although quite technical, the arguments are based on an appropriate combination of standard arguments from viscosity theory.

Proposition 2.4. *Let for all $x > 0$ the functions $u_1(x)$ and $u_2(x)$ be a viscosity sub- and supersolution, respectively, that satisfies the conditions fulfilled by the value function (locally Lipschitz, $u(y) - u(x) \geq y - x$ and some linear growth $u(x) \leq k_1 x + k_2$). If $u_1(0) \leq u_2(0)$, then $u_1(x) \leq u_2(x)$ for all $x \in [0, \infty)$.*

Proof. The result will be shown by contradiction. Assume there exists some $x_0 > 0$ such that $u_1(x_0) - u_2(x_0) > 0$. Let $\gamma > 0$ be a constant and define $\tilde{u}_1(x) = e^{-\gamma x} u_1(x)$ and $\tilde{u}_2(x) = e^{-\gamma x} u_2(x)$. Because u_1 and u_2 fulfill a linear growth condition, these functions are positive and bounded. If we choose γ small enough we get by continuity that $\tilde{u}_1(x_0) - \tilde{u}_2(x_0) > 0$. Therefore

$$0 < \max_{x \geq 0} \Big(\tilde{u}_1(x) - \tilde{u}_2(x) \Big) = M < \infty,$$

with a maximizing argument x^*. Further we have

$$\frac{\tilde{u}_1(y) - \tilde{u}_1(x)}{y - x} \leq m, \qquad \frac{\tilde{u}_2(y) - \tilde{u}_2(x)}{y - x} \leq m, \tag{10}$$

for some $m > 0$. Define the set A by

$$A = \{(x, y) \mid 0 \leq x \leq y\}.$$

In the following we need the function

$$\phi_\nu(x, y) := \tilde{u}_1(x) - \tilde{u}_2(y) - \frac{\nu}{2}(x - y)^2 - \frac{2m}{\nu^2 (y - x)^2 + \nu},$$

and

$$M_\nu := \max_{(x,y) \in A} \phi_\nu(x, y),$$

with the maximizer (x_ν, y_ν). We have

$$M_\nu \geq \phi_\nu(x^*, x^*) = M - \frac{2m}{\nu},$$

which is positive for ν large enough, leading to

$$\liminf_{\nu \to \infty} M_\nu \geq M > 0.$$

To ensure differentiability at the points x_ν and y_ν one needs to establish that (x_ν, y_ν) is not an element of the boundary of A (the proof of which is postponed to Lemma 2.5 after the end of this proof).
In the next step we define two test functions, such that we can use that $\tilde{u}_1$ and $\tilde{u}_2$ are viscosity sub- and supersolutions to a slightly modified problem

$$\psi(x) = \tilde{u}_2(y_\nu) + \frac{\nu}{2}(x - y_\nu)^2 + \frac{2m}{\nu^2 (y_\nu - x)^2 + \nu} + \phi_\nu(x_\nu, y_\nu),$$
$$\varphi(y) = \tilde{u}_1(x_\nu) - \frac{\nu}{2}(x_\nu - y)^2 - \frac{2m}{\nu^2 (y - x_\nu)^2 + \nu} - \phi_\nu(x_\nu, y_\nu).$$

ψ and φ are continuously differentiable functions. Further $\tilde{u}_1(x) - \psi(x) = \phi_\nu(x, y_\nu) - \phi_\nu(x_\nu, y_\nu)$ reaches a maximum equal to zero in x_ν. On the other hand $\tilde{u}_2(y) - \varphi(y) = -\phi_\nu(x_\nu, y) + \phi_\nu(x_\nu, y_\nu)$ reaches a minimum equal to zero in y_ν. Because u_1 and u_2 are viscosity sub- and supersolutions of the original HJB equation, $\tilde{u}_1$ and $\tilde{u}_2$ are viscosity sub- and supersolutions of the equation

$$\max\left\{1-e^{\gamma x}(\gamma u(x)+u'(x)),(c+ix)(\gamma u(x)+u'(x))-(\delta+\lambda)u(x)+\lambda\int_0^x u(x-y)e^{-\gamma y}dF_Y(y)\right\}=0.$$

In the points x_ν and y_ν we get

$$\max\left\{1-e^{\gamma x_\nu}(\gamma\tilde{u}_1(x_\nu)+\psi'(x_\nu)),(c+ix_\nu)(\gamma\tilde{u}_1(x_\nu)+\psi'(x_\nu))-(\delta+\lambda)\tilde{u}_1(x_\nu)+\lambda\int_0^{x_\nu}\tilde{u}_1(x_\nu-y)e^{-\gamma y}dF_Y(y)\right\}\geq 0,$$
$$\max\left\{1-e^{\gamma y_\nu}(\gamma\tilde{u}_2(y_\nu)+\varphi'(y_\nu)),(c+iy_\nu)(\gamma\tilde{u}_2(y_\nu)+\varphi'(y_\nu))-(\delta+\lambda)\tilde{u}_2(y_\nu)+\lambda\int_0^{y_\nu}\tilde{u}_2(y_\nu-y)e^{-\gamma y}dF_Y(y)\right\}\leq 0.$$

In addition we have that $\varphi'(y_\nu)=\psi'(x_\nu)=\nu(x_\nu-y_\nu)+\frac{4m\nu^2(y_\nu-x_\nu)}{\nu^2(y_\nu-x_\nu)^2+\nu}$.
Notice that $\max\{A,B\}\leq\max\{C,D\}$ implies $(A\leq C)\vee(B\leq D)$. We start with looking at $B\leq D$,

$$\begin{aligned}(c+iy_\nu)\left(\gamma\tilde{u}_2(y_\nu)+\nu(x_\nu-y_\nu)+\frac{4m\nu^2(y_\nu-x_\nu)}{\nu^2(y_\nu-x_\nu)^2+\nu}\right)-(c+ix_\nu)\left(\gamma\tilde{u}_1(x_\nu)+\nu(x_\nu-y_\nu)+\frac{4m\nu^2(y_\nu-x_\nu)}{\nu^2(y_\nu-x_\nu)^2+\nu}\right)\\ +(\delta+\lambda)(\tilde{u}_1(x_\nu)-\tilde{u}_2(y_\nu))\leq\lambda\left(\int_0^{x_\nu}\tilde{u}_1(x_\nu-y)e^{-\gamma y}dF_Y(y)-\int_0^{y_\nu}\tilde{u}_2(y_\nu-y)e^{-\gamma y}dF_Y(y)\right).\end{aligned}\tag{11}$$

From

$$\phi_\nu(x_\nu,x_\nu)+\phi_\nu(y_\nu,y_\nu)\leq 2\phi_\nu(x_\nu,y_\nu)$$

we immediately get

$$\tilde{u}_1(x_\nu)-\tilde{u}_2(x_\nu)+\tilde{u}_1(y_\nu)-\tilde{u}_2(y_\nu)-\frac{4m}{\nu}\leq 2\left(\tilde{u}_1(x_\nu)-\tilde{u}_2(y_\nu)-\frac{\nu}{2}(x_\nu-y_\nu)^2-\frac{2m}{\nu^2(y_\nu-x_\nu)^2+\nu}\right).$$

This yields, together with (10),

$$\begin{aligned}\nu(x_\nu-y_\nu)^2&\leq\tilde{u}_1(x_\nu)-\tilde{u}_1(y_\nu)+\tilde{u}_2(x_\nu)-\tilde{u}_2(y_\nu)+4m\frac{(y_\nu-x_\nu)^2}{\nu(y_\nu-x_\nu)^2+1}\\ &\leq 2m|y_\nu-x_\nu|+4m(|y_\nu-x_\nu|)^2\end{aligned}$$

and in particular, for ν large enough such that $\frac{4m}{\nu}<1$,

$$0\leq|y_\nu-x_\nu|\left(1-\frac{4m}{\nu}\right)\leq\frac{2m}{\nu}.\tag{12}$$

Now let $(\nu_n)_{n\in\mathbb{N}}$ be such that (x_ν,y_ν) converges to $(\overline{x},\overline{y})$ as $\nu_n\to\infty$. From (12) we get that $\overline{x}=\overline{y}$. Using (11) we get

$$(c+i\overline{x})\gamma(\tilde{u}_2(\overline{x})-\tilde{u}_1(\overline{x}))+(\delta+\lambda)(\tilde{u}_1(\overline{x})-\tilde{u}_2(\overline{x}))\leq\lambda\left(\int_0^{\overline{x}}e^{-\gamma y}(\tilde{u}_1(\overline{x}-y)-\tilde{u}_2(\overline{x}-y))dF_Y(y)\right).\tag{13}$$

The right-hand side of (13) is smaller than λM. If we choose γ small enough we derive

$$M\leq\liminf_{\nu\to\infty}M_\nu\leq\lim_{n\to\infty}M_{\nu_n}=\tilde{u}_1(\overline{x})-\tilde{u}_2(\overline{x})\leq\frac{\lambda}{\delta+\lambda}M,$$

which is a contradiction.
Now we concentrate on $A\leq C$ and observe that

$$e^{\gamma x_\nu}(\gamma\tilde{u}_1(x_\nu)+\varphi'(y_\nu))\leq e^{\gamma y_\nu}(\gamma\tilde{u}_2(y_\nu)+\varphi'(y_\nu)).$$

This implies

$$e^{\gamma x_\nu}\tilde{u}_1(x_\nu) - e^{\gamma y_\nu}\tilde{u}_2(y_\nu) \le \frac{1}{\gamma}\varphi'(y_\nu)\left(e^{\gamma y_\nu} - e^{\gamma x_\nu}\right).$$

For γ small enough we have $e^{\gamma y_\nu}\tilde{u}_2(y_\nu) - e^{\gamma x_\nu}\tilde{u}_1(x_\nu) \approx \tilde{u}_2(y_\nu) - \tilde{u}_1(x_\nu)$ so that

$$0 < M \le M_\nu = \phi_\nu(x_\nu, y_\nu) \le \tilde{u}_1(x_\nu) - \tilde{u}_2(y_\nu) \le \frac{1}{\gamma}\varphi'(y_\nu)(e^{\gamma y_\nu} - e^{\gamma x_\nu}). \tag{14}$$

If $\varphi'(y_\nu) \le 0$ for some $\nu > 0$ we are done, remember $(x_\nu, y_\nu) \in A$. Now look at $\varphi'(y_\nu) > 0$, we have

$$|\varphi'(y_\nu)| \le \nu|y_\nu - x_\nu| + \left|\frac{4m\nu(y_\nu - x_\nu)}{\nu(y_\nu - x_\nu)^2 + 1}\right|. \tag{15}$$

Choose again a sequence $(\nu_n)_{n\in\mathbb{N}}$ such that (x_ν, y_ν) converges to $(\overline{x}, \overline{y})$ as $\nu_n \to \infty$, (12) gives $\overline{x} = \overline{y}$. If $\lim_{\nu_n\to\infty}\varphi'(y_\nu)$ is bounded the right hand side of (14) converges to zero and also in this case we obtain a contradiction. Applying (12) to (15) we get the boundedness of $\varphi'(y_\nu)$ for large ν,

$$|\varphi'(y_\nu)| \le \frac{1}{1 - \frac{4m}{\nu}}(2m + 8m^2).$$

□

From the comments above, Proposition 2.4 implies the uniqueness of the viscosity solution for a given initial condition $v(0) = v_0$.

Lemma 2.5. *(x_ν, y_ν) is not an element of the boundary of A.*

Proof. First look at

$$\phi_\nu(0,0) = \tilde{u}_1(0) - \tilde{u}_2(0) - \frac{2m}{\nu} < 0,$$
$$\lim_{b\to\infty}\phi_\nu(x,b) = \tilde{u}_1(x) - \tilde{u}_2(b) - \frac{\nu}{2}(x-b)^2 - \frac{2m}{\nu^2(b-x)^2 + \nu} = -\infty.$$

The next step is to examine the right-hand derivative, in y, at the boundary of A along the diagonal. For all $x > 0$,

$$\limsup_{h\to 0+}\frac{\phi_\nu(x,x+h) - \phi_\nu(x,x)}{h} = \limsup_{h\to 0+}\frac{1}{h}\left(\tilde{u}_2(x) - \tilde{u}_2(x+h) + \frac{2m}{\nu} - \frac{\nu}{2}h^2 - \frac{2m}{\nu^2h^2 + \nu}\right)$$
$$\le \limsup_{h\to 0+}\left(-1 - \frac{\nu}{2}h + \frac{2mh}{\nu h^2 + 1}\right) = -1 < 0.$$

The last inequality holds because of the assumsptions on u_2 stated in Proposition 2.4. By continuity it follows from $\phi_\nu(0,0) < 0$ that $\phi_\nu(0,y) < 0$ for $y \in [0, \rho_\nu]$ and some $\rho_\nu > 0$. Now for $y > \rho_\nu$ we observe

$$\limsup_{h\to 0+}\frac{\phi_\nu(0,y) - \phi_\nu(h,y)}{h} = \limsup_{h\to 0+}\frac{1}{h}\left(\tilde{u}_1(0) - \tilde{u}_1(h) + \frac{\nu}{2}h^2 - \nu hy + \frac{2m\nu^2(y^2 - (h-y)^2)}{(\nu^2(h-y)^2 + \nu)(\nu y^2 + 1)}\right)$$
$$\le \limsup_{h\to 0+}\left(-e^{-\gamma h} + u_1(0)\frac{(1 - e^{-\gamma h})}{h} + \frac{\nu}{2}h - \nu y + \frac{1}{h}\frac{2m\nu^2(y^2 - (h-y)^2)}{(\nu^2(h-y)^2 + \nu)(\nu y^2 + 1)}\right)$$
$$= \gamma\tilde{u}_1(0) - 1 - \nu y + \frac{4my}{(\nu y^2 + 1)^2}, \tag{16}$$

which is negative for ν large enough and γ small enough. Here the inequality in (16) holds because the lower and upper linear growth conditions imply $-h \ge u_1(0) - u_1(h)$ and consequently

$$\tilde{u}_1(0) - \tilde{u}_1(h) = u_1(0) - e^{-\gamma h}u_1(h) = e^{-\gamma h}(u_1(0) - u_1(h)) + u_1(0)(1 - e^{-\gamma h})$$
$$\le -e^{-\gamma h}h + u_1(0)(1 - e^{-\gamma h}).$$

Hence we have proved that (x_ν, y_ν) does not belong to the boundary of A (negative value in $(0,0)$ and in every direction towards the boundary of A negative derivatives and a negative limit for the argument (x,b) if $b \to \infty$). □

2.4 Characterization of the value function

In contrast to some optimization problems in a diffusion framework the dividend maximization problem in our setup lacks an initial condition. In Proposition 2.6 we will prove that every viscosity supersolution to (5) which fufills a linear growth condition dominates the value function. This together with Proposition 2.4 allows us to define

$$V(0) = \inf\{u(0) \mid u \text{ is a viscosity solution to the HJB equation and fulfills a linear growth condition}\}.$$

Because of the comparison principle any other choice of an initial value will lead to a contradicition to Proposition 2.6, since for any suitable viscosity solution u with $u(0) < V(0)$ we would have $u(x) < V(x)$ for at least $x \in [0, \epsilon)$ and $u(x) \leq V(x)$ for all $x > 0$.
For a viscosity supersolution u_1 we have almost everywhere

$$u_1'(x) \leq \frac{1}{c+ix}\left((\delta+\lambda)u_1(x) - \lambda\int_0^x u_1(x-y)dF_Y(y)\right) \leq \frac{\delta+\lambda}{c+ix}u_1(x).$$

Throughout this section we need a sequence of non-negative functions $\{v_n(x)\}_{n\in\mathbb{N}}$ with the following properties:

- v_n is continuously differentiable with
$$1 \leq v_n'(x) \leq \frac{\delta+\lambda}{c+ix}v_n(x) \tag{17}$$
- $v_n(x) \leq k_1 x + k_2$ for some positive constants $k_1,\ k_2$
- v_n converges uniformly to the absolutely continuous supersolution u_1 of (5) on compact sets and v_n' converges to u_1' almost everywhere. Further $v_n(x) = 0$ for $x < 0$.

Such a sequence exists due to [15] and [3].

Proposition 2.6. *An absolutely continuous supersolution u_1 of the HJB equation (5) fulfilling a linear growth condition dominates the value function, $u_1(x) \geq V(x)$.*

Proof. Let $L = (L_t)_{t\geq 0}$ be an admissible strategy. The controlled process is $R^L = (R_t^L)_{t\geq 0}$, $R_0^L = x$ with ruin time τ. Let $v_n(x)$ be a continuously differentiable element from the sequence defined above. We have

$$v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)} = v_n(x) + \int_0^{(t\wedge\tau)} v_n'(R_s^L)\, e^{-\delta s}\, dR_s^L - \delta\int_0^{(t\wedge\tau)} v_n(R_s^L)\, e^{-\delta s}\, ds.$$

having in mind that claim occurrences lead to $R_{s-} \neq R_s$ and singular dividend payments (lump sums) lead to $R_{s+} \neq R_s$, we get from the construction of the reserve process

$$\begin{aligned}\int_0^{(t\wedge\tau)} v_n'(R_s^L)\, e^{-\delta s}\, dR_s^L = &\int_0^{(t\wedge\tau)} e^{-\delta s}\,(c+iR_s^L)v_n'(R_s^L)\, ds - \int_0^{(t\wedge\tau)} v_n'(R_s^L)\, dL_s^c \\ &+ \sum_{R_{s-}^L\neq R_s^L,\ s<(t\wedge\tau)} (v_n(R_s^L) - v_n(R_{s-}^L)\, e^{-\delta s} + \sum_{R_{s+}^L\neq R_s^L,\ s<(t\wedge\tau)} (v_n(R_{s+}^L) - v_n(R_s^L)\, e^{-\delta s}.\end{aligned}$$

Using the continuity of v_n' and $R_{s+}^L - R_s^L = -(L_{s+} - L_s)$ we can write

$$v_n(R_{s+}^L) - v_n(R_s^L) = -\int_0^{L_{s+}-L_s} v_n'(R_s^L - \gamma)\, d\gamma.$$

Further we use the martingale $(M_t)_{t\geq 0}$

$$M_t = \sum_{R_{s-}^L\neq R_s^L,\ s<t} (v_n(R_s^L) - v_n(R_{s-}^L)\, e^{-\delta s} - \lambda\int_0^t e^{-\delta s}\left(\int_0^{R_{s-}^L} v_n(R_{s-}^L - y)\, dF_Y(y) - v_n(R_{s-}^L)\right) ds,$$

which is the *compensated process*, see [5]. We arrive at

$$v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)} = v_n(x) + \int_0^{(t\wedge\tau)} e^{-\delta s}\left[(c+iR^L_s)v_n'(R^L_s) - (\delta+\lambda)V_n(R^L_s) + \lambda\int_0^{R^L_{s-}} v_n(R^L_{s-}-y)\,dF_Y(y)\right]ds$$
$$-\int_0^{(t\wedge\tau)} v_n'(R^L_s)\,dL^c_s - \sum_{L_{s+}\neq L_s,\, s<(t\wedge\tau)} e^{-\delta s}\int_0^{L_{s+}-L_s} v_n'(R^L_s-\gamma)\,d\gamma + M_{(t\wedge\tau)}.$$

Now we use $v_n' \geq 1$ and can estimate

$$-\int_0^{(t\wedge\tau)} v_n'(R^L_s)\,dL^c_s - \sum_{L_{s+}\neq L_s,\, s<(t\wedge\tau)} e^{-\delta s}\int_0^{L_{s+}-L_s} v_n'(R^L_s-\gamma)\,d\gamma$$
$$\leq -\int_0^{(t\wedge\tau)} dL^c_s - \sum_{L_{s+}\neq L_s,\, s<(t\wedge\tau)} e^{-\delta s}\int_0^{L_{s+}-L_s} d\gamma = -\int_0^{(t\wedge\tau)} e^{-\delta s}dL_s,$$

which leads to

$$v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)} \leq v_n(x) + \int_0^{(t\wedge\tau)} e^{-\delta s}\mathcal{L}_{v_n}(R^L_s)\,ds - \int_0^{(t\wedge\tau)} e^{-\delta s}dL_s + M_{(t\wedge\tau)}.$$

The next steps are taking expectations, examining the validity of taking the limit $t\to\infty$ and letting $n\to\infty$. This will give the desired result.
Starting with

$$\mathbb{E}\left(v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)}\right) \leq v_n(x) + \mathbb{E}\left(\int_0^{(t\wedge\tau)} e^{-\delta s}\mathcal{L}_{v_n}(R^L_s)\,ds\right) - \mathbb{E}\left(\int_0^{(t\wedge\tau)} e^{-\delta s}dL_s\right), \tag{18}$$

we have to find integrable bounds for every summand to justify the interchange of limit and integration. Because L_s is increasing, we get by monotone convergence

$$\lim_{t\to\infty}\mathbb{E}\left(\int_0^{(t\wedge\tau)} e^{-\delta s}dL_s\right) = \mathbb{E}\left(\int_0^{\tau} e^{-\delta s}dL_s\right) = V_L(x).$$

Next we look at the second summand on the right hand side, use the estimates for the first derivative (17), the linear growth and the reserve from above to get the integrable upper bound

$$(c+ix)v_n'(x) - (\lambda+\delta)v_n(x) + \lambda\int_0^y v_n(x-y)\,dF_Y(y) \leq \lambda\int_0^y v_n(x-y)\,dF_Y(y) \leq \lambda\, v_n(x),$$

which gives

$$\int_0^{(t\wedge\tau)} e^{-\delta s}\mathcal{L}_{v_n}(R^L_s)\,ds \leq \int_0^{(t\wedge\tau)} e^{-\delta s}\lambda v_n(R^L_s)\,ds \leq \int_0^{(t\wedge\tau)} e^{-\delta s}\lambda\left(k_1\, e^{is}\left(x + c\int_0^s e^{-ih}\,dh\right) + k_2\right)\,ds$$
$$< \int_0^{\infty} e^{-\delta s}\lambda\left(k_1\, e^{is}\left(x + c\int_0^s e^{-ih}\,dh\right) + k_2\right)\,ds < \infty$$

(recall that we have $i<\delta$), so that by dominated convergence

$$\lim_{t\to\infty}\mathbb{E}\left(\int_0^{(t\wedge\tau)} e^{-\delta s}\mathcal{L}_{v_n}(R^L_s)\,ds\right) = \mathbb{E}\left(\int_0^{\tau} e^{-\delta s}\mathcal{L}_{v_n}(R^L_s)\,ds\right).$$

The left hand side of (18 converges to zero by

$$0 \leq \mathbb{E}\left(v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)}\right) = \mathbb{E}\left(v_n(R^L_{(t\wedge\tau)})e^{-\delta(t\wedge\tau)}\, I_{\{t<\tau\}}\right)$$
$$\leq \mathbb{E}\left(v_n(R^L_t)e^{-\delta t}\right) \leq \mathbb{E}\left(e^{-\delta t}\left(k_1\, e^{is}\left(x + c\int_0^s e^{-ih}\,dh\right) + k_2\right)\right) \to 0$$

(recall $v_n(x) = 0$ for $x < 0$). Further $v_n' \to u_1'$ almost everywhere (at points where u_1 is differentiable) and $\lim_{n\to\infty} \mathcal{L}_{v_n}(x) = \mathcal{L}_{u_1}(x)$ holds.
We need again an integrable upper bound for $|\ \mathcal{L}_{v_n}(R_s^L) - \mathcal{L}_{u_1}(R_s^L)\ |$. This can be obtained from (17) and the linear growth conditions on v_n and u_1:

$$\begin{aligned}
&|\ \mathcal{L}_{v_n}(R_s^L) - \mathcal{L}_{u_1}(R_s^L)\ |\ e^{-\delta s} \\
&\qquad \le \Bigg((c + iR_s^L) v_n'(R_s^L) + (\delta+\lambda) v_n(R_s^L) + \lambda \int_0^{R_s^L} v_n(R_s^L - y)\ dF_Y(y) \\
&\qquad\quad + (c + iR_s^L) u_1'(R_s^L) + (\delta+\lambda) u_1(R_s^L) + \lambda \int_0^{R_s^L} u_1(R_s^L - y)\ dF_Y(y) \Bigg)\ e^{-\delta s} \\
&\le 2(\delta+\lambda) v_n(R_s^L) + \lambda \int_0^{R_s^L} v_n(R_s^L - y)\ dF_Y(y) + 2(\delta+\lambda) u_1(R_s^L) + \lambda \int_0^{R_s^L} u_1(R_s^L - y)\ dF_Y(y) \\
&\qquad \le K\ (u_1(R_s^L) + v_n(R_s^L))\ e^{-\delta s} \le K\ \left(k_1\ e^{is} \left(x + c \int_0^s e^{-ih}\ dh \right) + k_2 \right)\ e^{-\delta s}.
\end{aligned}$$

Altogether we arrive at

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^\tau e^{-\delta s}\ \mathcal{L}_{v_n}(R_s^L)\ ds \right) = \mathbb{E}\left(\int_0^\tau e^{-\delta s}\ \mathcal{L}_{u_1}(R_s^L)\ ds \right) \le 0.$$

Finally we arrive at

$$V_L(x) \le u_1(x) + \mathbb{E}\left(\int_0^\tau e^{-\delta s}\ \mathcal{L}_{u_1}(R_s^L)\ ds \right) \le u_1(x),$$

which holds for every admissible strategy L resulting in $V(x) \le u_1(x)$. □

The next proposition follows immediately.

Proposition 2.7. *An admissible strategy L with associated return function V_L which is an absolutely continuous supersolution of the HJB equation fulfills $V = V_L$. Consequently, L is an optimal dividend strategy.*

Now we state several auxiliary results which characterize the value function at points of potentially problematic differentiability behaviour. The proofs are in the spirit of Azcue and Muler in [3].
If it is optimal to pay out an amount a immediately, then $V(x) = a + V(x-a)$ so that $V'(x-) = 1$. If it is optimal to keep the surplus at a level x until the next claim occurrence at time τ_1 and pay out everything exceeding this level we have

$$V(x) = \mathbb{E}\left(\int_0^{\tau_1} (c + ix)\ e^{-\delta s}\ ds + e^{-\delta\tau_1} V(x - Y_1) \right) = \frac{1}{\delta+\lambda} \left(c + ix + \lambda \int_0^x V(x-y)\ dF_Y(y) \right).$$

The following assertions are needed to prove certain properties of the optimal strategy.
For some $z > 0$, the set Π_z will denote the set of admissible strategies $L \in \Pi$ for which the controlled reserve stays below z, i.e. $R_t^L \le z$ for $L \in \Pi_z$ and $t \ge 0$.
Define the operator

$$\Lambda(x) = c + ix - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)\ dF_Y(y).$$

Lemma 2.8. *If there is an $\overline{x} > 0$ such that $\Lambda(\overline{x}) = 0$, then $V(x) = \sup_{L\in\Pi_{\overline{x}}} V_L(x)$ for $x \in [0, \overline{x})$.*

Proof. The proof is done by induction. Let $\Pi_{(n)}$ be the set of admissible strategies such that for initial reserve $x < \overline{x}$ the claim process stays below $\overline{x}$ till the occurrence of the nth claim. The idea of the proof is to construct an ϵ-optimal strategy $\hat{L} \in \Pi_{\overline{x}}$ from a certain $\epsilon/2$-optimal strategy $L_n \in \Pi_{(n)}$ for some n large enough. Because of discounting and $\delta > i$ we get that $|V_{L_n}(x) - V_{\hat{L}}(x)|$ will be small enough to

derive the desired result.
First we want to show

$$V(x) = \sup_{L \in \Pi_{(n)}} V_L(x) \tag{19}$$

for all $n \geq 0$. This will be done by induction. Clearly $\Pi_{(0)} = \Pi$ and we have that $V(x) = \sup_{L \in \Pi_{(0)}} V_l(x)$. Let $n > 1$, $\epsilon > 0$ and (19) be fulfilled for $n-1$. By the induction hypothesis, $L_{n-1} \in \Pi_{(n-1)}$ such that $V(x) - V_{L_{n-1}}(x) < \frac{\epsilon}{2}$. Now we look for a strategy $L_n \in \Pi_{(n)}$ such that $0 \leq V_{L_{n-1}}(x) - V_{L_n}(x) \leq \frac{\epsilon}{2}$. In view of $\Lambda(\overline{x}) = 0$, L_n is defined as follows. Starting at $x < \overline{x}$ apply L_{n-1} as long as the reserve stays below $\overline{x}$. When reaching $\overline{x}$ pay out $c + i\overline{x}$ until a claim occurs and use again L_{n-1} with initial capital $\overline{x} - Y$, where Y denotes the random claim size.
As first step we show $V_{L_n}(\overline{x}) \geq V_{L_{n-1}}(\overline{x}) - \frac{\epsilon}{2}$. The initial capital is $R_0^{L_n} = \overline{x}$; Y_1, τ_1 denote amount and occurrence time of the first claim. For $0 \leq t < \tau_1$ we have $R_t^{L_n} = \overline{x}$, $L_{n,t} = (c + i\overline{x})t$ and $R_{\tau_1}^{L_n} = \overline{x} - Y_1$. We get

$$\begin{aligned}
V_{L_n}(\overline{x}) &= \mathbb{E}\left(\int_0^{\tau_1} e^{-\delta s}(c + i\overline{x})\, ds + e^{-\delta \tau_1} V_{L_{n-1}}(\overline{x} - Y_1)\right) \\
&= \frac{1}{\delta + \lambda}\left(c + i\overline{x} + \lambda \int_0^{\overline{x}} V_{L_{n-1}}(\overline{x} - y)\, dF_Y(y)\right) \\
&\geq \frac{1}{\delta + \lambda}\left(c + i\overline{x} + \lambda \int_0^{\overline{x}} (V(\overline{x} - y) - \frac{\epsilon}{2}\, dF_Y(y)\right) \\
&= \frac{1}{\delta + \lambda}\left(\Lambda(\overline{x}) + (\delta + \lambda)V(\overline{x}) - \frac{\lambda\epsilon}{2} F(\overline{x})\right) \\
&\geq V(\overline{x}) - \frac{\epsilon}{2}.
\end{aligned}$$

From the following two inequalities we get the required result,

$$\begin{aligned}
V(\overline{x}) &\geq V_{L_{n-1}}(\overline{x}) \geq V(\overline{x}) - \frac{\epsilon}{2}, \\
V(\overline{x}) &\geq V_{L_n}(\overline{x}) \geq V(\overline{x}) - \frac{\epsilon}{2},
\end{aligned}$$

which gives

$$V_{L_n}(\overline{x}) - V_{L_{n-1}}(\overline{x}) \geq V(\overline{x}) - \frac{\epsilon}{2} - V(\overline{x}) = -\frac{\epsilon}{2}.$$

Now we deal with the case $0 \leq x < \overline{x}$. We have to distiguish between paths of the process controlled by L_n which reach $\overline{x}$ in finite time (the set of these paths is denoted by $\mathcal{P}_1$) and those which do not. Let $\overline{\tau}$ be the first time a path from $\mathcal{P}_1$ reaches $\overline{x}$. We can split the value of the strategy L_n as follows

$$\begin{aligned}
V_{L_n}(x) &= \mathbb{E}\left(I_{\mathcal{P}_1} \int_0^{\tau^{L_n}} e^{-\delta s} dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_1^c} \int_0^{\tau^{L_n}} e^{-\delta s} dL_{n,s}\right) \\
&= \mathbb{E}\left(I_{\mathcal{P}_1} \int_0^{\overline{\tau}} e^{-\delta s} dL_{n,s}\right) + \mathbb{E}\left(e^{-\delta \overline{\tau}}\right) V_{L_n}(\overline{x}) + \mathbb{E}\left(I_{\mathcal{P}_1^c} \int_0^{\tau^{L_n}} e^{-\delta s} dL_{n,s}\right).
\end{aligned}$$

Because of the definition of the strategy L_n we have that in $\mathcal{P}_1^c$ and in $\mathcal{P}_1$ for $t < \overline{\tau}$ the paths $R_t^{L_n}$ and $R_t^{L_{n-1}}$ are identical. Therefore we arrive at

$$\begin{aligned}
V_{L_{n-1}}(x) - V_{L_n}(x) &= \mathbb{E}\left(I_{\mathcal{P}_1} e^{-\delta \overline{\tau}}\right)\left(V_{L_{n-1}}(\overline{x}) - V_{L_n}(\overline{x})\right) \\
&\leq \mathbb{E}\left(I_{\mathcal{P}_1} e^{-\delta \overline{\tau}}\right) \frac{\epsilon}{2} \leq \frac{\epsilon}{2}.
\end{aligned}$$

In the end we have to show that for every $\epsilon > 0$ there exists a strategy $\hat{L} \in \Pi_{\overline{x}}$ sucht that $V(x) - V_{\hat{L}}(x) < \epsilon$ for $x \in [0, \overline{x}]$. First define t_1 such that

$$e^{-\delta t_1} < \frac{\epsilon}{8V(\overline{x})},$$

and $n \geq 1$ large enough such that

$$P(N_{t_1} \geq n) = \sum_{k \geq n} \frac{e^{-\lambda t_1}(\lambda t_1)^k}{k!} \leq \frac{\epsilon}{8V(\overline{x})}.$$

Let $L_n \in \Pi_{(n)}$ be an $\epsilon/2$-optimal strategy for all $x \in [0, \overline{x}]$. Let $\hat{\tau}$ the first time a path of $(R_t^{L_n})_{t \geq 0}$ exceeds $\overline{x}$. The set $\mathcal{P}_2$ consists of all paths such that $\hat{\tau} < \infty$. For $t < \hat{\tau}$ we define $\hat{L} = L_n$, if $t = \hat{\tau}$ the strategy $\hat{L}$ pays out immediately $\overline{x}$ and the incoming premiums till the next claim occurrence which leads to ruin. As before the value of the strategy L_n as well as for $\hat{L}$ can be written in the following form,

$$\begin{aligned} V_{L_n}(x) &= \mathbb{E}\left(I_{\mathcal{P}_2^c}\int_0^{\tau^{L_n}} e^{-\delta s}\,dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_2}\int_0^{\tau^{L_n}} e^{-\delta s}\,dL_{n,s}\right) \\ &= \mathbb{E}\left(I_{\mathcal{P}_2^c}\int_0^{\tau^{L_n}} e^{-\delta s}\,dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_2}\int_0^{\hat{\tau}} e^{-\delta s}\,dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_2}e^{-\delta\hat{\tau}}\right)V_{L_n}(\overline{x}) \\ &\leq \mathbb{E}\left(I_{\mathcal{P}_2^c}\int_0^{\tau^{L_n}} e^{-\delta s}\,dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_2}\int_0^{\hat{\tau}} e^{-\delta s}\,dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_2}e^{-\delta\hat{\tau}}\right)V(\overline{x}). \end{aligned}$$

Since $R_t^{L_n}$ and $R_t^{\hat{L}}$ are identical on $\mathcal{P}_2^c$ and for $t < \hat{\tau}$ we get

$$|V_{L_n}(x) - V_{\hat{L}(x)}| \leq 2\,\mathbb{E}\left(I_{\mathcal{P}_2}e^{-\delta\hat{\tau}}\right)V(\overline{x}).$$

Because $L_n \in \Pi_{(n)}$ we have $\{\hat{\tau} < t_1\} \subset \{N_{t_1} \geq n\}$, furthermore we have $\mathcal{P}_2 = \{\hat{\tau} < \infty\} \subset \{\hat{\tau} \geq t_1\} \cup \{N_{t_1} \geq n\}$. We get

$$\begin{aligned} \mathbb{E}\left(I_{\mathcal{P}_2}e^{-\delta\hat{\tau}}\right) &\leq \mathbb{E}\left(I_{\{\hat{\tau}\geq t_1\}}e^{-\delta\hat{\tau}}\right) + \mathbb{E}\left(I_{\{N_{t_1}\geq n\}}e^{-\delta\hat{\tau}}\right) \\ &\leq e^{-\delta t_1} + P(\{N_{t_1} \geq n\}) < \frac{\epsilon}{4V(\overline{x})}, \end{aligned}$$

which gives

$$|V_{L_n}(x) - V_{\hat{L}(x)}| < \frac{\epsilon}{2}.$$

The required result follows from

$$V(x) - V_{\hat{L}}(x) \leq V(x) - V_{L_n}(x) + |V_{L_n}(x) - V_{\hat{L}(x)}| < \epsilon.$$

□

Lemma 2.9. *If there is an $\overline{x} > 0$ such that $V'(\overline{x}) = 1$, then $V(x) = \sup_{L \in \Pi_{\overline{x}}} V_L(x)$ for all $x \in [0, \overline{x}]$.*

Proof. We have to show that for every $\epsilon > 0$ we are able to find a strategy $\hat{L} \in \Pi_{\overline{x}}$ such that $0 \leq V(x) - V_{\hat{L}}(x) < \epsilon$ for all $x \in [0, \overline{x}]$. Let

$$D = \frac{c + ix}{\delta} \ln\left(\frac{2V(\overline{x})}{\epsilon}\right)$$

and define a sequence $\{x_n\}_{n \in \mathbb{N}}$ with

$$x_n = \overline{x} - \frac{D}{n}.$$

Further we need a sequence $\{h_n\}_{n \in \mathbb{N}}$ defined by

$$h_n = \frac{V(x_n) - V(\overline{x})}{x_n - \overline{x}} - 1.$$

Because of $V'(\overline{x}) = 1$ we have that $h_n \to 0$ for $n \to \infty$. Choose n_0 such that $h_{n_0} < \frac{\epsilon}{8D}$. A further specification of the size of n_0 will be needed in the end of the proof.

The proof needs two steps: first one fixes a sequence of strategies such that on a certain level we get an $\frac{\epsilon}{2}$-optimal strategy and the differences of the values of these strategies form a decreasing sequence. In a second step these ingredients are used to define an ϵ-optimal strategy within the set $\Pi_{\overline{x}}$.

Step 1:

Take a strategy $L \in \Pi$ such that $V(x) - V_L(x) < \frac{\epsilon}{8n_0}$. Now define in a recursive way the following set of strategies $(L_n)_{n \geq 0}$. For $n = 0$ set $L_0 = L$. For $n > 0$ and initial capital $x \leq x_{n_0}$ follow the strategy L as long as $R_t^L < \overline{x}$ and as R_t^L reaches $\overline{x}$, pay out immediately the difference $\overline{x} - x_{n_0}$ and follow L_{n-1} with initial capital x_{n_0}. If $x \in (x_{n_0}, \overline{x}]$, pay out $x - x_{n_0}$ and follow L_{n-1}.

The idea behind this procedure is to find an estimate for the time the process stays below $\overline{x}$ before crossing $\overline{x}$. Under the strategy L_n the intervall $[x_{n_0}, \overline{x}]$ has to be passed more than n times.

The first thing to show is $V(x) - V_{L_{n_0}}(x) < \frac{\epsilon}{2}$ for all $x \in [0, \overline{x}]$.

We start with showing that $V(x) - V_{L_1}(x) < \frac{\epsilon}{2n_0}$ for all $x \in [0, \overline{x}]$.

For $x = \overline{x}$ we have

$$\begin{aligned} V(\overline{x}) - V_{L_1}(\overline{x}) &\leq V(x_{n_0}) + (1 + h_{n_0})(\overline{x} - x_{n_0}) - ((\overline{x} - x_{n_0}) + V_{L_0}(x_{n_0})) \\ &= V(x_{n_0}) - V_L(x_{n_0}) + h_{n_0}(\overline{x} - x_{n_0}) \leq \frac{\epsilon}{4n_0}, \end{aligned}$$

because of $\overline{x} - x_{n_0} = \frac{D}{n_0}$, $h_{n_0} \leq \frac{\epsilon}{8D}$, $(1 + h_{n_0})(\overline{x} - x_{n_0}) = V(\overline{x}) - V(x_{n_0})$ and $V(\overline{x}) \geq \overline{x} - x + V(x)$.

If $x \in [x_{n_0}, \overline{x}]$ we get with $V(\overline{x}) = (1 + h_{n_0})(\overline{x} - x_{n_0}) + V(x_{n_0})$,

$$\begin{aligned} V(x) - V_{L_1}(x) &\leq V(\overline{x}) - (\overline{x} - x) - (x - x_{n_0} + V_L(x_{n_0})) \\ &= V(x_{n_0}) + h_{n_0}(\overline{x} - x_{n_0}) + \overline{x} - x_{n_0} - \overline{x} + x - x + x_{n_0} - V_L(x_{n_0}) \\ &= V(x_{n_0}) - V_L(x_{n_0}) + h_{n_0}(\overline{x} - x_{n_0}) \leq \frac{\epsilon}{4n_0}, \end{aligned}$$

with the same arguments as above.

In the end we look at $x \in [0, x_{n_0})$. Let $\mathcal{P}_3$ be the set of paths of R^L with initial capital x such that x_{n_0} is reached in finite time, let τ_x be the first time such that this is done by a path from $\mathcal{P}_3$. We derive

$$\begin{aligned} V_{L_1}(x) &= \mathbb{E}\left(I_{\mathcal{P}_3} \int_0^{\tau^{L_1}} e^{-\delta s}\, dL_{1,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_3^c} \int_0^{\tau^{L_1}} e^{-\delta s}\, dL_{1,s}\right) \\ &= \mathbb{E}\left(I_{\mathcal{P}_3} \int_0^{\tau_x} e^{-\delta s}\, dL_{1,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_3} e^{-\delta \tau_x}\right) V_{L_1}(x_{n_0}) + \mathbb{E}\left(I_{\mathcal{P}_3^c} \int_0^{\tau^{L_1}} e^{-\delta s}\, dL_{1,s}\right). \end{aligned}$$

Because the paths of R^L and R^{L_1} coincide in $\mathcal{P}_3^c$ and in $\mathcal{P}_3$ for $t < \tau_x$ we get

$$\mid V_{L_1}(x) - V_L(x) \mid = \mathbb{E}\left(I_{\mathcal{P}_3} e^{-\delta \tau_x}\right) \mid V_{L_1}(x_{n_0}) - V_L(x_{n_0}) \mid.$$

This together with the above estimates, $\mathbb{E}\left(I_{\mathcal{P}_3} e^{-\delta \tau_x}\right) \leq 1$, yields

$$\begin{aligned} \mid V(x) - V_{L_1}(x) \mid &\leq \mid V(x) - V_L(x) \mid + \mid V_L(x) - V_{L_1}(x) \mid \\ &\leq \mid V(x) - V_L(x) \mid + \mid V_L(x_{n_0}) - V_{L_1}(x_{n_0}) \mid \\ &\leq \mid V(x) - V_L(x) \mid + \mid V(x_{n_0}) - V_L(x_{n_0}) \mid + \mid V(x_{n_0}) - V_{L_1}(x_{n_0}) \mid \leq \frac{\epsilon}{2n_0}. \end{aligned}$$

Now we want for $n \geq 2$ and $x \in [0, \overline{x}]$ that $\mid V_{L_n}(x) - V_{L_{n-1}}(x) \mid \leq \mid V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0}) \mid$ holds. For $x \in [x_{n_0}, \overline{x}]$ and $n \geq 1$ we get the result immediately from $V_{L_n}(x) = x - x_{n_0} + V_{L_{n-1}}(x_{n_0})$.

Let $x \in [0, x_{n_0})$ and denote by $\mathcal{P}_4$ the set of paths of R^L such that $\overline{x}$ is reached in finite time, $\overline{\tau}_x$ denoting the first time of such an event. We obtain

$$\begin{aligned} V_{L_n}(x) &= \mathbb{E}\left(I_{\mathcal{P}_4} \int_0^{\tau^{L_n}} e^{-\delta s}\, dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_4^c} \int_0^{\tau^{L_n}} e^{-\delta s}\, dL_{n,s}\right) \\ &= \mathbb{E}\left(I_{\mathcal{P}_4} \int_0^{\overline{\tau}_x} e^{-\delta s}\, dL_{n,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_4} e^{-\delta \overline{\tau}_x}\right) (V_{L_{n-1}}(x_{n_0}) + \overline{x} - x_{n_0}) + \mathbb{E}\left(I_{\mathcal{D}^c} \int_0^{\tau^{L_n}} e^{-\delta s}\, dL_{n,s}\right). \end{aligned}$$

As before the paths of R^{L_n} and $R^{L_{n-1}}$ coincide on $\mathcal{P}_4^c$ and on $\mathcal{P}_4$ for $t < \overline{\tau}_x$. Therefore

$$\mid V_{L_n}(x) - V_{L_{n-1}}(x) \mid = \mathbb{E}\left(e^{-\delta\overline{\tau}_x}\right) \mid V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0}) \mid \le \mid V_{L_{n-1}}(x_{n_0}) - V_{L_{n-2}}(x_{n_0}) \mid.$$

We arrive at

$$\begin{aligned} V(x) - V_{L_{n_0}}(x) &= V(x) - V_{L_1}(x) + \sum_{n=2}^{n_0} \left(V_{L_{n-1}}(x) - V_{L_n}(x)\right) \\ &\le V(x) - V_{L_1}(x) + (n_0 - 1) \mid V_{L_1}(x_{n_0}) - V_L(x_{n_0}) \mid \\ &\le \frac{\epsilon}{2n_0} + (n_0 - 1)(\mid V_{L_1}(x_{n_0}) - V(x_{n_0}) \mid + \mid V(x_{n_0}) - V_L(x_{n_0}) \mid) \\ &\le \frac{\epsilon}{2n_0} + (n_0 - 1)(\frac{\epsilon}{4n_0} + \frac{\epsilon}{8n_0}) \le \frac{\epsilon}{2}. \end{aligned}$$

<u>Step 2:</u>
Now we identify a strategy $\overline{L} \in \Pi_{\overline{x}}$ such that $V_{L_{n_0}}(x) - V_{\overline{L}}(x) < \frac{\epsilon}{2}$ for all $x \in [0, \overline{x}]$. In order to reach $\overline{x}$ from x_{n_0} it takes at least $\frac{1}{i}\ln\left(\frac{i\overline{x}+c}{ix_{n_0}+c}\right)$ time units. For $x \in [0, \overline{x}]$ let $\overline{\tau} = \inf\{t > 0 \mid R_t^{L_{n_0}} > \overline{x}\}$. From the definition of the strategy L_{n_0} we get that the process has to go through the interval $[x_{n_0}, \overline{x}]$ at least n_0 times. We get

$$\delta\overline{\tau} \ge \frac{n_0\delta}{i}\ln\left(\frac{i\overline{x}+c}{ix_{n_0}+c}\right),$$

and subsequently

$$\mathbb{E}\left(e^{-\delta\overline{\tau}}\right) \le \left(\frac{i\overline{x}+c}{i(\overline{x}-\frac{D}{n_0})+c}\right)^{\frac{-\delta n_0}{i}} = \left(1 + \frac{iD}{n_0(i(\overline{x}-\frac{D}{n_0})+c)}\right)^{\frac{-\delta n_0}{i}} \approx e^{-\frac{D\delta}{c+i\overline{x}}} \le \frac{\epsilon}{2V(\overline{x})},$$

for n_0 large enough. Let $\mathcal{P}_5$ be the set of paths of $R^{L_{n_0}}$ with finite $\overline{\tau}$. Now we define the strategy $\overline{L} \in \Pi_{\overline{x}}$, with $\overline{L} = L_{n_0}$ as long as $t < \overline{\tau}$, and at $t = \overline{\tau}$ pay out $\overline{x}$ immediately and distribute the incoming premiums as dividends till the next claim occurrence causes ruin. Again we can write

$$\begin{aligned} V_{L_{n_0}}(x) &= \mathbb{E}\left(I_{\mathcal{P}_5^c}\int_0^{\tau^{L_{n_0}}} e^{-\delta s} dL_{n_0,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_5}\int_0^{\tau^{L_{n_0}}} e^{-\delta s} dL_{n_0,s}\right) \\ &\le \mathbb{E}\left(I_{\mathcal{P}_5^c}\int_0^{\tau^{L_{n_0}}} e^{-\delta s} dL_{n_0,s}\right) + \mathbb{E}\left(I_{\mathcal{P}_5}e^{-\delta\overline{\tau}}\right)V(\overline{x}) + \mathbb{E}\left(I_{\mathcal{P}_5}\int_0^{\overline{\tau}} e^{-\delta s} dL_{n_0,s}\right). \end{aligned}$$

Similarly we get

$$\begin{aligned} V_{\overline{L}}(x) &= \mathbb{E}\left(I_{\mathcal{P}_5^c}\int_0^{\tau^{\overline{L}}} e^{-\delta s} d\overline{L}_s\right) + \mathbb{E}\left(I_{\mathcal{P}_5}\int_0^{\tau^{\overline{L}}} e^{-\delta s} d\overline{L}_s\right) \\ &\ge \mathbb{E}\left(I_{\mathcal{P}_5^c}\int_0^{\tau^{\overline{L}}} e^{-\delta s} d\overline{L}_s\right) + \mathbb{E}\left(I_{\mathcal{P}_5}\int_0^{\overline{\tau}} e^{-\delta s} d\overline{L}_s\right) + \mathbb{E}\left(e^{-\delta\overline{\tau}}\right)\overline{x}. \end{aligned}$$

Because on the sets $\mathcal{P}_5^c$ and $\mathcal{P}$ the paths of $R^{L_{n_0}}$ and $R^{\overline{L}}$ coincide for $t < \bar{\tau}$, we arrive at

$$V_{L_{n_0}}(x) - V_{\overline{L}}(x) \le \mathbb{E}\left(e^{-\delta\overline{\tau}}\right)\left(V(\overline{x}) - \overline{x}\right) \le \mathbb{E}\left(e^{-\delta\overline{\tau}}\right)V(\overline{x}) \le \frac{\epsilon}{2}.$$

This finishes the proof since

$$0 \le V(x) - V_{\overline{L}}(x) = V(x) - V_{L_{n_0}}(x) + V_{L_{n_0}}(x) - V_{\overline{L}}(x) \le \epsilon.$$

□

Finally, the following is a consequence of the proof of Proposition 2.6:

Lemma 2.10. *Let $\overline{x} > 0$ and $u_1(x)$ be an absolutely continuous supersolution of the HJB equation for all $x \in [0, \overline{x})$. If L is an admissible strategy such that $R_t^L \leq \overline{x}$ for all $t \geq 0$ then $u_1(x) \geq V_L(x)$ for all $x \in [0, \overline{x})$.*

These three lemmas imply the following two propositions (the results resemble a similar local characterization of the value function in Shreve et al. [14], where the intermediate step with constrained controls Π_x were used for a dividend maximization problem in a general diffusion setup).

Proposition 2.11. *If either $\Lambda(\overline{x}) = 0$ or $V'(\overline{x}) = 1$ for some $\overline{x} > 0$ and $u_1(x)$ is an absolutely continuous supersolution of the HJB equation for all $x \in [0, \overline{x})$ then $u_1(x) \geq V(x)$ in $[0, \overline{x}]$. Hence, if $L \in \Pi_{\overline{x}}$ such that V_L is an absolutely continuous supersolution to the HJB equation for all $x \in [0, \overline{x})$ then $V(x) = V_L(x)$ for all $x \in [0, \overline{x}]$.*

Define for any $y > 0$

$$U_y(x) = \begin{cases} V(x) & x \leq y, \\ V(y) + x - y & x > y. \end{cases}$$

The following proposition will be the key in the numerical construction of a solution and we will see how it matches some properties of the optimal strategy.

Proposition 2.12. *(i) If U_y is a supersolution to the HJB equation in (y, ∞), then $U_y = V$ in $[0, \infty)$.*

(ii) If either $\Lambda(\overline{x}) = 0$ or $V'(\overline{x}) = 1$ for some $\overline{x} > 0$ and there exists $y < \overline{x}$ such that U_y is a supersolution of the HJB equation in $(y, \overline{x}]$, then $U_y = V$ in $[0, \overline{x}]$.

Proof. (i) If we prove that U_y is a supersolution in $y > 0$ we immediately have that $U_y \geq V$ in $[0, \infty)$. From the definition we have $U_y(y) = V(y)$ and therefore the supersolution property of V implies that $\mathcal{L}^*_{U_y,\phi}(y) \leq 0$ for an appropriate function ϕ. The right-hand derivative in y is given through

$$\lim_{x \downarrow y} \frac{U_y(x) - U_y(y)}{x - y} = 1.$$

Remark 2.2 shows that there exists a test function ϕ with the supersolution property if and only if

$$\lim_{x \uparrow y} \frac{V(x) - V(y)}{x - y} = \lim_{x \uparrow y} \frac{U_y(x) - U_y(y)}{x - y} = 1.$$

But in this case we get $\phi'(y) = 1$ showing in addition to V also U_y has the supersolution property. $U_y \leq V$ follows from the definition of U_y and Proposition 2.2. For (ii) use Proposition 2.11 instead of the general supersolution property. Then the same arguments as above give the desired result. ☐

The following settles the question of differentiability at points switching from the non-pay- to the pay-regime.

Remark 2.4. From the proof of Proposition 2.12 (i) and equation (7) of Remark 2.2, we obtain that at points $y > 0$ where a barrier strategy with height y is applied, we have differentiability of the value function: Below y we use V, in some interval above y we have V described by U_y. From Proposition 2.2 and the monotonicity of U_y we get (for $x < y < x'$ such $|x - y| \geq |x' - y|$),

$$1 \leq \frac{V(x) - V(y)}{x - y} = \frac{U_y(x) - U_y(y)}{x - y} \leq \frac{U_y(x') - U_y(y)}{x' - y} \to 1,$$

for $x' \to y$. This shows that in such *change* points the left-hand derivative is (by the viscosity solution property) bounded by the right-hand derivative, giving 1 as an upper and lower bound and therefore proving differentiability in these points.

3 Construction of the optimal strategy

3.1 The IDE part of the HJB equation

In intervals where V' exists and is greater than 1 we have to fulfill the second part of the HJB equation (5). Recall that in intervals where it is optimal to do nothing the generator $\mathbf{A}$ of the controlled process applied to V gives

$$\mathbf{A}V(x) = (c+ix)V'(x) - (\delta+\lambda)V(x) + \lambda \int_0^x V(x-y)\, dF_Y(y).$$

Let us therefore look for a solution of the following integro-differential equation with a given initial condition,

$$\begin{aligned} 0 &= (c+ix)f'(x) + \lambda \int_0^x f(x-y)dF_Y(y) - (\lambda+\delta)f(x), \qquad (20)\\ 1 &= f(0). \end{aligned}$$

As for each solution $f(x)$ of (20), $C\,f(x)$ is again a solution for arbitrary constant C, any boundary condition can be fulfilled.
Let $f(x)$ be a solution to (20) and define for some $b \geq 0$

$$V_b(x) = \begin{cases} f(x)/f'(b) & x \leq b, \\ x-b+V_b(b) & x > b. \end{cases} \qquad (21)$$

An analogue of [13, Lemma 2.49] shows that V_b is equal to the value of the expected discounted dividends when a constant barrier strategy with barrier height b is applied. Hence maximizing $V_b(x)$ over all $b \geq 0$ is equivalent to finding a minimum of $f'(x)$.

We will now prove the existence of a solution of a generalized version of (20). If it is optimal to pay out dividends following a barrier strategy only in a bounded interval ($V' = 1$) and for higher surplus $x > x_0$ it is optimal to pay nothing in some area ($V' > 1$), then we would need a solution to the equation

$$\begin{aligned} 0 &= (c+ix)u'(x) - (\delta+\lambda)u(x) + \lambda \int_0^{x-x_0} u(x-y)\, dF_Y(y) + \lambda \int_{x-x_0}^x f(x-y)\, dF_Y(y) \qquad (22)\\ f(x_0) &= u(x_0), \end{aligned}$$

where $f : [0, x_0] \to [0, \infty)$ is a given continuous and increasing function. Note that choosing $x_0 = 0$ and taking $u(0) = 1$ as initial condition leads to the existence proof of a solution to (20).

Lemma 3.1. *Let $x_0 \geq 0$. For a continuous and increasing function $f : [0, x_0] \to [0, \infty)$ there exists a unique, in (x_0, ∞) differentiable and strictly increasing solution $u : [x_0, \infty) \to [0, \infty)$ to (22) with $u(x_0) = f(x_0)$.*

Proof. For $\epsilon = \frac{c}{2(\delta+2\lambda)}$, we will show that there exists a solution with the required properties on $[x_0, x_0+\epsilon)$ and since ϵ does not depend on x_0 this will establish the existence on $[x_0, \infty)$.
The set of all continuous and increasing functions $u : [x_0, x_0+\epsilon) \to [0, \infty)$ is denoted by $CI[x_0, x_0+\epsilon)$, further let for a $u \in CI[x_0, x_0+\epsilon)$,

$$\overline{u}(x) = \frac{(\delta+\lambda)u(x) - \lambda \int_0^{x-x_0} u(x-y)\, dF_Y(y) - \lambda \int_{x-x_0}^x f(x-y)\, dF_Y(y)}{c+ix}.$$

As u and f are continuous, $\overline{u}$ is continuous for $x \geq 0$. Now we define for $u \in CI[x_0, x_0+\epsilon)$

$$T_u(x) = \int_{x_0}^x \overline{u}(s)\, ds + f(x_0).$$

Because of the monotonicity of u and f and $f(x_0) = u(x_0)$ we get

$$\begin{aligned} &\lambda \int_0^{x-x_0} u(x-y)\, dF_Y(y) + \lambda \int_{x-x_0}^x f(x-y)\, dF_Y(y) \\ &\qquad \leq \lambda u(x)\, F_Y(x-x_0) + \lambda f(x_0)(F_Y(x) - F_Y(x-x_0)) \leq \lambda\, u(x). \end{aligned}$$

This argument gives the following lower bound for $\overline{u}$

$$0 < \frac{\delta}{c+ix} u(x) \le \overline{u}(x) \le \frac{\delta+\lambda}{c+ix} u(x).$$

Here the upper bound follows from the fact that u and f are positive. This implies that T_u is increasing, positive and continuous for $x \in [x_0, x_0+\epsilon)$. Now for $u_1,\ u_2\ \in CI[x_0, x_0+\epsilon)$, we get

$$\begin{aligned}\overline{u}_1(x) - \overline{u}_2(x) &= \frac{(\delta+\lambda)(u_1(x)-u_2(x)) - \lambda \int_0^{x-x_0} (u_1(x-y)-u_2(x-y))\, dF_Y(y)}{c+ix} \\ &\le \frac{1}{c}\left((\delta+\lambda)\|u_1-u_2\| + \lambda\|u_1-u_2\| F_Y(x-x_0)\right) \le \frac{\delta+2\lambda}{c}\|u_1-u_2\|,\end{aligned}$$

where $\|\cdot\|$ denotes the supremum norm. This implies

$$T_{u_1}(x) - T_{u_2}(x) \le \epsilon\, \frac{\delta+2\lambda}{c}\|u_1-u_2\| \le \frac{1}{2}\|u_1-u_2\|.$$

Interchanging u_1 and u_2 results in $\|T_{u_1} - T_{u_2}\| \le \frac{1}{2}\|u_1-u_2\|$, proving that T is a contraction on $CI[x_0, x_0+\epsilon)$. Therefore there exists a $u \in CI[x_0, x_0+\epsilon)$ such that

$$u(x) = \int_{x_0}^{x} \frac{(\delta+\lambda)u(s) - \lambda\int_0^{s-x_0} u(s-y)\, dF_Y(y) - \lambda \int_{s-x_0}^{s} f(s-y)\, dF_Y(y)}{c+is}\, ds + f(x_0).$$

Further we have from above that $u'(x) = \overline{u}(x)$ holds everywhere in $[x_0, x_0+\epsilon)$. This gives the existence of a unique solution to (22) with the required properties on $[x_0, x_0+\epsilon)$. ☐

Remark 3.1. From the HJB (5) equation we get that at points of differentiablity we have that either $V'(x) = 1$ or $\mathcal{L}_V(x) = 0$ holds. Lemma 3.1 reveals that differentiability can only be violated at some switching points. Each equation part of (5) has a differentiable solution.

3.2 Crucial sets and the optimal strategy

This subsection deals with the construction of a candidate strategy L^* for the optimal one. Although it is not possible to directly show that V^{L^*} is a supersolution of (5) and verify its optimality with Proposition 2.6, it is possible to prove that $V^{L^*} = V$ via a fixed point argument, proving the optimality of the strategy L^*. Actually a full characterization of the value function is needed to obtain the correct solution with the construction of L^* (another solution of (5) with an arbitrary initial value for the definition of L^* would not lead to the solution of the maximization problem).
The following three sets will play a crucial role in the definition of the optimal strategy.

- $\mathcal{A} = \{x \in [0,\infty) \mid \Lambda(x) = 0\}$,
- $\mathcal{B} = \{x \in (0,\infty) \mid V'(x) = 1 \text{ and } \Lambda(x) < 0\}$,
- $\mathcal{C} = (\mathcal{A} \cup \mathcal{B})^c$.

Let us identify some properties of these sets.

Proposition 3.2. *1. $\mathcal{B}$ is a left-open set, i.e. for each $x \in \mathcal{B}\ \exists\ \delta > 0$ such that $(x-\delta, x] \subset \mathcal{B}$.*

2. $\mathcal{A}$ is a closed set.

3. If $(x_0, \overline{x}] \subset \mathcal{B}$ and $x_0 \notin \mathcal{B}$ then $x_0 \in \mathcal{A}$.

4. $\exists\ \hat{x}$ such that $(\hat{x}, \infty) \subset \mathcal{B}$.

5. $\mathcal{C}$ is a right-open set, i.e. for each $x \in \mathcal{C}\ \exists\ \delta > 0$ such that $[x, x+\delta) \subset \mathcal{C}$.

6. $\mathcal{A}, \mathcal{B} \neq \emptyset$.

Proof. 1. The idea is as follows: if for sufficiently small $h > 0$ we are able to show that U_{x-h} is a supersolution in $(x-h, x]$, then we get from Proposition 2.12 (ii) that $U_{x-h} = V$ in $[0, x]$, and hence $V' = 1$ in $(x-h, x]$ implying $(x-h, x] \subset \mathcal{B}$.
Let $y \in (x-h, x)$ and recall from the definition of $\mathcal{B}$ that $\mathcal{L}_V(x) < 0$,

$$\begin{aligned}\mathcal{L}_{U_{x-h}}(y) &= (c+iy) - (\delta+\lambda)(y-x+h+V(x-h)) + \lambda \int_0^y U_{x-h}(y-z)\, dF_Y(z) \\ &\le (c+ix)V'(x) - (\delta+\lambda)V(x) + (\delta+\lambda)(V(x) - (y-x+h+V(x-h)) + \lambda \int_0^y U_{x-h}(y-z)\, dF_Y(z) \\ &\le (c+ix)V'(x) - (\delta+\lambda)V(x) + (\delta+\lambda)(V(x) - V(x-h)) + \lambda \int_0^y U_{x-h}(y-z)\, dF_Y(z) \\ &\le \mathcal{L}_V(x) + (\delta+\lambda)(V(x) - V(x-h)) < 0.\end{aligned}$$

The last step holds for h small enough because of the continuity of V, $y < x$ and the following estimates,

$$\begin{aligned} V(y-z) &\ge y-z-x+h+V(x-h), \quad \text{for} \quad y-z \ge x-h, \\ \int_0^y U_{x-h}(y-z)\, dF_Y(z) &= \int_0^{y-x+h} (y-z-x+h+V(x-h))dF_Y(z) + \int_{y-x+h}^y V(y-z)\, dF_Y(z) \\ &\le \int_0^x V(x-z)\, dF_Y(z).\end{aligned}$$

We proved that U_{x-h} is indeed a supersolution in $(x-h, x]$ and therefore the statement holds.

2. Because Λ is continuous in x and $\Lambda(x) \le 0$ for all $x \in [0, \infty)$, the region where it equals 0 is closed. Assume that there is some x_0 such that $\Lambda(x_0) > 0$ then because of the continuity there is a $x_1 > x_0$ such that $\Lambda > 0$ in $[x_0, x_1)$. Let $y \in (x_0, x_1)$ such that $V'(y)$ exists. Because $V' \ge 1$ we get

$$\begin{aligned}\mathcal{L}_V(y) &= (c+iy)V'(y) - (\delta+\lambda)V(y) + \lambda \int_0^y V(y-z)\, dF_Y(z) \\ &\ge (c+iy) - (\delta+\lambda)V(y) + \lambda \int_0^y V(y-z)\, dF_Y(z) = \Lambda(y) > 0,\end{aligned}$$

which is a contradiction to the fact that V is a viscosity supersolution to the HJB equation (5).

3. First we deal with the case $x_0 = 0$. We know that $V(0) \ge \frac{c}{\delta+\lambda}$. This will also be an upper bound, implying that $\Lambda(0) = 0$ ($x_0 \in \mathcal{A}$). Because $(0, \overline{x}] \subset \mathcal{B}$ we have that $V(x) = x + V(0)$ in $[0, \overline{x}]$. For $x \in (0, \overline{x})$ we have from Lemma 2.9 that $V(0) = \sup_{L \in \Pi_x} V_L(0)$.
Let $L \in \Pi_x$, the time of the first claim occurrence be τ_1 and its size Y_1. For all $t < \tau_1$ we have $L_t \le \int_0^t (c+ix)ds$, $L_t \le ct + i\int_0^t R_s^L\, ds \le (c+ix)t$ due to $L \in \Pi_x$ and the definition of an admissible strategy. We get the obvious upper bound

$$\begin{aligned} V_L(0) &= \mathbb{E}\left(\int_0^{T_1} e^{-\delta s}\, dL_s + e^{-\delta T_1} V\left(\int_0^{T_1} (cT_1 + i\int_0^{T_1} R_s^L ds - L_{T_1} - Y_1\right)\right) \\ &\le \mathbb{E}\left(\int_0^{T_1} e^{-\delta s}(c+ix)ds\right) + \mathbb{E}\left(e^{-\delta T_1} V(x-Y_1)\right) \\ &= \int_0^\infty \lambda e^{-\lambda t} \int_0^t e^{-\delta s}(c+ix)ds\, dt + \int_0^\infty \lambda e^{-(\delta+\lambda)t} \int_0^x V(x-y)\, dF_Y(y)\, dt \\ &= \int_0^\infty e^{-(\delta+\lambda)t} \left((c+ix) + \lambda \int_0^x V(x-y)\, dF_Y(y)\right)\, dt.\end{aligned}$$

Using $V(x) = x + V(0)$ in the specific area, we arrive at

$$V(0) \le \liminf_{x \to 0} \frac{1}{\delta+\lambda} \left((c+ix) + \lambda V(0)F_Y(x) + \lambda x F_Y(x) - \lambda \int_0^x y\, F_Y(y)\right) = \frac{c}{\delta+\lambda},$$

which proves the statement for $x_0 = 0$.
Now we deal with the case $x_0 > 0$ following [3]. If $V'(x_0) = 1$ and $x_0 \notin \mathcal{B}$ we get that $\Lambda(x_0) = 0$ and therefore by definition $x_0 \in \mathcal{A}$. We have $\lim_{x\downarrow x_0} \frac{V(x)-V(x_0)}{x-x_0} = 1$. Suppose $\liminf_{x\uparrow x_0} \frac{V(x)-V(x_0)}{x-x_0} = q > 1$. Then we have from Remark 2.2 for all $1 < p \leq q$

$$\max\{1-p, (c+ix_0)p - (\delta+\lambda)V(x_0) + \lambda \int_0^{x_0} V(x_0-y)\, dF_Y(y)\} \geq 0,$$

which implies

$$(c+ix_0)p - (\delta+\lambda)V(x_0) + \lambda \int_0^{x_0} V(x_0-y)\, dF_Y(y) \geq 0.$$

The limit $p \to 1$ gives $\Lambda(x_0) \geq 0$ which implies $\Lambda(x_0) = 0$.
Now we assume $\liminf_{x\uparrow x_0} \frac{V(x)-V(x_0)}{x-x_0} = 1$. There is a sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n \to x_0$ such that $\lim_{n\to\infty} V'(x_n) = 1$. Choose a sequence $\{h_n\}_{n\in\mathbb{N}}$ with $h_n \downarrow 0$ such that $\lim_{n\to\infty} \frac{V(x_0)-V(x_0-h_n)}{h_n} = 1$. Take $a_n = \frac{V(x_0)-V(x_0-h_n)}{h_n} - 1$ and let A_n denote the set of all $x \in [0,h_n]$ such that V' exists and $V'(x) \geq 1+2a_n$. Because of the inequalities for the first derivative, see Proposition 2.2, we can assume $a_n \geq 0$. If for some n we would have $a_n = 0$ we get $V(x_0)-V(x) = x_0-x$ for $x \in [x_0-h_n, x_0]$ and therefore $V'(x_0) = 1$. Therefore assume $a_n > 0$, and we can write by the absolute continuity, $|A_n| \leq h_n$ and $A_n^c = [0,h_n]\backslash A_n$,

$$a_n + 1 = \frac{\int_{A_n} V'(z)\, dz + \int_{A_n^c} V'(z)\, dz}{h_n} \geq \frac{|A_n|(1+2a_n) + (h_n - |A_n|)}{h_n}.$$

This gives the estimates $|A_n| \leq \frac{h_n}{2} \to 0$. So we can choose a sequence $x_n \nearrow x_0$ with $1 \leq V'(x_n) \leq 1+2a_n$ such that $V'(x_n)$ exists. In the end we get $\lim_{n\to\infty} V'(x_n) = 1$.
If there is a subsequence $x_{n_j} \to x_0$ with $V'(x_{n_j}) > 1$ implying $\Lambda(x_{n_j}) = 0$ we would have $\Lambda(x_0) = 0$ because $\mathcal{A}$ is a closed set. Suppose $V'(x_n) = 1$ for all $n \in \mathbb{N}$ and $\Lambda(x_0) < 0$. Then we can find an x_n close enough to x_0 (Λ is continuous) such that U_{x_n} is a supersolution for $x \in [x_n, x_0]$ but Proposition 2.12 yields that $U_{x_n} = V$ in $[0,\overline{x}]$. This gives a contradiction because V would be differentiable at x_0,

$$\begin{aligned}
\mathcal{L}_{U_{x_n}}(x) &= (c+ix) - (\delta+\lambda)U_{x_n}(x) + \lambda \int_0^x U_{x_n}(x-y)\, dF_Y(y) \\
&\leq (c+ix_0) - (\delta+\lambda)U_{x_n}(x) + \lambda \int_0^{x_0} (V(x_0-y)\, dF_Y(y) \\
&= \Lambda(x_0) + (\delta+\lambda)(V(x_0) - (x - x_n + V(x_n))) \\
&\leq \Lambda(x_0) + (\delta+\lambda)(V(x_0) - V(x_n)) < 0.
\end{aligned}$$

The last inequality holds due to the continuity of V for n large enough. This proves the third point.

4. We want to show that for $y > 0$ large enough U_y is a supersolution for all $x \in (y,\infty)$. We already have $U_y' = 1$ in this interval, it is left to show that $\mathcal{L}_{U_y}(x) < 0$. We have

$$\begin{aligned}
\mathcal{L}_{U_y}(x) &= (c+ix) - (\delta+\lambda)(x-y+V(y)) + \lambda \int_0^x U_y(x-z)\, dF_Y(z) \\
&\leq (c+ix) - (\delta+\lambda)(x-y+V(y)) + \lambda U_y(x) \\
&= (c+ix) - (\delta+\lambda)(x-y+V(y)) + \lambda(x-y+V(y)) \\
&= c + (i-\delta)x + \delta(y - V(y)) \leq c + (i-\delta)x - \frac{c}{\delta+\lambda} < 0.
\end{aligned}$$

This holds for every $x \in (y,\infty)$ if y is large enough, because U_y is an increasing function and

$$\int_0^x U_y(x-z)\, dF_Y(z) \leq U_y(x),$$
$$y + \frac{c}{\delta+\lambda} \leq V(y).$$

5. For some $x \in \mathcal{C}$ we have $\Lambda(x) < 0$. Because of continuity we get the existence of a $\delta > 0$ such that $[x, x+\delta) \subset \mathcal{A}^c$. If there would be some $x_1 \in \mathcal{B}$ within this interval we would derive the existence of an $x_0 \in \mathcal{A}$ smaller than x_1 such that $(x_0, x_1] \subset \mathcal{B}$, but because $x \notin \mathcal{B}$ this x_0 also has to be in the interval $(x, x+\delta)$. Therefore we have $[x, x+\delta) \subset \mathcal{B}^c$ and $[x, x+\delta) \subset \mathcal{C}$.

6. The statement follows from the third and fourth point.

□

At this stage we are able to define the optimal strategy.

Definition 3.1. *The optimal strategy L^* is stationary, i.e. it depends only on $x = R_{t-}^{L^*} \geq 0$, and is given as follows:*

- *If $x \in \mathcal{A}$, everything exceeding x is paid out immediately as dividend (with rate $c + ix$).*
- *For $x \in \mathcal{B}$, we know from Proposition 3.2 that there is a $x_1 \in \mathcal{A}$ such that $(x_1, x] \subset \mathcal{B}$, and dividends are paid with the amount $x - x_1$.*
- *For $x \in \mathcal{C}$ no dividends are paid.*

From [3] one knows that the strategy as defined above is admissible.
The following proposition shows that this *band* strategy is indeed optimal.

Proposition 3.3. *The strategy L^* defined in Definition 3.1 is optimal, i.e. $V(x) = V_{L^*}(x)$ for all $x \geq 0$.*

Proof. From Proposition 3.2 we know that there exists some $\hat{x} = \inf\{x \mid (x, \infty) \subset \mathcal{B}\}$. We want to define a contraction map on the set of all functions $f : \mathbb{R} \to [0, \infty)$ with $f(x) = 0$ for $x < 0$ and $f(x) = x - \hat{x} + f(\hat{x})$ for $x > \hat{x}$ which are continuous on $\mathbb{R}^+$. The used distance measure is $d(f_1, f_2) = \max_{x \geq 0} |f_1(x) - f_2(x)|$. The operator T is defined as follows,

$$T_f(x) = \mathbb{E}\left(\int_0^{\tau_1} e^{-\delta s} dL_s^* + e^{-\delta \tau_1} f\left(e^{i\tau_1}(x + \int_0^{\tau_1} (c - l_s^*) e^{-is}\, ds) - \sum_{s<t} \Delta L_s^* - Y_1\right)\right),$$

where τ_1 denotes the time of the first claim occurrence and Y_1 its size.
Notice the similarity to the dynamic programming principle (4) with $R_t^{L^*} = e^{it}(x + \int_0^t (c - l^* s)\, ds) - \sum_{s<t} \Delta L_s^*$ where l^* denotes the density of the absolutely continuous part of L^*. From Definition 3.1 we have that $l^* = 0$ for $x \in \mathcal{B} \cup \mathcal{C}$ and $l^* = c + ix$ for $x \in \mathcal{A}$.
One gets

$$\begin{aligned} |T_{f_1} - T_{f_2}| &= \mathbb{E}\left(e^{-\delta\tau_1}\left(f_1\left(e^{i\tau_1}(x + \int_0^{\tau_1}(c - l_s^*)e^{-is}\, ds) - \sum_{s<t}\Delta L_s^* - Y_1\right)\right.\right. \\ &\quad \left.\left. -f_2\left(e^{i\tau_1}(x + \int_0^{\tau_1}(c - l_s^*)e^{-is}\, ds) - \sum_{s<t}\Delta L_s^* - Y_1\right)\right)\right) \\ &\leq \frac{\lambda}{\delta + \lambda} \max_{x \geq 0} |f_1(x) - f_2(x)|, \end{aligned}$$

therefore T is a contraction and has a unique fixed point. The definition of L^* ensures that T_f is in the same space as f. Clearly V_{L^*} is a fixed point because of the dynamic programming principle and the definition of L^*. Now we are going to show that V is also a fixed point which gives $V = V_{L*}$.
We start with $x \in \mathcal{A}$, then

$$\begin{aligned} T_V(x) &= \mathbb{E}\left(\int_0^{\tau_1} (c + ix) e^{-\delta s} ds + e^{-\delta T_1} V(x - Y_1)\right) \\ &= \frac{1}{\delta + \lambda}\left((c + ix) + \lambda \int_0^x V(x - y)\, dF_Y(y)\right) = V(x), \end{aligned}$$

because $\Lambda(x) = 0$ for $x \in \mathcal{A}$.
Next, we look at $x \in \mathcal{B}$. Let x_1 such that $(x_1, x] \subset \mathcal{B}$ and $x_1 \in \mathcal{A}$. We get from the definitions of L^* and $\mathcal{B}$,

$$T_V(x) = x - x_1 + T_V(x_1) = x - x_1 + V(x_1) = V(x).$$

Finally, we know that $\mathcal{C}$ is a right-open set. Therefore some x_1 exists such that $[x, x_1) \subset \mathcal{C}$ and $x_1 \notin \mathcal{C}$. Denote

$$x_t = e^{it}\left(x + c\int_0^t e^{-is} ds\right),$$

and let t_1 such that $x_{t_1} = x_1$.
Because V is a differentiable solution to $(c+iz)V'(z) - (\delta+\lambda)V(z) + \lambda\int_0^z V(z-y)\ dF_Y(y) = 0$ for $z \in (x, x_1)$, $\frac{d}{dt}x_t = c + ix_t$ and

$$\frac{d}{dt}e^{-(\delta+\lambda)t}V(x_t) = -(\delta+\lambda)e^{-(\delta+\lambda)t}V(x_t) + e^{-(\delta+\lambda)t}(c+ix_t)V'(x_t).$$

So we get

$$\begin{aligned} T(V)(x) &= \mathbb{E}\left(I_{\{\tau_1 \geq t_1\}}\right) e^{-\delta t_1}V(x_1) + \mathbb{E}\left(I_{\{\tau_1 < t_1\}}\ e^{-\delta\tau_1}V(x_{\tau_1} - Y_1)\right) \\ &= e^{-(\delta+\lambda)t_1}V(x_1) + \int_0^{t_1} e^{-(\delta+\lambda)t}\ \lambda \int_0^{x_t} V(x_t - y)\ dF_Y(y)\ dt \\ &= e^{-(\delta+\lambda)t_1}V(x_1) + \int_0^{t_1} e^{-(\delta+\lambda)t}\left((\delta+\lambda)V(x_t) - (c+ix_t)V'(x_t)\right) dt \\ &= e^{-(\delta+\lambda)t_1}V(x_1) + V(x) - e^{-(\delta+\lambda)t_1}V(x_1) = V(x). \end{aligned}$$

□

From Remark 2.4 we know that V' can not have any downward jumps and further that (22) has a differentiable solution. Therefore the only possibility of not being differentiable is at points where the optimal strategy changes from paying a lump sum to paying no dividends.
The similarity to the optimal strategy for the case $i = 0$ as it is dealt with in [3] and [13] allows us to use an algorithm from [13] to determine the value function piecewise. As mentioned in Section 3.1 and because of the construction of the band strategy there is a close relation to barrier strategies. For small initial capital the first thing to do is to find a local *optimal* barrier, i.e find the smallest point in the set $\mathcal{A}$ denoted by x_0. Notice that it is possible that $0 \in \mathcal{A}$. Let f_0 be the solution of (20) and choose the smallest point in $\mathcal{A}$ as $x_0 = \sup\{x \geq 0 \mid f_0'(x) = \inf_{y\geq 0} f_0'(y)\}$. Then define

$$v_0(x) = \begin{cases} f_0(x)/f_0'(x_0) & x \leq x_0, \\ x - x_0 + f_0(x_0)/f_0'(x_0) & x > x_0. \end{cases}$$

If v_0 fulfills the HJB equation (5) we are done, if not the solution is constructed recursively: In the nth step ($n \geq 1$), find some interval belonging to $\mathcal{B}$ of the form (x_n, a) (cf. Proposition 3.2). Then it is possible that some adjoining interval $[a, x_{n+1})$ belongs to the set $\mathcal{C}$; then it is necessary to calculate a solution to (22). The points a and x_{n+1} are determined in the following way. For given $v_n(x)$ and x_n, let $f_{n+1}(x; y)$ be a solution of (22) for $x \geq y$ and equal to $v_n(x)$ for $x < y$. We have to find the smallest $y > x_n$ such that $f_{n+1}'(\bar{x}; y) = 1$ for some $\bar{x} > y$,

$$a = \inf\{y \geq x_n \mid \inf_{z > y} f_{n+1}'(z, y) = 1\}.$$

If a is chosen too small or too large then the derivative of $f_{n+1}'(x; \cdot)$ will either take a minimum greater than 1 or smaller than 1. Due to Proposition 2.2 and the fact that V' can not have downward jumps a wrong choice would not lead to a solution of the maximization problem.
Then we obtain $x_{n+1} := \sup\{x \geq a \mid f'(x, a) = 1\}$ and

$$v_{n+1}(x) := \begin{cases} f_{n+1}(x, a), & x \leq x_{n+1}, \\ x - x_{n+1} + f_{n+1}(x_{n+1}, a), & x > x_{n+1}. \end{cases}$$

If $v_{n+1}(x)$ fulfills (5) we have constructed the value function, otherwise we restart the procedure.

4 Examples

4.1 $Exp(\alpha)$ distributed claim amounts

In the first example consider exponential claim amounts with $F_Y = 1 - e^{-\alpha y}$. We will see that in the case $0 < i < \delta$ a barrier strategy is optimal, an analogous result for $i = 0$ was first shown in [7]. To find an element of $\mathcal{A}$ we need to solve $\Lambda(x) = 0$, because of the properties of the set $\mathcal{B}$ some of these elements are lower boundaries of subsets of $\mathcal{B}$. Looking for a solution to $\Lambda(x) = 0$ we observe that we have to solve

$$V(x) = \frac{c + ix}{\delta + \lambda} + \frac{\lambda}{\delta + \lambda} e^{-\alpha x} \int_0^x V(y) \alpha e^{\alpha y}\, dy. \tag{23}$$

If a point $a \in \mathcal{A}$ is a boundary point of a connection component of $\mathcal{B}$ we have $V'(a+) = 1$. From $V' \geq 1$ and the fact that V' can not have downward jumps (see Remark 2.4) we get $V'(a) = 1$. Therefore, by $\mathcal{B} \neq \emptyset$ we can additionally use the condition $V'(x) = 1$ for at least one element of $\mathcal{A}$. From (23) and $V' = 1$ we get,

$$1 = V'(x) = \frac{i}{\delta + \lambda} - \frac{\alpha^2 \lambda}{\delta + \lambda}\, e^{-\alpha x} \int_0^x V(y) e^{\alpha y}\, dy + \frac{\alpha \lambda}{\delta + \lambda} V(x).$$

Using (23) again to eliminate the integral we derive,

$$\frac{\delta + \lambda}{\alpha} - \frac{i}{\alpha} = c + ix - \delta V(x).$$

Since $i < \delta$ and $V(x) \geq x + \frac{c}{\delta + \lambda}$ (Proposition 2.1) we further have that $ix - \delta V(x)$ is decreasing. There exists at most one positive point on the real axis which fulfills these conditions. This is equivalent to the statement that a barrier strategy b^* is the optimal one in the case of $Exp(\alpha)$ distributed claim amounts. In the following we identify the case $b^* > 0$. The case of an optimal barrier equal to zero is then treated in Section 4.1.2.

4.1.1 The case $b^* > 0$

For the determination of the optimal barrier we can use some results from [10]. As a by-product we can show why only the case $i < \delta$ makes sense mathematically. The structure of a constant barrier strategy is as follows. Given a barrier at level b, all surplus above this level will be immediately paid out as dividend. We denote the expected discounted dividends for a barrier b with $V_b(x)$. Assuming differentiability of $V_b(x)$ we get the following well-known IDE (see [10]), for $x < b$

$$0 = (c + ix)V'(x) + \lambda \int_0^x V(x - y) dF_Y(y) - (\lambda + \delta)V(x), \tag{24}$$

$$1 = V_b'(b). \tag{25}$$

From the nature of a barrier strategy we have for $x > b$

$$V_b(x) = x - b + V_b(b).$$

Because (24) is homogenous and linear in V we can look for a solution f of it with a modified initial condition $f(0) = 1$. By scaling we get that $V_b = f(x)/f'(b)$ for $0 \leq x \leq b$. Following [10] we have to solve

$$\begin{aligned} 0 &= (c + ix)f''(x) + (\alpha(c + ix) + i - (\delta + \lambda))f'(x) - \alpha\delta f(x) = 0, \\ 1 &= f(0), \\ 0 &= cf'(0) - (\delta + \lambda)f(0). \end{aligned}$$

The general solution is of the form

$$f(x) = e^{-\alpha x} \left(x + \frac{c}{i}\right)^{(\lambda+\delta)/i} \left(B_1\, F\left(1 + \frac{\delta}{i}, 1 + \frac{\lambda + \delta}{i}, \alpha(u + \frac{c}{i})\right) + B_2\, U\left(1 + \frac{\delta}{i}, 1 + \frac{\lambda + \delta}{i}, \alpha(u + \frac{c}{i})\right)\right),$$

where B_1 and B_2 are constants determined by the boundary conditions and F and U are confluent hypergeometric functions of the first and second kind, respectively. Because maximizing V_b is equivalent to minimizing f' we take a look on the asymptotics of f and f'. From [1] we have

$$\begin{aligned} F(a,b,z) &\sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b} \left(1 + O(|z|^{-1}\right), \\ U(a,b,z) &\sim z^{-a}\left(1+O(|z|^{-1}\right). \end{aligned}$$

So we get

$$f(x) \sim \frac{\Gamma(1+\frac{\delta+\lambda}{i})}{\Gamma(1+\frac{\delta}{i})} B_1\, e^{\alpha\frac{c}{i}}(x+\frac{c}{i})^{\frac{\delta}{i}}(1+O((\alpha(x+\frac{c}{i}))^{-1})).$$

We can use the same asymptotics to obtain the behaviour of $f'(x)$ for large x and it suffices to consider the terms in connection to $F(a,b,z)$. Therefore we get

$$f'(x) \sim B_1 \frac{\delta+\lambda}{i} \frac{\Gamma(1+\frac{\delta+\lambda}{i})}{\Gamma(1+\frac{\delta}{i})} e^{\alpha\frac{c}{i}} \alpha^{-\frac{\lambda}{i}} (x+\frac{c}{i})^{\frac{\delta}{i}-1}\ K + O((\alpha(x+\frac{c}{i}))^{\frac{\delta}{i}-2}),$$

with some constant K. Furthermore

$$\lim_{x\to\infty} f'(x) = \begin{cases} 0, & \delta < i, \\ \infty, & \delta > i, \\ const, & \delta = i. \end{cases}$$

and as a consequence for a fixed argument x

$$\lim_{b\to\infty} V_b(x) = \lim_{b\to\infty} \frac{f(x)}{f'(b)} = \begin{cases} \infty, & \delta < i, \\ 0 & \delta > i, \\ \frac{f(x)}{const}, & \delta = i. \end{cases}$$

Since $V_b(x) \le V(x)$ the value function is unbounded for $i > \delta$ and does not fulfill $\lim_{b\to\infty} V_b(x) = 0$ for $\delta = i$. Therefore only the case $\delta > i$ is interesting and leads to a well-formulated dividend maximization problem. If $b^* > 0$, then calculate f and determine $b^* = \text{argmax}\{f'(b) \mid b > 0\}$ numerically. Then

$$V(x) = V_{b^*}(x) = \begin{cases} f(x)/f'(b^*) & 0 \le x \le b^*, \\ x - b^* + f(b^*)/f'(b^*) & x > b^*. \end{cases}$$

As an illustration Figure 1 shows the value function when the optimal barrier strategy with height $b^* = 4.41$ is applied together with the two linear bounds from Proposition 2.1 (which are obviously not tight). The chosen parameters are $\alpha = 2,\ \lambda = 2,\ i = 0.05,\ \delta = 0.1$ and $c = 2.5$.

4.1.2 The case $b^* = 0$

We need to determine parameter settings for which $b^* = 0$ is optimal. For $b^* = 0$, $V(x) = V_0(x) = x + \frac{c}{\delta+\lambda}$. Because in this case $V_0' = 1$ for $x \ge 0$, we only have to check when

$$(c+ix) - (\delta+\lambda)V_0(x) + \lambda \int_0^x V_0(x-y)\alpha e^{-\alpha y}\, dy \le 0$$

holds. Evaluating this equation, it turns out that for

$$Z(x) := \frac{(\delta+\lambda-\alpha c)\lambda e^{-\alpha x} + x(i-\delta)\alpha(\delta+\lambda) - \lambda(\delta+\lambda-\alpha c)}{\alpha(\delta+\lambda)}, \tag{26}$$

we have to check when $Z(x) \le 0$ for all $x \ge 0$. Further

$$Z'(x) = (i-\delta) - \frac{\lambda}{\delta+\lambda} e^{-\alpha x}\,(\delta+\lambda-\alpha c).$$

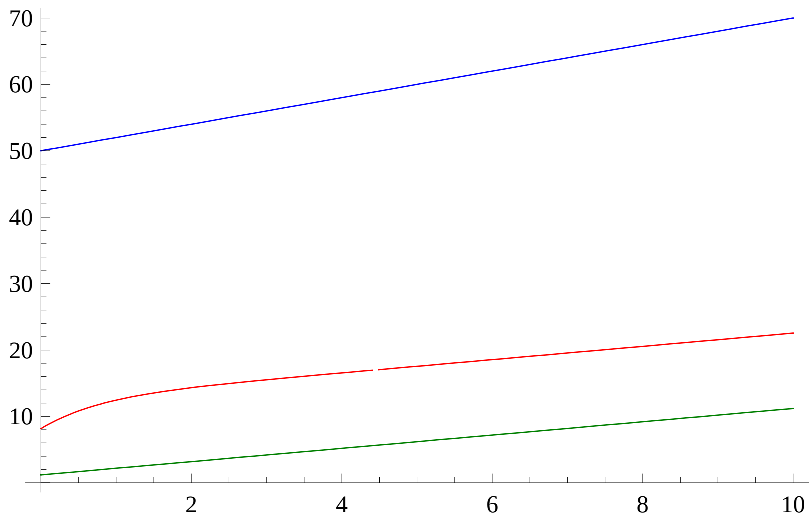

Figure 1: Value function for $\alpha = 2,\ \lambda = 2,\ i = 0.05,\ \delta = 0.1$ and $c = 2.5$.

If $\delta + \lambda \geq \alpha c$ we have $Z' \leq 0$ for all $x \geq 0$ and $Z(0) = 0$ is a maximum of Z. Therefore V_0 fulfills the HJB equation and $V_0 = V$.
If $\delta + \lambda < \alpha c$ we have that Z is concave,

$$Z''(x) = \frac{\alpha c(\delta + \lambda - \alpha c)}{\delta + \lambda} e^{-\alpha x}.$$

Therefore we get

$$Z(x) \leq 0 \quad \text{for } x \in [0, \infty) \iff Z'(0) \leq 0 \iff \alpha\lambda c + i(\delta + \lambda) \leq (\delta + \lambda)^2.$$

If on the other hand $\alpha\lambda c + i(\delta + \lambda) > (\delta + \lambda)^2$, we get that Z has a global maximum at

$$\hat{x} = \frac{1}{\alpha} \log\left(1 + \frac{\alpha\lambda c + i(\delta + \lambda) - (\delta + \lambda)^2}{(\delta - i)(\delta + \lambda)}\right) > 0.$$

We want to show that

$$Z(\hat{x}) = \frac{\alpha\lambda c + i(\delta + \lambda) - (\delta + \lambda)^2 - (\delta - i)(\delta + \lambda) \log\left(1 + \frac{\alpha\lambda c + i(\delta+\lambda) - (\delta+\lambda)^2}{(\delta - i)(\delta+\lambda)}\right)}{\alpha(\delta + \lambda)} > 0.$$

An easy discussion of the function $h(x) = x - \log(1 + x)$ for $x > 0$ gives the required result. Therefore $b^* = 0$ if and only if

- $\delta + \lambda \geq \alpha c$ or
- $\delta + \lambda < \alpha c$ and $Z'(0) \leq 0$.

If none of these cases holds, calculate $V = V_{b^*}$ as described in Section 4.1.1.

4.2 $Gamma(2, \gamma)$ distributed claim amounts

In this section we will identify an explicit example where a band strategy is optimal. In contrast to the case $i = 0$ (of [3] and [13]) an explicit solution to

$$(c + ix)f'(x) - (\delta + \lambda)f(x) + \lambda \int_0^x f(x - y)\, d_\gamma(y) dy = 0$$

is not available, where $d_\gamma(y) = y\gamma^2\, e^{\gamma y}$ denotes the $Gamma(2, \gamma)$ density function. Therefore we need numerical solutions to (20) and (22) for applying the algorithm presented in [13]. A natural approach is to use the contraction argument from Lemma 3.1 for determining a numerical solution but that turns out to be too time consuming and inaccurate. So here we implement another approach to obtain a reasonably accurate solution of (22).

Assume that the value function is determined up to a point x_n. Following the algorithm from [13] (see Section 3.2) we have to calculate $f_{n+1}(x;y)$ as a solution to (22) with x_0 replaced by y. In terms of the algorithm the initial condition is given by $v_n(y) = f_{n+1}(y;y)$. First we fix a step width $h > 0$ and choose a set of points $\{x_y\}_{0\leq k\leq K}$ with $y_k = y + kh$. Then we define piecewise linear functions $\{\omega_k(x)\}_{0\leq k\leq K}$ such that $\omega_k(y_{k-1}) = 0$, $\omega_k(y_k) = 1$, $\omega_k(y_{k+1}) = 0$ and $\omega_y(x) = 0$ for $x \notin [y_{k-1}, y_k + 1]$. Let the sequence $\{u_k\}_{0\leq k\leq K}$ denote the unknown values of a solution to (22) at the points y_k. The numerical solution we are looking for is of the form

$$u(x) = \sum_{k=0}^{K} u_k \omega_k(x).$$

Plugging $u(x)$ into (22) and evaluating this expression at every y_k leads to a linear system of equations for the unknowns u_k.

Finally we give a concrete example for a situation where a *band* strategy is optimal. Choose the parameters by $\lambda = 10,\ \delta = 0.1,\ \gamma = 1,\ c = 21.4$ (cf. [3]) but now with a positive interest rate $i = 0.02$. First observe that if we look at a solution to (20), the derivative is minimized in zero. On the other hand $x + \frac{c}{\delta+\lambda}$ does not fulfill (5) on $\mathbb{R}^+$. Therefore we have to choose $x_0 = 0$ and apply the numerical method presented above. We get that the sets $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given by

$$\begin{aligned} \mathcal{A} &= \{0, 12.96\}, \\ \mathcal{B} &= (0, 0.96) \cup (12.96, \infty) \\ \mathcal{C} &= [0.96, 12.96). \end{aligned}$$

A sample path of the reserve process controlled by the optimal strategy L^* is illustrated in Figure 2. Starting with initial capital $x \in \mathcal{B}$ the amount $x - x_1$, $x_1 = 12.96$, is immediately paid out as dividend (this lump sum payment is indicated as the left bold downward arrow). Then up to the first claim occurrence which puts the process into region $\mathcal{C}$, dividends are paid continuously at a rate $c + ix_1$. In the set $\mathcal{C}$ there are no control actions on the reserve process, so that (in the absence of further claims) it increases again to x_1 and stays there (with again dividends paid with intensity $c + ix_1$) until the second claim happens. As the second claim puts the reserve process into the set $\mathcal{B}$, the reserve is immediately further reduced by a dividend payment to the next point in the set $\mathcal{A}$ which is $x_0 = 0$. The process stays at this level, i.e. dividends are paid with intensity c, until ruin is caused by the third claim of the risk process. Figure 3 shows the value function for $i = 0.02$ in comparison to the value function with $i = 0$ (dashed line, as calculated in [3]). It can be observed that for low initial capital both follow the same strategy, but from 0.96 onwards, the case with $i > 0$ dominates the one with $i = 0$. Further we obtain that the value function is not differentiable at $x = 0.96$, $V'(0.96+) \approx 1.16 > 1 = V'(0.96-)$, where the derivative from the right is a numerical approximation calculated from the scheme described above.

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References

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] H. Albrecher, J. L. Teugels, and R. F. Tichy. On a gamma series expansion for the time-dependent probability of collective ruin. *Insurance Math. Econom.*, 29(3):345–355, 2001.

[3] P. Azcue and N. Muler. Optimal reinsurance and dividend distribution policies in the Cramér-Lundberg model. *Math. Finance*, 15(2):261–308, 2005.

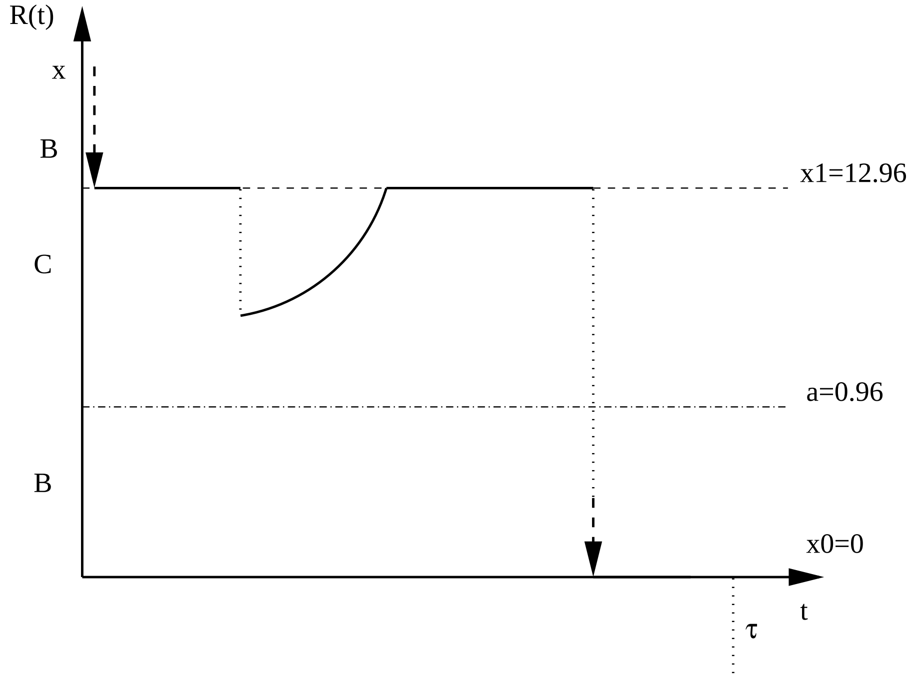

Figure 2: Path of R^{L^*}

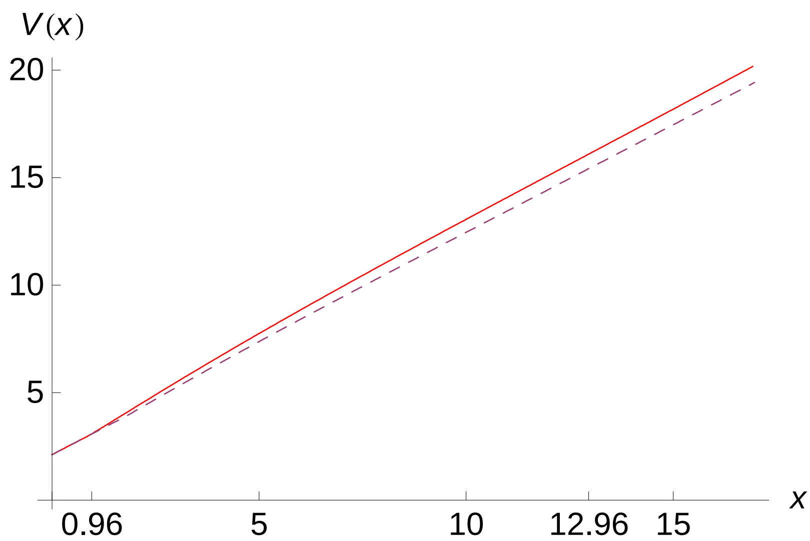

Figure 3: Value function for $\lambda = 10,\ \delta = 0.1,\ \gamma = 1,\ c = 21.4,\ i = 0.02$ and $i = 0$

[4] F. E. Benth, K. H. Karlsen, and K. Reikvam. Optimal portfolio selection with consumption and nonlinear integro-differential equations with gradient constraint: a viscosity solution approach. *Finance Stoch.*, 5(3):275–303, 2001.

[5] R. Cont and P. Tankov. *Financial modelling with jump processes.* Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.

[6] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Applications of Mathematics (New York).* Springer-Verlag, New York, 1993.

[7] H. U. Gerber. Entscheidungskriterien fuer den zusammengesetzten Poisson-prozess. *Schweiz. Aktuarver. Mitt.*, (1):185–227, 1968.

[8] M. Mnif and A. Sulem. Optimal risk control and dividend policies under excess of loss reinsurance. *Stochastics*, 77(5):455–476, 2005.

[9] J. Paulsen. Risk theory in a stochastic economic environment. *Stochastic Process. Appl.*, 46(2):327–361, 1993.

[10] J. Paulsen and H. K. Gjessing. Optimal choice of dividend barriers for a risk process with stochastic return on investments. *Insurance Math. Econom.*, 20(3):215–223, 1997.

[11] J. Paulsen and H. K. Gjessing. Ruin theory with stochastic return on investments. *Adv. in Appl. Probab.*, 29(4):965–985, 1997.

[12] A. Sayah. Equations d'hamiltonian-jacobi du premier ordre avec terms intégro-différentiels. partie i: Unicité des solutions de viscosité. *Comm. Partial Differential Equations*, (16 (6&7)):1057–1074, 1991.

[13] H. Schmidli. *Stochastic Control in Insurance.* Springer, 2008.

[14] S. E. Shreve, J. P. Lehoczky, and D. P. Gaver. Optimal consumption for general diffusions with absorbing and reflecting barriers. *SIAM J. Control Optim.*, 22(1):55–75, 1984.

[15] R. L. Wheeden and A. Zygmund. *Measure and integral.* Marcel Dekker Inc., New York, 1977. An introduction to real analysis, Pure and Applied Mathematics, Vol. 43.