Year : 2022

RUIN ANALYSIS FOR GAUSSIAN RISK MODELS

Kriukov Nikolai

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

RUIN ANALYSIS FOR GAUSSIAN RISK MODELS

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Docteur en Sciences actuarielles

par

Nikolai KRIUKOV

Directeur de thèse
Prof. Enkelejd Hashorva

Jury

Prof. Felicitas Morhart, Présidente
Prof. François Dufresne, expert interne
Prof. Thomas Mikosch, expert interne
Prof. Krzysztof Dębicki, expert externe
Dr. Alfred Kume, expert externe

LAUSANNE
2022

UNIL | Université de Lausanne

FACULTÉ DES HAUTES ÉTUDES COMMERCIALES

DÉPARTEMENT DE SCIENCES ACTUARIELLES

RUIN ANALYSIS FOR GAUSSIAN RISK MODELS

THÈSE DE DOCTORAT

présentée à la

Faculté des Hautes Études Commerciales
de l'Université de Lausanne

pour l'obtention du grade de
Docteur en Sciences actuarielles

par

Nikolai KRIUKOV

Directeur de thèse
Prof. Enkelejd Hashorva

Jury

Prof. Felicitas Morhart, Présidente
Prof. François Dufresne, expert interne
Prof. Thomas Mikosch, expert interne
Prof. Krzysztof Dębicki, expert externe
Dr. Alfred Kume, expert externe

LAUSANNE
2022

UNIL | Université de Lausanne
HEC Lausanne
Le Décanat
Bâtiment Internef
CH-1015 Lausanne

I M P R I M A T U R

Sans se prononcer sur les opinions de l'auteur, la Faculté des Hautes Etudes Commerciales de l'Université de Lausanne autorise l'impression de la thèse de Monsieur Nikolai KRIUKOV, titulaire d'un master en Mathématiques Fondamentales et Mécanique de l'Université d'État de Saint-Pétersbourg, en vue de l'obtention du grade de docteur en Sciences actuarielles.

La thèse est intitulée :

RUIN ANALYSIS FOR GAUSSIAN RISK MODELS

Lausanne, le 06 avril 2022

La Doyenne

Marianne SCHMID MAST

HEC Lausanne
Le Décanat
Tél. +41 21 692 33 40 | Fax +41 21 692 33 05
www.hec.unil.ch | hecdoyen@unil.ch

Members of the thesis committee

Prof. Enkelejd Hashorva

Professor, Doctor, University of Lausanne

Thesis supervisor

Prof. François Dufrense

Professor, Doctor, University of Lausanne

Internal member of the thesis committee

Prof. Krzysztof Dębicki

Professor, Doctor, University of Wroclaw

External member of the thesis committee

Dr. Alfred Kume

Doctor, University of Kent

External member of the thesis committee

Prof. Thomas Mikosch

Professor, Doctor, University of Copenhagen

External member of the thesis committee

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Nikolai KRIUKOV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: ______________________ Date: 17.02.2022

Prof. Enkelejd HASHORVA
Thesis supervisor

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Nikolai KRIUKOV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: ____________ Date: 17.02.2022

Prof. François DUFRESNE
Internal member of the doctoral committee

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Nikolai KRIUKOV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: Krzysztof Dębicki

Date: 17.02.2022

Prof. Krzysztof DĘBICKI
External member of the doctoral committee

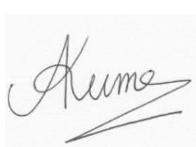
University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Nikolai KRIUKOV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: ____  __________ Date: __17/02/2022________

Dr. Alfred KUME
External member of the doctoral committee

University of Lausanne
Faculty of Business and Economics

PhD in Actuarial Science

I hereby certify that I have examined the doctoral thesis of

Nikolai KRIUKOV

and have found it to meet the requirements for a doctoral thesis.
All revisions that I or committee members
made during the doctoral colloquium
have been addressed to my entire satisfaction.

Signature: ____________________ Date: 21 FEB 2022

Prof. Thomas MIKOSCH
External member of the doctoral committee

ACKNOWLEDGMENTS

I am deeply indebted to my thesis supervisor, professor Enkelejd Hashorva. You significantly helped me to find the interesting directions for my research and aided to structure and improve the results of this thesis. In addition, you also supported me to maintain the motivation to proceed my research. I would be pleased to continue working with you.

I am also wery grateful to Professor Krzysztof Dębicki, who in fact became my second supervisor. I am very proud working with you and incredibly appreciate your support and the impulse you gave me after each our discussion.

Of course, I am also thankful for all my colleagues for having the most exciting time in my life. You supported me from my first appearance in this country and till now days.

Contents

Chapter 1

Introduction

For any real-valued separable stochastic process $X(t),\ t \in \mathbb{R}$ its classical ruin probability can be defined as follows

$$\mathbb{P}\left\{\exists t \in \mathcal{T} : X(t) > u\right\},$$

where $\mathcal{T}$ is some Borel set of $\mathbb{R}$, and u is some high threshold. An interesting and challenging problem in risk theory is to calculate, simulate or approximate the above probability of ruin. Even in the one dimensional classical models, exact formulas for ruin probabilities are typically only possible under very non-realistic assumptions. Therefore, often simulation or approximation as u increases is a reasonable task for dealing with the ruin probability. Both simulation and approximations require advanced techniques.

This dissertation has two main targets: On one side we consider approximations of various ruin probabilities. On the other side we shall also discuss interesting bounds for those probabilities, where the threshold u is fixed. Without loss of generality we may assume that all processes considered in this dissertation are separable and jointly measurable.

The other important problem is an extension of the classical notion of ruin, so-called the Parisian ruin. The core of the notion of the Parisian ruin is that now one allows the surplus process to spend a pre-specified time under the level zero before the ruin is recognized. Formally, the Parisian ruin can be defined as follows

$$\mathbb{P}\left\{\exists t \in \mathcal{T};\ \forall s \in [t, t+S] :\ X(s) > u\right\},$$

where $\mathcal{T}$ is again a Borel set, and S is some pre-specified positive time interval. Usually, S does not depend on the moment of time t, but depends on the threshold level u.

The multidimensional analog of classical ruin probability is also of interest. Such problem considers several processes $X_1(t),\dots,X_d(t)$ with the same domain which ruin simultaneously within some Borel set $\mathcal{T}$

$$\mathbb{P}\left\{\exists t\in\mathcal{T}: X_1(t)>u_1,\dots,X_d(t)>u_d\right\}.$$

The common choice of $\mathcal{T}$ is either an interval $[0,T]$, or a positive real halfline $[0,+\infty)$. This dissertation focused mainly on the first case.

We consider next the classical Brownian motion case. Let $B_1(t),\dots,B_d(t)$ be independent standard Brownian motions, and A is a non-singular $d\times d$ real matrix. Then processes $W_i(t)$ are defined as follows

$$(W_1(t),\dots,W_d(t))^\top := A(B_1(t),\dots,B_d(t))^\top$$

and all u_i have the same growing speed:

$$u_i=a_iu,$$

as u tends to infinity.

In Chapter 2 we obtain exact asymptotics for two-dimensional simultaneous Parisian ruin for two correlated Brownian motions with drifts:

$$\mathbb{P}\left\{\exists t\in[0,A-S/u^2];\ \forall s\in[t,t+S/u^2]:\ W_1(s)-c_1s>u, W_2(s)-c_2s>au\right\}.$$

for any real constants c_1,c_2,a, and any positive constant S as u tends to infinity. The Parisian ruin is focused on the probability of prolonged ruin period of company, accepting that the ruin may occurs for a tiny moment. Results, achived in Chapter 2, continues the study of Parisian ruin asymptotics started in the paper [49], and was further investigated in [13, 14].
In additional, this chapter considers cumulative Parisian ruin probability

$$\mathbb{P}\left\{\int_0^A \mathbb{I}(W_1(t)-c_1t>u, W_2(t)-c_2t>au)\mathrm{d}t>s/u^2\right\}.$$

again for any real constants c_1,c_1,a and any positive constant S. In fact, cumulative Parisian ruin is simply the tail of the sojourn of the underlying process. These type of investigations are of interest in various areas of probability beyond insurance mathematics, see e.g., the monograph by Berman [7].

In Chapter 3 we focused on the multidimensional simultaneous ruin probability, assuming that ruin occurs for at least k of d margins:

$$\mathbb{P}\left\{\exists t\in[S,T],\mathcal{I}\subset\{1,\dots,d\}:|\mathcal{I}|=k,\ \forall i\in\mathcal{I}\ W_i(t)-c_it>a_iu\right\}$$

for any real constants $c_1, \ldots, c_d$, $a_i, \ldots, a_d$. We derive both sharp bounds and asymptotic approximations of the probability of interest for the finite and the infinite time horizon. The results presented in this chapter extend previous findings of [10, 19].

Chapter 4 contains some generalisation of the results related to asymptotics of the simultaneous ruin probability for Brownian motion on more general class of Gaussian processes.

Let $\boldsymbol{Z}(t) = (Z_1(t), \ldots, Z_d(t))^\top, t \in \mathbb{R}$ where $Z_i(t), t \in \mathbb{R}$, $i = 1, ..., d$ are mutually independent centered Gaussian processes with continuous sample paths a.s. and stationary increments. For $\boldsymbol{X}(t) = A\boldsymbol{Z}(t)$, $t \in \mathbb{R}$, where A is as above, $\boldsymbol{u}, \boldsymbol{c} \in \mathbb{R}^d$ and $T > 0$ we derive tight bounds for the simultaneous ruin probability

$$\mathbb{P}\left\{\exists_{t\in[0,T]} : \cap_{i=1}^d \{X_i(t) - c_i t > u_i\}\right\}$$

and find its exact asymptotics as the thresholds tend to infinity.

Finally, in Chapter 5 we discuss another interesting ruin problem. The classical ruin probability can be represented as follows

$$\mathbb{P}\left\{\exists t \in [0,T] : \ W(t) > u\right\} = \mathbb{P}\left\{\exists t \in [0,T] : \ W(t) \in uS\right\},$$

where

$$S = \{x \in \mathbb{R} : x > 1\}.$$

Hence, we can generalise the ruin probability by putting an arbitrary set S. The same ruin probability may be defined in a multidimensional setup

$$\mathbb{P}\left\{\exists t \in [0,T] : (W_1(t), \ldots, W_d(t)) \in u\boldsymbol{S}\right\}.$$

In our multivariate setting, we shall allow S to be a general Borel set. This problem is already considered in the context of Shepp-statistics in [44]. In this Chapter we derive upper bounds for the ruin probability of interest extending in particular some results from [44].

Chapter 2

Parisian & Cumulative Parisian Ruin

1 Introduction

Calculation of Parisian ruin for Brownian risk model has been initially considered in [49]. For general Gaussian risk models Parisian ruin cannot be calculated explicitly. As shown in [13, 14] methods from the theory of extremes of Gaussian random fields can be successfully applied to approximate the Parisian ruin for general Gaussian risk models. In this chapter, we shall focus on the classical bivariate Brownian motion risk model, which in view of recent findings in [27], appears naturally as the limiting model of some general bivariate insurance risk model. Consider therefore two insurance risk portfolios with corresponding risk models

$$R_1(t) = u + c_1 t - W_1(t), \quad R_2(t) = au + c_2 t - W_2(t), \quad t \geq 0,$$

where W_1, W_2 are two standard Brownian motions and the initial capital for the first portfolio is $u > 0$, whereas for the second it is equal au for some real constant a. Further c_1 and c_2 are some constants which denote the premium rates of the first and the second portfolio, respectively. In this chapter we shall consider the benchmark model where $(W_1(t), W_2(t)), t \geq 0$ are assumed to be jointly Gaussian with the same law as

$$(B_1(t), \rho B_1(t) + \sqrt{1-\rho^2} B_2(t)), \quad t \geq 0, \quad \rho \in (-1,1), \tag{1.1}$$

where B_1, B_2 are two independent standard Brownian motions. As mentioned above, this model is supported by the findings of [27].

[1]This chapter is based on the paper [47].

Throughout the following we suppose without loss of generality (see [53]) that W_i's are jointly measurable and separable. This assumption is important for the definitions for classical ruin, Parisian ruin and cumulative Parisian ruin.

For given $A > 0$ and $H \geq 0$ define the simultaneous Parisian ruin probability on finite time horizon $[0, A]$ and $u > 0$

$$P_{H,A}(u, c_1, au, c_2) = \mathbb{P}\{\exists t \in [0, A], \forall s \in [t, t+H] : \ R_1(s) < 0, R_2(s) < 0\}. \quad (1.2)$$

When $H = 0$, the simultaneous Parisian ruin reduces to the simultaneous classical ruin. Such model has been recently studied in [18].

It follows that for any A, H, u positive

$$\begin{aligned} P_{H,A}(u, c_1, au, c_2) &\leq P_{0,A}(u, c_1, au, c_2) \\ &= \mathbb{P}\{\exists t \in [0, A] : R_1(t) < 0, R_2(t) < 0\} \end{aligned}$$

since $P_{H,A}(u, c_1, au, c_2)$ is monotone in H. In [18] it is shown that the simultaneous ruin probability can be bounded as follows

$$\begin{aligned} \mathbb{P}\{W_1^*(A) > u, W_2^*(A) > au\} &\leq P_{0,A}(u, c_1, au, c_2) \\ &\leq \frac{\mathbb{P}\{W_1^*(A) > u, W_2^*(A) > au\}}{\mathbb{P}\{W_1(A) > (c_1 A)^+, W_2(A) > (c_2 A)^+\}}, \end{aligned} \quad (1.3)$$

where we set

$$W_i^*(t) = W_i(t) - c_i t,$$

for i=1,2, and $x^+ = \max(x, 0)$.
A simple lower bound for $P_{H,A}$ is valid for any $u > 0$

$$\mathbb{P}\{\forall t \in [A, A+H] : R_1(t) < 0, R_2(t) < 0\} \leq P_{H,A}(u, c_1, au, c_2). \quad (1.4)$$

The above lower bound is very difficult to evaluate even asymptotically when u tends to infinity. The most simple case is when $a < \rho, \rho > 0$. We have (see Appendix) that for all large u and some $C \in (0, 1)$

$$\begin{aligned} C\mathbb{P}\{\forall t \in [A, A+H] : W_1^*(t) > u\} &\leq P_{H,A}(u, c_1, au, c_2) \\ &\leq \mathbb{P}\left\{ \sup_{t \in [0,A]} W_1^*(t) > u \right\}. \end{aligned} \quad (1.5)$$

Since $\mathbb{P}\left\{\sup_{t\in[0,A]} W_1^*(t) > u\right\}$ can be evaluated explicitly, it follows easily that as $u \to \infty$ it is asymptotically equal to $2\mathbb{P}\{W_1^*(A) > u\}$ and by [18][Thm 2.1] the lower bound is proportional to $\mathbb{P}\{W_1^*(A) > u\}/u$ as $u \to \infty$. Therefore, even for this simple case, the bounds derived above do not capture the exact decrease of the Parisian ruin probability as $u \to \infty$. The reason for this is that the interval $[A, A+H]$ is quite large. In the sequel, under the restriction that $H = S/u^2$ for any $S \geq 0$ we show that it is possible to derive the exact approximations of the Parisian ruin probability.

Motivated by [21] in this chapter we shall also investigate the so-called cumulative Parisian ruin probability on the finite time interval $[0, A]$, i.e.,

$$\Psi_{L,A}(u, au) = \mathbb{P}\left\{ \int_0^A \mathbb{I}(R_1(t) < 0, R_2(t) < 0) dt > L/f(u) \right\},$$

where $A > 0$, $L > 0$ are given constants and f is some positive function that depends on u. It is clear that the above is bounded by $P_{0,T}(u, c_1, au, c_2)$ and the calculation of the cumulative Parisian ruin probability is not possible for any fixed u and x positive. A natural question here is (see [21] for the infinite time-horizon case) if we can approximate the cumulative Parisian ruin probability as $u \to \infty$. This in particular requires to determine explicitly the function f. In the case of one-dimensional risk model it is shown in [21] that the cumulative Parisian ruin probability (or in the language of that chapter the tail of the sojourn time/occupation time) can be approximated exactly when u becomes large. In that aforementioned paper $f(u)$ equals u^2. We shall show that this is the right choice also for our setup.

Section 2 presents the exact asymptotics of both Parisian and cumulative Parisian ruin. Additionally, we discuss therein the approximation of the cumulative Parisian ruin time

$$\tau_L(u) = \inf_{A>0} \int_0^A \mathbb{I}(R_1(t) < 0, R_2(t) < 0) \mathrm{d}t > L/f(u). \tag{1.6}$$

Section 3 is dedicated to the proofs. We conclude this chapter with an Appendix composed of two auxiliary lemmas and a short discussion of general Parisian ruin.

2 Main results

Using the self-similarity of Brownian motion we have that

$$\begin{aligned}
P_{H,A}(u, c_1, au, c_2) &= \mathbb{P}\left\{\exists t \in [0,1], \forall s \in [At, At+H] : \ R_1(s) < 0, R_2(s) < 0\right\} \\
&= \mathbb{P}\left\{ \exists t \in [0,1], \forall s \in [t, t+(H/A)] : \begin{array}{c} R_1(As) < 0 \\ R_2(As) < 0 \end{array} \right\} \\
&= \mathbb{P}\left\{ \exists t \in [0,1], \forall s \in [t, t+(H/A)] : \begin{array}{c} u + Ac_1 s < W_1(As), \\ au + Ac_1 s < W_1(As) \end{array} \right\}
\end{aligned}$$

$$= \mathbb{P}\left\{ \exists t \in [0,1], \forall s \in [t, t+(H/A)] : \begin{array}{c} u/\sqrt{A} + \sqrt{A}c_1 s < W_1(s) \\ au/\sqrt{A} + \sqrt{A}c_1 s < W_1(s) \end{array} \right\}$$
$$= P_{H/A,1}(u/\sqrt{A}, \sqrt{A}c_1, au/\sqrt{A}, \sqrt{A}c_1).$$

In addition, we have

$$P_{H,A}(u, c_1, au, c_2) = P^{rev}_{H,A}(u, c_2, u/a, c_1),$$

where

$$P^{rev}_{H,A}(u, c_1, au, c_2) = \mathbb{P}\left\{ \exists t \in [0, A], \forall s \in [t, t+H] : \ R^{rev}_1(s) < 0, R^{rev}_2(s) < 0 \right\}$$

and

$$R^{rev}_1(t) = u + c_1 t - W_2(t), \quad R^{rev}_2(t) = au + c_2 t - W_1(t), \quad t \geq 0.$$

Hence, we can consider only the case $a \leq 1$ and $A = 1$. Let in the following

$$\lambda_1 = \frac{1 - a\rho}{1 - \rho^2}, \quad \lambda_2 = \frac{a - \rho}{1 - \rho^2}, \tag{2.1}$$

which are both positive if $a \in (\rho, 1]$. For the particular choice of $H = S/u^2$ we shall denote $P_{H,A}(u, au)$ simply as $\psi_S(u, au)$. We consider first the approximation of the Parisian ruin, recall $W^*_i(t) = W_i(t) - c_i t$.

Theorem 2.1 *Let c_1, c_2 be two given real constants and let $S \geq 0$ be given.*
i) If $a \in (\rho, 1]$, then as $u \to \infty$

$$\psi_S(u, au) \sim C_{a,\rho}(S)\mathbb{P}\left\{ W^*_1(1) > u, W^*_2(1) > au \right\},$$

where

$$C_{a,\rho}(S) = \lambda_1\lambda_2 \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists t \geq 0, \forall s \in [t-S, t] : \begin{array}{c} W_1(s) - s > x \\ W_2(s) - as > y \end{array} \right\} \times e^{\lambda_1 x + \lambda_2 y} \, \mathrm{d}x\mathrm{d}y \tag{2.2}$$

and $C_{a,\rho}(S) \in (0, \infty)$.
ii) If $a \leq \rho$, then as $u \to \infty$

$$\psi_S(u, au) \sim C(S)\mathbb{P}\left\{ W^*_1(1) > u, W^*_2(1) > au \right\},$$

where $C(S) = \mathbb{E}\left\{ e^{\sup_{t \geq 0} \inf_{s \in [t-S,t]} (W_1(s) - s)} \right\} \in (0, \infty)$.

The approximation of the cumulative Parisian ruin requires some different arguments since the sojourn functional is different from the supremum functional. In the following we shall choose the scaling function $f(u)$ to be equal to u^2. Since we consider $A = 1$, we can omit it and write simply $\Psi_L(u, au)$ instead of $\Psi_{L,A}(u, au)$.

Theorem 2.2 *Under the setup and the notation of Theorem 2.1 for any $L > 0$ we have:*
i) If $a \in (\rho, 1]$, then as $u \to \infty$

$$\Psi_L(u, au) \sim K_{a,\rho}(L)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where

$$K_{a,\rho}(L) = \lambda_1\lambda_2 \int_{\mathbb{R}^2} \mathbb{P}\left\{\int_0^\infty \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y)\mathrm{d}t > L\right\} \times e^{\lambda_1 x + \lambda_2 y}\mathrm{d}x\mathrm{d}y \in (0, \infty). \tag{2.3}$$

ii) If $a \le \rho$, then as $u \to \infty$

$$\Psi_L(u, au) \sim K(L)\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\},$$

where

$$K(L) = \int_{\mathbb{R}} e^x \mathbb{P}\left\{\int_0^\infty \mathbb{I}(W_1(t) - t > x)\mathrm{d}t > L\right\}\mathrm{d}x \in (0, \infty). \tag{2.4}$$

Remark 2.3 *Theorems 2.1 and 2.2 may be used also if c_1, c_2, S and L depend on u, but have finite limits as $u \to \infty$ ($c_1(u) \to c_1^*$, $c_2(u) \to c_2^*$, $S(u) \to S^*$, $L(u) \to L^*$). In this case all constants S and L on the right-hand sides should be replaced by S^* and L^*, respectively.*

The asymptotic distribution of the ruin time $\tau_L(u)$ defined in (1.6) may be explicitly calculated from Theorem 2.2 by using the self-similarity of Brownian motion.

Proposition 2.4 *i) If $a \in (\rho, 1]$, then for any $0 \le L_2 \le L_1 \le 1$ with $K_{a,\rho}$ defined in (2.3)*

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(1 - \tau_{L_1}(u)) \ge x | \tau_{L_2}(u) \le 1\right\} = \frac{K_{a,\rho}(L_1)}{K_{a,\rho}(L_2)} e^{-x\frac{1-2a\rho+a^2}{2-2\rho^2}}, \quad x \in (0, \infty).$$

ii) If $a \le \rho$, then for any $0 \le L_2 \le L_1 \le 1$ with K defined in (2.4)

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(1 - \tau_{L_1}(u)) \ge x | \tau_{L_2}(u) \le 1\right\} = \frac{K(L_1)}{K(L_2)} e^{-\frac{x}{2}}, \qquad x \in (0, \infty).$$

3 Proofs

Proof of Theorem 2.1: Let in the following $T>0$ and set $\delta(u,T)=1-Tu^{-2}$ for $T,u>0$.

For any S positive and all u large

$$\begin{aligned} m(u,S,T) &:= \mathbb{P}\left\{\exists_{t\in[0,\delta(u,T)]}, \forall s\in[t,t+S/u^2]: W_1^*(s)>u, W_2^*(s)>au\right\} \\ &\le \mathbb{P}\left\{\exists_{t\in[0,\delta(u,T)]}: W_1^*(t)>u, W_2^*(t)>au\right\} \\ &\le e^{-T/8}\frac{\mathbb{P}\left\{W_1^*(1)\ge u, W_2^*(1)\ge au\right\}}{\mathbb{P}\left\{W_1(1)>\max(c_1,0), W_2(1)>\max(c_2,0)\right\}}, \end{aligned} \tag{3.1}$$

where the upper bound follows from [18][Lemma 4.1].

We give below the exact asymptotics of

$$M(u,S,T) := \mathbb{P}\left\{\exists_{t\in[\delta(u,T),1]}, \forall s\in[t,t+S/u^2]: W_1^*(s)>u, W_2^*(s)>au\right\}$$

as u tends to infinity.

Lemma 3.1 *i) For any $a\in(\rho,1]$ and any positive S and T as $u\to\infty$*

$$M(u,S,T) \sim u^{-2}\varphi_\rho(u+c_1, au+c_2)I(S,T), \tag{3.2}$$

where

$$I(S,T) := \int_{\mathbb{R}^2}\mathbb{P}\left\{t\in[0,T], \forall s\in[t-S,t] \begin{array}{l} W_1(s)-s>x \\ W_2(s)-as>y \end{array}\right\} \times e^{\lambda_1 x+\lambda_2 y}\,\mathrm{d}x\mathrm{d}y,$$

and $I(S,T)\in(0,\infty)$.
ii) For any $a\le\rho$ and any positive S and T as $u\to\infty$

$$M(u,S,T)\sim u^{-1}\varphi_\rho(u+c_1,\rho u+c_2)I(S,T),$$

where

$$I(S,T) := \int_{\mathbb{R}^2}\mathbb{P}\left\{\exists t\in[0,T], \forall s\in[t-S,t]: W_1(s)-s>x\right\} \times\Big[\mathbb{I}(a<\rho)+\mathbb{I}(y<0,a=\rho)\Big]e^{x-\frac{y^2-2y(c_2-c_1\rho)}{2(1-\rho^2)}}\,\mathrm{d}x\mathrm{d}y.$$

The proof of Lemma 3.1 postponed to the Appendix.

In view of Lemma 3.1, inequality (3.1) and asymptotics of the probability $\mathbb{P}\{W_1^*(1) \geq u, W_2^*(1) \geq au\}$ (see Appendix, Lemma 4.1) we immediately obtain that

$$\lim_{T\to\infty}\lim_{u\to\infty}\frac{m(u,S,T)}{M(u,S,T)} = 0.$$

Hence, using that

$$M(u,S,T) \leq \psi_s(u,au) \leq m(u,S,T) + M(u,S,T)$$

we obtain

$$\lim_{T\to\infty}\lim_{u\to\infty}\frac{M(u,S,T)}{\psi_S(u,au)} = 1.$$

Consequently, it suffices to prove that

$$\lim_{T\to\infty} I(S,T) \in (0,\infty).$$

Since $I(S,T) \leq I(0,T)$, $I(S,T)$ is growing and the finiteness of $\lim_{T\to\infty} I(0,T)$ follows from [18], the claim follows according to the asymptotics of the probability $\mathbb{P}\{W_1^*(1) \geq u, W_2^*(1) \geq au\}$. □

Proof of Theorem 2.2: First recall $\delta(u,T) = 1 - Tu^{-2}$.
For given $L > 0$ if $\int_0^1 \mathbb{I}(R_1(t) < 0, R_2(t) < 0)\mathrm{d}t > L/f(u)$, then either the same integral but from $1-\delta$ to 1 is larger than $L/f(u)$, or for some point $t_1 \in [0, 1-\delta(u,T)]$ both $R_1(t_1)$ and $R_2(t_1)$ are smaller than zero. In terms of probabilities it means that for any $T > 0$

$$M(u,T) \leq \Psi_L(u,au) \leq M(u,T) + m(u,T), \tag{3.3}$$

where we set for $u > 0$

$$M(u,T) = \mathbb{P}\left\{\int_{1-\delta(u,T)}^1 \mathbb{I}(W_1^*(t) > u, W_2^*(t) > au)\mathrm{d}t > L/f(u)\right\},$$
$$m(u,T) = \mathbb{P}\left\{\exists_{t\in[0,1-\delta(u,T)]} : W_1^*(t) > u, W_2^*(t) > au\right\}.$$

In view of [18][Lemma 4.1] for all large enough u

$$m(u,T) \leq e^{-T/8}\frac{\mathbb{P}\{W_1^*(1) \geq u, W_2^*(1) \geq au\}}{\mathbb{P}\{W_1(1) > \max(c_1,0), W_2(1) > \max(c_2,0)\}}. \tag{3.4}$$

The following lemma establishes the exact asymptotics of $M(u,T)$.

Lemma 3.2 *i) For any $a \in (\rho, 1]$ and any $T > 0$ as $u \to \infty$*

$$M(u,T) \quad \sim \quad u^{-2}\varphi_\rho(u+c_1, au+c_2)I(T), \tag{3.5}$$

where

$$\begin{aligned} I(T) \quad &:= \quad \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}(W_1(t)-t>x, W_2(t)-at>y)\mathrm{d}t > L \right\} \\ &\quad \times e^{\lambda_1 x + \lambda_2 y}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

and $I(T) \in (0,\infty)$.
ii) For any $a \le \rho$ and any $T > 0$ as $u \to \infty$

$$M(u,T) \sim u^{-1}\varphi_\rho(u+c_1, \rho u + c_2)I(T),$$

where

$$\begin{aligned} I(T) \quad &:= \quad \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}\left(W_1(t)-t>x\right)\mathrm{d}t > L \right\} \\ &\quad \times \left[\mathbb{I}\{a<\rho\} + \mathbb{I}\{a=\rho,\ y<0\}\right] e^{x - \frac{y^2-2y(c_2-c_1\rho)}{2(1-\rho)}}\mathrm{d}x\mathrm{d}y, \end{aligned}$$

and $I(T) \in (0,\infty)$.

The proof of Lemma 3.2 postponed to the Appendix.

In view of Lemma 3.2, inequality (3.4) and asymptotics of the probability $\mathbb{P}\left\{W_1^*(1) \ge u, W_2^*(1) \ge au\right\}$ we immediately obtain that

$$\lim_{T\to\infty}\lim_{u\to\infty} \frac{m(u,T)}{M(u,T)} = 0.$$

Hence, using (3.3) we have

$$\lim_{T\to\infty}\lim_{u\to\infty} \frac{M(u,T)}{\Psi_L(u,au)} = 1.$$

Consequently, it suffices to show that

$$\lim_{T\to\infty} I(T) \in (0,\infty),$$

where $I(T)$ is defined in Lemma 3.2. Since $I(T) \le I(L,T)$ defined in Lemma 3.1, $I(T)$ is growing and $\lim_{T\to\infty} I(L,T) < \infty$, the claim follows. □

Proof of Proposition 2.4: Using the formula of conditional probability and the self-similarity of Brownian motion for L_1, L_2, u positive

$$\mathbb{P}\left\{u^2(1-\tau_{L_1}(u)) \ge x | \tau_{L_2}(u) \le 1\right\} = \frac{\mathbb{P}\left\{\tau_{L_1}(u) \le 1 - x/u^2\right\}}{\mathbb{P}\left\{\tau_{L_2}(u) \le 1\right\}}$$

$$
= \frac{\mathbb{P}\left\{\int_0^{1-x/u^2} \mathbb{I}\left(\begin{array}{c} W_1^*(t) > u \\ W_2^*(t) > au \end{array}\right) \mathrm{d}t > L_1/u^2\right\}}{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1^*(t) > u \\ W_2^*(t) > au \end{array}\right) \mathrm{d}t > L_2/u^2\right\}}
$$

$$
= \frac{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1^*((1-x/u^2)t) > u \\ W_2^*((1-x/u^2)t) > au \end{array}\right) \mathrm{d}(1-x/u^2)t > L_1/u^2\right\}}{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1^*(t) > u \\ W_2^*(t) > au \end{array}\right) \mathrm{d}t > L_2/u^2\right\}}
$$

$$
= \frac{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1(t) > \frac{u}{\sqrt{1-x/u^2}} + c_1\sqrt{1-x/u^2}t \\ W_2(t) > a\frac{u}{\sqrt{1-x/u^2}} + c_2\sqrt{1-x/u^2}t \end{array}\right) \mathrm{d}t > \frac{L_1/(1-x/u^2)^2}{(u/\sqrt{1-x/u^2})^2}\right\}}{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1^*(t) > u \\ W_2^*(t) > au \end{array}\right) \mathrm{d}t > L_2/u^2\right\}}.
$$

Applying Theorem 2.2 yields

$$
\frac{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1(t) > \frac{u}{\sqrt{1-x/u^2}} + c_1\sqrt{1-x/u^2}t \\ W_2(t) > a\frac{u}{\sqrt{1-x/u^2}} + c_2\sqrt{1-x/u^2}t \end{array}\right) \mathrm{d}t > \frac{L(1-x/u^2)^2}{(u/\sqrt{1-x/u^2})^2}\right\}}{\mathbb{P}\left\{\int_0^{1} \mathbb{I}\left(\begin{array}{c} W_1^*(t) > u \\ W_2^*(t) > au \end{array}\right) \mathrm{d}t > L/u^2\right\}}
$$

$$
\sim \frac{\mathbb{P}\left\{W_1(1) > \frac{u}{\sqrt{1-x/u^2}} + c_1\sqrt{1-x/u^2}, W_2(1) > a\frac{u}{\sqrt{1-x/u^2}} + c_2\sqrt{1-x/u^2}\right\}}{\mathbb{P}\left\{W_1^*(1) > u, W_2^*(1) > au\right\}/\Gamma(L_1, L_2)},
$$

where

$$
\Gamma(L_1, L_2) = \begin{cases} \frac{K_{a,\rho}(L_1)}{K_{a,\rho}(L_2)}, & a \in (\rho, 1], \\ \frac{K(L_1)}{K(L_2)}, & a \le \rho. \end{cases}
$$

Notice that (write $\varphi(x, y)$ for the pdf of vector $(W_1(1), W_2(1))$)

$$
\varphi\left(\frac{u}{\sqrt{1-x/u^2}} + c_1\sqrt{1-x/u^2}, \frac{u}{\sqrt{1-x/u^2}}a + c_2\sqrt{1-x/u^2}\right) = \varphi(u + c_1, au + c_2)\psi_u^*(a, c_1, c_2),
$$

where

$$
\lim_{u\to\infty} \log \psi_u^*(a, c_1, c_2) = -x\frac{1-2a\rho+a^2}{2-2\rho^2},
$$

hence by Lemma 4.1 the claim follows if $a > \rho$. For the case $a \le \rho$ notice that

$$\varphi\left(\frac{u}{\sqrt{1-x/u^2}} + c_1\sqrt{1-x/u^2}, \frac{u}{\sqrt{1-x/u^2}}\rho + c_2\sqrt{1-x/u^2}\right) \\ = \varphi(u+c_1, \rho u + c_2)\psi_u^*(\rho, c_1, c_2),$$

where

$$\lim_{u\to\infty} \log \psi_u^*(\rho, c_1, c_2) = -x/2.$$

This finishes the proof in the case $a \le \rho$ again using Lemma 4.1 □

4 Appendix

4.1 Parisian ruin for non-vanishing interval

Consider now the probability $P_{H,T}(u, au)$ with some fixed constant H. We can use the following upper bound:

$$\mathbb{P}\left\{\forall t \in [T, T+H] : W_1^*(t) > u, W_2^*(t) > au\right\} \le P_{H,T}(u, au).$$

We can present $W_2(t)$ using the correlation coefficient ρ as $\rho W_1(t) + \rho^* B(t)$, where $\rho^* = \sqrt{1-\rho^2}$, and $B(t)$ is an independent Brownian motion. Note that if $W_1^*(t) > u$ and $B(t) > (a-\rho)u + (c_2 - \rho c_1)t$, then also $W_2^*(t) > au$. Since W_1 and B are independent

$$\mathbb{P}\left\{\forall t \in [T, T+H] : R_1(t) < 0\right\} \\ \times \mathbb{P}\left\{\forall t \in [T, T+H] : B(t) > (a-\rho)u + (c_2 - \rho c_1)t\right\} \le P_{H,T}(u, au).$$

In case $\rho > 0$ and $\rho > a$, the probability

$$\mathbb{P}\left\{\forall t \in [T, T+H] : B(t) > (a-\rho)u + (c_2 - \rho c_1)t\right\}$$

tends to one when u tends to infinity. So, for any positive ε for large enough u we derived the following lower bound

$$(1-\varepsilon)\mathbb{P}\left\{\forall t \in [T, T+H] : W_1(t) - c_1 t > u\right\} \le P_{H,T}(u, au).$$

To find an upper bound we can put $H = 0$ and omit the restriction for W_2, namely

$$P_{H,T}(u, au) \le \mathbb{P}\left\{\sup_{t\in[0,T]} (W_1(t) - c_1 t) > u\right\}.$$

As u tends to infinity, the probability $\mathbb{P}\left\{\forall t \in [T, T+H] : W_1(t) - c_1 t > u\right\}$ is asymptotically equal to $\mathbb{P}\left\{W_1^*(T) > u\right\}$, and the probability $\mathbb{P}\left\{\sup_{t\in[0,T]}(W_1(t) - c_1 t) > u\right\}$ is asymptotically equal to $\mathbb{P}\left\{W_1^*(T) > u\right\}/u$.

4.2 Auxiliary lemmas

The following Lemma shows the exact asymptotics of the right-hand sides in Theorem 2.1 and Theorem 2.2.

Lemma 4.1 *Let X_1 and X_2 be Gaussian random variables with correlation coefficient $\rho \in (-1,1)$. Let also c_1, c_2 be two given real constants and $a \le 1$ be given. Write further $\varphi_\rho(x,y)$ for the joint density function of vector (X_1, X_2).*
i) If $a \in (\rho, 1]$, then as $u \to \infty$

$$\mathbb{P}\{X_1 > u + c_1, X_2 > au + c_2\} \sim \frac{u^{-2}}{\lambda_1 \lambda_2} \varphi_\rho(u + c_1, au + c_2),$$

where

$$\lambda_1 = \frac{1 - a\rho}{1 - \rho^2}, \quad \lambda_2 = \frac{a - \rho}{1 - \rho^2}.$$

ii) If $a \le \rho$, then we have as $u \to \infty$

$$\mathbb{P}\{X_1 > u + c_1, X_2 > au + c_2\}$$
$$\sim \sqrt{2\pi(1-\rho^2)}\Phi^*(c_1\rho - c_2) e^{\frac{(c_2 - \rho c_1)^2}{2(1-\rho^2)}} u^{-1} \varphi_\rho(u + c_1, \rho u + c_2),$$

where $\Phi^*(c_1\rho - c_2) = 1$ if $a < \rho$ and Φ^* is the df of $\sqrt{1-\rho^2}X_1$ when $a = \rho$.

Proof of Lemma 4.1: i) Using the dominated convergence theorem as $u \to \infty$

$$\begin{aligned}
\mathbb{P}\{X_1 > u + c_1, X_2 > au + c_2\} &= \frac{\varphi_\rho(u + c_1, au + c_2)}{u^2} \\
&\quad \times \int_{x,y>0} e^{-\lambda_1 x - \lambda_2 y} \frac{\varphi_\rho(c_1 + x/u, c_2 + y/u)}{\varphi_\rho(c_1, c_2)} \mathrm{d}x\mathrm{d}y \\
&\sim \frac{\varphi_\rho(u + c_1, au + c_2)}{u^2} \\
&\qquad \times \int_{x,y>0} e^{-\lambda_1 x - \lambda_2 y} \mathrm{d}x\mathrm{d}y.
\end{aligned}$$

ii) Again using the dominated convergence theorem as $u \to \infty$ (denote $C = 0$ if $a = \rho$ and $C = -\infty$ otherwise)

$$\begin{aligned}
\mathbb{P}\{X_1 > u + c_1, X_2 > au + c_2\} &\sim \frac{\varphi_\rho(u + c_1, \rho u + c_2)}{u} \\
&\qquad \times \int_{\substack{x>0 \\ y>C}} e^{-x} \frac{\varphi_\rho(c_1, c_2 + y)}{\varphi_\rho(c_1, c_2)} \mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\varphi_\rho(u+c_1,\rho u+c_2)}{u} e^{\frac{(c_2-\rho c_1)^2}{1-\rho^2}} \sqrt{2\pi(1-\rho^2)} \\
&\quad \times \int\limits_{y<\rho c_1 - c_2 - C} \frac{e^{-\frac{1}{2}\frac{y^2}{1-\rho^2}}}{\sqrt{2\pi(1-\rho^2)}} \mathrm{d}y \\
&= \frac{\varphi_\rho(u+c_1,\rho u+c_2)}{u} e^{\frac{(c_2-\rho c_1)^2}{1-\rho^2}} \sqrt{2\pi(1-\rho^2)} \\
&\quad \times \Phi^*(\rho c_1 - c_2).
\end{aligned}
$$

□

Then next lemma helps to go from "almost all L" to "all L" in Lemma 3.2

Lemma 4.2 *Let $X_1(t), X_2(t)$ for $t \geq 0$ satisfy the representation (1.1). Let also $\lambda_1, \lambda_2, a, T$ be positive constants and c_1, c_2 be real constants. Then the functions*

$$
\begin{aligned}
I_1(L) &= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}(X_1(t) - t > x, X_2(t) - at > y)\mathrm{d}t > L \right\} e^{\lambda_1 x + \lambda_2 y} \mathrm{d}x\mathrm{d}y, \\
I_2(L) &= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}\left(X_1(t) - t > x\right)\mathrm{d}t > L \right\} \\
&\qquad \times \left[\mathbb{I}\{a<\rho\} + \mathbb{I}\{a=\rho,\ y<0\}\right] e^{x - \frac{y^2 - 2y(c_2 - c_1\rho)}{2(1-\rho)}} \mathrm{d}x\mathrm{d}y
\end{aligned}
$$

are continuous for $L \in (0,\infty)$.

Proof of Lemma 4.2: Consider the function $I_1(L)$. The proof for $I_2(L)$ will be the same. To show the continuity of $I_1(L)$ it is sufficient to verify that

$$
\begin{aligned}
I_1^*(L) &:= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y)\mathrm{d}t = L \right\} e^{\lambda_1 x + \lambda_2 y} \mathrm{d}x\mathrm{d}y \\
&= 0
\end{aligned}
$$

for all positive L. Fix some $L > 0$ and let

$$
A_{x,y} = \left\{ f_1, f_2 \in C[0,T] : \int_0^T \mathbb{I}(f_1(t) - t > x, f_2(t) - at > y)\mathrm{d}t = L \right\}.
$$

For any fixed $y_0 \in \mathbb{R}$ the sets A_{x_1,y_0} and A_{x_2,y_0} are non-overlapping for $x_1 \neq x_2$. Define

$$
\mathcal{X} = \{x \in \mathbb{R} : \mathbb{P}\{A_{x,y_0}\} > 0\}, \qquad \mathcal{X}_n = \{x \in \mathbb{R} : \mathbb{P}\{A_{x,y_0}\} > 1/n\}.
$$

Since A_{x,y_0} are non-overlapping for different $x \in \mathbb{R}$, $|\mathcal{X}_n| < n$. In addition, $\mathcal{X} = \cup_{n\in\mathbb{N}} \mathcal{X}_n$. Thus, the set $\mathcal{X}$ is countable, establishing the proof.

□

4.3 Proofs of lemmas

This part contains proofs of all the lemmas presented above in this chapter.

Proof of Lemma 3.1: i) For any $x, y \in \mathbb{R}$ put

$$u_x = u + c_1 - x/u, \qquad u_y = au + c_2 - y/u.$$

Writing $\varphi(x, y)$ for the joint pdf of $(W_1(1), W_2(1))^\top$ we have

$$\varphi_\rho(u_x, u_y) =: \varphi_\rho(u + c_1, au + c_2)\psi_u(x, y), \tag{4.1}$$

where as $u \to \infty$ (write Σ for the covariance matrix of $(W_1(1), W_2(1))^\top$)

$$\begin{aligned}\log \psi_u(x, y) &= \frac{1}{u^2}(u + c_1, au + c_2)\Sigma^{-1}(x, y)^\top - \frac{1}{2u^2}(x, y)\Sigma^{-1}(x, y)^\top \\ &\to (1, a)\Sigma^{-1}(x, y)^\top \\ &= \frac{1 - a\rho}{1 - \rho^2}x + \frac{a - \rho}{1 - \rho^2}y = \lambda_1 x + \lambda_2 y.\end{aligned} \tag{4.2}$$

Denote further

$$u_{x,y} = u_y - \rho u_x = (a - \rho)u - (y - \rho x)/u + c_2 - \rho c_1.$$

Let B_1, B_2 be two independent Brownian motions. The representation of $(W_1(t), W_2(t))$ in terms of B_1 and B_2 is given in (1.1). Define the following transform

$$\bar{s}_u = 1 - s/u^2$$

and set $F(u) = u^{-2}\varphi_\rho(u + c_1, au + c_2)$.

For the function $M(u, S, T)$ we have using ψ_u defined in (4.1)

$$\begin{aligned}&M(u, S, T) \\ &= \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists t \in [0, T]\forall s \in [t - S, t] : \begin{array}{l} B_1(\bar{s}_u) - c_1\bar{s}_u > u \\ \rho B_1(\bar{s}_u) + \rho^* B_2(\bar{s}_u) - c_2\bar{s}_u > au \end{array} \middle| \begin{array}{l} B_1(1) = u_x \\ \rho^* B_2(1) = u_{x,y} \end{array}\right\} \\ &\qquad \times F(u)\psi_u(x, y)\mathrm{d}x\mathrm{d}y \\ &=: F(u)\int_{R^2} h_u(T, S, x, y)\psi_u(x, y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

Define two auxiliary processes for $s \in [-S, T]$ as follows

$$\begin{aligned}B_{u,1}(s) &:= \{B_1(\bar{s}_u)\,|B_1(1) = u_x\} - \bar{s}_u u_x, \\ B_{u,2}(s) &:= \{B_2(\bar{s}_u)\,|\rho^* B_2(1) = u_{x,y}\} - \bar{s}_u u_{x,y}/\rho^*.\end{aligned} \tag{4.3}$$

Represent the function $h_u(T,S,x,y)$ in terms of these processes as

$$\begin{aligned}
&h_u(T,S,x,y)\\
=\ &\mathbb{P}\left\{\exists t\in[0,T]\forall s\in[t-S,t]:\ \begin{array}{r} u(B_{u,1}(s)+\bar{s}_u u_x-c_1\bar{s}_u-u)>0\\ u\rho\left(B_{u,1}(s)+\bar{s}_u u_x-c_1\bar{s}_u-u\right)+\\ +u\rho^* B_{u,2}(s)+u[\bar{s}_u u_{x,y}-(c_2-\rho c_1)\bar{s}_u-u(a-\rho)]>0\end{array}\right\}.
\end{aligned}$$

We have the following weak convergence in the space $C([-S,T])$ as $u\to\infty$

$$uB_{u,1}(t)\to B_1(t),\quad uB_{u,2}(t)\to B_2(t),\qquad t\in[-S,T], \tag{4.4}$$

and further

$$\begin{aligned}
&u(\bar{s}_u u_x-c_1\bar{s}_u-u)=u\left[\left(1-\frac{s}{u^2}\right)\left(u+c_1-\frac{x}{u}\right)-c_1\left(1-\frac{s}{u^2}\right)-u\right]\\
&\qquad\qquad\to -s-x,\\
&u[\bar{s}_u u_{x,y}-(c_2-\rho c_1)\bar{s}_u-u(a-\rho)]\to -(a-\rho)s-(y-\rho x).
\end{aligned}$$

Consequently, as u tends to infinity

$$h_u(T,S,x,y)\to h(T,S,x,y),$$

where in view of (1.1)

$$\begin{aligned}
&h(T,S,x,y)\\
=\ &\mathbb{P}\left\{\exists t\in[0,T]\forall s\in[t-S,t]:\ \begin{array}{l} B_1(s)-s-x>0,\\ \rho(B_1(s)-s-x)+\rho^* B_2(s)-(a-\rho)s-(y-\rho x)>0\end{array}\right\}\\
=\ &\mathbb{P}\left\{\exists t\in[0,T]\forall s\in[t-S,t]:\ \begin{array}{l} W_1(s)-s>x,\\ W_2(s)-as>y\end{array}\right\}.
\end{aligned}$$

This convergence is justified by applying continuous mapping theorem for the continuous functional

$$H_{T,S}(F_1(t),F_2(t))=\sup_{t\in[0,T]}\inf\left(\inf_{s\in[t-S,t]}F_1(t),\inf_{s\in[t-S,t]}F_2(t)\right)$$

and random sequence $(F_{1,x,y,u},F_{2,x,y,u})\in C[-S,T]^2$

$$\begin{aligned}
F_{1,x,y,u}(s)&=u(B_{u,1}(s)+\bar{s}_u u_x-c_1\bar{s}_u-u),\\
F_{2,x,y,u}(s)&=u\rho\left(B_{u,1}(s)+\bar{s}_u u_x-c_1\bar{s}_u-u\right)+u\rho^* B_{u,2}(s)\\
&\quad +u[\bar{s}_u u_{x,y}-(c_2-\rho c_1)\bar{s}_u-u(a-\rho)].
\end{aligned}$$

To finish the proof it is enough to show the dominated convergence as $u \to \infty$ for

$$I_u(S,T) = \int_{\mathbb{R}^2} h_u(T,S,x,y)\psi_u(x,y)\mathrm{d}x\mathrm{d}y.$$

For $\psi_u(x,y)$ we can show the following upper bound. Fix some

$$0 < \varepsilon < \min(\lambda_1, \lambda_2)$$

(such constant exists as in our case both λ_1 and λ_2 are positive) and define constants $\lambda_{1,\varepsilon} = \lambda_1 + sign(x)\varepsilon$ and $\lambda_{2,\varepsilon} = \lambda_2 + sign(y)\varepsilon$. Hence for large enough u and all $x, y \in \mathbb{R}$

$$\psi_u(x,y) \le \bar{\psi} := e^{\lambda_{1,\varepsilon}x + \lambda_{2,\varepsilon}y}. \tag{4.5}$$

For $h_u(S,T,x,y)$ we use Piterbarg inequality (see [53], Thm 8.1), since for all t, s positive

$$u^2\mathbb{E}\left\{(B_{u,i}(t) - B_{u,i}(s))^2\right\} < Const|t-s| \tag{4.6}$$

for some positive constant and sufficiently large u. Thus, for such u we have for some positive constant $\bar{C}$

$$\begin{aligned} & h_u(T,S,x,y) \\ \le\ & \mathbb{P}\left\{ \exists s \in [0,T] : \begin{array}{r} u(B_{u,1}(s) + \bar{s}_u(u_x - c_1) - u) > 0 \\ u\rho\left(B_{u,1}(s) + \bar{s}_u(u_x - c_1) - u\right) + u\rho^* B_{u,2}(s) \\ + u[\bar{s}_u(u_{x,y} - c_2 + \rho c_1) - u(a-\rho)] > 0 \end{array} \right\} \\ \le\ & \bar{h} := \begin{cases} \bar{C}e^{-c(x^2+y^2)}, & x, y \ge 0, \\ \bar{C}e^{-cx^2}, & x \ge 0, y < 0, \\ \bar{C}e^{-cy^2}, & y \ge 0, x < 0, \\ 1, & x, y < 0. \end{cases} \end{aligned}$$

Since $\lambda_{1,\varepsilon}, \lambda_{2,\varepsilon}$ are positive

$$\begin{aligned} I_u(S,T) &\le \int_{\mathbb{R}^2} \bar{h}(T,S,x,y)\bar{\psi}(x,y)\mathrm{d}x\mathrm{d}y \\ &= \bar{C}\int_{x,y>0} e^{-c(x^2+y^2)+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y + \bar{C}\int_{x>0,y<0} e^{-cx^2+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y \\ &+ \bar{C}\int_{x<0,y>0} e^{-cy^2+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y + \bar{C}\int_{x,y<0} e^{\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y < \infty. \end{aligned}$$

Hence the proof follows from the dominated convergence theorem.

ii) In the case $a \le \rho$ we define

$$u_x = u + c_1 - x/u, \qquad u_y = \rho u + c_2 - y$$

and $u_{x,y} = u_y - \rho u_x = c_2 - y - \rho c_1 + \rho x/u$. In the previous notation

$$\varphi_\rho(u_x, u_y) =: \varphi_\rho(u + c_1, \rho u + c_2)\psi_u(x, y),$$

where as $u \to \infty$

$$\begin{aligned} \log \psi_u(x, y) &= (u + c_1, \rho u + c_2)\Sigma^{-1}(x/u, y)^\top - \frac{1}{2}(x/u, y)\Sigma^{-1}(x/u, y)^\top \\ &\to x - \frac{y^2 - 2y(c_2 - \rho c_1)}{2 - 2\rho^2}. \end{aligned} \tag{4.7}$$

Setting $F(u) = u^{-1}\varphi_\rho(u + c_1, \rho u + c_2)$, we have the following representation for the function $M(u, S, T)$ (write $\bar{s}_u$ for $1 - s/u^2$ and recall (1.1))

$$\begin{aligned} &M(u, S, T) = \\ &= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists t \in [\delta(u, T), 1] \forall s \in [t, t + S/u^2] : \begin{array}{l} W_1^*(s) > u \\ W_2^*(s) > au \end{array} \middle| \begin{array}{l} W_1(1) = u_x \\ W_2(1) = u_y \end{array} \right\} \\ &\qquad\qquad \times u^{-1}\varphi_\rho(u_x, u_y)\mathrm{d}x\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \mathbb{P}\left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{array}{l} B_1(\bar{s}_u) - c_1\bar{s}_u > u \\ \rho B_1(\bar{s}_u) + \rho^* B_2(\bar{s}_u) - c_2\bar{s}_u > au \end{array} \middle| \begin{array}{l} B_1(1) = u_x \\ \rho^* B_2(1) = u_{x,y} \end{array} \right\} \\ &\qquad\qquad \times F(u)\psi_u(x, y)\mathrm{d}x\mathrm{d}y \\ &=: F(u) \int_{R^2} h_u(T, S, x, y)\psi_u(x, y)\mathrm{d}x\mathrm{d}y. \end{aligned}$$

Using $B_{u,1}$ and $B_{u,2}$ defined in (4.3) we can represent the function $h_u(T, S, x, y)$ as

$$\begin{aligned} &h_u(T, S, x, y) \\ &= \mathbb{P}\left\{ \exists t \in [0, T] \forall s \in [t - S, t] : \begin{array}{r} u(B_{u,1}(s) + \bar{s}_u u_x - c_1\bar{s}_u - u) > 0 \\ u\rho\left(B_{u,1}(s) + \bar{s}_u u_x - c_1\bar{s}_u - u\right) + \\ + u\rho^* B_{u,2}(s) + u[\bar{s}_u u_{x,y} - (c_2 - \rho c_1)\bar{s}_u - u(a - \rho)] > 0 \end{array} \right\}. \end{aligned}$$

As u tends to infinity we have

$$u(\bar{s}_u u_x - c_1\bar{s}_u - u) = u\left[\left(1 - \frac{s}{u^2}\right)\left(u + c_1 - \frac{x}{u}\right) - c_1\left(1 - \frac{s}{u^2}\right) - u\right]$$

$$
\begin{aligned}
&\to -s - x,\\
u[\bar{s}_u u_{x,y} - (c_2 - \rho c_1)\bar{s}_u - u(a-\rho)] &= -u^2(a-\rho) - uy + \rho x\\
&\qquad\qquad +ys/u + \rho xs/u^2.
\end{aligned}
$$

If $a < \rho$, then the above tends to ∞, and if $a = \rho$ then it tends to ∞ only if $y < 0$ and to $-\infty$ if $y > 0$. Finally, if $a = \rho$ and $y = 0$, then the above tends to ρx. Again using continuous mapping theorem, since (4.4) holds, we have the following convergence (except if $y = 0$)

$$
h_u(T,S,x,y) \to h(T,S,x,y), \qquad u \to \infty,
$$

where

$$
\begin{aligned}
&h(T,S,x,y) =\\
&\quad = \mathbb{P}\left\{ \exists t \in [0,T] \forall s \in [t-S,t] : \begin{array}{l} B_1(s) - s - x > 0,\\ \rho(B_1(s) - s - x) + \rho^* B_2(s) + \infty > 0 \end{array} \right\}\\
&\qquad \times (\mathbb{I}\{a < \rho\} + \mathbb{I}\{a = \rho,\ y < 0\})\\
&\quad = \mathbb{P}\{\exists t \in [0,T] \forall s \in [t-S,t] : W_1(s) - s > x\}\\
&\qquad \times (\mathbb{I}\{a < \rho\} + \mathbb{I}\{a = \rho,\ y < 0\}).
\end{aligned}
$$

To show the claim we can apply the dominated convergence theorem. Note that for large enough u and all $x, y \in \mathbb{R}$

$$
\log \psi_u(x,y) \le \bar{\varphi}(x,y) = (1 + sgn(x)/2)x + \frac{c_2 - \rho c_1}{1-\rho^2} y - \frac{y^2}{2}.
$$

By Piterbarg inequality (as (4.6) holds here for $i = 1$) we can establish that for some positive constant $\bar{C}$

$$
\begin{aligned}
h_u(T,S,x,y) &\le \mathbb{P}\{\exists s \in [0,T] : u(B_{u,1}(s) + \bar{s}_u(u_x - c_1) - u) > 0\}\\
&\le \bar{h} := \begin{cases} \bar{C} e^{-cx^2}, & x \ge 0,\\ 1, & x < 0. \end{cases}
\end{aligned}
$$

Since $(1 + sign(x)/2) > 0$, then

$$
\int_{\mathbb{R}^2} \bar{h}(x,y)\bar{\varphi}(x,y) \mathrm{d}x\mathrm{d}y < \infty
$$

and by the dominated convergence theorem the claim follows. □

Proof of Lemma 3.2: i) We use the same notation as in Lemma 3.1 i). Hence the convergence (4.2) holds. For the function $M(u,T)$ we have

$$
M(u,T)
$$

$$
\begin{aligned}
= \quad & \int_{\mathbb{R}^2} \mathbb{P}\left\{ \int_0^T \mathbb{I}\begin{pmatrix} B_1(\bar{t}_u) - c_1\bar{t}_u > u \\ \rho B_1(\bar{t}_u) + \rho^* B_2(\bar{t}_u) - c_2\bar{t}_u > au \end{pmatrix} \mathrm{d}t > L \,\middle|\, \begin{matrix} B_1(1) = u_x \\ \rho^* B_2(1) = u_{x,y} \end{matrix} \right\} \\
& \quad \times F(u)\psi_u(x,y)\mathrm{d}x\mathrm{d}y \\
=: \quad & F(u) \int_{R^2} h_u(L,T,x,y)\psi_u(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Recalling the processes $B_{u,1}$ and $B_{u,2}$ from (4.3) we can represent the function $h_u(T,S,x,y)$ as follows

$$
\begin{aligned}
& h_u(L,T,x,y) \\
& = \mathbb{P}\left\{ \int_0^T \mathbb{I}\begin{pmatrix} u(B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u - u) > 0 \\ u\rho\,(B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u - u) + \\ \quad + u\rho^* B_{u,2}(t) + u[\bar{t}_u u_{x,y} - (c_2 - \rho c_1)\bar{t}_u - u(a-\rho)] > 0 \end{pmatrix} \mathrm{d}t > L \right\}.
\end{aligned}
$$

We have the same weak convergence as in (4.4) and further as u tends to infinity

$$
\begin{aligned}
u(\bar{t}_u u_x - c_1\bar{t}_u - u) &= u\left[\left(1 - \frac{t}{u^2}\right)\left(u + c_1 - \frac{x}{u}\right) - c_1\left(1 - \frac{t}{u^2}\right) - u\right] \\
&\to -t - x, \\
u[\bar{t}_u u_{x,y} - (c_2 - \rho c_1)\bar{t}_u - u(a-\rho)] &\to -(a-\rho)t - (y - \rho x).
\end{aligned} \tag{4.8}
$$

Now we want to apply the continuous mapping theorem to the function

$$
H_T(F_1, F_2) = \int_0^T \mathbb{I}\left(F_1(t) > 0, F_2(t) > 0\right)\mathrm{d}t
$$

and a random sequence $(F_{1,x,y,u}, F_{2,x,y,u}) \in C([0,T] \to \mathbb{R}^2)$ defined as

$$
\begin{aligned}
F_{1,x,y,u} &= u(B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u - u), \\
F_{2,x,y,u} &= u\rho\,(B_{u,1}(t) + \bar{t}_u u_x - c_1\bar{t}_u - u) \\
&\qquad + u\rho^* B_{u,2}(t) + u[\bar{t}_u u_{x,y} - (c_2 - \rho c_1)\bar{t}_u - u(a-\rho)],
\end{aligned}
$$

with exception set

$$
\Lambda = \{F \in C([0,T]) : \mu(F^{-1}(\partial\{(x,y) \in \mathbb{R}^2 | x > 0,\ y > 0\})) > 0\}.
$$

First we need to show that $H_T(F_1, F_2)$ is continuous for $(F_1, F_2) \notin \Lambda$. Define an area

$$
\lambda = (F_1, F_2)^{-1}(\partial\{(x,y) \in \mathbb{R}^2 | x > 0,\ y > 0\}).
$$

For any sequence $(F_{1,n}, F_{2,n})$ converging in $C([0,T] \to \mathbb{R}^2)$ to some function (F_1, F_2) as $n \to \infty$ we can define

$$(F'_{1,n}(t), F'_{2,n}(t)) = \begin{cases} (F_{1,n}(t), F_{2,n}(t)), & t \notin \lambda, \\ (F_1(t), F_2(t)), & t \in \lambda. \end{cases}$$

In this case for all $t \in [0,T]$ as $n \to \infty$

$$\mathbb{I}(F'_{1,n} > 0, F'_{2,n} > 0) \to \mathbb{I}(F_1 > 0, F_2 > 0).$$

Since $\mu(\lambda) = 0$, we have $H_T(F'_{1,n}, F'_{2,n}) = H_T(F_{1,n}, F_{2,n})$. Hence, the dominated convergence theorem establishes the continuity of the function H_T at the point (F_1, F_2).

From (4.4) and (4.8) we can establish that as u tends to infinity

$$\begin{aligned} F_{1,x,y,u}(t) &\to B_1(t) - t - x = W_1(t) - t - x, \\ F_{2,x,y,u}(t) &\to \rho(B_1(t) - t - x) + \rho^* B_2(t) - (a-\rho)t - (y - \rho x) \\ &= W_2(t) - at - y. \end{aligned}$$

Since W_1 and W_2 are standard Brownian motions

$$\mathbb{P}\left\{\mu((W_1(\cdot) - \cdot)^{-1}(x)) > 0\right\} = 0, \qquad \mathbb{P}\left\{\mu((W_2(\cdot) - a\cdot)^{-1}(y)) > 0\right\} = 0.$$

Consequently, $\mathbb{P}\left\{(W_1(\cdot) - x - \cdot, W_2(\cdot) - a \cdot - y) \in \Lambda\right\} = 0$, and we can apply continuous mapping theorem, which establish that for almost all L positive

$$h_u(L,T,x,y) \to h(L,T,x,y), \qquad u \to \infty,$$

where

$$\begin{aligned} & h(L,T,x,y) \\ = \ & \mathbb{P}\left\{\int_0^T \mathbb{I}\begin{pmatrix} B_1(t) - t - x > 0 \\ \rho(B_1(t) - t - x) + \rho^* B_2(t) - (a-\rho)t - (y - \rho x) > 0 \end{pmatrix} \mathrm{d}t > L\right\} \\ = \ & \mathbb{P}\left\{\int_0^T \mathbb{I}(W_1(t) - t > x, W_2(t) - at > y)\mathrm{d}t > L\right\}. \end{aligned}$$

To finish the proof it is enough to show the dominated convergence for the integrals

$$I_u(T) = \int_{\mathbb{R}^2} h_u(L,T,x,y)\psi_u(x,y)\mathrm{d}x\mathrm{d}y.$$

In view of (4.5) and (4.6) for large enough u we have for some positive constant $\bar{C}$ such that for all $x, y \in \mathbb{R}$

$$h_u(L,T,x,y)$$

$$
\leq \mathbb{P}\left\{ \exists t\in[0,T]:\ \begin{array}{c} u(B_{u,1}(t)+\bar{t}_u(u_x-c_1)-u)>0 \\ u\rho\left(B_{u,1}(t)+\bar{t}_u(u_x-c_1)-u\right)+u\rho^* B_{u,2}(t) \\ \qquad +u[\bar{t}_u(u_{x,y}-c_2+\rho c_1)-u(a-\rho)]>0 \end{array} \right\}
$$

$$
\leq \bar{h}(T,x,y):=\begin{cases} \bar{C}e^{-c(x^2+y^2)}, & x,y\geq 0,\\ \bar{C}e^{-cx^2}, & x\geq 0, y<0,\\ \bar{C}e^{-cy^2}, & y\geq 0, x<0,\\ 1, & x,y<0. \end{cases}
$$

Since $\lambda_{1,\varepsilon},\lambda_{2,\varepsilon}$ are positive,

$$
\begin{aligned}
I_u(T) &\leq \int_{\mathbb{R}^2}\bar{h}(T,x,y)\bar{\psi}(x,y)\mathrm{d}x\mathrm{d}y\\
&= \bar{C}\int_{x,y>0} e^{-c(x^2+y^2)+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y+\bar{C}\int_{x>0,y<0} e^{-cx^2+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y\\
&+\bar{C}\int_{x<0,y>0} e^{-cy^2+\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y+\bar{C}\int_{x,y<0} e^{\lambda_{1,\varepsilon}x+\lambda_{2,\varepsilon}y}\mathrm{d}x\mathrm{d}y<\infty.
\end{aligned}
$$

Thus the dominated convergence theorem may be applied and provides us with the claimed assertion. (The constant $I(T)$ is continuous with respect to L (see Appendix, Lemma 4.2), so it holds for all L positive).

ii) We keep the same notation as in Lemma 3.1 i).
The following representation for the function $M(u,T)$ holds (write $\bar{t}_u$ for $1-t/u^2$ and recall (1.1))

$$
\begin{aligned}
& M(u,T)\\
= & \int_{\mathbb{R}^2}\mathbb{P}\left\{\int_0^T \mathbb{I}\begin{pmatrix} B_1(\bar{t}_u)-c_1\bar{t}_u>u, \\ \rho B_1(\bar{t}_u)+\rho^* B_2(\bar{t}_u)-c_2\bar{t}_u>au\end{pmatrix}\mathrm{d}t>L \,\middle|\, \begin{array}{l} B_1(1)=u_x \\ \rho^* B_2(1)=u_{x,y}\end{array}\right\}\\
& \qquad\qquad \times F(u)\psi_u(x,y)\mathrm{d}x\mathrm{d}y\\
=: & F(u)\int_{R^2} h_u(L,T,x,y)\psi_u(x,y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Using again $B_{u,1}$ and $B_{u,2}$ as in (4.3) we can represent the function $h_u(L,T,x,y)$ as

$$
\begin{aligned}
& h_u(L,T,x,y)\\
= & \mathbb{P}\left\{\int_0^T \mathbb{I}\begin{pmatrix} u(B_{u,1}(t)+\bar{t}_u u_x-c_1\bar{t}_u-u)>0 \\ u\rho\left(B_{u,1}(t)+\bar{t}_u u_x-c_1\bar{t}_u-u\right) \\ \qquad +u\rho^* B_{u,2}(t)+u[\bar{t}_u u_{x,y}-(c_2-\rho c_1)\bar{t}_u-u(a-\rho)]>0\end{pmatrix}\mathrm{d}t>L\right\}.
\end{aligned}
$$

We have the same weak convergence as in (4.4). Moreover, in this case we may use the convergence (4.7). With the same arguments as in i) we can apply the continuous mapping theorem and establish the following convergence for almost all L positive and all $x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\}$

$$h_u(L,T,x,y) \to h(L,T,x,y), \qquad u \to \infty,$$

where

$$h(L,T,x,y) = \mathbb{P}\left\{ \int_0^T \mathbb{I}\left(B_1(t) - t > x\right) \mathrm{d}t > L \right\} (\mathbb{I}\{a < \rho\} + \mathbb{I}\{a = \rho,\ y < 0\}).$$

To show the claim we can apply the dominated convergence theorem. Note that for large enough u

$$\log \psi_u(x,y) \le \bar{\varphi}(x,y) = (1 + sign(x)/2)x + \frac{c_2 - \rho c_1}{1-\rho^2} y - \frac{y^2}{2}.$$

By Piterbarg inequality we can establish that for some positive constant $\bar{C}$

$$\begin{aligned} h_u(L,T,x,y) &\le \mathbb{P}\left\{\exists t \in [0,T] : u(B_{u,1}(t) + \bar{t}_u(u_x - c_1) - u) > 0\right\} \\ &\le \bar{h}(x) := \begin{cases} \bar{C} e^{-cx^2}, & x \ge 0, \\ 1, & x < 0. \end{cases} \end{aligned}$$

Since $(1 + sign(x)/2) > 0$, then

$$\int_{\mathbb{R}^2} \bar{h}(x) \bar{\varphi}(x,y) \mathrm{d}x \mathrm{d}y < \infty,$$

and by the dominated convergence theorem the claim follows for almost all $L \in (0,\infty)$. The function $I(T)$ is continuous with respect to L, so the claimed assertion holds for all $L \in (0,\infty)$. □

Chapter 3

Multivariate Pandemic-type Failures

1 Introduction

In this chapter we are interested in the probabilistic aspects of multiple simultaneous failures typically occurring due to pandemic-type events. A key benchmark risk model considered here is the d-dimensional Brownian risk model (Brm)

$$\boldsymbol{R}(t,\boldsymbol{u})=(R_1(t,u_1),\ldots,R_d(t,u_d))^\top=\boldsymbol{u}+\boldsymbol{c}t-\boldsymbol{W}(t),\quad t\geq 0,$$

where $\boldsymbol{c}=(c_1,\ldots,c_d)^\top, \boldsymbol{u}=(u_1,\ldots,u_d)^\top$ are vectors in $\mathbb{R}^d$ and random process $\boldsymbol{W}(t)=\Gamma\boldsymbol{B}(t),\ t\in\mathbb{R}$, with Γ a $d\times d$ real-valued non-singular matrix and $\boldsymbol{B}(t)=(B_1(t),\ldots,B_d(t))^\top, t\in\mathbb{R}$ a d-dimensional Brownian motion with independent components which are standard Brownian motions. By bold symbols we denote column vectors, operations with vectors are meant component-wise and $a\boldsymbol{x}=(ax_1,\ldots,ax_d)^\top$ for any scalar $a\in\mathbb{R}$ and any $\boldsymbol{x}\in\mathbb{R}^d$.

Indeed, Brm is a natural limiting model in many statistical applications. Moreover, as shown in [27] such a risk model appears naturally in insurance applications. Since Brm is a natural limiting model, it can be used as a benchmark for various complex models. Given the fundamental role of Brownian motion in applied probability and statistics, it is also of theoretical interest to study failure events arising from this model. Specifically, in this chapter we are interested in the behaviour of the probability of multiple simultaneous failures occurring in a given time horizon $[S,T]\subset[0,\infty]$.

[2]This chapter is based on the joint work [15] with Krzysztof Dębicki and Enkelejd Hashorva.

In our settings failures can be defined in various ways. Let us consider first the failure of a given component of our risk model. Namely, we say that the ith component of our Brm has a failure (or ruin occurs) if $R_i(t,u_i)= u_i+c_it - W_i(t) < 0$ for some $t \in [S,T]$. The extreme case of a catastrophic event is when d multiple simultaneous failures occurs. Typically, for pandemic-type events there are at least k components of the model with simultaneous failures and k is large with the extreme case $k=d$. In mathematical notation, for given positive integer $k \le d$ of interest is the calculation of the following probability

$$\begin{aligned}\psi_k(S,T,\boldsymbol{u})\\ &= \mathbb{P}\left\{\exists t \in [S,T], \mathcal{I} \subset \{1,\dots,d\}, |\mathcal{I}| = k : \cap_{i\in\mathcal{I}}\{R_i(t,u_i) < 0\}\right\}\\ &= \mathbb{P}\left\{\exists t \in [S,T], \mathcal{I} \subset \{1,\dots,d\}, |\mathcal{I}| = k : \cap_{i\in\mathcal{I}}\left\{W_i(t) - c_it > u_i\right\}\right\},\end{aligned}$$

where $|\mathcal{I}|$ denotes the cardinality of the set $\mathcal{I}$. If T is finite, by the self-similarity property of the Brownian motion $\psi_k(S,T,\boldsymbol{u})$ can be derived from the case $T=1$, whereas $T=\infty$ has to be treated separately.

There are no results in the literature investigating $\psi_k(S,T,\boldsymbol{u})$ for general k. The particular case $k=d$, for which $\psi_d(S,T,\boldsymbol{u})$ coincides with the simultaneous ruin probability has been studies in different contexts, see e.g., [2, 3, 9, 10, 28, 30, 36, 40, 41, 48, 52, 54]. The case $d=2$ of Brm has been recently investigated in [19].

Although the probability of multiple simultaneous failures seems very difficult to compute, our first result below, motivated by [44][Thm 1.1], shows that $\psi_k(S,T,\boldsymbol{u})$ can be bounded by the multivariate Gaussian survival probability, namely by

$$p_T(\boldsymbol{u}) = \mathbb{P}\left\{(W_1(T) - c_1T, \dots, W_d(T) - c_dT) \in \boldsymbol{E}(\boldsymbol{u})\right\},$$

where

$$\boldsymbol{E}(\boldsymbol{u})= \bigcup_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \boldsymbol{E}_{\mathcal{I}}(\boldsymbol{u}) = \bigcup_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \{\boldsymbol{x} \in \mathbb{R}^d : \forall i \in \mathcal{I} :\ \boldsymbol{x}_i \ge \boldsymbol{u}_i\}. \tag{1.1}$$

When $u \to \infty$ we can approximate $p_T(\boldsymbol{u})$ utilising Laplace asymptotic method, see e.g., [43], whereas for small and moderate values of u it can be calculated or simulated with sufficient accuracy. Our next result gives bounds for $\psi_k(S,T,\boldsymbol{u})$ in terms of $p_T(\boldsymbol{u})$.

Theorem 1.1 *If the matrix Γ is non-singular, then for any positive integer $k \le d$, all constants $0 \le S < T < \infty$ and all $\mathbf{c}, \boldsymbol{u} \in \mathbb{R}^d$*

$$p_T(\boldsymbol{u}) \le \psi_k(S,T,\boldsymbol{u}) \le K p_T(\boldsymbol{u}), \tag{1.2}$$

where $K = 1/\min_{\mathcal{I}\subset\{1,\dots,d\},|\mathcal{I}|=k} \mathbb{P}\left\{\forall_{i\in\mathcal{I}} : W_i(T) > \max(0, c_iT)\right\} > 0$.

The bounds in (1.2) indicate that it might be possible to derive an approximations of $\psi_k(S, T, \boldsymbol{u})$ for large threshold $\boldsymbol{u}$, which has been already shown for $k = d = 2$ in [19]. In this chapter we consider the general case $k \le d, d > 2$ discussing both the finite time interval (i.e., $T = 1$) and the infinite time horizon case with $T = \infty$ extending the results of [10] where $d = k$ is considered.

In Section 2 we explain the main ideas that lead to the approximation of $\psi_k(S, T, \boldsymbol{u})$. Section 3 discusses some interesting special cases, whereas the proofs are postponed to Section 4. Some technical calculations are displayed in Section 5.

2 Main Results

In this section $\boldsymbol{W}(t), t \ge 0$ is as in the Introduction and for a given positive integer $k \le d$ we shall investigate the approximation of $\psi_k(S, T, \boldsymbol{u})$ where we fix $\boldsymbol{u} = \boldsymbol{a}u$, with $\boldsymbol{a}$ in $\mathbb{R}^d \setminus (-\infty, 0]^d$ and u is sufficiently large.

Let hereafter $\mathcal{I}$ denote a non-empty index set of $\{1, \ldots, d\}$. For a given vector, say $\boldsymbol{x} \in \mathbb{R}^d$ we shall write $\boldsymbol{x}_{\mathcal{I}}$ to denote a subvector of $\boldsymbol{x}$ obtained by dropping its components not in $\mathcal{I}$. Set further

$$\psi_{\mathcal{I}}(S, T, \boldsymbol{a}_{\mathcal{I}} u) = \mathbb{P}\left\{\exists t \in [S, T] : A_{\mathcal{I}}(t)\right\},$$

with

$$A_{\mathcal{I}}(t) = \{\boldsymbol{W}(t) - \boldsymbol{c}t \in \boldsymbol{E}_{\mathcal{I}}(\boldsymbol{a}u)\} = \{\forall i \in \mathcal{I} : \quad W_i(t) - c_i t \ge a_i u\}, \tag{2.1}$$

where $E_{\mathcal{I}}(\boldsymbol{a}u)$ was defined in (1.1). In vector notation for any $u \in \mathbb{R}$

$$\begin{aligned}
\psi_k(S, T, \boldsymbol{a}u) &= \mathbb{P}\left\{\exists t \in [S, T] : \bigcup_{\substack{\mathcal{I} \subset \{1,\ldots,d\} \\ |\mathcal{I}| = k}} A_{\mathcal{I}}(t)\right\} \\
&= \mathbb{P}\left\{\bigcup_{\substack{\mathcal{I} \subset \{1,\ldots,d\} \\ |\mathcal{I}| = k}} \{\exists t \in [S, T] : \ A_{\mathcal{I}}(t)\}\right\}.
\end{aligned}$$

The following lower bound (by Bonferroni inequality)

$$\begin{aligned}\psi_k(S,T,\boldsymbol{a}u) \geq & \sum_{\substack{\mathcal{I}\subset\{1,\ldots,d\}\\|\mathcal{I}|=k}} \psi_{\mathcal{I}}(S,T,\boldsymbol{a}_{\mathcal{I}}u) \\ & - \sum_{\substack{\mathcal{I},\mathcal{J}\subset\{1,\ldots,d\}\\|\mathcal{I}|=|\mathcal{J}|=k\\ \mathcal{I}\neq\mathcal{J}}} \mathbb{P}\left\{\exists t,s\in[S,T]:\ A_{\mathcal{I}}(t)\cap A_{\mathcal{J}}(s)\right\}\end{aligned} \tag{2.2}$$

together with the upper bound

$$\psi_k(S,T,\boldsymbol{a}u) \quad\leq \sum_{\substack{\mathcal{I}\subset\{1,\ldots,d\}\\|\mathcal{I}|=k}} \psi_{\mathcal{I}}(S,T,\boldsymbol{a}_{\mathcal{I}}u) \tag{2.3}$$

are crucial for the derivation of the exact asymptotics of $\psi_k(S,T,\boldsymbol{a}u)$ as $u\to\infty$. As we shall show below, the upper bound (2.3) turns out to be exact asymptotically as $u\to\infty$. The following theorem constitutes the main finding of this chapter.

Theorem 2.1 *Suppose that the square $d\times d$ real-valued matrix Γ is non-singular. If $\boldsymbol{a}$ has no more than $k-1$ non-positive components, where $k\leq d$ is a positive integer, then for all $0\leq S<T<\infty, \boldsymbol{c}\in\mathbb{R}^d$*

$$\psi_k(S,T,\boldsymbol{a}u) \quad\sim \sum_{\substack{\mathcal{I}\subset\{1,\ldots,d\}\\|\mathcal{I}|=k}} \psi_{\mathcal{I}}(0,T,\boldsymbol{a}_{\mathcal{I}}u), \quad u\to\infty. \tag{2.4}$$

Moreover, (2.4) holds also if $T=\infty$, provided that $\boldsymbol{c}$ and $\boldsymbol{a}+\boldsymbol{c}t$ have no more than $k-1$ non-positive components for all $t\geq 0$.

Essentially, the above result is the claim that the second term in the Bonferroni lower bound (2.2) is asymptotically negligible. In order to prove that, the asymptotics of $\psi_{|\mathcal{I}|}(S,T,\boldsymbol{a}_{\mathcal{I}}u)$ has to be derived. For the special case that $\mathcal{I}$ has only two elements and $S=0$, its approximation has been obtained in [19]. Note in passing that the assumption in Theorem 2.1 that $\boldsymbol{a}$ has no more than $k-1$ non-positive components excludes the case that there exists a set $\mathcal{I}\subset\{1,\ldots,d\},\ |\mathcal{I}|=k$ such that $\psi_{\mathcal{I}}(0,T,\boldsymbol{a}_{\mathcal{I}}u)$ does not tend to 0 as $u\to\infty$, which due to its non-rare event nature is out of interest in this chapter.

The next result extends the findings of [19] to the case $d>2$. For notational simplicity we consider the case $\mathcal{I}$ has d elements and thus avoid indexing by $\mathcal{I}$. Recall that in our model $\boldsymbol{W}(t)=\Gamma\boldsymbol{B}(t)$ where $\boldsymbol{B}(t)$ has independent standard Brownian motion components and Γ is a $d\times d$ non-singular real-valued matrix. Consequently $\Sigma=\Gamma\Gamma^\top$ is a positive definite matrix.

Hereafter $\mathbf{0} \in \mathbb{R}^d$ is the column vector with all elements equal 0. Denote by $\Pi_\Sigma(\boldsymbol{a})$ the quadratic programming problem:

$$\text{minimise } \boldsymbol{x}^\top \Sigma^{-1} \boldsymbol{x}, \text{ for all } \boldsymbol{x} \geq \boldsymbol{a}. \tag{2.5}$$

Its unique solution $\tilde{\boldsymbol{a}}$ is such that

$$\tilde{\boldsymbol{a}}_I = \boldsymbol{a}_I, \quad (\Sigma_{II})^{-1}\boldsymbol{a}_I > \mathbf{0}_I, \quad \tilde{\boldsymbol{a}}_J = \Sigma_{JI}(\Sigma_{II})^{-1}\boldsymbol{a}_I \geq \boldsymbol{a}_J, \tag{2.6}$$

where $\tilde{\boldsymbol{a}}_J$ is defined if $J = \{1, \ldots, d\} \setminus I$ is non-empty. The index set I is unique with $m = |I| \geq 1$ elements (see Lemma 4.6 in chapter 4, or [10][Lem 2.1]) for more details).

In the following we set

$$\boldsymbol{\lambda} = \Sigma^{-1}\tilde{\boldsymbol{a}}.$$

It is known that

$$\boldsymbol{\lambda}_I = (\Sigma_{II})^{-1}\boldsymbol{a}_I > \mathbf{0}_I, \quad \boldsymbol{\lambda}_J \geq \mathbf{0}_J, \tag{2.7}$$

with the convention that when J is empty the indexing should be disregarded so that the last inequality above is irrelevant.

The next theorem extends the main result in [19] and further complements findings presented in Theorem 2.1 showing that the simultaneous ruin probability (i.e., $k = d$) behaves up to some constant, asymptotically as $u \to \infty$ the same as $p_T(\boldsymbol{u})$. For notational simplicity and without loss of generality we consider next $T = 1$.

Theorem 2.2 *If $\boldsymbol{a} \in \mathbb{R}^d$ has at least one positive component and Γ is non-singular, then for all $S \in [0, 1)$*

$$\psi_d(S, 1, \boldsymbol{a}u) \quad \sim \quad C(\boldsymbol{a}) p_1(\boldsymbol{a}u), \quad u \to \infty, \tag{2.8}$$

where $C(\boldsymbol{a}) = \prod_{i \in I} \lambda_i \int_{\mathbb{R}^m} \mathbb{P}\{\exists_{t \geq 0} : \boldsymbol{W}_I(t) - t\boldsymbol{a}_I > \boldsymbol{x}_I\} e^{\boldsymbol{\lambda}_I^\top \boldsymbol{x}_I} \mathrm{d}\boldsymbol{x}_I \in (0, \infty)$.

Remarks 2.3 *i) By Lemma 4.6 below taking $T = 1$ therein (hereafter φ denotes the probability density function (pdf) of $\Gamma\boldsymbol{B}(1)$)*

$$\begin{aligned} p_1(\boldsymbol{a}u) &= \mathbb{P}\{\boldsymbol{W}(1) - \boldsymbol{c} > u\boldsymbol{a}\} \\ &\sim \prod_{i \in I} \lambda_i^{-1} \mathbb{P}\{\boldsymbol{W}_U(1) > \boldsymbol{c}_U | \boldsymbol{W}_I(1) > \boldsymbol{c}_I\} u^{-|I|} \varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c}) \end{aligned} \tag{2.9}$$

as $u \to \infty$, where $\boldsymbol{\lambda} = \Sigma^{-1}\tilde{\boldsymbol{a}}$ and if $J = \{1, \ldots, d\} \setminus I$ is non-empty, then $U = \{j \in J : \tilde{a}_j = a_j\}$. When J is empty the conditional probability related to U

above is set to 1.
ii) Combining Theorem 2.1 and 2.2 for all $S \in [0,1)$ and all $\boldsymbol{a} \in \mathbb{R}^d$ with no more than $k-1$ non-positive components we have as $u \to \infty$

$$\begin{aligned}\psi_k(S,1,\boldsymbol{a}u) &\sim \sum_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} C(\boldsymbol{a}_{\mathcal{I}})\psi_{|\mathcal{I}|}(0,1,\boldsymbol{a}_{\mathcal{I}}u) \\ &\sim C\mathbb{P}\left\{\forall_{i\in\mathcal{I}^*} : W_i(1) > ua_i + c_i\right\}\end{aligned} \tag{2.10}$$

for some $C>0$ and some $\mathcal{I}^* \subset \{1,\dots,d\}$ with k elements.
iii) Comparing the results of Theorem 2.2 and [10] we obtain

$$\limsup_{u\to\infty} \frac{(-\ln \psi_k(S_1,1,\boldsymbol{a}u))^{1/2}}{-\ln \psi_k(S_2,\infty,\boldsymbol{a}u)} < \infty$$

for all $S_1 \in [0,T], S_2 \in [0,\infty)$.
iv) Define the failure time (consider for simplicity $k=d$) for our multidimensional model by

$$\tau(u) = \inf\{t \geq 0 : \boldsymbol{W}(t) - t\boldsymbol{c} > \boldsymbol{a}u\}, \qquad u>0.$$

If $\boldsymbol{a}$ has at least one positive component, then for all $T>S\geq 0, x>0$

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(T-\tau(u)) \geq x | \tau(u) \in [S,T]\right\} = e^{-x\frac{\tilde{\boldsymbol{a}}^\top \Sigma^{-1}\tilde{\boldsymbol{a}}}{2T^2}}, \tag{2.11}$$

see the proof in Section 4.

3 Examples

In order to illustrate our findings we shall consider three examples assuming that $\Gamma\Gamma^\top$ is a positive definite correlation matrix. The first example is dedicated to the simplest case $k=1$. In the second one we discuss $k=2$ restricting $\boldsymbol{a}$ to have all components equal to 1 followed then by the last example where only the assumption $\Gamma\Gamma^\top$ is an equi-correlated correlation matrix is imposed. In this section $T=1$ and $S \in [0,1)$ is fixed.
Example 1 ($k=1$): Suppose that $\boldsymbol{a}$ has all components positive. In view of Theorem 2.1 we have that

$$\psi_k(S,1,\boldsymbol{a}u) \sim \sum_{i=1}^d \psi_{\{i\}}(0,1,a_iu)$$

as $u \to \infty$. Note that for any positive integer $i \le d$

$$\psi_{\{i\}}(0,1,a_i u) = \mathbb{P}\left\{\exists_{t\in[0,1]} : B(t) - c_i t > a_i u\right\},$$

where B is a standard Brownian motion. It follows easily that

$$\psi_k(S,1,\boldsymbol{a}u) \sim 2\sum_{i=1}^{d} \mathbb{P}\left\{B(1) > a_i u + c_i\right\}, \quad u \to \infty.$$

Example 2 ($k=2$ and $\boldsymbol{a}=\mathbf{1}$): Suppose next $k=2$ and $\boldsymbol{a}$ has all components equal 1. By Theorems 2.1 and 2.2 we have that

$$\psi_k(S,1,\mathbf{1}u) \sim \sum_{\{i,j\}\subset\{1,\dots,d\}} C_{i,j}(\mathbf{1})\mathbb{P}\left\{\min_{k\in\{i,j\}} (W_k(1) - c_k) > u)\right\}$$

as $u \to \infty$, where $\mathbf{1} \in \mathbb{R}^d$ has all components equal to 1. Using further Remark 2.3 we obtain as $u \to \infty$

$$\mathbb{P}\left\{\min_{k\in\{i,j\}} (W_k(1) - c_k) > u)\right\}$$
$$\sim \frac{u^{-2}}{(1-\rho_{i,j})^2\sqrt{2\pi(1-\rho_{i,j}^2)}} e^{-\frac{u^2}{1+\rho_{i,j}} - \frac{(c_i+c_j)u}{1+\rho_{i,j}} - \frac{c_i^2 - 2\rho_{i,j}c_ic_j + c_j^2}{2(1-\rho_{i,j}^2)}}.$$

Here we set $\rho_{i,j} = corr(W_i(1), W_j(1))$. Consequently, if $\rho_{i,j} > \rho_{i^*,j^*}$, then as $u \to \infty$

$$\mathbb{P}\left\{\min_{k\in\{i^*,j^*\}} (W_k(1) - c_k > u)\right\} = o\left(\mathbb{P}\left\{\min_{k\in\{i,j\}} (W_k(1) - c_k > u)\right\}\right).$$

The same holds also if $\rho_{i,j} = \rho_{i^*,j^*}$ and $c_i + c_j > c_{i^*} + c_{j^*}$. If we denote by τ the maximum of all $\rho_{i,j}$'s and by c_* the maximum of $c_i + c_j$ for all i,j's such that $\rho_{i,j} = \tau$, then we conclude that

$$\psi_k(S,1,\boldsymbol{a}u) \sim \sum_{i,j\in\{1,\dots,d\},\rho_{i,j}=\tau,\ c_i+c_j=c_*} C_{i,j}(\mathbf{1})\mathbb{P}\left\{\min_{k\in\{i,j\}} (W_k(1) - c_k > u)\right\}.$$

Note that in this case $C_{i,j}(\mathbf{1})$ does not depend on i and j and is equals to

$$(1-\tau)^2 \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\ge 0} : B_1(t) - t > x, B_2(t) - t > y\right\} e^{(1-\rho^2)(x+y)} \mathrm{d}x\mathrm{d}y,$$

where $(B_1(t), B_2(t)), t \geq 0$ is a 2-dimensional Gaussian process with B_i's being standard Brownian motions with constant correlation τ. Consequently, as $u \to \infty$

$$\psi_2(S, 1, \mathbf{1}u) \sim C_* u^{-2} e^{-\frac{u^2}{1+\tau} - \frac{c_* u}{2(1+\tau)}},$$

where

$$C_* = \frac{e^{-\frac{c_*^2}{2(1-\tau^2)}}}{\sqrt{2\pi(1-\tau^2)}} \sum_{i,j \in \{1,\dots,d\}, \rho_{i,j}=\tau,\ c_i+c_j=c_*} e^{\frac{c_i c_j}{1-\tau}}$$
$$\times \int_{\mathbb{R}^2} \mathbb{P}\left\{\exists_{t\geq 0} : B_1(t) - t > x, B_2(t) - t > y\right\} e^{(1-\tau^2)(x+y)} \mathrm{d}x\mathrm{d}y \in (0, \infty).$$

Example 3 (Equi-correlated risk model): We consider the matrix Γ such that $\Sigma = \Gamma\Gamma^\top$ is an equi-correlated non-singular correlation matrix with off-diagonal entries equal to $\rho \in (-1/(d-1), 1)$. Let $\boldsymbol{a} \in \mathbb{R}^d$ with at least one positive component and assume for simplicity that its components are ordered, i.e., $a_1 \geq a_2 \geq \cdots \geq a_d$ and thus $a_1 > 0$. The inverse of Σ equals

$$\left[J_d - \mathbf{1}\mathbf{1}^\top \frac{\rho}{1+\rho(d-1)}\right] \frac{1}{1-\rho},$$

where J_d is the identity matrix. First we determine the index set I corresponding to the unique solution of $\Pi_\Sigma(\boldsymbol{a})$. We have for this case that I with m elements is unique and in view of (2.6)

$$\boldsymbol{\lambda}_I = (\Sigma_{II})^{-1}\boldsymbol{a}_I = \frac{1}{1-\rho}\left[\boldsymbol{a}_I - \rho \frac{\sum_{i\in I} a_i}{1+\rho(m-1)} \mathbf{1}_I\right] > \mathbf{0}_I, \tag{3.1}$$

with $\mathbf{0} \in \mathbb{R}^d$ the origin. From the above $m = |I| = d$ if and only if

$$a_d > \rho \frac{\sum_{i=1}^d a_i}{1+\rho(d-1)},$$

which holds in the particular case that all a_i's are equal and positive.

When the above does not hold, the second condition on the index set I given in (2.6) reads

$$\Sigma_{JI}\Sigma_{II}^{-1}\boldsymbol{a}_I = \rho(\mathbf{1}\mathbf{1}^\top)_{JI}\Sigma_{II}^{-1}\boldsymbol{a}_I \geq \boldsymbol{a}_J.$$

Next, suppose that $a_i = a > 0, c_i = c \in \mathbb{R}$ for all $i \leq d$. In view of (2.10) for any positive integer $k \leq d$ and any $S \in [0, 1)$ we have

$$\psi_k(S, 1, \boldsymbol{a}u) \quad \sim \quad C\mathbb{P}\left\{\forall_{i\leq k} : W_i(1) > ua + c\right\}, \quad u \to \infty, \tag{3.2}$$

where (set $I=\{1,\ldots,k\}$)

$$C=\frac{d!}{k!(d-k)!}\prod_{i\in I}\lambda_i\int_{\mathbb{R}^k}\mathbb{P}\left\{\exists_{t\geq 0}:\boldsymbol{W}_I(t)-t\boldsymbol{a}_I>\boldsymbol{x}_I\right\}e^{\boldsymbol{\lambda}_I^\top x_I}\mathrm{d}\boldsymbol{x}_I\in(0,\infty).$$

Note that the case $\rho=0$ is treated in [5][Prop. 3.6] and follows as a special case of this example.

4 Proofs

4.1 Proof of Theorem 1.1

Our proof below is based on the idea of the proof of [44][Thm 1.1], where $\boldsymbol{c}$ has zero components, $k=d$ and $S=0$ has been considered. Recall the definition of sets $\boldsymbol{E}_{\mathcal{I}}(\boldsymbol{u})$ and $E(\boldsymbol{u})$ introduced in (1.1) for any non-empty $\mathcal{I}\subset\{1,\ldots,d\}$ such that $|\mathcal{I}|=k\leq d$. With this notation we have

$$\psi_k(S,T,\boldsymbol{u})=\mathbb{P}\left\{\exists t\in[S,T]:\boldsymbol{W}(t)-\boldsymbol{c}t\in\boldsymbol{E}(\boldsymbol{u})\right\}=\mathbb{P}\left\{\tau_k(\boldsymbol{u})\leq T\right\},$$

where $\tau_k(\boldsymbol{u})$ is the ruin time defined by

$$\tau_k(\boldsymbol{u})=\inf\{t\geq S:\boldsymbol{W}(t)-\boldsymbol{c}t\in\boldsymbol{E}(\boldsymbol{u})\}.$$

For the lower bound, we note that

$$\psi_k(S,T,\boldsymbol{u})=\mathbb{P}\left\{\exists t\in[S,T]:\boldsymbol{W}(t)-\boldsymbol{c}t\in\boldsymbol{E}(\boldsymbol{u})\right\}\geq\mathbb{P}\left\{\boldsymbol{W}(T)-\boldsymbol{c}T\in\boldsymbol{E}(\boldsymbol{u})\right\}.$$

By the fact that Brownian motion has continuous sample paths

$$\boldsymbol{W}(\tau_k(\boldsymbol{u}))-\boldsymbol{c}\tau_k(\boldsymbol{u})\in\partial\boldsymbol{E}(\boldsymbol{u})\tag{4.1}$$

almost surely, where ∂A stands for the topological boundary (frontier) of the set $A\subset\mathbb{R}^d$.
Consequently, by the strong Markov property of the Brownian motion, we can write further

$$\begin{aligned}\mathbb{P}&\left\{\boldsymbol{W}(T)-\boldsymbol{c}T\in\boldsymbol{E}(\boldsymbol{u})\right\}\\&=\int_0^T\int_{\partial\boldsymbol{E}(\boldsymbol{u})}\mathbb{P}\left\{\boldsymbol{W}(t)-\boldsymbol{c}t\in\mathrm{d}\boldsymbol{x}|\tau_k(\boldsymbol{u})\right\}\\&\quad\times\mathbb{P}\left\{\boldsymbol{W}(T)-\boldsymbol{c}T\in\boldsymbol{E}(\boldsymbol{u})|\boldsymbol{W}(t)-\boldsymbol{c}t=\boldsymbol{x}\right\}\mathbb{P}\{\tau_k(\boldsymbol{u})\in\mathrm{d}t\}.\end{aligned}$$

Crucial is that the boundary $\partial \boldsymbol{E}(\boldsymbol{u})$ can be represented as the following union

$$\partial \boldsymbol{E}(\boldsymbol{u}) = \bigcup_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} (\partial \boldsymbol{E}_{\mathcal{I}}(\boldsymbol{u}) \cap \partial \boldsymbol{E}(\boldsymbol{u})) =: \bigcup_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} F_{\mathcal{I}}(\boldsymbol{u}).$$

For every $\boldsymbol{x} \in F_{\mathcal{I}}(\boldsymbol{u})$ using the self-similarity of Brownian motion for all non-empty index sets $\mathcal{I} \subset \{1,\dots,d\}$ and all $t \in (S,T)$

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{W}(T) - \boldsymbol{c}T \in \boldsymbol{E}(\boldsymbol{u}) | \boldsymbol{W}(t) - \boldsymbol{c}t = \boldsymbol{x}\right\}\\
&\qquad \geq \mathbb{P}\left\{\boldsymbol{W}(T) - \boldsymbol{c}T \in \boldsymbol{E}_{\mathcal{I}}(\boldsymbol{u}) | \boldsymbol{W}(t) - \boldsymbol{c}t = \boldsymbol{x}\right\}\\
&\qquad = \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) - \boldsymbol{c}_{\mathcal{I}}T \geq \boldsymbol{u}_{\mathcal{I}} | \boldsymbol{W}(t) - \boldsymbol{c}t = \boldsymbol{x}\right\}\\
&\qquad \geq \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T-t) - \boldsymbol{c}_{\mathcal{I}}(T-t) \geq \boldsymbol{0}\right\}\\
&\qquad \geq \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T-t) \geq \boldsymbol{c}_{\mathcal{I}}(T-t)\right\}\\
&\qquad = \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(1) \geq \boldsymbol{c}_{\mathcal{I}}\sqrt{T-t}\right\}\\
&\qquad \geq \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(1) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}\sqrt{T}\right\}\\
&\qquad = \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}T\right\}\\
&\qquad \geq \min_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}T\right\},
\end{aligned}$$

where $\tilde{c}_i = \max(0, c_i)$, hence for all $\boldsymbol{x} \in \partial \boldsymbol{E}(\boldsymbol{u})$

$$\mathbb{P}\left\{\boldsymbol{W}(T) - \boldsymbol{c}T \in \boldsymbol{E}(\boldsymbol{u}) | \boldsymbol{W}(t) - \boldsymbol{c}t = \boldsymbol{x}\right\} \geq \min_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}T\right\}.$$

Consequently, using further (4.1) we obtain

$$\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{W}(T) - \boldsymbol{c}T \in \boldsymbol{E}(\boldsymbol{u})\right\}\\
&\quad \geq \min_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}T\right\}\\
&\qquad \times \int_S^T \int_{\partial \boldsymbol{E}(u)} \mathbb{P}\left\{\boldsymbol{W}(t) - \boldsymbol{c}t \in \mathrm{d}\boldsymbol{x} | \tau_k(\boldsymbol{u}) = t\right\} \mathbb{P}\left\{\tau_k(\boldsymbol{u}) \in \mathrm{d}t\right\}\\
&\quad = \min_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\ |\mathcal{I}|=k}} \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(T) \geq \tilde{\boldsymbol{c}}_{\mathcal{I}}T\right\} \psi_k(S,T,\boldsymbol{u}),
\end{aligned}$$

establishing the proof. $\square$

4.2 Proof of Theorem 2.1

The results in this section hold under the assumption that $\Sigma = \Gamma\Gamma^\top$ is positive definite, which is equivalent with our assumption that Γ is non-singular. The next lemma is a consequence of [34][Lem 2]. We recall that φ denotes the probability density function of $\Gamma \boldsymbol{B}(1)$.

Lemma 4.1 *For any $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ we have for some positive constants C_1, C_2 as $u \to \infty$*

$$\mathbb{P}\left\{\boldsymbol{W}(1) - \boldsymbol{c} > \boldsymbol{a}u\right\} \sim C_1 \mathbb{P}\left\{\forall_{i\in I} : W_i(1) - c_i > a_i u\right\} \sim C_2 u^{-\alpha} \varphi(\tilde{\boldsymbol{a}}u + \boldsymbol{c}),$$

where α is some integer and $\tilde{\boldsymbol{a}}$ is the solution of quadratic programming problem $\Pi_\Sigma(\boldsymbol{a}), \Sigma = \Gamma\Gamma^\top$ and I is the unique index set that determines the solution of $\Pi_\Sigma(\boldsymbol{a})$.

We agree in the following that if $\mathcal{I}$ is empty, then the term $A_{\mathcal{I}}(t)$ should be simply deleted from the expressions below; recall that $A_{\mathcal{I}}(t)$ is defined in (2.1).

We state next three lemmas utilised in the case $T < \infty$. Their proofs are displayed Section 5.

Lemma 4.2 *Let $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$ be two index sets such that $\mathcal{I} \neq \mathcal{J}$ and $|\mathcal{I}| = |\mathcal{J}| = k \geq 1$. If $\boldsymbol{a}_{\mathcal{I}\cup\mathcal{J}}$ has at least two positive components, then for any $s, t \in [0, 1]$ there exists some $\nu = \nu(s, t) > 0$ such that as $u \to \infty$*

$$\mathbb{P}\left\{A_{\mathcal{I}}(t) \cap A_{\mathcal{J}}(s)\right\} = o\left(e^{-\nu u^2}\right) \sum_{\substack{\mathcal{I}^* \subset \{1,\ldots,d\} \\ |\mathcal{I}^*| = k}} \mathbb{P}\left\{A_{\mathcal{I}^*}(1)\right\}, \tag{4.2}$$

and

$$\mathbb{P}\left\{A_{\mathcal{I}\setminus\mathcal{J}}(t), A_{\mathcal{J}\setminus\mathcal{I}}(s), A_{\mathcal{I}\cap\mathcal{J}}(\min(t, s))\right\} = o\left(e^{-\nu u^2}\right) \sum_{\substack{\mathcal{I}^* \subset \{1,\ldots,d\} \\ |\mathcal{I}^*| = k}} \mathbb{P}\left\{A_{\mathcal{I}^*}(1)\right\}. \tag{4.3}$$

Lemma 4.3 *Let $S > 0$, $k \leq d$ be a positive integer and let $\boldsymbol{a} \in \mathbb{R}^d$ be given. If $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$ are two different index sets with $k \geq 1$ elements such that $\boldsymbol{a}_{\mathcal{I}\cup\mathcal{J}}$ has at least one positive component, then there exist $s_1, s_2 \in [S, 1]$ and some positive constant τ such that as $u \to \infty$*

$$\begin{aligned} &\mathbb{P}\left\{\exists s, t \in [S, 1] : A_{\mathcal{I}}(s) \cap A_{\mathcal{J}}(t)\right\} \\ &\qquad = o\left(e^{\tau u}\right) \mathbb{P}\left\{A_{\mathcal{I}\setminus\mathcal{J}}(s_1) \cap A_{\mathcal{J}\setminus\mathcal{I}}(s_2) \cap A_{\mathcal{I}\cap\mathcal{J}}(\min(s_1, s_2))\right\}. \end{aligned} \tag{4.4}$$

Case $T<\infty$. According to Theorem 1.1 and Lemma 4.1 it is enough to show the proof for $S\in(0,T)$. In view of the self-similarity of Brownian motion we assume for simplicity $T=1$. Recall that in our notation $\Sigma=\Gamma\Gamma^\top$ is the covariance matrix of $\boldsymbol{W}(1)$ which is non-singular and we denote its pdf by φ. In view of (4.3) and (4.4) for all $S\in(0,1)$ there exists some $\nu>0$ such that as $u\to\infty$

$$\sum_{\substack{\mathcal{I},\mathcal{J}\subset\{1,\dots,d\}\\|\mathcal{I}|=|\mathcal{J}|=k,\mathcal{I}\neq\mathcal{J}}}\mathbb{P}\left\{\exists s,t\in[S,1]:A_{\mathcal{I}}(s)\cap A_{\mathcal{J}}(t)\right\}=o\left(e^{-\nu u^2}\right)\sum_{\substack{\mathcal{I}\subset\{1,\dots,d\}\\|\mathcal{I}|=k}}\mathbb{P}\left\{A_{\mathcal{I}}(1)\right\}.$$

Note that we may utilise (4.3) and (4.4) for sets $\mathcal{I}$ and $\mathcal{J}$ of length k, because of the assumption that $\boldsymbol{a}$ has no more than $k-1$ non-positive components. Hence any vector $\boldsymbol{a}_{\mathcal{I}}$ has at least one positive component.

Further, by Theorem 1.1 and the inclusion-exclusion formula we have that for some $K>0$ and all u sufficiently large

$$\psi_k(S,1,\boldsymbol{u})\le K\sum_{\substack{I\subset\{1,\dots,d\}\\|I|=k}}\mathbb{P}\left\{A_{\mathcal{I}}(1)\right\}.$$

Hence the claim follows from (2.2) and (2.3).

Case $T=\infty$. Using the self-similarity of Brownian motion we have

$$\begin{aligned}\mathbb{P}\left\{\exists t>0:A_{\mathcal{I}}(t)\right\}&=\mathbb{P}\left\{\exists t>0:\boldsymbol{W}_{\mathcal{I}}(ut)\ge(\boldsymbol{a}+\boldsymbol{c}t)_{\mathcal{I}}u\right\}\\&=\mathbb{P}\left\{\exists t>0:\boldsymbol{W}_{\mathcal{I}}(t)\ge(\boldsymbol{a}+\boldsymbol{c}t)_{\mathcal{I}}\sqrt{u}\right\}\\&=\mathbb{P}\left\{\exists t>0:A^*_{\mathcal{I}}(t)\right\},\end{aligned}$$

where

$$A^*_{\mathcal{I}}(t)=\{\boldsymbol{W}_{\mathcal{I}}(t)\ge(\boldsymbol{a}+\boldsymbol{c}t)_{\mathcal{I}}\sqrt{u}\}.\tag{4.5}$$

For $t>0$ define

$$r_{\mathcal{I}}(t)=\min_{\boldsymbol{x}\ge\boldsymbol{a}_{\mathcal{I}}+\boldsymbol{c}_{\mathcal{I}}t}\frac{1}{t}\boldsymbol{x}^\top\Sigma^{-1}_{\mathcal{I}\mathcal{I}}\boldsymbol{x},\quad\Sigma_{\mathcal{I}\mathcal{I}}=Var(\boldsymbol{W}_{\mathcal{I}}(1)),\quad\Sigma^{-1}_{\mathcal{I}\mathcal{I}}=(\Sigma_{\mathcal{I}\mathcal{I}})^{-1}.\tag{4.6}$$

Since $\lim_{t\downarrow0}r_{\mathcal{I}}(t)=\infty$ we set below $r_{\mathcal{I}}(0)=\infty$.

In view of Lemma 4.1 we have as $u\to\infty$

$$\mathbb{P}\left\{A^*_{\mathcal{I}}(t)\right\}\sim C_1u^{-\alpha/2}\varphi_{\mathcal{I},t}((\widetilde{\boldsymbol{a}_{\mathcal{I}}+\boldsymbol{c}_{\mathcal{I}}}t)\sqrt{u})=C_2u^{-\alpha/2}e^{-\frac{r_{\mathcal{I}}(t)u}{2}},$$

where $\widetilde{\boldsymbol{a}_{\mathcal{I}}+\boldsymbol{c}_{\mathcal{I}}}t$ is the solution of quadratic programming problem

$$\Pi_{t\Sigma_{\mathcal{I}\mathcal{I}}}(\boldsymbol{a}_{\mathcal{I}}+\boldsymbol{c}_{\mathcal{I}}t)$$

and $\varphi_{\mathcal{I},t}(\boldsymbol{x})$ is the pdf of $\boldsymbol{W}_{\mathcal{I}}(t)$, α is some integer and C_1, C_2 are positive constant that do not depend on u. For notational simplicity we shall omit below the subscript $\mathcal{I}$.

The rest of the proof is established by utilising the following lemmas, whose proofs are displayed in Section 5.

Lemma 4.4 *Let $k \le d$ be a positive integer and let $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^d$. Consider two different sets $\mathcal{I}, \mathcal{J} \subset \{1 \dots d\}$ of cardinality k. If both $\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} t$ and $\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} t$ have at least one positive component for all $t > 0$ and both $\boldsymbol{c}_{\mathcal{I}}$ and $\boldsymbol{c}_{\mathcal{J}}$ also have at least one positive component, then if*

$$\hat{t}_{\mathcal{I}} := arg \min_{t>0} \ r_{\mathcal{I}}(t) \neq \hat{t}_{\mathcal{J}} := arg \min_{t>0} \ r_{\mathcal{J}}(t)$$

we have

$$\mathbb{P}\left\{\exists s, t > 0 : \ A^*_{\mathcal{I}}(t) \cap A^*_{\mathcal{J}}(s)\right\} = o(\mathbb{P}\left\{A^*_{\mathcal{I}}(\hat{t}_{\mathcal{I}})\right\} + \mathbb{P}\left\{A^*_{\mathcal{J}}(\hat{t}_{\mathcal{J}})\right\}), \quad u \to \infty.$$

Lemma 4.5 *Under the settings of Lemma 4.4, if $\boldsymbol{a} + \boldsymbol{c}t$ has no more than $k-1$ non-positive component for all $t > 0$ and $\boldsymbol{c}$ has no more than $k-1$ non-positive components, then in case $\hat{t}_{\mathcal{I}} := arg \min_{t>0} \ r_{\mathcal{I}}(t) = \hat{t}_{\mathcal{J}} := arg \min_{t>0} \ r_{\mathcal{J}}(t)$*

$$\mathbb{P}\left\{\exists s, t > 0 : \ A^*_{\mathcal{I}}(t) \cap A^*_{\mathcal{J}}(s)\right\} = o\left(\sum_{\substack{\mathcal{K} \subset \{1 \dots d\} \\ |\mathcal{K}| = k}} \mathbb{P}\left\{A^*_{\mathcal{K}}(\hat{t}_{\mathcal{K}})\right\}\right), \quad u \to \infty.$$

Combining the above two lemmas we have that for any two index sets $\mathcal{I}, \mathcal{J} \subset \{1, \dots, d\}$ of cardinality k, there is some index set $\mathcal{K} \subset \{1, \dots, d\}$ such that as $u \to \infty$

$$\mathbb{P}\left\{\exists s, t > 0 : A^*_{\mathcal{I}}(s) \cap A^*_{\mathcal{J}}(t)\right\} = o\left(\mathbb{P}\left\{\exists t > 0 : A^*_{\mathcal{K}}(t)\right\}\right),$$

which is equivalent with

$$\mathbb{P}\left\{\exists s, t > 0 : A_{\mathcal{I}}(s) \cap A_{\mathcal{J}}(t)\right\} = o\left(\mathbb{P}\left\{\exists t > 0 : A_{\mathcal{K}}(t)\right\}\right).$$

The proof follows now by (2.2) and (2.3). □

4.3 Proof of Theorem 2.2

Below we set

$$\delta(u,\Lambda) := 1 - \Lambda u^{-2}$$

and denote by $\tilde{\boldsymbol{a}}$ the unique solution of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{a})$.

We denote below by I the index set that determines the unique solution of $\Pi_\Sigma(\boldsymbol{a})$, where $\boldsymbol{a} \in \mathbb{R}^d$ has at least one positive component. If $J = \{1,\ldots,d\} \setminus I$ is non-empty, then we set below $U = \{j \in J : \tilde{a}_j = a_j\}$. The number of elements $|I|$ of I is denoted by m, which is a positive integer.

The next lemma is proved in Section 5.

Lemma 4.6 *For any $\Lambda > 0$, $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d, \boldsymbol{c} \in \mathbb{R}^d$ and all sufficiently large u there exist $C > 0$ such that*

$$\begin{aligned} m(u,\Lambda) &:= \mathbb{P}\left\{\exists_{t\in[0,\delta(u,\Lambda)]} : \boldsymbol{W}(t) - t\boldsymbol{c} > u\boldsymbol{a}\right\} \\ &\le e^{-\Lambda/C} \frac{\mathbb{P}\{\boldsymbol{W}(1) \ge \boldsymbol{a}u + \boldsymbol{c}\}}{\mathbb{P}\{\boldsymbol{W}(1) > \max(\boldsymbol{c}, 0)\}} \end{aligned} \tag{4.7}$$

and further

$$\begin{aligned} M(u,\Lambda) &:= \mathbb{P}\left\{\exists_{t\in[\delta(u,\Lambda),1]} : \boldsymbol{W}(t) - t\boldsymbol{c} > u\boldsymbol{a}\right\} \\ &\sim C(\boldsymbol{c}) K([0,\Lambda]) u^{-m} \varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c}), \end{aligned} \tag{4.8}$$

where $C(\boldsymbol{c}) = \mathbb{P}\{\boldsymbol{W}_U(1) > \boldsymbol{c}_U | \boldsymbol{W}_I(1) > \boldsymbol{c}_I\}$ and for $\boldsymbol{\lambda} = \Sigma^{-1}\tilde{\boldsymbol{a}}$

$$E([\Lambda_1,\Lambda_2]) = \int_{\mathbb{R}^m} \mathbb{P}\left\{\exists_{t\in[\Lambda_1,\Lambda_2]} : \boldsymbol{W}_I(t) - t\boldsymbol{a}_I > \boldsymbol{x}_I\right\} e^{\boldsymbol{\lambda}_I^\top \boldsymbol{x}_I} \mathrm{d}\boldsymbol{x}_I \in (0,\infty)$$

for all constants $\Lambda_1 < \Lambda_2$. We set $C(\boldsymbol{c})$ equal 1 if U defined in Remark 2.3 is empty. Further we have

$$\lim_{\Lambda\to\infty} E([0,\Lambda]) = \int_{\mathbb{R}^m} \mathbb{P}\{\exists_{t\ge 0} : \boldsymbol{W}_I(t) - t\boldsymbol{a}_I > \boldsymbol{x}_I\} e^{\boldsymbol{\lambda}_I^\top \boldsymbol{x}_I} \mathrm{d}\boldsymbol{x}_I \in (0,\infty). \tag{4.9}$$

First note that for all Λ, u positive

$$M(u,\Lambda) \le \mathbb{P}\left\{\exists_{t\in[0,1]} : \boldsymbol{W}(t) - t\boldsymbol{c} > u\boldsymbol{a}\right\} \le M(u,\Lambda) + m(u,\Lambda).$$

In view of Lemmas 4.6 and 4.1

$$\lim_{\Lambda\to\infty} \lim_{u\to\infty} \frac{m(u,\Lambda)}{M(u,\Lambda)} = 0,$$

hence

$$\lim_{\Lambda\to\infty} \lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists_{t\in[0,1]} : \boldsymbol{W}(t) - t\boldsymbol{c} > u\boldsymbol{a}\right\}}{M(u,\Lambda)} = 1,$$

and the proof follows applying (4.8). □

4.4 Proof of Eq. (2.11)

The proof is similar to that of [12][Thm 2.5] and therefore we highlight only the main steps. If $T > S \ge 0$ by the definition of $\tau(u)$ and the self-similarity of Brownian motion

$$\begin{aligned}\frac{\tau(u)}{T} &= \inf\{t \ge 0 : \boldsymbol{W}(Tt) - tT\boldsymbol{c} > \boldsymbol{a}u\}\\ &= \inf\{t \ge 0 : \boldsymbol{W}(t) - t\sqrt{T}\boldsymbol{c} > \boldsymbol{a}u/\sqrt{T}\}.\end{aligned}$$

Thus, without loss of generality in the rest of the proof we suppose that $T = 1 > S \ge 0$.

We note that

$$\begin{aligned}\mathbb{P}\left\{u^2(1-\tau(u)) \ge x | \tau(u) \in [S,1]\right\} &= \frac{\mathbb{P}\left\{u^2(1-\tau(u)) \ge x, \tau(u) \in [S,1]\right\}}{\mathbb{P}\left\{\tau(u) \in [S,1]\right\}}\\ &= \frac{\mathbb{P}\left\{u^2(1-\tau(u)) \ge x, \tau(u) \le 1\right\}}{\mathbb{P}\left\{\tau(u) \in [S,1]\right\}} - \frac{\mathbb{P}\left\{u^2(1-\tau(u)) \ge x, \tau(u) \le S\right\}}{\mathbb{P}\left\{\tau(u) \in [S,1]\right\}}\\ &= P_1(u) - P_2(u).\end{aligned}$$

Next, for $\tilde{x}(u) = 1 - \frac{x}{u^2}$

$$\begin{aligned}P_1(u) &= \frac{\mathbb{P}\left\{\tau(u) \le \tilde{x}(u)\right\}}{\mathbb{P}\left\{\tau(u) \in [S,1]\right\}} \sim \frac{\mathbb{P}\left\{\exists_{t\in[0,\tilde{x}(u)]} : \boldsymbol{W}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\exists_{t\in[0,1]} : \boldsymbol{W}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}\\ &= \frac{\mathbb{P}\left\{\exists_{t\in[0,1]} : \boldsymbol{W}(t) - (\boldsymbol{c}\sqrt{\tilde{x}(u)})t > \frac{u}{\sqrt{\tilde{x}(u)}}\boldsymbol{a}\right\}}{\mathbb{P}\left\{\exists_{t\in[0,1]} : \boldsymbol{W}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}, \quad u \to \infty.\end{aligned}$$

Hence by Theorem 2.2, using the fact that

$$\varphi\left(\frac{u}{\sqrt{\tilde{x}(u)}}\tilde{\boldsymbol{a}} + (\boldsymbol{c}\sqrt{\tilde{x}(u)})\right) = \varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c})e^{-\frac{1}{2}\left(\frac{1}{\tilde{x}(u)}-1\right)u^2\tilde{\boldsymbol{a}}^\top\Sigma^{-1}\tilde{\boldsymbol{a}}}e^{-\frac{1}{2}(\tilde{x}(u)-1)\boldsymbol{c}^\top\Sigma^{-1}\boldsymbol{c}}$$

and

$$\lim_{u\to\infty} e^{-\frac{1}{2}\left(\frac{1}{\tilde{x}(u)}-1\right)u^2\tilde{\boldsymbol{a}}^\top\Sigma^{-1}\tilde{\boldsymbol{a}}} = e^{-x\frac{\tilde{\boldsymbol{a}}^\top\Sigma^{-1}\tilde{\boldsymbol{a}}}{2}}, \quad \lim_{u\to\infty} e^{-\frac{1}{2}(\tilde{x}(u)-1)\boldsymbol{c}^\top\Sigma^{-1}\boldsymbol{c}} = 1$$

we obtain

$$\lim_{u\to\infty} P_1(u) = e^{-x\frac{\tilde{\boldsymbol{a}}^\top\Sigma^{-1}\tilde{\boldsymbol{a}}}{2}}. \tag{4.10}$$

Moreover, following the same reasons as above

$$P_2(u) = \frac{\mathbb{P}\left\{\tau(u) \le S\right\}}{\mathbb{P}\left\{\tau(u) \in [S,1]\right\}} \quad \sim \quad \frac{\mathbb{P}\left\{\tau(u) \le S\right\}}{\mathbb{P}\left\{\tau(u) \le 1\right\}} \to 0 \tag{4.11}$$

as $u \to \infty$. Thus, combination of (4.10) with (4.11) leads to

$$\lim_{u\to\infty} \mathbb{P}\left\{u^2(1-\tau(u)) \ge x | \tau(u) \in [S,1]\right\} = e^{-x\frac{\tilde{\boldsymbol{a}}^\top \Sigma^{-1}\tilde{\boldsymbol{a}}}{2}}.$$

□

5 Appendix

Lemma 5.1 *If for $\boldsymbol{a} \in (\mathbb{R} \cup \{-\infty\})^d$ and $\mathcal{I} \subset \{1,\dots,d\}$ such that $\boldsymbol{a}_{\mathcal{I}}$ has at least two positive components and Γ is non-singular, then for all $t > 0$*

$$\mathbb{P}\left\{A_{\mathcal{I}}(t)\right\} = o\left(e^{-\nu u^2}\right) \sum_{i\in\mathcal{I}} \mathbb{P}\left\{A_{\mathcal{I}\setminus\{i\}}(t)\right\}, \quad u \to \infty,$$

where $\nu = \nu(t, \mathcal{I}) > 0$ does not depend on u.

Remark 5.2 *Lemma 5.1 implies that for any vector $\boldsymbol{a} \in (\mathbb{R} \cup \{-\infty\})^d$ and for any d-dimensional Gaussian random vector $\boldsymbol{W}$, if $\boldsymbol{a}$ has at least two positive components, there exists some positive constant η and $i \in \{1 \dots d\}$ such that as $u \to \infty$*

$$\mathbb{P}\left\{\boldsymbol{W} > \boldsymbol{a}u\right\} = o(e^{-\eta u^2})\mathbb{P}\left\{\boldsymbol{W}_K > \boldsymbol{a}_K u\right\}, \quad K = \{1,\dots,d\} \setminus \{i\}.$$

Proof of Lemma 5.1: For notational simplicity we shall assume that $\mathcal{I} = \{1,\dots,d\}$ and set $K_i = \mathcal{I} \setminus \{i\}$. By the assumption for all $i \in \mathcal{I}$ the vector $\boldsymbol{a}_{K_i}$ has at least one positive component and $\Sigma = \Gamma\Gamma^\top$ is positive definite. In view of Lemma 4.1 for any fixed $t > 0$ and some C_1, C_2 two positive constants we have

$$\mathbb{P}\left\{A_{\mathcal{I}}(t)\right\} \sim C_1 u^{\alpha_1} \varphi_t(\tilde{\boldsymbol{a}}u + \boldsymbol{c}), \quad \mathbb{P}\left\{A_{K_i}(t)\right\} \sim C_2 u^{\alpha_2} \varphi_t(\bar{\boldsymbol{a}}_i u + \boldsymbol{c}), \quad u \to \infty,$$

where φ_t is the pdf of $\boldsymbol{W}(t)$ with covariance matrix $\Sigma(t) = t\Sigma$ and

$$\tilde{\boldsymbol{a}} = arg \min_{\boldsymbol{x} \ge \boldsymbol{a}} \boldsymbol{x}^\top \Sigma^{-1}(t)\boldsymbol{x},$$
$$\bar{\boldsymbol{a}}_i = arg \min_{\boldsymbol{x} \in S_i} \boldsymbol{x}^\top \Sigma^{-1}(t)\boldsymbol{x},$$

with $S_i = \{\boldsymbol{x} \in \mathbb{R}^d : \ \forall j \in K_i : x_j \ge a_j\}$. Since $\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \ge \boldsymbol{a}\} \subset S_i$, it is clear that

$$\tilde{\boldsymbol{a}}^\top \Sigma^{-1}(t)\tilde{\boldsymbol{a}} \ge \bar{\boldsymbol{a}}_i{}^\top \Sigma^{-1}(t)\bar{\boldsymbol{a}}_i$$

for any $i \leq d$. Next, if we have strict inequality for some $i \leq d$, i.e., $\tilde{\boldsymbol{a}}^\top \Sigma^{-1}(t)\tilde{\boldsymbol{a}} > \bar{\boldsymbol{a}}_i^\top \Sigma^{-1}(t)\bar{\boldsymbol{a}}_i$, then it follows that

$$\mathbb{P}\left\{A_{\mathcal{I}}(t)\right\} \sim Cu^{\alpha_1}\varphi_t(\tilde{\boldsymbol{a}}u + \boldsymbol{c}) = o\left(e^{-\nu u^2}\mathbb{P}\left\{A_{K_i}(t)\right\}\right), \quad u \to \infty$$

for $\nu = \frac{1}{2}\left(\tilde{\boldsymbol{a}}^\top \Sigma^{-1}(t)\tilde{\boldsymbol{a}} - \bar{\boldsymbol{a}}_i^\top \Sigma^{-1}(t)\bar{\boldsymbol{a}}_i\right) > 0$, hence the claim follows.
Let us consider now the extreme case that for all $i \leq d$ we have $\tilde{\boldsymbol{a}}^\top \Sigma^{-1}\tilde{\boldsymbol{a}} = \bar{\boldsymbol{a}}_i^\top \Sigma^{-1}\bar{\boldsymbol{a}}_i$. As we know that each $\bar{\boldsymbol{a}}_i$ is unique, then $\bar{\boldsymbol{a}}_i = \tilde{\boldsymbol{a}}$ for all $i \in \mathcal{I}$. Consider set

$$E = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x}^\top \Sigma^{-1}(t)\boldsymbol{x} \leq \tilde{\boldsymbol{a}}^\top \Sigma^{-1}(t)\tilde{\boldsymbol{a}}\}.$$

Since $\Sigma(t)$ is positive definite, E is a full dimensional ellipsoid in $\mathbb{R}^d$. By the definition, $E \cap S_i = \{\tilde{\boldsymbol{a}}\}$. Define the following lines in $\mathbb{R}^d$

$$l_i = \{\boldsymbol{x} \in \mathbb{R}^d : \forall i \in K_i, x_i = \tilde{a}_i\}$$

and observe that since $l_i \in S_i$, we have $l_i \cap E = \{\tilde{\boldsymbol{a}}\}$, and they are linearly independent. Since E is smooth, there can not be more than $d-1$ linearly independent tangent lines at the point $\tilde{\boldsymbol{a}}$, which leads to a contradiction.

□

Proof of Lemma 4.2: First note that since $\mathcal{I} \neq \mathcal{J}$, then $|\mathcal{I} \cup \mathcal{J}| \geq k+1$. Consequently, we can find some index set $\mathcal{K}$ such that

$$|\mathcal{K}| = k+1, \quad \mathcal{K} \subset \mathcal{I} \cup \mathcal{J}$$

and further $\boldsymbol{a}_{\mathcal{K}}$ has at least two positive components. Applying Lemma 5.1 for any $t \in [0,1]$ and some $\nu > 0$

$$\mathbb{P}\left\{A_{\mathcal{K}}(t)\right\} = o\left(e^{-\nu u^2}\right)\sum_{j \in \mathcal{K}} \mathbb{P}\left\{A_{\mathcal{K}\setminus\{j\}}(t)\right\}, \quad u \to \infty.$$

If $s = t$, then applying Lemma 4.1

$$\begin{aligned} 0 \leq \mathbb{P}\left\{A_{\mathcal{I}}(t) \cap A_{\mathcal{J}}(t)\right\} &= \mathbb{P}\left\{A_{\mathcal{I}\cup\mathcal{J}}(t)\right\} \leq \mathbb{P}\left\{A_{\mathcal{K}}(t)\right\} \\ &= o\left(e^{-\nu u^2}\right) \sum_{\substack{\mathcal{I}^* \subset \{1,\dots,d\} \\ \mathcal{I}^*|=k}} \mathbb{P}\left\{A^*_{\mathcal{I}}(t)\right\}. \end{aligned}$$

Next, if $s < 1$, then applying Lemma 4.1 we obtain

$$0 \leq \mathbb{P}\left\{A_{\mathcal{I}}(t) \cap A_{\mathcal{J}}(s)\right\} \leq \mathbb{P}\left\{A_{\mathcal{J}}(s)\right\} = o\left(e^{-\nu u^2}\mathbb{P}\left\{A_{\mathcal{J}}(1)\right\}\right)$$

$$= o\left(e^{-\nu u^2}\right) \sum_{\substack{\mathcal{I}^* \subset \{1,\ldots,d\} \\ |\mathcal{I}^*|=k}} \mathbb{P}\left\{A_{\mathcal{I}^*}(1)\right\}.$$

A similar asymptotic bound follows for $t < 1$, whereas if $s = t = 1$, the first claim follows directly from the case $s = t$ discussed above. We show next (4.3). If $s < t$, then $s < 1$ and applying Lemma 4.1 we obtain

$$\begin{aligned} 0 &\leq \mathbb{P}\left\{A_{\mathcal{I}\setminus\mathcal{J}}(t), A_{\mathcal{J}\setminus\mathcal{I}}(s), A_{\mathcal{I}\cap\mathcal{J}}(\min(t,s))\right\} \\ &\leq \mathbb{P}\left\{A_{\mathcal{J}}(s)\right\} = o\left(e^{-\nu u^2}\mathbb{P}\left\{A_{\mathcal{J}}(1)\right\}\right) \\ &= o\left(e^{-\nu u^2}\right) \sum_{\substack{\mathcal{K} \subset \{1,\ldots,d\} \\ |\mathcal{K}|=k}} \mathbb{P}\left\{A_{\mathcal{K}}(1)\right\}. \end{aligned}$$

A similar asymptotic bound follows for $t < s$ or $s = t \leq 1$ by applying (4.2) establishing the proof. □

Proof of Lemma 4.3: Define for $s, t \in [S, 1]$ the Gaussian random vector

$$\mathcal{W}(s,t) = (\boldsymbol{W}_{\mathcal{I}\setminus\mathcal{J}}(s)^\top, \boldsymbol{W}_{\mathcal{J}\setminus\mathcal{I}}(t)^\top, \boldsymbol{W}_{\mathcal{I}\cap\mathcal{J}}(\min(s,t))^\top)^\top,$$

with covariance matrix $D(s,t)$. We show first that this matrix is positive definite. For this we assume that $s \leq t$. As $D(s,t)$ is some covariance matrix, we know that it is non-negative definite. Choose some vector $\boldsymbol{v} \in \mathbb{R}^d$. It is sufficient to show that if $\boldsymbol{v}^\top D(s,t)\boldsymbol{v} = \boldsymbol{0}$, then $\boldsymbol{v} = \boldsymbol{0}$ ($\boldsymbol{0} := (0,\ldots,0)^\top \in \mathbb{R}^d$). Note that

$$\begin{aligned} \boldsymbol{v}^\top D(s,t)\boldsymbol{v} &= Var(\langle \mathcal{W}(s,t), \boldsymbol{v}\rangle) \\ &= Var(\langle \boldsymbol{W}(s), \boldsymbol{v}\rangle + \langle \boldsymbol{W}_{\mathcal{J}\setminus\mathcal{I}}(t) - \boldsymbol{W}_{\mathcal{J}\setminus\mathcal{I}}(s), \boldsymbol{v}_{\mathcal{J}\setminus\mathcal{I}}\rangle). \end{aligned}$$

Using that $\boldsymbol{W}(t)$ has independent increments, this variance is equal to the sum of the variances. Hence, both of them should be equal to zero. In particular it means that $Var(\langle \boldsymbol{W}(s), \boldsymbol{v}\rangle) = 0$. Hence, as $s \geqslant S > 0$, we have that $\boldsymbol{v} = \boldsymbol{0}$. Thus, $D(s,t)$ is positive definite and $D^{-1}(s,t)$ exists.
Set further

$$\mathfrak{a} = (\boldsymbol{a}_{\mathcal{I}\setminus\mathcal{J}}^\top, \boldsymbol{a}_{\mathcal{J}\setminus\mathcal{I}}^\top, \boldsymbol{a}_{\mathcal{I}\cap\mathcal{J}}^\top)^\top, \quad \mathfrak{c}(s,t) = (s\boldsymbol{c}_{\mathcal{I}\setminus\mathcal{J}}^\top, t\boldsymbol{c}_{\mathcal{J}\setminus\mathcal{I}}^\top, \min(s,t)\boldsymbol{c}_{\mathcal{I}\cap\mathcal{J}}^\top)^\top.$$

With this notation we have

$$\mathbb{P}\left\{\exists s,t \in [S,1] : A_{\mathcal{I}}(s) \cap A_{\mathcal{J}}(t)\right\} \leq \mathbb{P}\left\{\exists s,t \in [S,1] : \mathcal{W}(s,t) - \mathfrak{c}(s,t) \geq \mathfrak{a}u\right\}.$$

Let $\tilde{\mathfrak{a}}(s,t) = arg\min_{\boldsymbol{x}\geq\mathfrak{a}} \boldsymbol{x}^\top D^{-1}(s,t)\boldsymbol{x}$ be the unique solution of $\Pi_{D(s,t)}(\mathfrak{a})$ and let further $\mathfrak{w}(s,t) = D^{-1}(s,t)\tilde{\mathfrak{a}}(s,t)$ be the solution of the dual problem. We denote by $I(s,t)$ the index set related to the quadratic programming problem $\Pi_{D(s,t)}(\mathfrak{a})$. Then $\mathfrak{w}(s,t)$ has non-negative components and according to the properties of quadratic programming problems, since both $s,t \ge S > 0$ we have

$$\mathfrak{a}^\top \mathfrak{w}(s,t) = \tilde{\mathfrak{a}}^\top(s,t)\mathfrak{w}(s,t) = \tilde{\mathfrak{a}}^\top(s,t)D^{-1}(s,t)\tilde{\mathfrak{a}}(s,t) > 0.$$

Consequently, we have

$$\begin{aligned}
&\mathbb{P}\left\{\exists s,t \in [S,1] : \mathcal{W}(s,t) - \mathfrak{c}(s,t) \ge \mathfrak{a}u\right\}\\
&\quad \le \mathbb{P}\left\{\exists s,t \in [S,1] : \mathfrak{w}^\top(s,t)\left(\mathcal{W}(s,t) - \mathfrak{c}(s,t)\right) \ge u\mathfrak{w}^\top(s,t)\tilde{\mathfrak{a}}(s,t)\right\}\\
&\quad = \mathbb{P}\left\{\exists s,t \in [S,1] : \frac{\mathfrak{w}^\top(s,t)\left(\mathcal{W}(s,t) - \mathfrak{c}(s,t)\right)}{\mathfrak{w}^\top(s,t)\tilde{\mathfrak{a}}(s,t)} \ge u\right\}\\
&\quad \le \mathbb{P}\left\{\exists s,t \in [S,1] : \frac{\mathfrak{w}^\top(s,t)\mathcal{W}(s,t)}{\mathfrak{w}^\top(s,t)\tilde{\mathfrak{a}}(s,t)} \ge u + \mathfrak{C}\right\}
\end{aligned}$$

for any positive u, where $\mathfrak{C} = \min_{s,t\in[S,1]} \frac{\mathfrak{w}^\top(s,t)\mathfrak{c}(s,t)}{\mathfrak{w}^\top(s,t)\tilde{\mathfrak{a}}(s,t)}$. Moreover, for some $s_1, s_2 \in [S,1]$

$$\begin{aligned}
\sigma^2 = \sup_{s,t\in[S,1]} \mathbb{E}\left\{\left(\frac{\mathfrak{w}^\top(s,t)\mathcal{W}(s,t)}{\mathfrak{w}^\top(s,t)\tilde{\mathfrak{a}}(s,t)}\right)^2\right\} &= \sup_{s,t\in[S,1]} \frac{1}{\tilde{\mathfrak{a}}^\top(s,t)D^{-1}(s,t)\tilde{\mathfrak{a}}(s,t)}\\
&= \frac{1}{\tilde{\mathfrak{a}}^\top(s_1,s_2)D^{-1}(s_1,s_2)\tilde{\mathfrak{a}}(s_1,s_2)},
\end{aligned}$$

since $[S,1]^2$ is compact. Moreover, one can check that for some positive constant G and $s_1, s_2, t_1, t_2 \in [S,1]$

$$\begin{aligned}
\mathbb{E}\left\{\left(\frac{\mathfrak{w}^\top(s_1,t_1)\mathcal{W}(s_1,t_1)}{\mathfrak{w}^\top(s_1,t_1)\tilde{\mathfrak{a}}(s,t)} - \frac{\mathfrak{w}^\top(s_2,t_2)\mathcal{W}(s_2,t_2)}{\mathfrak{w}^\top(s_2,t_2)\tilde{\mathfrak{a}}(s,t)}\right)^2\right\}&\\
\le G[|s_1 - s_2| + |t_1 - t_2|].&
\end{aligned}$$

Thus, utilizing Piterbarg inequality, see e.g., [53][Thm 8.1], we have that there exist positive constants C, γ such that

$$\mathbb{P}\left\{\exists s,t \in [S,1] : \mathcal{W}(s,t) - \mathfrak{c}(s,t) \ge \mathfrak{a}u\right\} \le Cu^\gamma e^{-(u+\mathfrak{C})^2/2\sigma^2}$$

for all u positive. Further, by Lemma 4.1 for some constants α, C^*, C^+ as $u \to \infty$

$$\begin{aligned}
&\mathbb{P}\left\{A_{\mathcal{I}\setminus\mathcal{J}}(s_1), A_{\mathcal{J}\setminus\mathcal{I}}(s_2), A_{\mathcal{I}\cap\mathcal{J}}(\min(s_1,s_2))\right\}\\
&\quad = \mathbb{P}\left\{\mathcal{W}(s_1,s_2) - \mathfrak{c}(s_1,s_2) \ge \mathfrak{a}u\right\}\\
&\quad \sim C^* u^{-\alpha} e^{-\frac{1}{2}(\tilde{\mathfrak{a}}(s_1,s_2)u + \mathfrak{c}(s_1,s_2))^\top D^{-1}(s_1,s_2)(\tilde{\mathfrak{a}}(s_1,s_2)u + \mathfrak{c}(s_1,s_2))}\\
&\quad = C^+ u^{-\alpha} e^{-\frac{u^2}{2\sigma^2}} e^{-u(\tilde{\mathfrak{a}}_{s_1,s_2})^\top D^{-1}(s_1,s_2)(\mathfrak{c}(s_1,s_2))}.
\end{aligned}$$

Hence the claim follows for $\tau = |\mathfrak{C}/\sigma^2| + \sup_{s,t\in[S,1]} |\tilde{\mathfrak{a}}(s,t)D^{-1}(s,t)\mathfrak{c}(s,t)| + 1$. □

Lemma 5.3 *The function $r_{\mathcal{I}}(t), t > 0$ defined in (4.6) is convex and if $\boldsymbol{c}_{\mathcal{I}}$ has at least one positive component, then there exists $T > 0$ such that for some positive s and any $t > 0$*

$$r_{\mathcal{I}}(T + t) \geq r_{\mathcal{I}}(T) + st. \tag{5.1}$$

Moreover, if $\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}}t$ for any $t > 0$ have at least one positive component, then $r_{\mathcal{I}}(t), t > 0$ has a unique point of minimum.

The proof of Lemma 5.3 is purely analytical, thus we skip the details.

Lemma 5.4 *Suppose that $\Sigma = \Gamma\Gamma^\top$ is positive definite. For any non-empty subset $\mathcal{I} \subset \{1, \ldots, d\}$ if $\boldsymbol{c}_{\mathcal{I}}$ and $\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}}t$ for all $t \geq 0$ have at least one positive component, then for any point $0 < t \neq \hat{t} = arg\min_{t>0} r_{\mathcal{I}}(t)$ there exists some positive constant ν such that as $u \to \infty$*

$$\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(t) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}}t)\sqrt{u}\right\} = o\left(e^{-\nu u}\right)\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(\hat{t}) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}}\hat{t})\sqrt{u}\right\}.$$

Proof of Lemma 5.4: For notational simplicity we omit below the subscript $\mathcal{I}$. Since for any $t > 0$ we have $Var(\boldsymbol{W}(t)) = t\Sigma$, then by Lemma 4.1

$$\mathbb{P}\left\{\boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\} \sim Cu^{-\alpha(t)/2}e^{-\frac{u}{2t}\tilde{\mathbf{p}}(t)^\top\Sigma^{-1}\tilde{\mathbf{p}}(t)},$$

where C is some positive constant, $\alpha(t)$ is an integer and $\tilde{\mathbf{p}}(t)$ is the unique solution of $\Pi_{t\Sigma}(\boldsymbol{a} + \boldsymbol{c}t)$, which can be reformulated also as

$$\mathbb{P}\left\{\boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\} \sim Cu^{-\alpha(t)/2}e^{-\frac{u}{2}r(t)}, \quad u \to \infty.$$

If $t \neq \hat{t}$, then $r(t) - r(\hat{t}) = \tau > 0$ and

$$\frac{\mathbb{P}\left\{\boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\}}{\mathbb{P}\left\{\boldsymbol{W}(\hat{t}) > (\boldsymbol{a} + \boldsymbol{c}\hat{t})\sqrt{u}\right\}} \sim C^* u^{(\alpha(\hat{t})-\alpha(t))/2}e^{-\frac{\tau u}{2}} = o\left(e^{-\frac{\tau}{3}u}\right)$$

as $u \to \infty$. ☐

Lemma 5.5 *Let $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^d$ be such that $\boldsymbol{a} + \boldsymbol{c}t$ has at least one positive component for all t in a compact set $\mathcal{T} \subset (0, \infty)$. If $\Sigma = \Gamma\Gamma^\top$ is positive definite, then there exist constants $C > 0$, $\gamma > 0$ and $\mathfrak{t} \in \mathcal{T}$ such that for all $u > 0$*

$$\mathbb{P}\left\{\exists t \in \mathcal{T} : \ \boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\} \leq Cu^\gamma e^{-\frac{u}{2}r(\mathfrak{t})}.$$

If we also have that for some non-overlapping index sets $\mathcal{I}, \mathcal{J} \subset \{1, \ldots, d\}$ and some compact subset $\mathcal{T} \subset [0, \infty)^2$ both $((\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}}t_1)^\top, (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}}t_2)^\top)^\top$ have at least *one positive component for all* $(t_1, t_2) \in \mathcal{T}$*, then for some* $\mathfrak{t} = (\mathfrak{t}_1, \mathfrak{t}_2) \in \mathcal{T}$ *as* $u \to \infty$

$$\begin{aligned}&\mathbb{P}\left\{\exists t \in \mathcal{T}: \ \boldsymbol{W}_{\mathcal{I}}(t_1) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} t_1)\sqrt{u}, \ \boldsymbol{W}_{\mathcal{J}}(t_2) > (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} t_2)\sqrt{u}\right\} \\ &\quad = o(e^{\sqrt{u}}\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(\mathfrak{t}_1) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} \mathfrak{t}_1)\sqrt{u}, \ \boldsymbol{W}_{\mathcal{J}}(\mathfrak{t}_2) > (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} \mathfrak{t}_2)\sqrt{u}\right\}).\end{aligned}$$

Moreover, the same estimate holds if $\mathcal{I}$ *and* $\mathcal{J}$ *are overlapping and for all* $(\mathfrak{t}_1, \mathfrak{t}_2) \in \mathcal{T}$ *we have* $\mathfrak{t}_1 \neq \mathfrak{t}_2$*.*

Proof of Lemma 5.5: Denote by $D(t)$ the covariance matrix of $\boldsymbol{W}(t)$, which by assumption on Γ is positive definite. Let

$$\tilde{\mathfrak{a}}(t) = arg \min_{\boldsymbol{x} \geq \boldsymbol{a} + \boldsymbol{c} t} \boldsymbol{x}^\top D^{-1}(t) \boldsymbol{x}$$

be the solution of $\Pi_D(\boldsymbol{a} + \boldsymbol{c}t), t > 0$ and let further

$$\mathfrak{w}(t) = D^{-1}(t)\tilde{\mathfrak{a}}(t)$$

be the solution of the dual optimization problem. In view of (2.7) $\mathfrak{w}_I(t)$ has positive components and moreover

$$f(t) = \mathfrak{w}^\top(t)(\boldsymbol{a} + \boldsymbol{c}t) = \tilde{\mathfrak{a}}^\top(t) D^{-1}(t)\tilde{\mathfrak{a}}(t) > 0$$

implying

$$\mathbb{P}\left\{\exists t \in \mathcal{T} : \boldsymbol{W}(t) \geq (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\} \quad \leq \quad \mathbb{P}\left\{\exists t \in \mathcal{T} : \frac{\mathfrak{w}^\top(t)\boldsymbol{W}(t)}{\mathfrak{w}^\top(t)(\boldsymbol{a} + \boldsymbol{c}t)} \geq \sqrt{u}\right\}.$$

We have further that

$$\begin{aligned}\sigma^2 = \sup_{t \in \mathcal{T}} \mathbb{E}\left\{\left(\frac{\mathfrak{w}^\top(t)\boldsymbol{W}(t)}{\mathfrak{w}^\top(t)(\boldsymbol{a} + \boldsymbol{c}t)}\right)^2\right\} \quad &= \quad \sup_{t \in \mathcal{T}} \frac{1}{\tilde{\mathfrak{a}}^\top(t) D^{-1}(t)\tilde{\mathfrak{a}}(t)} \\ &= \quad \frac{1}{\tilde{\mathfrak{a}}^\top(\mathfrak{t}) D^{-1}(\mathfrak{t})\tilde{\mathfrak{a}}(\mathfrak{t})} > 0\end{aligned}$$

for some $\mathfrak{t} \in \mathcal{T}$, since $\mathcal{T}$ is compact. Since $f(t) > 0, t \in \mathcal{T}$ is continuous, we may apply Piterbarg inequality (as in the proof of (4.4)) and obtain

$$\mathbb{P}\left\{\exists t \in \mathcal{T} : \boldsymbol{W}(t) \geq (\boldsymbol{a} + \boldsymbol{c}t)\sqrt{u}\right\} \leq C u^\gamma e^{-u/2\sigma^2}$$

for some positive constants γ and C, which depend only on $\boldsymbol{W}(t)$ and d. Since, by the definition we have $r(\mathfrak{t}) = 1/\sigma^2$, the proof of the first inequality is complete.

The next assertion may be obtained with the same arguments but for vector-valued random process

$$\mathcal{W}(s,t) = (\boldsymbol{W}_{\mathcal{I}}^{\top}(s), \boldsymbol{W}_{\mathcal{J}}^{\top}(t))^{\top}.$$

By the definition of $\mathcal{T}$, for any $(s,t) \in \mathcal{T}$ we have $|Var(\mathcal{W}(s,t))| > 0$, thus we can apply Piterbarg inequality and in consequence, using Lemma 4.1, the claim follows. □

Lemma 5.6 *Suppose that $\Sigma = \Gamma\Gamma^{\top}$ is positive definite. For any non-empty subset $\mathcal{I} \subset \{1,\dots,d\}$ if $\boldsymbol{c}_{\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|}$ has at least one positive component and $\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} t \in \mathbb{R}^{|\mathcal{I}|}$ has at least one positive component for all non-negative t, then for some positive constants ν, $\hat{t} = arg\min_{t>0} r_{\mathcal{I}}(t)$ and all T large, as $u \to \infty$*

$$\mathbb{P}\left\{\exists t > T: \ \boldsymbol{W}_{\mathcal{I}}(t) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} t)\sqrt{u}\right\} = o(e^{-\nu u})\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(\hat{t}) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} \hat{t})\sqrt{u}\right\}.$$

Proof of Lemma 5.6: For notational simplicity we omit below the subscript $\mathcal{I}$. For some given $T > \hat{t}$ we have using Lemmas 5.5, 5.3

$$\begin{aligned}
\mathbb{P}&\left\{\exists t > T: \ \boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c} t)\sqrt{u}\right\} \\
&\le \sum_{i=0}^{\infty} \mathbb{P}\left\{\exists t \in [T+i, T+i+1]: \ \boldsymbol{W}(t) > (\boldsymbol{a} + \boldsymbol{c} t)\sqrt{u}\right\} \\
&\le \sum_{i=0}^{\infty} C u^{\gamma} e^{-\frac{r(t_i)}{2} u} \\
&\le C u^{\gamma} e^{-\frac{r(T)}{2} u} \sum_{i=0}^{\infty} e^{-isu} \\
&\le C u^{\gamma} e^{-\frac{r(T)}{2} u} \left(1 + \int_0^{\infty} e^{-sux} \mathrm{d}x\right),
\end{aligned}$$

where $s > 0$ and $t_i \in [T+i, T+i+1]$. The last integral is finite and decreasing for sufficiently large u. Hence the claim follows with the same arguments as in the proof of Lemma 5.4. □

Proof of Lemma 4.4: Using Lemma 5.6 we know that there exist points $t_{\mathcal{I}}$, $t_{\mathcal{J}}$ such that as $u \to \infty$

$$\mathbb{P}\left\{\exists t \ge T_{\mathcal{I}}: \ A_{\mathcal{I}}^*(t)\right\} = o(\mathbb{P}\left\{A_{\mathcal{I}}^*(\hat{t}_{\mathcal{I}})\right\}),$$
$$\mathbb{P}\left\{\exists t \ge T_{\mathcal{J}}: \ A_{\mathcal{J}}^*(t)\right\} = o(\mathbb{P}\left\{A_{\mathcal{J}}^*(\hat{t}_{\mathcal{J}})\right\}).$$

Next, for some positive $\varepsilon < |\hat{t}_{\mathcal{I}} - \hat{t}_{\mathcal{J}}|/3$ we have

$$\mathbb{P}\left\{\exists s,t > 0: \ A_{\mathcal{I}}^*(t) \cap A_{\mathcal{J}}^*(s)\right\}$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\exists (s,t) \in [\hat{t}_{\mathcal{I}}-\varepsilon, \hat{t}_{\mathcal{I}}+\varepsilon] \times [\hat{t}_{\mathcal{J}}-\varepsilon, \hat{t}_{\mathcal{J}}+\varepsilon] :\ A^*_{\mathcal{I}}(t) \cap A^*_{\mathcal{J}}(s)\right\} \\
&+\mathbb{P}\left\{\exists t \in [0, \hat{t}_{\mathcal{I}}-\varepsilon] :\ A^*_{\mathcal{I}}(t)\right\} + \mathbb{P}\left\{\exists t \in [\hat{t}_{\mathcal{I}}+\varepsilon, T_{\mathcal{I}}] :\ A^*_{\mathcal{I}}(t)\right\} \\
&+\mathbb{P}\left\{\exists t \in [0, \hat{t}_{\mathcal{J}}-\varepsilon] :\ A^*_{\mathcal{J}}(t)\right\} + \mathbb{P}\left\{\exists t \in [\hat{t}_{\mathcal{J}}+\varepsilon, T_{\mathcal{J}}] :\ A^*_{\mathcal{J}}(t)\right\} \\
&+\mathbb{P}\left\{\exists t \ge T_{\mathcal{I}} :\ A^*_{\mathcal{I}}(t)\right\} + \mathbb{P}\left\{\exists t \ge T_{\mathcal{J}} :\ A^*_{\mathcal{J}}(t)\right\}.
\end{aligned}
$$

Using Lemmas 5.5, 5.6 and

$$
\mathbb{P}\left\{A^*_{\mathcal{I}}(t)\right\} \sim Cu^{-\alpha} e^{-r(t)u/2}, \quad \mathbb{P}\left\{A^*_{\mathcal{I}}(t)\right\} = o(ue^{-r(t)u/2}), \quad u \to \infty
$$

we obtain

$$
\begin{aligned}
\mathbb{P}\left\{\exists s,t > 0 :\ A^*_{\mathcal{I}}(t) \cap A^*_{\mathcal{J}}(s)\right\} &= o(e^{\sqrt{u}} \mathbb{P}\left\{A^*_{\mathcal{I}}(s_1) \cap A^*_{\mathcal{J}}(s_2)\right\}) \\
&+ o(u^{\tau_3} \mathbb{P}\left\{A^*_{\mathcal{I}}(t_3)\right\}) + o(u^{\tau_4} \mathbb{P}\left\{A^*_{\mathcal{I}}(t_4)\right\}) \\
&+ o(u^{\tau_5} \mathbb{P}\left\{A^*_{\mathcal{J}}(t_5)\right\}) + o(u^{\tau_6} \mathbb{P}\left\{A^*_{\mathcal{J}}(t_6)\right\}) \\
&+ o(\mathbb{P}\left\{A^*_{\mathcal{I}}(\hat{t}_{\mathcal{I}})\right\}) + o(\mathbb{P}\left\{A^*_{\mathcal{J}}(\hat{t}_{\mathcal{J}})\right\})
\end{aligned}
$$

for some positive constants $t_i, 3 \le i \le 6$, where

$$
\begin{aligned}
&t_3 \in [0, \hat{t}_{\mathcal{I}}-\varepsilon], \quad t_4 \in [\hat{t}_{\mathcal{I}}+\varepsilon, T_{\mathcal{I}}], \quad t_5 \in [0, \hat{t}_{\mathcal{J}}-\varepsilon], \\
&t_6 \in [\hat{t}_{\mathcal{J}}+\varepsilon, T_{\mathcal{J}}] \quad s_1 \in [\hat{t}_{\mathcal{I}}-\varepsilon, \hat{t}_{\mathcal{I}}+\varepsilon] \quad s_2 \in [\hat{t}_{\mathcal{J}}-\varepsilon, \hat{t}_{\mathcal{J}}+\varepsilon].
\end{aligned}
$$

Note that for $i = 3, 4$, $t_i \neq \hat{t}_{\mathcal{I}}$. Hence by Lemma 5.4

$$
u^{\tau_i} \mathbb{P}\left\{A^*_{\mathcal{I}}(t_i)\right\} = o(\mathbb{P}\left\{A^*_{\mathcal{I}}(\hat{t}_{\mathcal{I}})\right\}).
$$

The same works also for $j = 5, 6$

$$
u^{\tau_j} \mathbb{P}\left\{A^*_{\mathcal{J}}(t_j)\right\} = o(\mathbb{P}\left\{A^*_{\mathcal{J}}(\hat{t}_{\mathcal{J}})\right\}).
$$

Thus we can focus only on the first probability. By the definition of $A^*_{\mathcal{I}}$ and $A^*_{\mathcal{J}}$ in (4.5)

$$
\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1) \cap A^*_{\mathcal{J}}(s_2)\right\} = \mathbb{P}\left\{\mathcal{W}(s_1, s_2) > \boldsymbol{b}\sqrt{u}\right\},
$$

where $\boldsymbol{b} = ((\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} s_1)^\top, (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} s_2)^\top)$ and $\mathcal{W}(s,t) = (\boldsymbol{W}_{\mathcal{I}}(s)^\top, \boldsymbol{W}_{\mathcal{J}}(t)^\top)^\top$. Define $\widehat{i} = \mathcal{I} \cup \mathcal{J} \setminus \{i\}$. Applying Remark 5.2, there exists an index i and a constant $\eta > 0$ such that

$$
\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1) \cap A^*_{\mathcal{J}}(s_2)\right\} = o\left(e^{-\eta u}\right) \mathbb{P}\left\{(\mathcal{W}(s_1, s_2))_{\widehat{i}} > \boldsymbol{b}_{\widehat{i}}\sqrt{u}\right\}.
$$

If $i \in \mathcal{I}$, then

$$
\mathbb{P}\left\{(\mathcal{W}(s_1, s_2))_{\widehat{i}} > \boldsymbol{b}_{\widehat{i}}\sqrt{u}\right\} \le \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{J}}(s_2) > (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} s_2)u\right\},
$$

or

$$\mathbb{P}\left\{(\mathcal{W}(s_1,s_2))_{\widehat{i}} > \boldsymbol{b}_{\widehat{i}}\sqrt{u}\right\} \le \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(s_1) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} s_1)u\right\}.$$

In both cases

$$\begin{aligned} & e^{\sqrt{u}}\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1)\cap A^*_{\mathcal{J}}(s_2)\right\} \\ &\quad = o\left(\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(s_1) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} s_1)u\right\} + \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{J}}(s_1) > (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} s_1)u\right\}\right) \\ &\quad = o\left(\mathbb{P}\left\{\boldsymbol{W}_{\mathcal{I}}(\hat{t}_{\mathcal{I}}) > (\boldsymbol{a}_{\mathcal{I}} + \boldsymbol{c}_{\mathcal{I}} \hat{t}_{\mathcal{I}})u\right\} + \mathbb{P}\left\{\boldsymbol{W}_{\mathcal{J}}(\hat{t}_{\mathcal{I}}) > (\boldsymbol{a}_{\mathcal{J}} + \boldsymbol{c}_{\mathcal{J}} \hat{t}_{\mathcal{J}})u\right\}\right) \end{aligned}$$

establishing the proof. □

Proof of Lemma 4.5: Using Lemma 5.6 we have

$$\begin{aligned} \mathbb{P}\left\{\exists s,t>0:\ A^*_{\mathcal{I}}(s)\cap A^*_{\mathcal{J}}(t)\right\} \quad \le \quad & \mathbb{P}\left\{\exists (s,t)\in\mathbb{T}_1:\ A^*_{\mathcal{I}}(s)\cap A^*_{\mathcal{J}}(t)\right\} \\ & +\mathbb{P}\left\{\exists (s,t)\in\mathbb{T}_2:\ A^*_{\mathcal{I}}(s)\cap A^*_{\mathcal{J}}(t)\right\} \\ & +o(\mathbb{P}\left\{A^*_{\mathcal{I}}(\hat{t}_{\mathcal{I}})\right\}) + o(\mathbb{P}\left\{A^*_{\mathcal{J}}(\hat{t}_{\mathcal{J}})\right\}), \end{aligned}$$

where

$$\begin{aligned} \mathbb{T}_1 &= \{(s,t)\in[0,T_{\mathcal{I}}]\times[0,T_{\mathcal{J}}] : |s-\hat{t}_{\mathcal{I}}| \ge |t-\hat{t}_{\mathcal{I}}|\}, \\ \mathbb{T}_2 &= \{(s,t)\in[0,T_{\mathcal{I}}]\times[0,T_{\mathcal{J}}] : |s-\hat{t}_{\mathcal{I}}| \le |t-\hat{t}_{\mathcal{I}}|\}, \end{aligned}$$

and $T_{\mathcal{I}}$ and $T_{\mathcal{J}}$ are the constants from (5.1). According to Lemma 5.5 for some $(s_i,t_i)\in\mathbb{T}_i$

$$\mathbb{P}\left\{\exists (s,t)\in\mathbb{T}_i : A^*_{\mathcal{I}}(s)\cap A^*_{\mathcal{J}}(t)\right\} = o\left(e^{\sqrt{u}}\right)\mathbb{P}\left\{A^*_{\mathcal{I}}(s_i)\cap A^*_{\mathcal{J}\setminus\mathcal{I}}(t_i)\right\}.$$

If $s_1 \ne \hat{t}_{\mathcal{I}}$, then according to Lemma 5.4

$$e^{\sqrt{u}}\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1)\cap A^*_{\mathcal{J}\setminus\mathcal{I}}(t_1)\right\} \le e^{\sqrt{u}}\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1)\right\} = o(\mathbb{P}\left\{A^*_{\mathcal{I}}(\hat{t}_{\mathcal{I}})\right\}).$$

Otherwise, using the definition of $\mathbb{T}_1$, $|t_1-\hat{t}_{\mathcal{I}}| \le |s_1-\hat{t}_{\mathcal{I}}| = 0$, so $t_1 = \hat{t}_{\mathcal{I}}$ and thus

$$\mathbb{P}\left\{A^*_{\mathcal{I}}(s_1)\cap A^*_{\mathcal{J}\setminus\mathcal{I}}(t_1)\right\} = \mathbb{P}\left\{A^*_{\mathcal{I}\cup\mathcal{J}}(\hat{t}_{\mathcal{I}})\right\}.$$

This probability can be bounded using Remark 5.2, namely we have

$$\mathbb{P}\left\{A^*_{\mathcal{I}\cup\mathcal{J}}(\hat{t}_{\mathcal{I}})\right\} = o\left(e^{-\nu u}\right)\mathbb{P}\left\{A^*_{\mathcal{I}\cup\mathcal{J}\setminus\{i\}}(\hat{t}_{\mathcal{I}})\right\}$$

for some $i\in\mathcal{I}\cup\mathcal{J}$ and $\eta>0$. As $|\mathcal{I}| = |\mathcal{J}| = k$, and $\mathcal{I}\ne\mathcal{J}$, then $|\mathcal{I}\cup\mathcal{J}| \ge k+1$ and thus $|\mathcal{I}\cup\mathcal{J}\setminus\{i\}| \ge k$. Consequently, we have

$$e^{\sqrt{u}}\mathbb{P}\left\{A^*_{\mathcal{I}\cup\mathcal{J}}(\hat{t}_{\mathcal{I}})\right\} = o\left(\mathbb{P}\left\{A^*_{\mathcal{I}\cup\mathcal{J}\setminus\{i\}}(\hat{t}_{\mathcal{I}})\right\}\right) = o\left(\sum_{\substack{\mathcal{K}\subset\{1\ldots d\}\\ |\mathcal{K}|=k}} \mathbb{P}\left\{A^*_{\mathcal{K}}(\hat{t}_{\mathcal{K}})\right\}\right).$$

With similar arguments we obtain further

$$\mathbb{P}\left\{\exists(s,t)\in\mathbb{T}_2:\ A^*_{\mathcal{I}}(s)\cap A^*_{\mathcal{J}}(t)\right\} = o\left(\sum_{\substack{\mathcal{K}\subset\{1...d\}\\|\mathcal{K}|=k}}\mathbb{P}\left\{A^*_{\mathcal{K}}(\hat{t}_{\mathcal{K}})\right\}\right).$$

Hence the claim follows.

□

Recall that $\tilde{\boldsymbol{a}}$ stands for the unique solution of the quadratic programming problem $\Pi_\Sigma(\boldsymbol{a})$.

Proof of Lemma 4.6: By the self-similarity of Brownian motion for all $u>0$

$$\begin{aligned} m(u,\Lambda) &:= \mathbb{P}\left\{\exists_{t\in[0,\delta(u,\Lambda)]}: \boldsymbol{W}(t)-t\boldsymbol{c}>u\boldsymbol{a}\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[0,1]}: \boldsymbol{W}(t)-\delta^{1/2}(u,\Lambda)t\boldsymbol{c}>\delta^{-1/2}(u,\Lambda)u\boldsymbol{a}\right\}. \end{aligned}$$

Hence, applying Theorem 1.1 we obtain

$$m(u,\Lambda)\le\frac{\mathbb{P}\left\{\boldsymbol{W}(1)\ge\delta^{-1/2}(u,\Lambda)u\boldsymbol{a}+\delta^{1/2}(u,\Lambda)\boldsymbol{c}\right\}}{\mathbb{P}\left\{\boldsymbol{W}(1)>\max(\boldsymbol{c},\boldsymbol{0})\right\}},$$

which after some standard algebraic manipulations, straightforwardly implies inequality (4.7).

Asymptotics (4.8) and limit (4.9) follow by the same idea as the proof of "Pickands' lemma" in e.g. [10]; see Lemmas 4.2 and 4.3 therein. We skip long but standard proof, referring for details to the extended version of contribution [11].
□

Chapter 4

Multivariate Gaussian Risk Model

1 Introduction

Let $\boldsymbol{Z}(t) = (Z_1(t), \ldots, Z_d(t))^\top, t \in \mathbb{R}$ be a d-dimensional Gaussian process, where $Z_i(t), t \in \mathbb{R}$, $i = 1, ..., d$ are mutually independent centered Gaussian processes with continuous sample paths a.s. and stationary increments. For $\boldsymbol{u}, \boldsymbol{c} \in \mathbb{R}^d$ and $T > 0$ we consider

$$\begin{aligned} p_T(\boldsymbol{u}) &:= \mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{u}\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[0,T]} : \cap_{i=1}^d \{X_i(t) - c_i t > u_i\}\right\}, \end{aligned}$$

where $\boldsymbol{X}(t) = A\boldsymbol{Z}(t)$, with A a nonsingular $d \times d$ real-valued matrix.

In the main result of this chapter, which is Theorem 3.3, we derive exact asymptotics of $p_T(\boldsymbol{u})$ for $\boldsymbol{u} = u\boldsymbol{a} = (a_1 u, ..., a_d u)^\top$, as $u \to \infty$, where the vector $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$. The core assumption that we work with is the so-called *Berman condition*

$$v_i(t) := \mathbb{V}\mathrm{ar}(Z_i(t)) = o(t), \text{ as } t \to 0$$

for $i = 1, ..., d$. Interestingly, while in the one-dimensional case, under Berman condition

$$p_T(u) \sim \mathbb{P}\{X(T) - cT > u\}, \text{ as } u \to \infty$$

(see the seminal paper by Berman [6] and [26] for the non-centered case), the vector-valued case considered in this chapter leads to more diverse scenarios that

[3]This chapter is based on the joint work [8] with Krzysztof Dębicki and Krzysztof Bisewski.

can be captured in the form

$$p_T(u\boldsymbol{a}) \sim \mathcal{C} \cdot \mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\},$$

as $u \to \infty$, where $\mathcal{C} \geq 1$ is a constant depending on the model parameters; see Eq. (3.1) below.

We note that for $\boldsymbol{Z}$ being a two-dimensional standard Brownian motion, the asymptotic behavior of $p_T(u\boldsymbol{a})$ was recently analyzed in [20], where the strategy of the proof was based on the independence of increments and self-similarity of Brownian motion. In general, Gaussian processes with stationary increments do not have these properties and thus the proof of the main result of this chapter needs more subtle and refined analysis than the one used in [20]. More precisely, the idea of the proof of Theorem 3.3 is based on two steps: (i) showing that

$$\lim_{L\to\infty} \lim_{u\to\infty} \frac{p_T(u\boldsymbol{a})}{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-2},T]} : \cap_{i=1}^{d}\{X_i(t) - c_i t > u_i\}\right\}} = 1$$

and (ii) finding the exact asymptotics of the denominator above. In the first step, particularly precise analysis is needed for the neighbourhood of the right end of the parameter set $[0, T - Lu^{-2}]$, see Lemma 4.3.

Complementary to the exact asymptotics derived in Theorem 3.3, in Theorem 3.1 we establish uniform upper and lower bounds for $p_T(\boldsymbol{u})$. This result extends recently derived bounds for Z being a d-dimensional standard Brownian motion [11, 20, 45].

The quantity $p_T(\boldsymbol{u})$ has been already introduced in Chapter 2 and has a natural interpretation as the *simultaneous ruin probability* in time horizon $[0, T]$ of an insurance portfolio represented by d mutually dependent risk processes $(R_i(t), ..., R_d(t))^\top = \boldsymbol{R}(t, \boldsymbol{u})$, where $\boldsymbol{R}(t, \boldsymbol{u}) = \boldsymbol{u} - \boldsymbol{X}(t) + \boldsymbol{c}t,\ t \in \mathbb{R}$, since

$$p_T(\boldsymbol{u}) = \mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{R}(t, \boldsymbol{u}) < \boldsymbol{0}\right\},$$

where for the i-th business line, u_i is the initial capital, $X_i(t)$ is the accumulated clam size in time interval $[0, t]$ and c_i is the premium rate. In this context, our results complement work [42], where the particular case $d = 2, T = \infty$ with $X_2(t) = \sigma_2 X_1(t)$ where X_1 is a fractional Brownian motion, was analyzed. We refer to, e.g., [4, 29, 37, 51] for recent works on simultaneous ruin probability for Lévy processes and to recently derived asymptotics for centered vector valued Gaussian processes; see [23, 24].

Our findings cover two special cases that play important role in the literature on the Gaussian risk models, i.e. fractional Brownian motion risk model and Gaussian integrated risk model; see Section 3 for details. We refer to [22, 38, 39, 50] for the analysis of Gaussian risk models in $d = 1$ dimensional setting.

2 Notation

We follow the notational convention of [25]. All vectors in $\mathbb{R}^d$ are written in bold letters, for instance $\boldsymbol{b} = (b_1, \ldots, b_d)^\top$, $\boldsymbol{0} = (0, \ldots, 0)^\top$, $\boldsymbol{1} = (1, \ldots, 1)^\top$. We follow the convention that 1-dimensional vectors are vertical. For two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, we write $\boldsymbol{x} > \boldsymbol{y}$ if $x_i > y_i$ for all $1 \le i \le d$. For any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ we use $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for scalar product and $\boldsymbol{xy}$ for a component-wise product.

Given a real-valued matrix A we shall write A_{IJ} for the submatrix of A determined by keeping the rows and columns of A with row indices in the non-empty set I and column indices in the non-empty set J, respectively. In our notation $\mathcal{I}_d$ is the $d \times d$ identity matrix and $\mathrm{diag}(\boldsymbol{x}) = \mathrm{diag}(x_1, \ldots, x_d)$ stands for the diagonal matrix with entries x_i, $i = 1, \ldots, d$ on the main diagonal, respectively.

Let in the sequel $\Sigma \in \mathbb{R}^{d \times d}$ be any positive definite matrix. We write $\Sigma_{IJ}^{-1} := (\Sigma_{IJ})^{-1}$ for the inverse matrix of Σ_{IJ} whenever it exists. For any vector $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, let $\Pi_\Sigma(\boldsymbol{a})$ we will use the definition of quadratic programming problem $\Pi_\Sigma(\boldsymbol{a})$ given in Chapter 23 (2.5), definitions of its solution $\tilde{\boldsymbol{a}}$ of sets I, J given in Chapter 3 (2.6), and definition of $\boldsymbol{\lambda}$, given in Chapter 3, (2.9). Additionally, we define $U := \{i \in J : \tilde{a}_i = a_i\}$. We refer to Lemma 4.6 below for more details.

Throughout the chapter, let $\Sigma(t)$ denote the variance matrix of process $\boldsymbol{X}$ at time $t \in [0, T]$, that is

$$\Sigma(t) := \mathbb{E}\left\{\boldsymbol{X}(t)\boldsymbol{X}^\top(t)\right\} = A\mathbb{E}\left\{\boldsymbol{Z}(t)\boldsymbol{Z}^\top(t)\right\} A^\top = A\,\mathrm{diag}(\boldsymbol{v}(t))A^\top,$$

where $\boldsymbol{v}(t) = (v_1(t) \ldots v_d(t))^\top$. Moreover, for all $i \in \{1, \ldots, d\}$ let

$$\rho_i(t, s) := \mathbb{C}\mathrm{ov}(Z_i(t), Z_i(s)) = \frac{v_i(s) + v_i(t) - v_i(|s - t|)}{2}, \qquad (2.1)$$

where in the second equality we used the fact that Z_i has stationary increments. For all $t \in (0, T]$, let $\tilde{\boldsymbol{a}}(t)$ be the solution of quadratic programming problem $\Pi_{\Sigma(t)}(\boldsymbol{a})$ and $D(t) := \tilde{\boldsymbol{a}}(t)^\top \Sigma^{-1}(t) \tilde{\boldsymbol{a}}(t)$. Moreover, let $\boldsymbol{\lambda}(t) := \Sigma^{-1}(t)\tilde{\boldsymbol{a}}(t)$, and $I_t := \{i : \lambda_i(t) > 0\}$, $J_t := \{1, \ldots, d\} \setminus I_t$ (which can be empty). Throughout this chapter we slightly abuse the notation by writing $\lambda_i(t)$ instead of $\lambda(t)_i$, $a_i(t)$ instead of $a(t)_i$, and $\tilde{a}_i(t)$ instead of $\tilde{a}(t)_i$.

3 Main results

Consider a centered, $d-$dimensional Gaussian process with stationary increments, continuous sample paths and mutually independent components $\boldsymbol{Z}(t), t \ge 0$. Let $v_i(t) := \mathbb{V}\mathrm{ar}\, Z_i(t)$ be the variance function of process Z_i. Due to the stationarity of increments, the covariance structure of Z_i is determined by its variance function v_i, see (2.1). We shall establish the following conditions for each $i \in \{1, \ldots, d\}$.

B0. $v_i \in C^1([0,T])$ is strictly increasing and $v_i(0) = 0$.

BI. The first derivative $\dot{v}_i(T) > 0$.

BII. $v_i(t) = o(t)$, as $t \to 0$.

The following families of Gaussian processes satisfy assumptions **B0-BII**:

$\diamond$ *fractional Brownian motions*: $\boldsymbol{Z}(t) = (B_{\alpha_1}(t), ..., B_{\alpha_d}(t))^\top, t \geq 0$, where $B_{\alpha_i}(t), t \geq 0$, $i = 1, ..., d$ are mutually independent standard fractional Brownian motions with Hurst parameters $\alpha_i/2 \in (1/2, 1)$, that is centered Gaussian processes with stationary increments, continuous sample paths a.s. and variance function $v_i(t) = t^{\alpha_i}$ respectively. We refer to, e.g., [26, 39, 50] for the motivation and relations of this class of stochastic processes in risk theory.

$\diamond$ *integrated stationary processes*: $\boldsymbol{Z}(t) = (Z_1(t), ..., Z_d(t))^\top, t \geq 0$, where $Z_i(t) = \int_0^t \eta_i(s)ds$, with $\eta_i(t), t \geq 0$, $i = 1, ..., d$ mutually independent centered stationary Gaussian processes with continuous sample paths a.s. and continuous strictly positive covariance function $R_i(t) := \mathbb{C}\text{ov}(Z_i(s), Z_i(s+t))$. One can check that $v_i(t) = 2\int_0^t ds \int_0^s R(w)dw$ in this case. We refer to [22, 26, 38] for the analysis of extremes of this class of processes in the context of Gaussian risk theory and its relations to Gaussian fluid queueing models.

In the following, $\mathcal{N}_d(\mu, \Sigma)$ stands for the law of a d-dimensional normal distribution with mean $\mu \in \mathbb{R}^d$ and covariance matrix $\Sigma \in \mathbb{R}^{d\times d}$.

Theorem 3.1 *Let $\boldsymbol{X}(t) = A\boldsymbol{Z}(t)$, $t \geq 0$ be such that $\boldsymbol{Z}$ satisfies* **B0** *with $v_i(t) = v(t)$ for all i, $v(t)$ is convex, and $A \in \mathbb{R}^{d\times d}$ is a non-singular matrix satisfying $(A^\top A)_{ij} \geq 0$. Then, for each $\boldsymbol{u} \in \mathbb{R}^d$ and $\boldsymbol{c} \geq \boldsymbol{0}$,*

$$\begin{aligned} \mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > \boldsymbol{u}\right\} &\leq \mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{u}\right\} \\ &\leq \frac{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > \boldsymbol{u}\right\}}{\mathbb{P}\left\{\mathcal{N}_d(0, A^\top A) > \boldsymbol{0}\right\}}. \end{aligned}$$

Remarks 3.2 *In the case when $\boldsymbol{Z}$ is a standard d-dimensional Brownian motion the assumption $(A^\top A)_{ij} \geq 0$ can be lifted and the upper bound in Theorem 3.1 holds for any non-singular matrix A; see [11]. It can be verified that this bound holds also for all u large enough for the process $\boldsymbol{Z}$ considered in Example 3.4 below, which suggests that the upper bound in Theorem 3.1 holds for any non-singular matrix A.*

To the end of this chapter, let

$$\mathcal{C} := \frac{\sum_{i=1}^{d} \max\left(\lambda_i \cdot (AQA^{-1}\tilde{\boldsymbol{a}})_i, 0\right)}{\sum_{i=1}^{d} \lambda_i \cdot (AQA^{-1}\tilde{\boldsymbol{a}})_i} \tag{3.1}$$

where $Q = \mathrm{diag}(\dot{v}_i(T)/v_i(T))$, and $\boldsymbol{\lambda}$, $\tilde{\boldsymbol{a}}$ correspond to the solution of the quadratic problem $\Pi_{\Sigma(T)}(\boldsymbol{a})$; see also Lemma 4.6 below.

Theorem 3.3 *Let $\boldsymbol{X}(t) = A\boldsymbol{Z}(t)$, $t \geq 0$ be such that $\boldsymbol{Z}$ satisfies* **B0-BII**, *$\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, $\boldsymbol{c} \in \mathbb{R}^d$ and A is a non-singular matrix. Then,*

$$\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} \sim \mathcal{C} \cdot \mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}, \quad u \to \infty,$$

where $\mathcal{C}$ defined in (3.1) is a positive constant.

The heuristic interpretation of the bounds and asymptotics derived in Theorems 3.1, 3.3 is that only a small area around the end point T of the parameter set $[0, T]$ contributes to the tail distribution of the analyzed problem. We refer to [33, 35] and references therein for the analysis of the exact form of the asymptotics for $\mathbb{P}\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\}$, as $u \to \infty$; see also Lemma 4.5.

The following example illustrates the main findings of this section.

Example 3.4 *Suppose that $d = 2$, and $Z_1(t), Z_2(t)$ are mutually independent and identically distributed centered Gaussian processes that satisfy* **B0-BII**. *Then the constant $\mathcal{C}$ has the following form*

$$\mathcal{C} = \frac{\sum_{i=1}^{2} \max\left(\lambda_i \cdot \tilde{\boldsymbol{a}}_i, 0\right)}{\sum_{i=1}^{2} \lambda_i \cdot \tilde{\boldsymbol{a}}_i}.$$

We assume further that $A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$, where $\rho \in (-1, 1)$ and $\boldsymbol{a} = (1, a)^\top$, with $a \leq 1$.

◇ *If $a < \rho$, then $I = \{1\}$, $J = \{2\}$ and hence, as $u \to \infty$,*

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} &\sim \mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\right\} \\ &\sim \mathbb{P}\left\{X_1(T) - c_1 T > u\right\};\end{aligned}$$

◇ *If $a = \rho$, then $I = \{1\}$, $U = \{2\}$ and as $u \to \infty$,*

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} &\sim \mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\right\} \\ &\sim \mathbb{P}\left\{X_2(T) > c_2 T | X_1(T) = c_1 T\right\} \\ &\quad \times \mathbb{P}\left\{X_1(T) - c_1 T > u\right\};\end{aligned}$$

$\diamond$ *If* $a > \rho$, *then* $I = \{1, 2\}$, $\tilde{\boldsymbol{a}} = \boldsymbol{a}$ *and* $\boldsymbol{\lambda} = \Sigma^{-1}(T)\boldsymbol{a} = \frac{v(T)}{1-\rho^2}\binom{1-a\rho}{a-\rho}$, *see also* [20]. *Thus, if further* $a \geq 0$, *then* $\mathcal{C} = 1$ *and*

$$\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} \sim \mathbb{P}\left\{X_1(T) - c_1T > u, X_2(T) - c_2T > au\right\},$$

as $u \to \infty$. *Otherwise, if* $a < 0$, *then* $\mathcal{C} = \frac{1-a\rho}{1-a\rho+a^2-a\rho} = \frac{1-a\rho}{1-2a\rho+a^2} > 1$ *and hence, as* $u \to \infty$,

$$\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} \sim \frac{1-a\rho}{1-2a\rho+a^2}\mathbb{P}\left\{X_1(T) - c_1T > u, X_2(T) - c_2T > au\right\}.$$

In the above we used [35, *Lemma 4.2*] *for the exact asymptotics of the probability* $\mathbb{P}\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\}$.

4 Proofs of main results

Proof of Theorem 3.1: Using the fact that $\boldsymbol{c} \geq \boldsymbol{0}$ and $\boldsymbol{u} > \boldsymbol{0}$, we have

$$\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,T]}\boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{u}\right\} &= \mathbb{P}\left\{\bigcup_{t\in[0,T]}\bigcap_{i=1}^{d}\{X_i(t) > u_i + c_it\}\right\} \\
&= 1 - \mathbb{P}\left\{\bigcap_{t\in[0,T]}\bigcup_{i=1}^{d}\left\{\frac{X_i(t)}{u_i + c_it} \leq 1\right\}\right\} \\
&= 1 - \mathbb{P}\left\{\bigcap_{t\in[0,T]}\bigcup_{i=1}^{d}\left\{\frac{-X_i(t)}{u_i + c_it} \geq -1\right\}\right\} \\
&= 1 - \mathbb{P}\left\{\bigcap_{t\in[0,T]}\bigcup_{i=1}^{d}\left\{\frac{X_i(t)}{u_i + c_it} \geq -1\right\}\right\},
\end{aligned}$$

where in the last equality above we used that X_i are centered.

Let $B_i(t), t \geq 0$, $i = 1, 2, ..., d$ mutually independent standard Brownian motions, and $\boldsymbol{B}^*(t) = A\boldsymbol{B}(t)$.

Next, we show that

$$\mathbb{P}\left\{\bigcap_{t\in[0,T]}\bigcup_{i=1}^{d}\left\{\frac{X_i(t)}{u_i + c_it} \geq -1\right\}\right\} \geq \mathbb{P}\left\{\bigcap_{t\in[0,T]}\bigcup_{i=1}^{d}\left\{\frac{B_i^*(v(t))}{u_i + c_it} \geq -1\right\}\right\},$$

for which by Gordon inequality (see e.g. [31] or [1, page 55]) it suffices to check that

$$\begin{aligned}\mathbb{E}\left\{X_i(t)^2\right\} &= \mathbb{E}\left\{B_i^*(v(t))^2\right\}, &(4.1)\\ \mathbb{E}\left\{X_i(t)X_j(t)\right\} &= \mathbb{E}\left\{B_i^*(v(t))B_j^*(v(t))\right\}, &(4.2)\\ \mathbb{E}\left\{X_i(t)X_j(s)\right\} &\geq \mathbb{E}\left\{B_i^*(v(t))B_j^*(v(s))\right\}, \quad \text{for } t \neq s. &(4.3)\end{aligned}$$

For all $i, j \in \{1, \ldots, d\}$ and $t \in [0, T]$ we have

$$\begin{aligned}\mathbb{E}\left\{X_i(t)X_j(t)\right\} &= \mathbb{E}\left\{(AZ)_i(t)(AZ)_j(t)\right\}\\ &= \mathbb{E}\left\{\sum_{k=1}^d a_{ik}Z_k(t)\sum_{k=1}^d a_{jk}Z_k(t)\right\}\\ &= \mathbb{E}\left\{\sum_{k=1}^d a_{ik}a_{jk}Z_k^2(t)\right\}\\ &= \sum_{k=1}^d a_{ik}a_{jk}\mathbb{E}\left\{Z_k^2(t)\right\}\\ &= \sum_{k=1}^d a_{ik}a_{jk}v(t) = (AA^\top)_{i,j}v(t).\end{aligned}$$

Analogously,

$$\begin{aligned}\mathbb{E}\left\{B_i^*(v(t))B_j^*(v(t))\right\} &= \mathbb{E}\left\{(AB)_i(v(t))(AB)_j(v(t))\right\}\\ &= \mathbb{E}\left\{\sum_{k=1}^d a_{ik}B_k(v(t))\sum_{k=1}^d a_{jk}B_k(v(t))\right\}\\ &= \mathbb{E}\left\{\sum_{k=1}^d a_{ik}a_{jk}B_k^2(v(t))\right\}\\ &= \sum_{k=1}^d a_{ik}a_{jk}\mathbb{E}\left\{B_k^2(v(t))\right\}\\ &= \sum_{k=1}^d a_{ik}a_{jk}v(t) = (AA^\top)_{i,j}v(t).\end{aligned}$$

Hence, equalities (4.1), (4.2) are satisfied. For $t \neq s$ we obtain that

$$\begin{aligned}\mathbb{E}\left\{X_i(t)X_j(s)\right\} &= (AA^\top)_{i,j}\mathbb{E}\left\{Z_1(t)Z_1(s)\right\}\\ &= (AA^\top)_{i,j}\frac{v(s)+v(t)-v(|s-t|)}{2},\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left\{B_i^*(v(t))B_j^*(v(s))\right\} &= (AA^\top)_{i,j}\mathbb{E}\left\{B_1(v(t))B_1(v(s))\right\}\\
&= (AA^\top)_{i,j}\min(v(t),v(s)).
\end{aligned}
$$

As $(AA^\top)_{i,j} \geq 0$, it is enough to show that

$$
\frac{v(s)+v(t)-v(|s-t|)}{2} \geq \min(v(t),v(s)).
$$

Using the convexity of $v(\cdot)$, we have for all $s<t$

$$
\begin{aligned}
v(t-s) &= \int_0^{t-s} v'(x)\mathrm{d}x\\
&\leq \int_s^t v'(x)\mathrm{d}x = v(t)-v(s)
\end{aligned}
$$

hence

$$
\frac{v(s)+v(t)-v(|s-t|)}{2} \geq \frac{v(s)+v(t)-|v(s)-v(t)|}{2}.
$$

Thus, inequality (4.3) holds which jointly with (4.1) and (4.2) implies

$$
\begin{aligned}
\mathbb{P}\left\{\exists_{t\in[0,T]}:\boldsymbol{X}(t)-\boldsymbol{c}t>\boldsymbol{u}\right\} &\leq \mathbb{P}\left\{\exists_{t\in[0,T]}:A\boldsymbol{B}(v(t))-\boldsymbol{c}t>\boldsymbol{u}\right\}\\
&= \mathbb{P}\left\{\exists_{t\in[0,v(T)]}:A\boldsymbol{B}(t)-\boldsymbol{c}w(t)>\boldsymbol{u}\right\}\\
&\leq \mathbb{P}\left\{\exists_{t\in[0,v(T)]}:A\boldsymbol{B}(t)-\boldsymbol{c}\frac{T}{v(T)}t>\boldsymbol{u}\right\},
\end{aligned}
$$

where $w(t)$ is the inverse function of $v(t)$. In the second line we used the fact that $v(\cdot)$ is continuous and strictly increasing, while the inequality in the third line follows by concavity of $w(t)$ (recall that $v(t)$ is supposed to be convex). Finally, by [11, Theorem 1.1] (see also [45] for the centred case) the above is bounded by

$$
\frac{\mathbb{P}\left\{\boldsymbol{AB}(v(T))-\boldsymbol{c}T>\boldsymbol{u}\right\}}{\mathbb{P}\left\{\boldsymbol{AB}(v(T))>\boldsymbol{0}\right\}} = \frac{\mathbb{P}\left\{\boldsymbol{X}(T)-\boldsymbol{c}T>\boldsymbol{u}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T)>\boldsymbol{0}\right\}}.
$$

This completes the proof. □

Proof of Theorem 3.3: For any $L>0$ we have

$$
\begin{aligned}
P_3(u,L) &\leq \frac{\mathbb{P}\left\{\exists_{t\in[0,T]}:\boldsymbol{X}(t)-\boldsymbol{c}t>\boldsymbol{u}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T)-\boldsymbol{c}T>u\boldsymbol{a}\right\}}\\
&\leq \sum_{n=0}^{3} P_n(u,L),
\end{aligned}
$$

with

$$P_n(u,L) := \frac{\mathbb{P}\left\{\exists_{t\in[T_n(u),T_{n+1}(u)]} : \boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{u}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}},$$

where $T_0(u) = 0$, $T_1(u) = T_1 > 0$ is chosen small enough to satisfy the conditions in Lemma 4.1,

$$T_2(u,L) := T - Lu^{-2}\ln^2 u, \quad T_3(u,L) := T - Lu^{-2}, \quad \text{and} \quad T_4(u) = T.$$

The proof consists of several steps which follow by lemmas displayed and proved in the rest of this section. It turns out, that asymptotically as $u \to \infty$, only $P_3(u,L)$ contributes to the asymptotics, while $\sum_{n=0}^{2} P_n(u,L)$ is asymptotically negligible. Since each term in $\sum_{n=0}^{2} P_n(u,L)$ needs a different argument for its negligibility, we provide detailed justification in separate lemmas. Namely,

- ⋄ For any $L > 0$ it holds that $\lim_{u\to\infty} P_0(u,L) = 0$ due to Lemma 4.1,
- ⋄ For any $L > 0$ it holds that $\lim_{u\to\infty} P_1(u,L) = 0$ due to Lemma 4.2,
- ⋄ $\lim_{L\to\infty}\lim_{u\to\infty} P_4(u,L) = 0$ due to Lemma 4.3,
- ⋄ $\lim_{L\to\infty}\lim_{u\to\infty} P_5(u,L) = \mathcal{C}$ due to Lemma 4.4.

This completes the proof. □

Lemma 4.1 *Under the assumptions of Theorem 3.3 it holds that*

$$\frac{\mathbb{P}\left\{\exists_{t\in[0,T_1]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \to 0, \quad u \to \infty$$

for all $T_1 \in (0,T)$ small enough.

Lemma 4.2 *Under the assumptions of Theorem 3.3, for any $L > 0$ it holds that*

$$\frac{\mathbb{P}\left\{\exists_{t\in[T_1,T-Lu^{-2}\ln^2 u]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \to 0, \quad u \to \infty$$

for all $T_1 \in (0,T) > 0$ small enough.

Lemma 4.3 *Under the assumptions of Theorem 3.3, there exist positive constants $C, \beta > 0$, such that*

$$\lim_{u\to\infty} \frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-2}\ln^2 u, T-Lu^{-2}]} \boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{a}u\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \le Ce^{-\beta L}$$

for all $L > 0$ large enough.

Lemma 4.4 *Under the assumptions of Theorem 3.3, for any $L > 0$ there exists a positive constant $\mathcal{C}(L)$ such that*

$$\frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-2},T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \to \mathcal{C}(L), \quad u \to \infty.$$

Moreover, $\lim_{L\to\infty} \mathcal{C}(L) = \mathcal{C}$*, with $\mathcal{C}$ defined in* (3.1)*.*

The proofs of the four lemmas above are given in the following three subsections. The proof of each result is located at the very end of these subsections and is preceded by additional preparatory results.

4.1 Proof of Lemma 4.1

Let $\varphi_{\boldsymbol{X}}(\cdot)$ be the pdf of random variable $\boldsymbol{X}$. Before giving the proof of Lemma 4.1, we need the following lemma, which can be deduced from [35, Lemma 4.2]. Since we need a bit different (although equivalent) form of the derived below asymptotics, we provide an independent short proof of the following lemma.

Lemma 4.5 *Let $\boldsymbol{X} \in \mathbb{R}^d$ be a centered Gaussian vector with an arbitrary, non-singular covariance matrix Σ. Then, for any $\boldsymbol{c} \in \mathbb{R}^d$, $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$ we have, as $u \to \infty$*

$$\mathbb{P}\left\{\boldsymbol{X} - \boldsymbol{c} > u\boldsymbol{a}\right\} \sim \frac{u^{-|I|}\varphi_{\boldsymbol{X}}(u\tilde{\boldsymbol{a}} + \boldsymbol{c})}{\prod_{i\in I}\lambda_i} \times \int_{\mathbb{R}^{|J|}} \mathbb{I}\left\{\boldsymbol{x}_U < \boldsymbol{0}_U\right\} e^{-\frac{1}{2}\boldsymbol{x}_J^\top(\Sigma^{-1})_{JJ}\boldsymbol{x}_J} e^{\langle\tilde{\boldsymbol{c}}_J,\boldsymbol{x}_J\rangle} \mathrm{d}\boldsymbol{x}_J,$$

where $\tilde{\boldsymbol{c}} := \boldsymbol{c}^\top\Sigma^{-1}$, and $\tilde{\boldsymbol{a}}, \boldsymbol{\lambda}$, and index sets I, J, U corresponding to the quadratic programming problem $\Pi_\Sigma(\boldsymbol{a})$.

Proof of Lemma 4.5: Let $\bar{u} \in \mathbb{R}^d$ be such that $\bar{u}_i = u$ when $i \in I$ and $\bar{u}_i = 1$ when $i \in J$. We apply substitution $\boldsymbol{w} = u\tilde{\boldsymbol{a}} + \boldsymbol{c} - \boldsymbol{x}/\bar{u}$ and obtain

$$\begin{aligned}\mathbb{P}\left\{\boldsymbol{X} - \boldsymbol{c} > u\boldsymbol{a}\right\} &= \int_{\boldsymbol{w}>u\boldsymbol{a}+\boldsymbol{c}T} \varphi_{\boldsymbol{X}}(\boldsymbol{w})\mathrm{d}\boldsymbol{w} \\ &= u^{-|I|}\int_{\boldsymbol{x}<u\bar{u}(\tilde{\boldsymbol{a}}-\boldsymbol{a})} \varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c} - \boldsymbol{x}/\bar{\boldsymbol{u}})\mathrm{d}\boldsymbol{x} \\ &= u^{-|I|}\varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c})\int_{\mathbb{R}^d} \mathbb{I}\left\{\boldsymbol{x} < u\bar{u}(\tilde{\boldsymbol{a}} - \boldsymbol{a})\right\}\theta_u(\boldsymbol{x})\mathrm{d}\boldsymbol{x},\end{aligned}$$

where $\theta_u(\boldsymbol{x}) := \varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c} - \boldsymbol{x}/\bar{\boldsymbol{u}})/\varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c})$. We have as $u \to \infty$

$$\mathbb{I}_{\{\boldsymbol{x} < u\bar{u}(\tilde{\boldsymbol{a}} - \boldsymbol{a})\}} \to \mathbb{I}\left\{\boldsymbol{x}_{I \cup U} < \boldsymbol{0}\right\},$$

and

$$\begin{aligned}\theta_u(\boldsymbol{x}) &= \exp\left\{u\tilde{\boldsymbol{a}}^\top \Sigma^{-1}(\boldsymbol{x}/\bar{\boldsymbol{u}}) + \boldsymbol{c}^\top \Sigma^{-1}(\boldsymbol{x}/\bar{\boldsymbol{u}}) - \frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top \Sigma^{-1}(\boldsymbol{x}/\bar{\boldsymbol{u}})\right\} \\ &\to e^{\langle \boldsymbol{\lambda}_I, \boldsymbol{x}_I \rangle} \cdot e^{-\frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J} e^{\langle \tilde{\boldsymbol{c}}_J, \boldsymbol{x}_J \rangle} =: \theta(\boldsymbol{x}),\end{aligned}$$

as $u \to \infty$. So, applying the dominated convergence theorem, with dominating, integrable function

$$e^{\langle \boldsymbol{\lambda}_I, \boldsymbol{x}_I \rangle} e^{\frac{1}{2}\langle \boldsymbol{\lambda}_I, |\boldsymbol{x}_I| \rangle} \cdot e^{-\frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J} e^{\langle \tilde{\boldsymbol{c}}_J, \boldsymbol{x}_J \rangle},$$

we obtain

$$\begin{aligned}\frac{\mathbb{P}\left\{\boldsymbol{X} - \boldsymbol{c} > u\boldsymbol{a}\right\}}{u^{-|I|}\varphi(u\tilde{\boldsymbol{a}} + \boldsymbol{c})} &\to \int_{\mathbb{R}^d} \mathbb{I}\left\{\boldsymbol{x}_{I \cup U} < \boldsymbol{0}\right\} \theta(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \\ &= \int_{\mathbb{R}^{|I|}} \mathbb{I}\{\boldsymbol{x}_I < \boldsymbol{0}_I\} e^{\sum_{i \in I} \lambda_i x_i} \mathrm{d}\boldsymbol{x}_I \\ &\quad \times \int_{\mathbb{R}^{|J|}} \mathbb{I}\left\{\boldsymbol{x}_U < \boldsymbol{0}_U\right\} e^{-\frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J} e^{\langle \tilde{\boldsymbol{c}}_J, \boldsymbol{x}_J \rangle} \mathrm{d}\boldsymbol{x}_J \\ &= \frac{1}{\prod_{i \in I} \lambda_i} \int_{\mathbb{R}^{|J|}} e^{-\frac{1}{2}\boldsymbol{x}_J^\top (\Sigma^{-1})_{JJ} \boldsymbol{x}_J} e^{\langle \tilde{\boldsymbol{c}}_J, \boldsymbol{x}_J \rangle} \mathrm{d}\boldsymbol{x}_J,\end{aligned}$$

which concludes the proof. □

Proof of Lemma 4.1: First, using Lemma 4.5, we know that there exist some $C > 0$, $k \in \mathbb{N}$ such that

$$\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > \boldsymbol{a}u\right\} \sim Cu^{-k}\varphi_T(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T),$$

as $u \to \infty$, where φ_T is the density of $\boldsymbol{X}(T)$. Second, fix some $i \in \{1, \ldots, d\}$ such that $a_i > 0$. Then

$$\begin{aligned}\mathbb{P}\left\{\exists_{t \in [0,T_1]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} &\leq \mathbb{P}\left\{\exists_{t \in [0,T_1]} : X_i(t) - c_i t > a_i u\right\} \\ &\leq \mathbb{P}\left\{\exists_{t \in [0,T_1]} : X_i(t) > a_i u - |c_i| T_1\right\}.\end{aligned}$$

Using assumption **B0**, we can apply Piterbarg's inequality [53, Thm 8.1], receiving for some positive constant C_1 and all sufficiently large u

$$\begin{aligned}&\mathbb{P}\left\{\exists_{t \in [0,T_1]} : X_i(t) > a_i u - |c_i| T_1\right\} \\ &\qquad \leq \mathbb{P}\left\{\exists_{t \in [0,T_1]} : |X_i(t)| > a_i u - |c_i| T_1\right\}\end{aligned}$$

$$\leq C_1(a_i u - |c_i|\, T_1)^2 \mathbb{P}\left\{X_i(T_1) > a_i u - |c_i|\, T_1\right\}$$

for $u > 0$. Hence, for all u large enough we have

$$\frac{\mathbb{P}\left\{\exists_{t\in[0,T_1]} : X_i(t) > a_i u - |c_i|\, T_1\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > \boldsymbol{a}u\right\}} \leq 2C_1(a_i u - |c_i|\, T_1)^2 \times u^k \frac{\mathbb{P}\left\{X_i(T_1) > a_i u - |c_i|\, T_1\right\}}{\varphi_T(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)}.$$

Since $\mathbb{P}\left\{\mathcal{N}(0,1) > u\right\} \sim \frac{1}{\sqrt{2\pi}u} \exp(-u^2/2)$ as $u \to \infty$, it is left to show that

$$\lim_{u\to\infty} u^{k-1} \frac{\exp\left(-(a_i u - |c_i|\, T_1)^2/(2v_i(T_1))\right)}{\exp\left(-\frac{1}{2}(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)^\top \Sigma^{-1}(T)(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)\right)} = 0.$$

We have

$$\frac{\exp\left(-(a_i u - |c_i|\, T_1)^2/(2v_i(T_1))\right)}{\exp\left(-\frac{1}{2}(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)^\top \Sigma^{-1}(T)(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)\right)} = \exp\left(-\frac{1}{2}\left(\frac{a_i^2}{v_i(T_1)} - \tilde{\boldsymbol{a}}^\top \Sigma^{-1}(T)\tilde{\boldsymbol{a}}\right) u^2 + O(u)\right).$$

Finally, since $v_i(0) = 0$ and $v_i(\cdot)$ is continuous, then $\frac{a_i^2}{v_i(T_1)} > \tilde{\boldsymbol{a}}^\top \Sigma^{-1}(T)\tilde{\boldsymbol{a}}$ for all T_1 small enough, which completes the proof. □

4.2 Proof of Lemma 4.2

Before giving the proof, we need to layout preliminary results. Below, we cite the result from [25, Lemma 4]. In the following, $J = \{1, \dots, d\} \setminus I$ can be empty; the claim in Lemma 4.6(ii) is formulated under the assumption that J is non-empty.

Lemma 4.6 *Let $d \geq 2$ and Σ a $d \times d$ symmetric positive definite matrix with inverse Σ^{-1}. If $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$, then the quadratic programming problem $\Pi_\Sigma(\boldsymbol{b})$ has a unique solution $\tilde{\boldsymbol{a}}$ and there exists a unique non-empty index set $I \subset \{1, \dots, d\}$ with $|I| \leq d$ elements such that*

(i) $\tilde{\boldsymbol{a}}_I = \boldsymbol{a}_I \neq \boldsymbol{0}_I$;

(ii) $\tilde{\boldsymbol{a}}_J = \Sigma_{IJ}^{-1}\Sigma_{II}^{-1}\boldsymbol{a}_I \geq \boldsymbol{a}_J$, *and* $\Sigma_{II}^{-1}\boldsymbol{a}_I > \boldsymbol{0}_I$;

(iii) $\min_{\boldsymbol{x}\geq\boldsymbol{a}} \boldsymbol{x}^\top \Sigma^{-1}\boldsymbol{x} = \tilde{\boldsymbol{a}}^\top \Sigma^{-1}\tilde{\boldsymbol{a}} = \boldsymbol{a}^\top \Sigma^{-1}\tilde{\boldsymbol{a}} = \boldsymbol{a}_I^\top \Sigma_{II}^{-1}\boldsymbol{a}_I > 0$,

with $\boldsymbol{\lambda} = \Sigma^{-1}\tilde{\boldsymbol{a}}$ *satisfying* $\boldsymbol{\lambda}_I = \Sigma_{II}^{-1}\boldsymbol{a}_I > \boldsymbol{0}_I$, *and* $\boldsymbol{\lambda}_J = \boldsymbol{0}_J$.

Remark 4.7 *Using Lemma 4.6 it can be found that* $\tilde{\boldsymbol{a}}^\top \Sigma^{-1} \tilde{\boldsymbol{a}} = \boldsymbol{a}^\top \Sigma^{-1} \tilde{\boldsymbol{a}}$.

To the end of this chapter, let $D(t) := \tilde{\boldsymbol{a}}(t)^\top \Sigma^{-1}(t) \tilde{\boldsymbol{a}}(t)$.

Lemma 4.8 *Assuming conditions* ***B0-BII*** *hold, then* $D(t)$ *is positive and strictly decreasing on* $t \in (0, T]$. *Moreover,*

$$\dot{D}(T) = -\left\| \operatorname{diag}\left(\sqrt{\dot{\boldsymbol{v}}(T)} / \boldsymbol{v}(T) \right) A^{-1} \tilde{\boldsymbol{a}}(T) \right\|_2^2 < 0.$$

Proof of Lemma 4.8: Let $0 < t_1 < t_2 \le T$. Then

$$\begin{aligned} D(t_2) &= \tilde{\boldsymbol{a}}(t_2)^\top \Sigma^{-1}(t_2) \tilde{\boldsymbol{a}}(t_2) \le \tilde{\boldsymbol{a}}(t_1)^\top \Sigma^{-1}(t_2) \tilde{\boldsymbol{a}}(t_1) \\ &= \tilde{\boldsymbol{a}}(t_1)^\top A^{-\top} \operatorname{diag}(1/\boldsymbol{v}(t_2)) A^{-1} \tilde{\boldsymbol{a}}(t_1) \\ &= D(t_1) - \tilde{\boldsymbol{a}}(t_1)^\top {A^{-1}}^\top \operatorname{diag}\left(\frac{\boldsymbol{v}(t_2) - \boldsymbol{v}(t_1)}{\boldsymbol{v}(t_2)\boldsymbol{v}(t_1)} \right) A^{-1} \tilde{\boldsymbol{a}}(t_1) \\ &= D(t_1) - \left\| \operatorname{diag}\left(\frac{\sqrt{\boldsymbol{v}(t_2) - \boldsymbol{v}(t_1)}}{\sqrt{\boldsymbol{v}(t_2)\boldsymbol{v}(t_1)}} \right) A^{-1} \tilde{\boldsymbol{a}}(t_1) \right\|_2^2 < D(t_1), \end{aligned}$$

because $\tilde{\boldsymbol{a}}(t_1) \neq \boldsymbol{0}$ and $\boldsymbol{v}(t)$ is strictly increasing. This shows that $D(t)$ is strictly decreasing. Furthermore, we have

$$\begin{aligned} &\tilde{\boldsymbol{a}}(T)^\top A^{-\top} \operatorname{diag}\left(\frac{\boldsymbol{v}(t) - \boldsymbol{v}(T)}{(T-t)\boldsymbol{v}(t)\boldsymbol{v}(T)} \right) A^{-1} \tilde{\boldsymbol{a}}(T) \\ &\qquad \le \frac{D(T) - D(t)}{T - t} \\ &\qquad \le \tilde{\boldsymbol{a}}(t)^\top A^{-\top} \operatorname{diag}\left(\frac{\boldsymbol{v}(t) - \boldsymbol{v}(T)}{(T-t)\boldsymbol{v}(t)\boldsymbol{v}(T)} \right) A^{-1} \tilde{\boldsymbol{a}}(t) \end{aligned}$$

using that $\tilde{\boldsymbol{a}}(t)$ and $\boldsymbol{v}(t)$ are continuous and $\boldsymbol{v}(t)$ has a positive derivative at the point $t = T$, we have as $t \to T$

$$\begin{aligned} \frac{D(T) - D(t)}{T - t} &\to -\tilde{\boldsymbol{a}}(T)^\top A^{-\top} \left(\frac{\dot{\boldsymbol{v}}(T)}{\boldsymbol{v}^2(T)} \right) A^{-1} \tilde{\boldsymbol{a}}(T) \\ &= -\left\| \operatorname{diag}\left(\sqrt{\dot{\boldsymbol{v}}(T)} / \boldsymbol{v}(T) \right) A^{-1} \tilde{\boldsymbol{a}}(T) \right\|_2^2 < 0, \end{aligned}$$

hence the claim follows. □

Lemma 4.9 *Let* $T_0 \in (0, T]$. *Then* $D(t)$, *and* $\tilde{a}_i(t)$, $\lambda_i(t)$, $\frac{\lambda_i(t)}{D(t)}$ *are Lipschitz continuous functions on* $t \in [T_0, T]$ *for all* $i \in \{1, \ldots, d\}$.

Proof of Lemma 4.9: Let $T_0 \in (0, T]$ be fixed. According to [32, Theorem 3.1], $\tilde{a}_i(\cdot)$ and $\lambda_i(\cdot)$ are Lipschitz continuous, provided that conditions [32, A1-A3] are satisfied. First, let us note that the conditions A1-A2 are clearly satisfied in our setting so will will focus only on condition A3. In order to state what is condition A3, let $M(t) \in \mathbb{R}^{I(t)\times d}$ such that $M(t) := (-\mathcal{I}_d)_{I(t)}$, where $(-\mathcal{I}_d)_{I(t)}$ is the submatrix of $-\mathcal{I}_d$ consisting of rows corresponding to the indices of $I(t)$. Then, condition [32, A3] states that that there exist $\alpha, \beta > 0$ such that for all $t \in [T_0, T]$:

(i) $x^\top \Sigma^{-1}(t) x \geq \alpha \|x\|^2$ for all $x \in \mathbb{R}^d$ satisfying $M(t)x = 0$,

(ii) $\|M(t)^\top x\| \geq \beta \|x\|$ for all $x \in \mathbb{R}^{I(t)}$.

Since $\|M(t)^\top x\| = \|x\|$, then (ii) is satisfied with $\beta = 1$. To see that (i) holds, we have

$$x^\top \Sigma^{-1}(t) x \geq \sigma_1(t) \|x\|^2,$$

where $\sigma_1(t)$ is the smallest eigenvalue of $\Sigma^{-1}(t)$. The matrix $\Sigma^{-1}(t)$ is symmetric and positive definite for $t > 0$, thus it has only real positive eigenvalues $\sigma_1(t) \dots \sigma_d(t)$. So the related characteristic polynomial $p_{\Sigma^{-1}(t)}$ has continuous monoms and always has d real solutions. It means that we can order the eigenvalues $\sigma_1(t), \dots, \sigma_d(t)$ in such way that this functions will be continuous by t and thus we can take $\alpha := \min_{t\in[T_0,T]} \sigma_1(t) > 0$, which concludes the proof of (i) and of the Lipschitz continuity of $\tilde{a}_i(\cdot)$ and $\lambda_i(\cdot)$.

Now, the fact that $\tilde{a}_i(\cdot)$ is Lipschitz continuous immediately implies the Lipschitz continuity of $D(\cdot)$. Lastly, we need to show the Lipschitz continuity of $\lambda_i(t)/D(t)$. For $t, s \in [T_0, T]$ we have

$$\left| \frac{\lambda_i(s)}{D(s)} - \frac{\lambda_i(t)}{D(t)} \right| \leq \left| \frac{\lambda_i(s) - \lambda_i(t)}{D(s)} \right| + \lambda_i(t) \left| \frac{D(t) - D(s)}{D(t)D(s)} \right|.$$

The proof is concluded by the Lipschitz continuity of $\lambda_i(\cdot)$, $D(\cdot)$, and by noting that $\min_{t\in[T_0,T]} D(t) = D(T) > 0$; see Lemma 4.8. □

In the following, $\mathrm{argmin}_{\,\mathrm{t\in[a,b]}}\mathrm{f(t)}$ is the smallest minimizer of function $f(\cdot)$ over set $[a, b]$. For $t \in (0, T]$ we define function

$$\mathcal{G}(t) := \frac{\langle \boldsymbol{\lambda}(t), \boldsymbol{c} \rangle t}{D(t)}. \tag{4.4}$$

Lemma 4.10 *Let $\boldsymbol{X}(t) = A\boldsymbol{Z}(t)$, $t \in [0, T]$ be such that $\boldsymbol{Z}(t)$ satisfies the conditions* **B0-BII** *and $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$. Then, for any $T_1 \in (0, T)$ there exists a constant $C > 0$ such that for any $T_1 \leq L < R \leq T$ and $u > -\mathcal{G}(T^*)$*

$$\mathbb{P}\left\{\exists_{t\in[L,R]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} \leq C(u + \mathcal{G}(T^*)) \exp\bigl(-D(R)(u + \mathcal{G}(T^*))^2/2\bigr),$$

where $T^* := \mathrm{argmin}_{\,\mathrm{t\in[L,R]}}\mathcal{G}(\mathrm{t})$.

Proof of Lemma 4.10: Recall that $\Sigma(t) := \mathbb{V}\mathrm{ar}(\boldsymbol{X}(t))$ and $\tilde{\boldsymbol{a}}(t)$ is the solution to the quadratic programming problem $\Pi_{\Sigma(t)}(\boldsymbol{a})$ for each $t > 0$. According to Lemma 4.6(iii), for $t > 0$ we have $D(t) > 0$. Hence

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[L,R]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}& \\ &\le \mathbb{P}\left\{\exists_{t\in[L,R]} : \langle \boldsymbol{\lambda}(t), (\boldsymbol{X}(t) - \boldsymbol{c}t)\rangle > \langle \boldsymbol{\lambda}(t), \boldsymbol{a}\rangle u\right\} \\ &= \mathbb{P}\left\{\exists_{t\in[L,R]} : \frac{\langle \boldsymbol{\lambda}(t), \boldsymbol{X}(t)\rangle}{D(t)} > u + \mathcal{G}(t)\right\}, \end{aligned} \tag{4.5}$$

with $\mathcal{G}(\cdot)$ defined in (4.4). In the following let $Y(t) := \frac{\langle \boldsymbol{\lambda}(t), \boldsymbol{X}(t)\rangle}{D(t)}$. Using the inequality $(\sum_{i=1}^d a_i)^2 \le d\sum_{i=1}^d a_i^2$ we find that

$$\begin{aligned}\mathbb{E}\left\{(Y(t) - Y(s))^2\right\} &\le 2\mathbb{E}\left\{\left(\frac{\langle \boldsymbol{\lambda}(t), \boldsymbol{X}(t)\rangle}{D(t)} - \frac{\langle \boldsymbol{\lambda}(s), \boldsymbol{X}(t)\rangle}{D(s)}\right)^2\right\} \\ &\quad + 2\mathbb{E}\left\{\left(\frac{\langle \boldsymbol{\lambda}(s), \boldsymbol{X}(t)\rangle}{D(s)} - \frac{\langle \boldsymbol{\lambda}(s), \boldsymbol{X}(s)\rangle}{D(s)}\right)^2\right\} \\ &\le 2d\sum_{i=1}^d \left(\frac{\lambda_i(t)}{D(t)} - \frac{\lambda_i(s)}{D(s)}\right)^2 v_i(t) + 2d\frac{\lambda_i^2(s)}{D^2(s)} v_i(|t-s|).\end{aligned}$$

Now, the functions $v_i(\cdot)$ and $\lambda_i(\cdot)/D(\cdot)$ are Lipshitz continuous due to **B0** and Lemma 4.9 respectively, so there exists $C_1 > 0$ such that the inequalities $|v_i(|t - s|)| \le C_1|t-s|$, and $|\frac{\lambda_i(t)}{D(t)} - \frac{\lambda_i(s)}{D(s)}| \le C_1|t-s|$ hold for all $i \in \{1, \dots, d\}$. Thus

$$\mathbb{E}\left\{(Y(t) - Y(s))^2\right\} \le 2d^2C_1^2|t-s|^2 + 2d \max_{i\in\{1,\dots,d\}} \max_{t\in[L,R]} \left\{\frac{\lambda_i^2(t)}{D^2(t)}\right\} C_1|t-s|.$$

We conclude that there exists $C_2 > 0$ such that

$$\mathbb{E}\left\{(Y(t) - Y(s))^2\right\} \le C_2|t-s|$$

for all $t, s \in [L, R]$. Since $\mathbb{V}\mathrm{ar}(Y(t)) = \mathbb{V}\mathrm{ar}(\tilde{\boldsymbol{a}}^\top(t)\Sigma^{-1}\boldsymbol{X}(t))/D^2(t) = 1/D(t)$ and $D(t)$ is strictly decreasing, see Lemma 4.8, then the maximum of $\mathbb{V}\mathrm{ar}(Y(t))$ over $[L, R]$ is attained at $t = R$. According to Piterbarg inequality [53, Thm 8.1], there exists a constant C_3 such that for any $0 < L < R \le T$ we have

$$\mathbb{P}\left\{\exists_{t\in[L,R]} : Y(t) > u\right\} \le C_3(R-L)u^2\mathbb{P}\left\{Y(R) > u\right\} \tag{4.6}$$

for $u > 0$. Finally, since $\mathbb{P}\{\mathcal{N}(0,1) > u\} \le \frac{1}{\sqrt{2\pi}u}e^{-u^2/2}$ for $u > 0$, then upon combining (4.5), (4.6) we obtain

$$\mathbb{P}\left\{\exists_{t\in[L,R]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\} \le C_3(R-L)\sqrt{\frac{D(R)}{2\pi}}(u + \mathcal{G}(T^*))$$

$$\times \exp\{-D(R)(u+\mathcal{G}(T^*))^2/2\}.$$

Since function $D(\cdot)$ is decreasing (Lemma 4.8), we conclude the proof by taking $C := C_3(R-L)\sqrt{\frac{D(T_1)}{2\pi}}$. □

Lemma 4.11 *Let* ***B0-BII*** *hold and let $T_1 \in (0,T]$ and $T_1 \le L(u) < R(u) \le T$. If either of the following two conditions is satisfied:*

(i) $R(u) \to T$, and $u(T-R(u)) \to \infty$, or

(ii) $T - L(u) = o(u(T-R(u)))$, and $u^2(T-R(u))/\ln(u) \to \infty$,

then

$$\lim_{u\to 0} \frac{\mathbb{P}\left\{\exists_{t\in[L(u),R(u)]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} = 0$$

as $u \to \infty$.

Proof of Lemma 4.11: Using Lemma 4.5, we know that there exist some $C_1 > 0$, $k \in \mathbb{N}$ such that

$$\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > \boldsymbol{a}u\right\} \sim C_1 u^{-k} \varphi_T(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T), \tag{4.7}$$

as $u \to \infty$, where φ_T is the density of $\boldsymbol{X}(T)$. According to Lemma 4.10 there exist $C_2 > 0, K \in \mathbb{R}$ such that, for all $u > -\mathcal{G}(T^*(u))$ we have

$$\begin{aligned}\mathbb{P}&\left\{\exists_{t\in[L(u),R(u)]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}\\ &\le C_2(u+\mathcal{G}(T^*(u)))\\ &\quad \times \exp\bigl(-D(R(u))(u+\mathcal{G}(T^*(u)))^2/2\bigr),\end{aligned} \tag{4.8}$$

where $K(\cdot)$ is defined in Lemma 4.10 and $T^*(u) := \text{argmin}_{\,\mathrm{t\in[L(u),R(u)]}}\mathcal{G}(\mathrm{T})$. From now on, we take $u > -\inf_{t\in[T_1,T]}\mathcal{G}(t)$, which is finite due to Lemma 4.9.

For brevity, in the following we denote $\Sigma := \Sigma(T)$, $\tilde{\boldsymbol{a}} := \tilde{\boldsymbol{a}}(T)$. In light of (4.7) and (4.8), it suffices to show that, for any $\beta \in \mathbb{R}$ we have

$$\begin{aligned}u^\beta & \frac{\exp(-D(R(u))(u+\mathcal{G}(T^*(u)))^2/2)}{\exp\left(-(\tilde{\boldsymbol{a}}u+\boldsymbol{c}T)^\top\Sigma^{-1}(\tilde{\boldsymbol{a}}u+\boldsymbol{c}T)/2\right)}\\ &= \exp\Bigl(-\bigl(u^2h_2(u)+uh_1(u)+h_0(u)\bigr)+\beta\ln(u)\Bigr)\\ &\to 0,\end{aligned} \tag{4.9}$$

as $u \to \infty$, where

$$h_2(u) := \frac{1}{2}\Bigl(D(R(u)) - D(T)\Bigr), \quad h_1(u) := D(R(u))\mathcal{G}(T^*(u)) - D(T)\mathcal{G}(T),$$

$$h_0(u) := \frac{1}{2}\Big(D(R(u))K^2(T^*(u)) - T^2\boldsymbol{c}^\top\Sigma^{-1}\boldsymbol{c}\Big).$$

We notice that functions $|h_i(u)|$ are all bounded for u large enough.

Suppose that $L(u), R(u)$ satisfy conditions (i). Due to the continuity of $D(\cdot)$, we have that $D(R(u)) \to D(T)$. Using the assumption **B1**, that $D(t)$ is differentiable at the point $t = T$, with $\dot{D}(T) < 0$, we have

$$-uh_2(u) - h_1(u) = u(T - R(u)) \cdot \frac{D(R(u)) - D(T)}{2(R(u) - T)} - h_1(u) \to -\infty,$$

which implies (4.9) under conditions in item (i).

Next, suppose that $L(u), R(u)$ satisfy conditions (ii). We have

$$h_1(u) = \tfrac{D(R(u))}{D(T^*(u))} \cdot \Big(D(T^*(u))\mathcal{G}(T^*(u)) - D(T)\mathcal{G}(T)\Big) + \tfrac{D(T)}{D(T^*(u))}\mathcal{G}(T)\Big(D(R(u)) - D(T^*(u))\Big).$$

Functions $D(t)$ and $D(t)\mathcal{G}(t) = \langle \boldsymbol{\lambda}(t), c\rangle t$ are Lipschitz continuous due to Lemma 4.9 and the fact that sums and products of Lipschitz continuous functions are Lipschitz continuous. This implies that there exists $C_3 > 0$ such that

$$\Big|D(T^*(u))\mathcal{G}(T^*(u)) - D(T)\mathcal{G}(T)\Big| \le C_3|T - T^*(u)| \le C_3|T - L(u)|,$$

as well as

$$\Big|D(R(u)) - D(T^*(u))\Big| \le C_3|R(u) - T^*(u)| \le C_3|T - L(u)|.$$

Hence, there exists $C_4 > 0$ such that $|h_1(u)| \le C_4|T - L(u)|$ for all u large enough and

$$-u^2h_2(u) - uh_1(u) \le u^2(T - R(u)) \cdot \left[\frac{D(R(u)) - D(T)}{2(R(u) - T)} + C_4 \cdot \frac{T - L(u)}{u(T - R(u))}\right].$$

Since $D(R(u)) < D(T)$ and $\dot{D}(T) < 0$ and $T - L(u) = o(u(T - R(u)))$ then the term in the square brackets above is eventually negative and bounded away from 0 for u large enough. Finally, (4.9) follows from the fact that $u^2(T - R(u))/\ln(u) \to \infty$.
□

Proof of Lemma 4.2: Consider the following upper bound

$$\frac{\mathbb{P}\left\{\exists_{t\in[T_1, T - Lu^{-2}\ln^2 u]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}}$$

$$\leq \frac{\mathbb{P}\left\{\exists_{t\in[T_1,T-Lu^{-1}\ln u]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} + \frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-1}\ln u, T-Lu^{-1}\ln^{-1} u]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} + \frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-1}\ln^{-1} u], T-Lu^{-2}\ln^2 u]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}}.$$

Now, the first and second term above satisfy condition (i) of Lemma 4.11, while the third term satisfies condition (ii) of Lemma 4.11. Thus the righthand side of the above inequality converges to 0, as $u \to \infty$. □

4.3 Proofs of Lemma 4.3 and Lemma 4.4

Before proceeding to the proof of Lemma 4.3 and Lemma 4.4 we need some auxiliary lemmas. The following result generalizes [25, Lemma 5.3].

Lemma 4.12 *For any* $\boldsymbol{f} := (f_1, \ldots, f_d) \in R^d$, *if* $\sum_{i=1}^d f_i = 0$

$$\int_{\mathbb{R}^d} \mathbb{I}\left\{\exists_{t\in[0,L]} :\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} = 1 + \sum_{i=1}^d f_i^+ L,$$

otherwise

$$\int_{\mathbb{R}^d} \mathbb{I}\left\{\exists_{t\in[0,L]} :\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} = \frac{\sum_{i=1}^d f_i^-}{\sum_{i=1}^d f_i} + \frac{\sum_{i=1}^d f_i^+}{\sum_{i=1}^d f_i} e^{\sum_{i=1}^d f_i L},$$

where $f_i^+ := \max\{f_i, 0\}$ *and* $f_i^- := \min\{f_i, 0\}$.

Proof of Lemma 4.12: Define $S_+ := \sum_{i=1}^d f_i^+$, $S_- := -\sum_{i=1}^d f_i^-$, and $S := \sum_{i=1}^d f_i = S_+ - S_-$. Without the loss of generality, let $k \in \{0, \ldots, d\}$ be such that $f_1 \geq 0, \ldots, f_k \geq 0$, and $f_{k+1} < 0, \ldots, f_d < 0$. We distinguish three cases: (i) $k = d$ (all f_is are non-negative, which is equivalent to $S_- = 0$), (ii) $k = 0$ (all f_is are negative, which is equivalent to $S_+ = 0$), and (iii) $0 < k < d$. It can be easily seen that in case (i) we have

$$\int_{\mathbb{R}^d} \mathbb{I}\left\{\exists_{t\in[0,L]} :\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} = \prod_{i=1}^d \int_{-\infty}^{f_i L} e^{x_i} \mathrm{d}x_i = e^{SL},$$

and in case (ii) we have

$$\int_{\mathbb{R}^d} \mathbb{I}\left\{\exists_{t\in[0,L]}:\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} = \prod_{i=1}^d \int_0^\infty e^{-x_i} \mathrm{d}x_i = 1.$$

Till the end we consider case (iii). Let us define

$$Q_1 := \{\boldsymbol{x} \in \mathbb{R}^d : \forall_{i\in\{1,\dots,d\}} \text{ if } f_i < 0 \text{ then } x_i < 0\},$$
$$Q_2 := \{x \in \mathbb{R}^d : \exists_{i\in\{1,\dots,d\}} \text{ if } f_i < 0 \text{ then } x_i \geq 0\},$$

so that $Q_1 \cup Q_2 = \mathbb{R}^d$ and $Q_1 \cap Q_2 = \emptyset$. It can be seen that

$$\int_{Q_2} \mathbb{I}\left\{\exists_{t\in[0,L]}:\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} = 0.$$

Furthermore, with $m := m(x_{k+1}, \dots, x_d) = \min\{\frac{x_{k+1}}{-f_{k+1}}, \dots, \frac{x_d}{-f_d}\}$, we have

$$\begin{aligned}
&\int_{Q_1} \mathbb{I}\left\{\exists_{t\in[0,L]}:\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}\boldsymbol{x} \\
&\quad= \int_0^\infty \cdots \int_0^\infty \left[\prod_{i=1}^k \int_{-\infty}^{f_i(L\wedge m)} e^{x_i} \mathrm{d}x_i\right] e^{-\sum_{i=k+1}^d x_i}\, \mathrm{d}x_{k+1} \cdots \mathrm{d}x_d \\
&\quad= \int_0^\infty \cdots \int_0^\infty \exp\left\{S_+(L\wedge m)\right\} e^{-\sum_{i=k+1}^d x_i} \mathrm{d}x_{k+1} \cdots \mathrm{d}x_d.
\end{aligned}$$

We recognize that $\exp\{-\sum_{i=k+1}^d x_i\} \cdot \mathbb{I}\left\{x_i \geq 0\right\}$ is the density of minimum of $d-k$ independent exponential distributions with rate 1; using that such minimum is again exponentially distributed with rate $(d-k)$, we find that, with $Y \sim \mathrm{Exp}(S_-)$,

$$\begin{aligned}
\int_{Q_1} \mathbb{I}\left\{\exists_{t\in[0,L]}:\ \boldsymbol{x} < \boldsymbol{f}t\right\} e^{\sum_{i=1}^d x_i} \mathrm{d}x &= \mathbb{E}\left\{e^{S_+(L\wedge Y)}\right\} \\
&= S_- \int_0^\infty e^{S_+(L\wedge y) - S_- y} \mathrm{d}y \\
&= S_- \int_0^L e^{(S_+ - S_-)y} \mathrm{d}y \\
&\qquad + e^{S_+ L} \int_L^\infty S_- e^{-S_- y} \mathrm{d}y \\
&= \begin{cases} L \cdot S_- + e^{SL}, & S = 0, \\ \frac{S_-}{S} \cdot (e^{S\cdot L} - 1) + e^{SL}, & \text{otherwise}, \end{cases}
\end{aligned}$$

which completes the proof. ∎

Lemma 4.13 *There exist $\bar{\tau} \in (0,T)$, $\lambda^* > 0$, $\eta > 0$ such that:*

(i) $I_T \subseteq I_t$ *for all* $t \in [\bar{\tau}, T]$,

(ii) $\lambda_i(t) > \lambda^*$ *for all* $i \in I_t$, $t \in [\bar{\tau}, T]$,

(iii) $\Sigma^{-1}(t) - \eta \mathcal{I}_d$ *is positive definite for all* $t \in [\bar{\tau}, T]$.

Proof of Lemma 4.13: According to Lemma 4.6, $i \in I_t$ if and only if $\lambda_i(t) > 0$. Since $\lambda_i(t)$ is a continuous function for any $i \in \{1, \dots, d\}$ (see Lemma 4.9), then for any $i \in I_T$ there must exist $\tau_i < T$, and $\lambda_i^* > 0$ such that $\lambda_i(t) > \lambda_i^*$ for all $t \in [\tau_i, T]$. for all $t \in [\tau_i, T]$ we have $\lambda_i(t) > 0$. Thus the claims in (i) and (ii) follow by taking $\tau = \max_{i \in I_T}(\tau_i)$ and $\lambda^* = \min_{i \in I_T} \lambda_i^*$.

The matrix $\Sigma^{-1}(t)$ is symmetric and positive definite for $t > 0$, thus it has only real positive eigenvalues $\sigma_1(t) \dots \sigma_d(t)$. So the related characteristic polynomial $p_{\Sigma^{-1}(t)}$ has continuous monoms and always has d real solutions. It means that we can order the eigenvalues $\sigma_1(t), \dots, \sigma_d(t)$ in such way that this functions will be continuous by t and thus $\eta = \min\limits_{i \in \{1,\dots,d\}} \min\limits_{t \in [\tau, T]} \sigma_i(t) > 0$. This concludes the proof of (iii). □

In the following, for all $u > 0, \tau \in (0, T]$, and $\boldsymbol{x} \in \mathbb{R}^d$ we define:

$$\boldsymbol{w}_{u,\tau}(\boldsymbol{x}) := u\tilde{\boldsymbol{a}}(\tau) + \boldsymbol{c}\tau - \frac{\boldsymbol{x}}{\bar{\boldsymbol{u}}(\tau)}, \tag{4.10}$$

where $\bar{\boldsymbol{u}}(\tau) \in \{u, 1\}^d$ such that $\bar{\boldsymbol{u}}_{I_\tau}(\tau) := u \cdot \mathbf{1}_{|I_\tau|}$, and $\bar{\boldsymbol{u}}_{J_\tau}(\tau) := \mathbf{1}_{|J_\tau|}$, that is $\bar{\boldsymbol{u}}(\tau)$ has the components in the set I_τ equal to u and the other components equal to 1. Further, for all $L > 0$, $\tau \in (0, T]$, $\boldsymbol{x} \in \mathbb{R}^d$ and $u > \sqrt{L/\tau}$ consider a Gaussian process $\{\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t), t \in [0, L]\}$ defined conditionally:

$$\left(\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t)\right)_{t \in [0,L]} \overset{d}{:=} \left(\boldsymbol{Z}(\tau - \tfrac{t}{u^2}) \mid \boldsymbol{Z}(\tau) = A^{-1}\boldsymbol{w}_{u,\tau}(\boldsymbol{x})\right)_{t \in [0,L]}. \tag{4.11}$$

Since the components of $\boldsymbol{Z}(t)$ are mutually independent, then the components of $\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t)$ are mutually independent as well, i.e.

$$\mathbb{C}\mathrm{ov}((\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(s))_i, (\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t))_j) = 0$$

for $i \neq j$. By the definition in (4.11), for any $i \in \{1, \dots, d\}$, $t, s \in [0, L]$ we have

$$\mathbb{E}\left\{(\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t))_i\right\} = \frac{\rho_i(\tau - \frac{t}{u^2}, \tau)}{v_i(\tau)} \left(A^{-1}\boldsymbol{w}_{u,\tau}(\boldsymbol{x})\right)_i,$$

$$\mathbb{C}\mathrm{ov}\left((\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(s))_i, (\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t))_i\right) = \rho_i(\tau - \tfrac{s}{u^2}, \tau - \tfrac{t}{u^2}) - \frac{\rho_i(\tau - \frac{s}{u^2}, \tau)\rho_i(\tau - \frac{t}{u^2}, \tau)}{v_i(\tau)}$$

with ρ_i defined in (2.1). In the following let $\widehat{\boldsymbol{Z}}_{u,\tau}(t) := \boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t) - \mathbb{E}\left\{\boldsymbol{Z}_{u,\tau}^{\boldsymbol{x}}(t))\right\}$. It is noted that the distribution of $\widehat{\boldsymbol{Z}}_{u,\tau}(t)$ does not depend on $\boldsymbol{x}$.

Lemma 4.14 *There exists a constant $C > 0$ such that for all $L > 0$, $\tau \in (0, T]$, and $t, s \in [0, L]$ we have*

$$u^2 \mathbb{E}\left\{ \left((A\hat{\boldsymbol{Z}}_{u,\tau}(t))_i - (A\hat{\boldsymbol{Z}}_{u,\tau}(s))_i \right)^2 \right\} \le C u^2 \max_{j \in \{1,\dots,d\}} v_j(Lu^{-2})$$

for all $i \in \{1, \dots, d\}$ and u large enough.

Proof of Lemma 4.14: For brevity, in the following denote $\bar{t} := \tau - \frac{t}{u^2}$, $\bar{s} := \tau - \frac{s}{u^2}$, and $\hat{\boldsymbol{Z}}_u(t) := \hat{\boldsymbol{Z}}_{u,\tau}(t)$. We have

$$\begin{aligned}
\mathbb{E}\left\{ \left((\hat{\boldsymbol{Z}}_u(t))_i - (\hat{\boldsymbol{Z}}_u(s))_i \right)^2 \right\} &= \mathbb{V}\mathrm{ar}\{(\hat{\boldsymbol{Z}}_u(t))_i\} + \mathbb{V}\mathrm{ar}\{(\hat{\boldsymbol{Z}}_u(s))_i\} \\
&\quad - 2\,\mathbb{C}\mathrm{ov}\{(\hat{\boldsymbol{Z}}_u(t))_i, (\hat{\boldsymbol{Z}}_u(s))_i\} \\
&= v_i(\bar{t}) - \frac{\rho_i^2(\bar{t}, \tau)}{v_i(\tau)} + v_i(\bar{s}) - \frac{\rho_i^2(\bar{s}, \tau)}{v_i(\tau)} \\
&\quad - 2\left(\rho_i(\bar{s}, \bar{t}) - \frac{\rho_i(\bar{t}, \tau)\rho_i(\bar{s}, \tau)}{v_i(\tau)} \right) \\
&= v_i(|\bar{s} - \bar{t}|) - \frac{\left(\rho_i(\bar{t}, \tau) - \rho_i(\bar{s}, \tau)\right)^2}{v_i(\tau)}.
\end{aligned}$$

Now, the above is not greater than $v_i(|\bar{s} - \bar{t}|) = v_i(|s - t|/u^2) \le v_i(Lu^{-2})$. Furthermore, we have

$$\begin{aligned}
&u^2 \mathbb{E}\left\{ \left((A\hat{\boldsymbol{Z}}_u(s))_i - (A\hat{\boldsymbol{Z}}_u(t))_i \right)^2 \right\} \\
&\qquad = u^2 \mathbb{E}\left\{ \left(\sum_{j=1}^d a_{ij} \left((\hat{\boldsymbol{Z}}_u(s))_j - (\hat{\boldsymbol{Z}}_u(t))_j \right) \right)^2 \right\} \\
&\qquad \le u^2 \left(\sum_{j=1}^d a_{ij}^2 \right) \left(\sum_{j=1}^d \mathbb{E}\left\{ (\hat{\boldsymbol{Z}}_u(s))_j - (\hat{\boldsymbol{Z}}_u(t))_j)^2 \right\} \right),
\end{aligned}$$

where we used CauchySchwarz inequality. This completes the proof. □

The following corollary to Lemma 4.14 is a straightforward application of Piterbarg inequality [53, Theorem 8.1] and Lipshitz continuity of functions $v_i(\cdot)$.

Corollary 4.15 *There exists $C > 0$ such that for all $L > 0$, $\tau \in (0, T]$, $z > 0$ we have*

$$\mathbb{P}\left\{ \sup_{t \in [0,L]} u(A\hat{\boldsymbol{Z}}_{u,\tau}(t))_i > z \right\} \le C z^2 e^{-z^2/(2u^2 \max_{j\in\{1,\dots,d\}} v_j(Lu^{-2}))}$$

for $i \in \{1, \dots, d\}$ and all u large enough.

In the following, for any $i \in \{1, \ldots, d\}$, $t, s \in [0, L]$ we define

$$\begin{aligned} h_{u,\tau}(L, \boldsymbol{x}) &:= \mathbb{P}\left\{\exists_{s\in[\tau - Lu^{-2}, \tau]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a} \mid \boldsymbol{X}(\tau) = \boldsymbol{w}_{u,\tau}(\boldsymbol{x})\right\}, \\ \theta_{u,\tau}(\boldsymbol{x}) &:= \varphi_\tau(\boldsymbol{w}_{u,\tau}(\boldsymbol{x}))/\varphi_\tau(u\tilde{\boldsymbol{a}}(\tau) + \boldsymbol{c}\tau). \end{aligned} \tag{4.12}$$

Lemma 4.16 *There exists $\tau_0 \in (0, T)$ and function $H : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}_+$ satisfying $\int_{\mathbb{R}^d} H(L, \boldsymbol{x})\mathrm{d}\boldsymbol{x} =: C^*(L) < \infty$ for all $L > 0$, and $u_0 : \mathbb{R}_+ \to \mathbb{R}_+$ such that for all $L > 0$, $\tau \in [\tau_0, T]$, and $\boldsymbol{x} \in \mathbb{R}^d$ we have*

$$h_{u,\tau}(L, \boldsymbol{x})\theta_{u,\tau}(\boldsymbol{x}) \le H(L, \boldsymbol{x})$$

for all $u > u_0(L)$. Moreover, there exists $C^* > 0$ such that

$$\limsup_{L\to\infty} C^*(L) < C^*.$$

Proof of Lemma 4.16: For $u > 0, \tau \in (0, T]$, and $\boldsymbol{x} \in \mathbb{R}^d$ let

$$\theta_{u,\tau}(\boldsymbol{x}) := \varphi_\tau(\boldsymbol{w}_{u,\tau}(\boldsymbol{x}))/\varphi_\tau(u\tilde{\boldsymbol{a}}(\tau) + \boldsymbol{c}\tau).$$

Then

$$\begin{aligned} &\theta_{u,\tau}(\boldsymbol{x}) \\ &\quad = \exp\left\{u\tilde{\boldsymbol{a}}^\top(\tau)\Sigma^{-1}(\tau)(\boldsymbol{x}/\bar{\boldsymbol{u}}) + \tau\boldsymbol{c}^\top\Sigma^{-1}(\tau)(\boldsymbol{x}/\bar{\boldsymbol{u}}) - \frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\Sigma^{-1}(\tau)(\boldsymbol{x}/\bar{\boldsymbol{u}})\right\} \\ &\quad = e^{<\boldsymbol{\lambda}_I(\tau), \boldsymbol{x}_I>} e^{<\tilde{\boldsymbol{c}}_I(\tau)/u, \boldsymbol{x}_I>} e^{<\tilde{\boldsymbol{c}}_J(\tau), \boldsymbol{x}_J>} e^{-\frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\Sigma^{-1}(\tau)(\boldsymbol{x}/\bar{\boldsymbol{u}})}, \end{aligned}$$

where $\tilde{\boldsymbol{c}}(\tau) := \boldsymbol{c}^\top\Sigma^{-1}(\tau)$. From Lemma 4.13(iii) we know that there exists $\eta > 0$ such that $\Sigma^{-1}(\tau) - \eta\mathcal{I}_d$ is positive definite for all $\tau < T$, thus

$$e^{-\frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\Sigma^{-1}(\tau)(\boldsymbol{x}/\bar{\boldsymbol{u}})} \le e^{-\frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\left(\Sigma^{-1}(\tau) - \eta I_d\right)(\boldsymbol{x}/\bar{\boldsymbol{u}})} e^{-\frac{\eta}{2}||\boldsymbol{x}/\bar{\boldsymbol{u}}||^2} \le e^{-\frac{\eta}{2}\|\boldsymbol{x}_J\|^2}.$$

Furthermore, due to the continuity of $v_i(\cdot)$ and $\lambda_i(\cdot)$ (see Lemma 4.9), for all $\varepsilon > 0$ there exists $\tau_0 < T$ large enough such that

$$\lambda_i(T)(1 - \varepsilon) < \lambda_i(\tau) < \lambda_i(T)(1 + \varepsilon),$$

and

$$c_i^*(T)(1 - \varepsilon) \le |c_i^*(\tau)| \le c_i^*(T)(1 + \varepsilon)$$

for all $i \in \{1, \ldots, d\}$ and $\tau \in [\tau_0, T]$. Moreover, for all $\varepsilon > 0$ small enough and $\tau_0 > \bar{\tau}$, where $\bar{\tau}$ is defined in Lemma 4.13 we also have

$$\lambda_i^{\varepsilon,\boldsymbol{x}}(t) := \lambda_i(t) - \mathrm{sgn}(x_i)\varepsilon > 0$$

for all $i \in I$, thus for every $\varepsilon > 0$ small enough there exists τ_0 such that

$$\theta_u(\boldsymbol{x}) \le e^{<\boldsymbol{\lambda}_I(T)(1+\varepsilon\cdot \mathrm{sgn}(\boldsymbol{x}_I)),\boldsymbol{x}_I>} e^{<\boldsymbol{c}_J^*(T)(1+\varepsilon\cdot \mathrm{sgn}(\boldsymbol{x}_J)),\boldsymbol{x}_J>} e^{-\frac{\eta}{2}||\boldsymbol{x}_J||^2} =: \bar{\theta}(\boldsymbol{x}).$$

Now, let $\boldsymbol{Z}^{\boldsymbol{x}}_{u,\tau}(t)$, and $\hat{\boldsymbol{Z}}_{u,\tau}(t)$ be defined as in (4.11). Since $\boldsymbol{X} = A\boldsymbol{Z}$, then

$$\begin{aligned}
h_{u,\tau}(L,\boldsymbol{x}) &= \mathbb{P}\left\{\exists_{t\in[\tau-Lu^{-2},\tau]} : A\boldsymbol{Z}(t) - \boldsymbol{c}t > u\boldsymbol{a} \mid A\boldsymbol{Z}(\tau) = \boldsymbol{w}_{u,\tau}(\boldsymbol{x})\right\}, \qquad (4.13)\\
&= \mathbb{P}\left\{\exists_{t\in[0,L]} : A\boldsymbol{Z}(\tau - \tfrac{t}{u^2}) - \boldsymbol{c}(\tau - \tfrac{t}{u^2}) > u\boldsymbol{a} \mid A\boldsymbol{Z}(\tau) = \boldsymbol{w}_{u,\tau}(\boldsymbol{x})\right\},\\
&= \mathbb{P}\left\{\exists_{t\in[0,L]} : A\boldsymbol{Z}^{\boldsymbol{x}}_{u,\tau}(t) - \boldsymbol{c}\tau + \boldsymbol{c}t/u^2 > u\boldsymbol{a}\right\}\\
&= \mathbb{P}\left\{\exists_{t\in[0,L]}\forall_{i\in\{1,\dots,d\}} : u(A\hat{\boldsymbol{Z}}_{u,\tau}(t))_i + (\boldsymbol{\mu}_{u,\tau}(t,\boldsymbol{x}))_i > 0\right\},
\end{aligned}$$

where, with defining $R_{u,\tau}(t) := \mathrm{diag}(\rho_i(\tau - \frac{t}{u^2}, \tau)/v_i(\tau))$ for breviety, we have

$$\begin{aligned}
\boldsymbol{\mu}_{u,\tau}(t,\boldsymbol{x}) &:= uA\,R_{u,\tau}(t)A^{-1}\boldsymbol{w}_{u,\tau}(\boldsymbol{x}) - u\boldsymbol{c}\tau + \boldsymbol{c}t/u - u^2\boldsymbol{a}\\
&= uA\,R_{u,\tau}(t)A^{-1}(u\tilde{\boldsymbol{a}} + \boldsymbol{c}\tau - \boldsymbol{x}/\bar{\boldsymbol{u}}) - u\boldsymbol{c}\tau + \boldsymbol{c}t/u - u^2\boldsymbol{a}\\
&= u^2A\left(R_{u,\tau}(t) - \mathcal{I}_d\right)A^{-1}\cdot(\tilde{\boldsymbol{a}} + \boldsymbol{c}\tau/u - \boldsymbol{x}/(u\bar{\boldsymbol{u}}))\\
&\qquad + \boldsymbol{c}t/u + u^2(\tilde{\boldsymbol{a}} - \boldsymbol{a}) - u\boldsymbol{x}/\bar{\boldsymbol{u}}.
\end{aligned}$$

Notice that

$$\begin{aligned}
&u^2A\left(R_{u,\tau}(t) - \mathcal{I}_d\right)A^{-1}\\
&\qquad = u^2A\cdot\mathrm{diag}\left(\frac{\rho_i(\tau-\frac{t}{u^2},\tau) - v_i(\tau)}{v_i(\tau)}\right)\cdot A^{-1}\\
&\qquad = tA\cdot\mathrm{diag}\left(\frac{1}{2v_i(\tau)}\left[\frac{v_i(\tau-\frac{t}{u^2}) - v_i(\tau)}{t/u^2} + \frac{v_i(\frac{t}{u^2})}{t/u^2}\right]\right)\cdot A^{-1}\\
&\qquad \to -\frac{t}{2}AQ(\tau)A^{-1}, \quad u\to\infty \qquad (4.14)
\end{aligned}$$

for any fixed τ, T, where $Q(\tau) := \mathrm{diag}(\dot{v}_i(\tau)/v_i(\tau))$. Moreover, applying the mean value theorem yields

$$\inf_{s\in[\tau_0-Lu^{-2},T]}|\dot{v}_i(s)| \le \left|\frac{v_i(\tau-\frac{t}{u^2}) - v_i(\tau)}{t/u^2}\right| \le \sup_{s\in[\tau_0-Lu^{-2},T]}|\dot{v}_i(s)|,$$

so using the assumption **B0**, for every $\varepsilon > 0$ there exists $\tau_0 < T$ such that

$$-(1+\varepsilon)\dot{v}_i(T) \le \frac{v_i(\tau-\frac{t}{u^2}) - v_i(\tau)}{t/u^2} \le -(1-\varepsilon)\dot{v}_i(T)$$

for all $\tau \in [\tau_0, T]$, $t \in [0, L]$ and u large enough. The bound above implies that for any $\varepsilon > 0$, we can find $\tau_0 < T$ such that for all $i \in I$ we have

$$(\boldsymbol{\mu}_{u,\tau}(t, \boldsymbol{x}))_i \leq \left(-\tfrac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon\right) t - x_i(1 - \varepsilon \cdot \operatorname{sgn}(x_i)),$$

for all $t \in [\tau_0, T]$ and u large enough, where $Q := Q(T) = \operatorname{diag}(\dot{v}_i(T)/v_i(T))$. In the following define $W_{u,\tau} := \max_{i \in \{1,\dots,d\}} \sup_{t \in [0,L]} u(A\hat{\boldsymbol{Z}}_{u,\tau}(t))_i$ and see that

$$\begin{aligned}
&h_{u,\tau}(L, \boldsymbol{x}) \\
&\quad \leq \mathbb{P}\left\{\exists_{t \in [0,L]} \forall_{i \in I} : W_{u,\tau} + \left(-\tfrac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon\right) t - x_i(1 - \varepsilon \cdot \operatorname{sgn}(x_i)) > 0\right\} \\
&\quad \leq \mathbb{P}\left\{\exists_{t \in [0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon \cdot \operatorname{sgn}(x_i)} \cdot t > x_i - \frac{W_{u,\tau}}{1 - \varepsilon}\right\} \\
&\quad \leq \sum_{k=0}^{\infty} \mathbb{P}\left\{\exists_{t \in [0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon \cdot \operatorname{sgn}(x_i)} \cdot t > x_i - \frac{W_{u,\tau}}{1 - \varepsilon}; W_{u,\tau} \in (\varepsilon k, \varepsilon(k+1)]\right\} \\
&\quad \leq \sum_{k=0}^{\infty} \mathbb{I}\left\{\exists_{t \in [0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon \cdot \operatorname{sgn}(x_i)} \cdot t > x_i - \frac{\varepsilon(k+1)}{1 - \varepsilon}\right\} \mathbb{P}\left\{W_{u,\tau} > \varepsilon k\right\}.
\end{aligned}$$

Furthermore, due to Corollary 4.15 and assumption **BII**, we have

$$\mathbb{P}\left\{W_{u,\tau} > \varepsilon k\right\} \leq e^{-(\varepsilon k)^2}$$

for all $\tau \in [\tau_0, T]$ and u large enough. Thus,

$$\begin{aligned}
&h_{u,\tau}(L, \boldsymbol{x}) \\
&\qquad \leq \sum_{k=0}^{\infty} \mathbb{I}\left\{\exists_{t \in [0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon \cdot \operatorname{sgn}(x_i)} \cdot t > x_i - \frac{\varepsilon(k+1)}{1 - \varepsilon}\right\} e^{-(\varepsilon k)^2} \\
&\qquad := \bar{h}(L, \boldsymbol{x})
\end{aligned}$$

for all u large enough. Furthermore, define

$$\begin{aligned}
E_k(L) := \int_{\mathbb{R}^{|I|}} \mathbb{I}&\left\{\exists_{t \in [0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon \cdot \operatorname{sgn}(x_i)} \cdot t > x_i - \frac{\varepsilon(k+1)}{1 - \varepsilon}\right\} \\
&\times e^{<\boldsymbol{\lambda}_I(T)(1 + \varepsilon \cdot \operatorname{sgn}(\boldsymbol{x}_I)), \boldsymbol{x}_I>} \mathrm{d}\boldsymbol{x}_I.
\end{aligned}$$

Then

$$\int_{\mathbb{R}^d} \bar{h}(L, \boldsymbol{x}) \bar{\theta}(\boldsymbol{x}) = \sum_{k=0}^{\infty} E_k(L) e^{-(\varepsilon k)^2} \cdot \int_{\mathbb{R}^{|J|}} e^{<\boldsymbol{c}_J^*(T)(1 + \varepsilon \cdot \operatorname{sgn}(\boldsymbol{x}_J)), \boldsymbol{x}_J>} e^{-\frac{\eta}{2}||\boldsymbol{x}_J||^2} \mathrm{d}\boldsymbol{x}_J.$$

Now, the integral over $\mathbb{R}^{|J|}$ above is bounded for all ε small enough because it does not depend on τ and u. We now focus on the sum $\sum_{k=0}^{\infty} E_k(L) e^{-(\varepsilon k)^2}$. Let $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_d) \in \{-1,1\}^{|I|}$. For each $k \in \mathbb{N}$ we have

$$
\begin{aligned}
&E_k(L) \\
&\quad \leq \sum_{\boldsymbol{\delta} \in \{-1,1\}^{|I|}} \int_{\mathbb{R}^{|I|}} \mathbb{I}\left\{ \exists_{t\in[0,L]} \forall_{i \in I} : \frac{-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon}{1 - \varepsilon\delta_i} \cdot t > x_i - \frac{\varepsilon(k+1)}{1-\varepsilon} \right\} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times e^{<\boldsymbol{\lambda}_I(T)(1+\varepsilon\delta_i), \boldsymbol{x}_I>} \mathrm{d}\boldsymbol{x}_I.
\end{aligned}
$$

After applying substitution $x_i := \lambda_i(1+\varepsilon\delta_i)\left[x_i - \frac{\varepsilon(k+1)}{1-\varepsilon}\right]$, each term of the sum above is bounded from above by $\mathcal{C}(L; \varepsilon, \boldsymbol{\delta}) \cdot e^{g_i(\varepsilon)}$, where

$$
\mathcal{C}(L; \varepsilon, \boldsymbol{\delta}) := \frac{1}{\prod_{i \in I} \lambda_i (1+\varepsilon\delta_i)} \int_{\mathbb{R}^{|I|}} \mathbb{I}\left\{ \exists_{t \in [0,L]} \forall_{i\in I} : f_i(\varepsilon, \delta_i) t > x_i \right\} e^{\sum_{i\in I} x_i} \mathrm{d}\boldsymbol{x}_I,
$$

with $f_i(\varepsilon, \delta_i)$ and $g_i(\varepsilon)$ defined below

$$
f_i(\varepsilon, \delta_i) := \frac{\left(-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i + \varepsilon\right)\lambda_i(1+\varepsilon\delta_i)}{1-\varepsilon\delta_i}, \quad g_i(\varepsilon) := \frac{\lambda_i(1+\varepsilon)\varepsilon(k+1)}{1-\varepsilon}.
$$

It is straightforward to see that $f_i(\varepsilon, \delta_i) \to f_i := (-A\frac{\dot{\boldsymbol{v}}(T)}{2\boldsymbol{v}(T)} A^{-1}\boldsymbol{a})_i \lambda_i$, as $\varepsilon \to 0$ for any $\boldsymbol{\delta}$ and similarly $g_i(\varepsilon) \to 0$. Therefore,

$$
\mathcal{C}(L; \varepsilon, \boldsymbol{\delta}) \to \mathcal{C}(L), \quad \varepsilon \to 0,
$$

where

$$
\begin{aligned}
\mathcal{C}(L) &:= \prod_{i\in I} \lambda_i \int_{\mathbb{R}^{|I|}} \mathbb{I}\left\{ \exists_{t\in[0,L]} : -\tfrac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_I - \boldsymbol{x}_I > \boldsymbol{0}_I \right\} e^{\sum\limits_{i\in I} \lambda_i x_i} \mathrm{d}\boldsymbol{x}_I \qquad (4.15) \\
&= \int_{\mathbb{R}^{|I|}} \mathbb{I}\left\{ \exists_{t\in[0,L]} \forall_{i\in I} : -\tfrac{1}{2}\lambda_i \cdot (AQA^{-1}\tilde{\boldsymbol{a}})_i - x_i > 0 \right\} e^{\sum\limits_{i\in I} x_i} \mathrm{d}\boldsymbol{x}.
\end{aligned}
$$

Now, since $\Sigma^{-1} = A^{-\top} \operatorname{diag}(v_i(T)) A^{-1}$ and $\boldsymbol{\lambda} = \Sigma^{-1}\tilde{a}$, then

$$
\begin{aligned}
\sum_{i\in I} \lambda_i \cdot (AQA^{-1}\tilde{\boldsymbol{a}})_i &= \langle AQA^{-1}\tilde{\boldsymbol{a}}, \boldsymbol{\lambda} \rangle = \langle AQA^{-1}\tilde{\boldsymbol{a}}, \boldsymbol{\lambda} \rangle \\
&\qquad = \boldsymbol{\lambda}^\top \operatorname{diag}\left(\frac{\dot{v}_i(T)}{v_i(T)}\right) A^\top A^{-\top} \operatorname{diag}\left(\frac{1}{v_i(T)}\right) A^{-1}\tilde{\boldsymbol{a}} \\
&\qquad = -\dot{D}(T),
\end{aligned}
$$

where in the last line we used Lemma 4.8. Applying Lemma 4.12 using $\boldsymbol{f} = (f_i)$, $f_i = -\frac{1}{2}\sum_{i\in I}\lambda_i \cdot (AQA^{-1}\tilde{\boldsymbol{a}})_i$ and the fact that, $\sum_i f_i = \frac{1}{2}\dot{D}(T) < 0$, see Lemma 4.8, we conclude that $\mathcal{C}(L) \to \mathcal{C}$, as $L \to \infty$, with $\mathcal{C}$ defined in (3.1).

Now, since $\mathcal{C}(L) \to \mathcal{C}$, then there must exist some $c_1 > 0$ such that for all ε small enough and all $\boldsymbol{\delta} \in \{-1, 1\}^{|I|}$, we have $\mathcal{C}(L; \varepsilon, \boldsymbol{\delta}) \le (1 + c_1)2^{d+1}\mathcal{C}$. Finally, notice that for each $\varepsilon > 0$,

$$\sum_{k=0}^{\infty} \exp\left\{\frac{\lambda_i(1+\varepsilon)\varepsilon(k+1)}{1-\varepsilon}\right\} e^{-(\varepsilon k)^2} < \infty.$$

These observations combined give us that there exists a constant $c_2 > 0$ such that

$$\int_{\mathbb{R}^d} \bar{h}(L, \boldsymbol{x})\bar{\theta}(\boldsymbol{x}) < c_2 \cdot \mathcal{C}$$

for all u large enough. This completes the proof. □

In the following, for any $\tau \in (0, T]$, $L > 0$ and $u > \sqrt{L/\tau}$ let

$$M_\tau(u, L) := \mathbb{P}\left\{\exists_{t\in[\tau - Lu^{-2}, \tau]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a}\right\}. \tag{4.16}$$

Proof of Lemma 4.4: For any $T > 0$, with $M_T(u, L)$ defined in (4.16), we have

$$M_T(u, L) = \int_{\mathbb{R}^d} \mathbb{P}\left\{\exists_{t\in[T-Lu^{-2},T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a} \mid \boldsymbol{X}(T) = \boldsymbol{x}\right\}\varphi_T(\boldsymbol{x})\mathrm{d}\boldsymbol{x},$$

where φ_T be the pdf of $\boldsymbol{X}(T)$. After applying substitution $\boldsymbol{w}_{u,T}(\boldsymbol{x})$; see (4.10) we obtain

$$\begin{aligned} M_T(u, L) &= u^{-|I_T|}\int_{\mathbb{R}^d} \mathbb{P}\left\{\exists_{t\in[T-Lu^{-2},T]} : \boldsymbol{X}(t) - \boldsymbol{c}t > u\boldsymbol{a} \mid \boldsymbol{X}(T) = \boldsymbol{w}_{u,T}(\boldsymbol{x})\right\} \\ &\qquad\qquad \times \varphi(\boldsymbol{w}_{u,T}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x} \\ &= u^{-|I_T|}\int_{\mathbb{R}^d} h_{u,T}(L, \boldsymbol{x})\varphi(\boldsymbol{w}_{u,T}(\boldsymbol{x}))\mathrm{d}\boldsymbol{x}. \end{aligned} \tag{4.17}$$

Now, let $\theta_{u,T}(\boldsymbol{x}) := \varphi_T(\boldsymbol{w}_{u,T}(\boldsymbol{x}))/\varphi_T(u\tilde{\boldsymbol{a}}(T) + \boldsymbol{c}T)$. Then

$$\begin{aligned} &\theta_{u,T}(\boldsymbol{x}) \\ &\quad = \exp\left\{u\tilde{\boldsymbol{a}}^\top(T)\Sigma^{-1}(T)(\boldsymbol{x}/\bar{\boldsymbol{u}}) + T\boldsymbol{c}^\top\Sigma^{-1}(T)(\boldsymbol{x}/\bar{\boldsymbol{u}}) - \frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\Sigma^{-1}(T)(\boldsymbol{x}/\bar{\boldsymbol{u}})\right\}, \end{aligned}$$

with the three terms under the exponent exhibit the following behavior:

$$e^{-\frac{1}{2}(\boldsymbol{x}/\bar{\boldsymbol{u}})^\top\Sigma^{-1}(T)(\boldsymbol{x}/\bar{\boldsymbol{u}})} \to e^{-\frac{1}{2}\boldsymbol{x}_J^\top(\Sigma^{-1}(T))_{JJ}\boldsymbol{x}_J}, \quad u \to \infty$$

$$e^{T\boldsymbol{c}^\top\Sigma^{-1}(T)(\boldsymbol{x}/\bar{\boldsymbol{u}})} \to e^{T\langle(\boldsymbol{c}^\top\Sigma^{-1}(T))_J,\boldsymbol{x}_J\rangle}, \quad u\to\infty$$
$$e^{\tilde{\boldsymbol{a}}^\top(T)\Sigma^{-1}(T)(u\boldsymbol{x}/\bar{\boldsymbol{u}})} = e^{\boldsymbol{\lambda}^\top(T)(u\boldsymbol{x}/\bar{u})} = e^{\langle\boldsymbol{\lambda}_I(T),\boldsymbol{x}_I\rangle}.$$

Letting $\tilde{\boldsymbol{c}} := Tc^\top\Sigma^{-1}(T)$ we obtain

$$\theta(\boldsymbol{x}) := \lim_{u\to\infty}\theta_u(\boldsymbol{x}) = e^{\langle\boldsymbol{\lambda}_I(T),\boldsymbol{x}_I\rangle}\cdot e^{-\frac{1}{2}\boldsymbol{x}_J^\top(\Sigma^{-1}(T))_{JJ}\boldsymbol{x}_J}e^{\langle\tilde{\boldsymbol{c}}_J,\boldsymbol{x}_J\rangle}.$$

Let $\hat{\boldsymbol{Z}}_{u,\tau}(t)$ be defined as in (4.11). Repeating steps from the proof of Lemma 4.16, cf. Eq. (4.13) and below, we find that

$$h_{u,T}(L,\boldsymbol{x}) = \mathbb{P}\left\{\exists_{t\in[0,L]} : uA\hat{\boldsymbol{Z}}_{u,\tau}(t) + \boldsymbol{\mu}_{u,T}(t,\boldsymbol{x}) > 0\right\},$$

where

$$\begin{aligned}\boldsymbol{\mu}_{u,T}(t,\boldsymbol{x}) = {} & u^2A\left(R_{u,\tau}(t)-\mathcal{I}_d\right)A^{-1}\\ & \times\left(\tilde{\boldsymbol{a}}+\boldsymbol{c}\tau/u - x/(u\bar{\boldsymbol{u}})\right) + \boldsymbol{c}t/u + u^2(\tilde{\boldsymbol{a}}-\boldsymbol{a}) - u\boldsymbol{x}/\bar{\boldsymbol{u}}\end{aligned}$$

with $R_{u,\tau}(t) := \mathrm{diag}(\rho_i(\tau-\frac{t}{u^2},\tau)/v_i(\tau))$. Let $U_T := \{i\in J_T : \tilde{a}_i(T) = a_i\}$ be the subset of J_T. Repeating steps from the proof of Lemma 4.16, cf. Eq. (4.14) and below, we obtain

$$(\boldsymbol{\mu}_{u,T}(t,\boldsymbol{x}))_i \to \mu(T,\boldsymbol{x}) := \begin{cases}-\frac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_i - x_i, & i\in I,\\ \mathrm{sgn}(-x_i)\cdot\infty, & i\in U,\\ \infty, & i\in J\setminus U\end{cases}$$

as $u\to\infty$, where $Q := \mathrm{diag}(\dot{v}_i(T)/v_i(T))$. We can see that

$$\begin{aligned}&\mathbb{P}\left\{\exists_{t\in[0,L]}\forall_{i\in\{1,\dots,d\}} : (\boldsymbol{\mu}_{u,T}(t,\boldsymbol{x}))_i > 0\right\} \le h_{u,T}(L,\boldsymbol{x})\\ &\quad\le \mathbb{P}\left\{\exists_{t\in[0,L]}\forall_{i\in\{1,\dots,d\}} : \sup_{t\in[0,L]}(uA\hat{\boldsymbol{Z}}_{u,\tau}(t))_i + (\boldsymbol{\mu}_{u,T}(t,\boldsymbol{x}))_i > 0\right\}.\end{aligned}$$

Corollary 4.15 implies that $\sup_{t\in[0,L]}(uA\hat{\boldsymbol{Z}}_{u,\tau}(t))_i \to 0$, as $u\to\infty$, so for every $\boldsymbol{x}\in\mathbb{R}$ we have

$$\begin{aligned}\lim_{u\to\infty}h_{u,T}(L,\boldsymbol{x}) &= \mathbb{I}\left\{\exists_{t\in[0,L]} : -\tfrac{1}{2}(AQA^{-1}\tilde{\boldsymbol{a}})_I - \boldsymbol{x}_I > \boldsymbol{0}_I\right\}\cdot\mathbb{I}\{\boldsymbol{x}_U < \boldsymbol{0}_U\}\\ &=: h(L,\boldsymbol{x}).\end{aligned}$$

Thanks to Lemma 4.16, we may apply Lebesgue's dominated convergence theorem and obtain as $u\to\infty$

$$\int_{\mathbb{R}^d}h_{u,T}(L,\boldsymbol{x})\theta_{u,T}(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \to \int_{\mathbb{R}^d}h(L,\boldsymbol{x})\theta(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

$$= \mathcal{C}(L) \cdot \frac{1}{\prod_{i\in I} \lambda_i(T)} \int_{\mathbb{R}^{|J|}} e^{-\frac{1}{2}(\boldsymbol{x}_J - \tilde{\boldsymbol{c}})^\top (\Sigma^{-1}(T))_{JJ}(\boldsymbol{x}_J-\tilde{\boldsymbol{c}})} \times e^{\frac{1}{2}\tilde{\boldsymbol{c}}^\top(\Sigma^{-1}(T))_{JJ}\tilde{\boldsymbol{c}}} \mathbb{I}_{\{\boldsymbol{x}_U<\boldsymbol{0}_U\}} \mathrm{d}\boldsymbol{x}_J,$$

with $\mathcal{C}(L)$ defined in (4.15). Finally, using Lemma 4.5 yields

$$\frac{M_T(u,L)}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \to \mathcal{C}(L)$$

as $u \to \infty$. Repeating the reasoning from the proof of Lemma 4.16, we conclude that $\mathcal{C}(L) \to \mathcal{C}$, as $L \to \infty$, with $\mathcal{C}$ defined in (3.1). □

Proof of Lemma 4.3: Define a sequence $\tau_k := T - kLu^{-2}$ and a constant $K(u) = [\ln^2(u) + 1]$. Then

$$\frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-2}\ln^2(u),T-Lu^{-2}]}\boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{a}u\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \le \sum_{k=1}^{K(u)} \frac{M_{\tau_k}(u,L)}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}},$$

with $M_\tau(u,L)$ defined in (4.16). Similarly to (4.17), for any $\tau > 0$ we have

$$M_\tau(u,L) = u^{-|I_\tau|} \int_{\mathbb{R}^d} h_{u,\tau}(L,\boldsymbol{x})\varphi_\tau(\boldsymbol{w})\mathrm{d}\boldsymbol{x},$$

with $h_{u,\tau}$ defined as in (4.12). Let τ_0 be as in Lemma 4.16. Then $\tau_k \in [\tau_0, T]$ for all $k \in \{1,\dots,K(u)\}$ we may apply Lemma 4.16 and obtain

$$M_{\tau_k}(u,L) \le u^{-|I_T|}\varphi_{\tau_k}(u\tilde{\boldsymbol{a}}(\tau_k) + \boldsymbol{c}\tau_k) \cdot \int_{\mathbb{R}^d} H(L,\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$

for all u large enough. Furthermore, according to Lemma 4.16, there exists a constant C_1 such that $\int_{\mathbb{R}^d} H(L,\boldsymbol{x})\mathrm{d}\boldsymbol{x} \le C_1$ for all u large enough. Moreover, according to Lemma 4.5, there exist $C_2 > 0$, such that

$$\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\} \ge C_2 u^{-|I_T|}\varphi_T(\tilde{\boldsymbol{a}}u + \boldsymbol{c}T)$$

for all u large enough. We thus have

$$\frac{\mathbb{P}\left\{\exists_{t\in[T-Lu^{-2}\ln^2(u),T-Lu^{-2}]}\boldsymbol{X}(t) - \boldsymbol{c}t > \boldsymbol{a}u\right\}}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \le \frac{C_1}{C_2}\sum_{k=1}^{K(u)} \frac{\varphi_{\tau_k}(u\tilde{\boldsymbol{a}}(\tau_k) + \boldsymbol{c}\tau_k)}{\varphi_T(u\tilde{\boldsymbol{a}}(T) + \boldsymbol{c}T)}.$$

Consider one of the terms in the sum above. We have

$$\frac{\varphi_{\tau_k}(u\tilde{\boldsymbol{a}}(\tau_k) + \boldsymbol{c}\tau_k)}{\varphi_T(u\tilde{\boldsymbol{a}}(T) + \boldsymbol{c}T)} = \sqrt{\frac{|\Sigma(T)|}{|\Sigma(\tau_k)|}} \cdot e^{-\frac{u^2}{2}(D(\tau_k)-D(T))} e^{-u\langle \boldsymbol{\lambda}(\tau_k) - \boldsymbol{\lambda}(T), \boldsymbol{c}\rangle}$$

$$\times e^{-\frac{1}{2}c^\top(\Sigma^{-1}(\tau_k)-\Sigma^{-1}(T))c}.$$

Now, using the fact that $D(t)$ has a negative derivative at $t = T$ (cf. Lemma 4.8), and the fact that $\tau_{K(u)} \to T$ we know that if u large enough, then $u^2(D(\tau_k) - D(T)) \geq |\dot{D}(T)|kL/2$. Using Lipschitz continuity of λ_i we also find that there exists a constant $C_1 > 0$ such that $|\langle\lambda_i(\tau_k) - \lambda_i(T), \boldsymbol{c}\rangle| \leq C_1 kL$. Both these observations combined imply that there exist some constants C_3, $\beta > 0$ such that

$$\frac{\varphi_{\tau_k}(u\tilde{\boldsymbol{a}}(\tau_k) + \boldsymbol{c}\tau_k)}{\varphi_T(u\tilde{\boldsymbol{a}}(T) + \boldsymbol{c}T)} \leq C_3 e^{-\beta kL}$$

for all u large enough. Finally, we have

$$\sum_{k=1}^{K(u)} \frac{M_{\tau_k}(u, L)}{\mathbb{P}\left\{\boldsymbol{X}(T) - \boldsymbol{c}T > u\boldsymbol{a}\right\}} \leq \frac{C_1 C_3}{C_2} \sum_{k=1}^{\infty} e^{-\beta kL} \leq \frac{C_1 C_3}{C_2(1 - e^{-\beta L})} \cdot e^{-\beta L},$$

which completes the proof. □

Chapter 5

Uniform Bounds for Ruin Probability

1 Introduction and first result

Let $\boldsymbol{B}(t), t \geq 0$ be a d-dimensional Brownian motion with independent standard Brownian motion components and set $\boldsymbol{Z}(t) = A\boldsymbol{B}(t), t \geq 0$ with A a $d \times d$ real non-singular matrix. The recent contribution [44] derived the following remarkable inequality

$$1 \leq \frac{\mathbb{P}\{\exists t \in [0,T] : \boldsymbol{Z}(t) \geq \boldsymbol{b}\}}{\mathbb{P}\{\boldsymbol{Z}(T) \geq \boldsymbol{b}\}} \quad \leq \quad K(T),\ K(T) = \frac{1}{\mathbb{P}\{\boldsymbol{Z}(T) \geq \boldsymbol{0}\}}. \tag{1.1}$$

valid for all $\boldsymbol{b} \in \mathbb{R}^d, T > 0$. In our notation bold symbols are column vectors with d rows and all operations are meant component-wise, for instance $\boldsymbol{x} \geq \boldsymbol{0}$ means $x_i \geq 0$ for all $i \leq d$ with $\boldsymbol{0} = (0, \ldots, 0) \in \mathbb{R}^d$.
The special and crucial feature of (1.1) is that the bounds are uniform with respect to $\boldsymbol{b}$. Moreover, if at least one component of $\boldsymbol{b}$ tends to infinity, then $\mathbb{P}\{\exists t \in [0,T] : \boldsymbol{Z}(t) \geq \boldsymbol{b}\}$ can be accurately approximated (up to some constant) by the survival probability $\mathbb{P}\{\boldsymbol{Z}(T) \geq \boldsymbol{b}\}$.
Inequality (1.1) has been crucial in the context of Shepp-statistics investigated in [44]. It is also of great importance in the investigation of simultaneous ruin probabilities in vector-valued risk models (see [16, 17, 47]). Specifically, consider the multidimensional risk model

$$\boldsymbol{R}(t,u) = \boldsymbol{a}u - \boldsymbol{X}(t),\ \boldsymbol{X}(t) = \boldsymbol{Z}(t) - \boldsymbol{c}t$$

[4]This chapter is based on the paper [46].

for some vectors $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^d$ and $\boldsymbol{Z}(t), t \ge 0$ defined above. Typically, $\boldsymbol{R}$ models the surplus of all d-portfolios of an insurance company, where $a_i u, u > 0$ plays the role of the initial capital, the component Z_i models the accumulated claim amount up to time t and $c_i t$ is the premium income for the ith portfolio.

Given a positive integer $k \le d$, of interest is the k-th simultaneous ruin probability, i.e., at least k out of d portfolios are ruined on a given time interval $[0, T]$ with T possibly also infinite. That ruin probability can be written as

$$\mathbb{P}\left\{\exists_{t \in [0,T]} : \boldsymbol{Z}(t) - \boldsymbol{c}t \in u\boldsymbol{S}\right\}, \qquad u > 0,$$

where

$$\boldsymbol{S} := \bigcup_{\substack{I \subset \{1 \dots d\} \\ |I| = k}} \boldsymbol{S}_I, \qquad \boldsymbol{S}_I = \{\boldsymbol{x} \in \mathbb{R}^d : \ \forall i \in I, \ x_i > a_i\}.$$

The particular case $\boldsymbol{Z}(t) = A\boldsymbol{B}(t), t \ge 0$ with A a $d \times d$ non-singular matrix is of special importance for insurance risk models, see e.g., [27]. Clearly, this instance is also of great importance in statistics and probability given the central role of the $\mathbb{R}^d$-valued Brownian motion.

In [15] it has been shown that (1.1) can be extended for this model, i.e., for all $u > 0$, $T > 0$, and any compact set $L_T \subset [0, T]$, such that $T \in L_T$

$$1 \le \frac{\mathbb{P}\{\exists_{t \in L_T} : \boldsymbol{X}(t) \in u\boldsymbol{S}\}}{\mathbb{P}\{\boldsymbol{X}(T) \in u\boldsymbol{S}\}} \quad \le \quad K_{\boldsymbol{S}}(L_T), \ \boldsymbol{X}(t) = \boldsymbol{Z}(t) - \boldsymbol{c}(t), \qquad (1.2)$$

with $\boldsymbol{c}(t) = \boldsymbol{c}t, t \ge 0$ and some known constant $K_{\boldsymbol{S}}(L_T) > 0$. Again the bounds are uniform with respect to u.

It is clear that the inequality above does not hold for an arbitrary set $\boldsymbol{S} \subset \mathbb{R}^d$. For example, taking $\boldsymbol{S} = \{\boldsymbol{x} \in \mathbb{R}^d : x_1 = 1\}$ we have that

$$\begin{aligned} \mathbb{P}\{\exists_{t \in L_T} : \boldsymbol{X}(t) \in u\boldsymbol{S}\} &= \mathbb{P}\{\exists_{t \in L_T} : X_1(t) > u\} \ge \mathbb{P}\{X_1(T) > u\} > 0, \\ \mathbb{P}\{\boldsymbol{X}(T) \in u\boldsymbol{S}\} &= 0, \end{aligned}$$

and (1.2) does not hold for any constant $K_{\boldsymbol{S}}(L_T)$.

We consider further only sets $\boldsymbol{S}$ which satisfy the following condition:

Definition 1.1 *Let $\boldsymbol{X}$ and $\boldsymbol{Z}$ are as defined above. The Borel set $\boldsymbol{S} \subset \mathbb{R}^d$ satisfies cone condition with respect to the vector-valued process $\boldsymbol{X}$ if there exists a strictly positive function $\varepsilon_{\boldsymbol{S}}(t), t > 0$ such that for any point $\boldsymbol{x} \in \boldsymbol{S}$ and any $t > 0$ there exists a Borel set $\boldsymbol{V}_{\boldsymbol{x}} \subset \boldsymbol{S}$ that contains $\boldsymbol{x}$ satisfying $\boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x} \subset c(\boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x})$ for all $c > 1$ and $\mathbb{P}\{\boldsymbol{Z}(t) \in \boldsymbol{V}_x - \boldsymbol{x}\} \ge \varepsilon_{\boldsymbol{S}}(t)$.*

It is of interest to consider a general trend function in (1.2). We consider below a large class of trend functions which is tractable if $\boldsymbol{Z}$ has self-similar coordinates with index $\alpha > 0$. This is in particular the case when $\boldsymbol{Z} = A\boldsymbol{B}$.

Definition 1.2 *A vector-valued measurable function* $\boldsymbol{c} : [0, +\infty) \to \mathbb{R}^d$ *belongs to* $RV_{t_0}(\alpha)$ *for some* $\alpha > 0$ *and* $t_0 \in [0, T]$ *if for some* $M > 0$*, all* $i \in \{1 \dots d\}$ *and all* $t \in [0, T]$

$$|c_i(t) - c_i(t_0)| \leq M|t - t_0|^\alpha.$$

We state next our first result. Below $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$ growing means that for any $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$ such that for all $i \in \{1, \dots, d\}$ $x_i \geq y_i$ we have that for all $i \in \{1, \dots, d\}$ $F_i(\boldsymbol{x}) \geq F_i(\boldsymbol{y})$.

Theorem 1.3 *If* $\boldsymbol{S} \subset \mathbb{R}^d$ *satisfies the cone condition with respect to the process* $\boldsymbol{Z} = A\boldsymbol{B}$ *such that* $\boldsymbol{0} \notin \boldsymbol{S}$ *and* $\boldsymbol{c} \in RV_T(1/2)$*, then for all constants* $T > 0, u > 1$ *the inequality* (1.2) *holds and*

$$K(L_T) = \frac{2^{d/2}}{\mathfrak{C}(L_T)\varepsilon_{\boldsymbol{S}}(T)}, \quad \mathfrak{C}(L_T) = \inf_{t \in L_T \setminus \{T\}} e^{-T\left(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}}\right)^\top \Sigma^{-1}\left(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}}\right)} > 0,$$

where Σ *is the covariance function of* $\boldsymbol{Z}(T)$*. In particular, for any growing function* $\boldsymbol{F} : \mathbb{R}^d \to \mathbb{R}^d$ *we have*

$$\mathbb{P}\left\{\exists_{t \in L_T} : \boldsymbol{F}(\boldsymbol{Z}(t) - \boldsymbol{c}(t)) > u\boldsymbol{a}\right\} \leq C_T \mathbb{P}\left\{\boldsymbol{F}(\boldsymbol{Z}(T) - \boldsymbol{c}(T)) > u\boldsymbol{a}\right\}$$

for all $\boldsymbol{a} \in \mathbb{R}^d \setminus (-\infty, 0]^d$*,* $u > 1$ *and some constant* C_T *which does not depend on* u*.*

If $\boldsymbol{Z} : \mathbb{R}^n \to \mathbb{R}^d$ is a given separable random field, it is of interest to determine conditions such that (1.2) can be extended to

$$1 \leq \frac{\mathbb{P}\left\{\exists_{\boldsymbol{t} \in L_{\boldsymbol{T}}} : \boldsymbol{Z}(\boldsymbol{t}) - \boldsymbol{c}(\boldsymbol{t}) \in u\boldsymbol{S}\right\}}{\mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{T}) - \boldsymbol{c}(\boldsymbol{T}) \in u\boldsymbol{S}\right\}} \quad \leq \quad K_{\boldsymbol{S}}(L_{\boldsymbol{T}}), \tag{1.3}$$

where $L_{\boldsymbol{T}} \subset [0, T_1] \times \ldots \times [0, T_n]$ is compact, $\boldsymbol{T} \in L_{\boldsymbol{T}}$ and $\boldsymbol{T} = (T_1, \dots, T_n)$ has positive components. For the case $\boldsymbol{Z}(\boldsymbol{t}) = \sum_{i=1}^n \boldsymbol{Z}_i(t_i)$, where $\boldsymbol{Z}_i$ are independent copies of $\boldsymbol{Z}$, and $\boldsymbol{c}(\boldsymbol{t}) = 0$ the result (1.3) was shown in [44][Thm 1.1] for some special set $\boldsymbol{S}$. For more general set $\boldsymbol{S}$ we can show the following result:

Theorem 1.4 *If* $\boldsymbol{S} \subset \mathbb{R}^d$ *satisfies the cone condition with respect to the process* $\boldsymbol{Z}(\boldsymbol{t})$*,* $\boldsymbol{0} \notin \boldsymbol{S}$ *and all* $\boldsymbol{c}_i \in RV_T(1/2)$*, then for any* $T_1, \dots, T_n > 0, u > 1$ *the inequality* (1.3) *holds with* $\boldsymbol{Z}(\boldsymbol{t}) = \sum_{k=1}^n \boldsymbol{Z}_k(t_k)$ *and* $\boldsymbol{c}(\boldsymbol{t}) = \sum_{k=1}^n \boldsymbol{c}_k(t_k)$ *and*

$$\begin{aligned} K_{\boldsymbol{S}}(L_{\boldsymbol{T}}) &= \prod_{k=1}^n \frac{2^{d/2}}{\mathfrak{C}_k(T_k)\varepsilon_{\boldsymbol{S}}(T_k)}, \\ \mathfrak{C}_k(T_k) &= \inf_{t \in [0, T_k)} e^{-T_k\left(\frac{\boldsymbol{c}_k(T_k)-\boldsymbol{c}_k(t)}{\sqrt{T_k-t}}\right)^\top \Sigma^{-1}(T_k)\left(\frac{\boldsymbol{c}_k(T_k)-\boldsymbol{c}_k(t)}{\sqrt{T_k-t}}\right)} > 0, \end{aligned}$$

where $\varepsilon_{\boldsymbol{S}}$ *is any function satisfies the claims of Definition 1.1.*

2 Discussion

In this section, as in Introduction, we consider

$$\boldsymbol{Z}(t) = A\boldsymbol{B}(t), \qquad t \geq 0,$$

with A non-singular and $\boldsymbol{B}$ a d-dimensional Brownian motion with independent components. We are investigating the generalisation of the upper bound (1.2).

2.1 Order statistics

The classical multidimensional Brownian motion risk model (see [27]) is formulated in terms of risk process $\boldsymbol{R}$ specified by

$$\boldsymbol{R}(t,u) = \boldsymbol{a}u - \boldsymbol{Z}(t) + \boldsymbol{c}t$$

for some vectors $\boldsymbol{a}, \boldsymbol{c} \in \mathbb{R}^d$. We are interested in the finite-time simultaneous ruin probability for k out of d portfolios, i.e., probability that at least k portfolios are ruined. In other words, we are investigating the probability

$$\mathbb{P}\left\{\exists_{t\in[0,T]}, \exists_{\mathcal{I}\subset\{1,\ldots,d\}} : |\mathcal{I}| = k,\ \forall i \in \mathcal{I}\ Z_i(t) - c_i t > a_i u\right\}.$$

The probability above can be represented as follows

$$\mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{Z}(t) - \boldsymbol{c}t \in \boldsymbol{S}_u\right\}, \qquad u > 0,$$

where

$$\boldsymbol{S}_u := \bigcup_{\substack{I\subset\{1\ldots d\} \\ |I|=k}} \boldsymbol{S}_{I,u}, \qquad \boldsymbol{S}_{I,u} = \{\boldsymbol{x} \in \mathbb{R}^d :\ \forall i \in I\ x_i > a_i u\}.$$

Asymptotics of such probability was already investigated in [15]. Now we want to show a uniform bound. It is clear that all sets $\boldsymbol{S}_{I,u}$ satisfy the cone condition with respect to the process $\boldsymbol{Z}(t)$. Thus, $\boldsymbol{S}_u$ also satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$, hence we can use Theorem 1.3 and write for some positive constant C

$$\mathbb{P}\left\{\boldsymbol{Z}(T) - \boldsymbol{c}T \in \boldsymbol{S}_u\right\} \leq \mathbb{P}\left\{\exists_{t\in[0,T]} : \boldsymbol{Z}(t) - \boldsymbol{c}t \in \boldsymbol{S}_u\right\} \leq C\mathbb{P}\left\{\boldsymbol{Z}(T) - \boldsymbol{c}T \in \boldsymbol{S}_u\right\}.$$

2.2 Fractional Brownian motion

Consider next the risk model $d = 1$ driven by the one-dimensional fractional Brownian motion $B_H(t)$ for $t > 0$, i.e., the risk process is

$$R(u,t) = u - B_H(t) + ct,\ t > 0.$$

We are interested in the calculation of the finite-time ruin probability for given $T > 0$. The inequalities below have already been shown in [17]. We show now the way to obtain them using the general theorem presented above. Using Slepian inequality, we can write for $H > \frac{1}{2}$

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,T]}R(u,t)<0\right\} &\leq \mathbb{P}\left\{\exists_{t\in[0,T]}B_{1/2}\left(t^{2H}\right)-ct>u\right\}\\ &= \mathbb{P}\left\{\exists_{t\in[0,T^{2H}]}B_{1/2}\left(t\right)-ct^{1/2H}>u\right\}\\ &= \mathbb{P}\left\{\exists_{t\in[0,1]}B_{1/2}\left(T^{2H}t\right)-cTt^{1/2H}>u\right\}\\ &= \mathbb{P}\left\{\exists_{t\in[0,1]}B_{1/2}\left(t\right)-cT^{1-H}t^{1/2H}>u/T^{H}\right\}.\end{aligned}$$

Since $cT^{1-H}t^{1/2H} \in RV_1(1/2)$, using Theorem 1.3, for some positive constant C we can write

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,1]}B_{1/2}\left(t\right)-cT^{1-H}t^{1/2H}>u/T^{H}\right\} &\leq C\mathbb{P}\left\{B_{1/2}\left(1\right)-cT^{1-H}>u/T^{H}\right\}\\ &= C\mathbb{P}\left\{B_{1/2}\left(T^{2H}\right)-cT>u\right\}\\ &= C\mathbb{P}\left\{R(u,T)<0\right\}.\end{aligned}$$

The above can be extended considering the convolution of n independent one-dimensional fractional Brownian motions $B_i^{H_i}(t)$, for $t > 0, i \leq n$. Let $H_i > 1/2$ and define the risk processes

$$R_i(u,t) = u/n - B_i^{H_i}(t) + c_i t,\ i \leq n.$$

Consider the convolution of processes $R_i(u,t)$. Using Slepian inequality, as all $H_i > \frac{1}{2}$ we can write

$$\begin{aligned}\mathbb{P}\left\{\exists_{\boldsymbol{t}\in\prod_{i=1}^{n}[0,T_i]}\sum_{i=1}^{n}R_i(u,t_i)<0\right\} &\leq \mathbb{P}\left\{\exists_{\boldsymbol{t}\in\prod_{i=1}^{n}[0,T_i]}\sum_{i=1}^{n}B_i\left(t^{2H_i}\right)-c_it_i>u\right\}\\ &= \mathbb{P}\left\{\exists_{\boldsymbol{t}\in\prod_{i=1}^{n}[0,T_i^{2H_i}]}\sum_{i=1}^{n}B_i\left(t\right)-c_it_i^{1/2H_i}>u\right\}.\end{aligned}$$

Here B_i stands for an independent copy of Brownian motion. As it is clear that $c_i t^{1/2H_i} \in RV_{T_i}(1/2, 1)$, using Theorem 1.4, for some positive constant C we can write

$$\mathbb{P}\left\{\exists_{\boldsymbol{t}\in\prod_{i=1}^{n}[0,T_i^{2H_i}]}\sum_{i=1}^{n}B_i\left(t\right)-c_it_i^{1/2H_i}>u\right\} \leq C\mathbb{P}\left\{\sum_{i=1}^{n}B_i\left(T_i^{2H_i}\right)-c_iT_i>u\right\}$$

$$
\begin{aligned}
&= C\mathbb{P}\left\{\sum_{i=1}^{n} B_i^{H_i}(T_i) - c_i T_i > u\right\}\\
&= C\mathbb{P}\left\{\sum_{i=1}^{n} R_i(u, T_i) < 0\right\}.
\end{aligned}
$$

The same approach may be applied for a different Gaussian process with convex variance.

3 Vector-valued time-transform

Finally, we discuss some extensions of (1.2) under different time transformations. We use the notations from Section 2 and define the following time transform. Let $\boldsymbol{f}(t) : [0, +\infty) \in \mathbb{R}^d$ be a growing vector-valued function, and define

$$
\boldsymbol{Z}(\boldsymbol{f}(t)) = (Z_1(f_1(t)), \ldots, Z_d(f_d(t)))^\top.
$$

Hence $\boldsymbol{f}(t)$ can be considered as a generalised transformation of time.

Theorem 3.1 *Let $\boldsymbol{c}(t), \boldsymbol{f}(t) : [0, T] \to \mathbb{R}^d$. Suppose that all $f_i(t)$ are continuous, strictly growing and for all $i \in \{1, \ldots, d\}$ we have $f_i(0) = 0$ and function $\delta_i(t) = \frac{f_i(T) - f_i(t)}{f_1(T) - f_1(t)}$ has a positive finite limit as $t \to T$. Let also $|c_i(T) - c_i(t)| < M\sqrt{f_1(T) - f_1(t)}$ for all $t \in [0, T]$, all $i \in \{1, \ldots, d\}$, some $M > 0$, and $\boldsymbol{S}$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(\boldsymbol{t})$. If $\boldsymbol{0} \notin \boldsymbol{S}$, then for all constants $T > 0, u > 1$ the inequality (1.2) holds with $\boldsymbol{X}(t) = \boldsymbol{Z}(\boldsymbol{f}(t))$ and*

$$
\begin{aligned}
K^*(L_T) &= \frac{(2f_1(T))^{d/2}}{\mathfrak{C}(L_T)\bar{\varepsilon}_{\boldsymbol{S}}},\\
\mathfrak{C}(L_T) &= \min_{t \in L_T} e^{-\left(\frac{\boldsymbol{c}(T_k) - \boldsymbol{c}(t)}{\sqrt{f_1(T) - f_1(t)}}\right)^\top \Sigma^{-1}(\boldsymbol{\delta}(t))\left(\frac{\boldsymbol{c}(T_k) - \boldsymbol{c}(t)}{\sqrt{f_1(T) - f_1(t)}}\right)} > 0,
\end{aligned}
$$

and

$$
\bar{\varepsilon}_{\boldsymbol{S}} = \left(\frac{\inf\limits_{\substack{i \in \{1,\ldots,d\}\\ t \in L_T}} \delta_i(t)}{\sup\limits_{\substack{i \in \{1,\ldots,d\}\\ t \in L_T}} \delta_i(t)}\right)^{d/2} \varepsilon_{\boldsymbol{S}}\left(\inf_{\substack{i \in \{1,\ldots,d\}\\ t \in L_T}} \delta_i(t)\right) > 0.
$$

Remark 3.2 *The function $\boldsymbol{f}(t)$ in Theorem 3.1 may also be an almost surely growing stochastic process, independent of $\boldsymbol{Z}(t)$, satisfying the following conditions:*

$$
\max_{i \in \{1,\ldots,d\}} f_i(T) < F,
$$

$$\max_{i\in\{1,\dots,d\}} \sup_{t\in L_T\setminus\{T\}} \left| \frac{c_i(T_k)-c_i(t)}{\sqrt{f_1(T)-f_1(t)}} \right| < M,$$
$$\delta < \inf_{\substack{i\in\{1,\dots,d\}\\ t\in L_T}} \delta_i(t) \le \sup_{\substack{i\in\{1,\dots,d\}\\ t\in L_T}} \delta_i(t) < \Delta,$$

almost surely with some positive constants F, M, δ, Δ. *In this case the inequality* (1.2) *holds with*

$$K^*(L_T) = \frac{(2F)^{d/2}}{\mathfrak{C}(L_T)\bar{\varepsilon}_{\boldsymbol{S}}}, \quad \mathfrak{C}(L_T) = \min_{\substack{\boldsymbol{x}\in[-M,M]^d\\ \boldsymbol{t}\in[\delta,\Delta]^d}} e^{-\boldsymbol{x}^\top \Sigma^{-1}(\boldsymbol{t})\boldsymbol{x}} > 0,$$

and

$$\bar{\varepsilon}_{\boldsymbol{S}} = \left(\frac{\delta}{\Delta}\right)^{d/2} \varepsilon_{\boldsymbol{S}}(\delta) > 0.$$

We illustrate the above findings considering again d independent one-dimensional fractional Brownian motions $B_{H_i}(t), t > 0$ with Hurst parameters $H_i > \frac{1}{2}, i \le d$. Define d ruin portfolios

$$R_i(u,t) = u - B_{H_i}(t) + c_i t.$$

We are interested in probability that all of them will be simultaneously ruined in $[0,T]$.
Using Gordon inequality (see [1, page 55]), we obtain

$$\mathbb{P}\left\{\exists_{t\in[0,T]}\forall_{i\in\{1,\dots,d\}} R_i(u,t) < 0\right\} \le \mathbb{P}\left\{\exists_{t\in[0,T]}\forall_{i\in\{1,\dots,d\}} B_i\left(t^{2H_i}\right) - c_i t > u\right\}.$$

where $B_i(t)$ are intependent Brownian motions. Since

$$\lim_{t\to T} \frac{T^{2H_i}-t^{2H_i}}{T^{2H_1}-t^{2H_1}} = \frac{2H_i}{2H_1}\frac{T^{2H_i-1}}{T^{2H_1-1}} > 0,$$

using Theorem 3.1, for some positive constant C, which does not depend on u we can write

$$\begin{aligned}\mathbb{P}\left\{\exists_{t\in[0,T]}\forall_{i\in\{1,\dots,d\}} B_i\left(t^{2H_i}\right) - c_i t > u\right\} &\le C\mathbb{P}\left\{\forall_{i\in\{1,\dots,d\}} B_i\left(T^{2H_i}\right) - c_i T > u\right\}\\ &= C\mathbb{P}\left\{\forall_{i\in\{1,\dots,d\}} B_{H_i}(T) - cT > u\right\}\\ &= C\mathbb{P}\left\{\forall_{i\in\{1,\dots,d\}} R_i(u,T) < 0\right\}.\end{aligned}$$

4 Proofs

Let us note the following property of the function $\varepsilon_{\boldsymbol{S}}(t)$.

Lemma 4.1 *If set $\boldsymbol{S}$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$ with some function $\varepsilon_{\boldsymbol{S}}(t)$, then for any constant $u > 1$ set $u\boldsymbol{S}$ also satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$, and for any function $\varepsilon_{\boldsymbol{S}}(t)$ exists a function $\varepsilon_{u\boldsymbol{S}}(t)$ such that*

$$\varepsilon_{u\boldsymbol{S}}(t) \geq \varepsilon_{\boldsymbol{S}}(t).$$

Proof of Lemma 4.1: Fix some $\boldsymbol{x} \in u\boldsymbol{S}$. Then we know that $\boldsymbol{y} = \boldsymbol{x}/u \in \boldsymbol{S}$. As $\boldsymbol{S}$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$, there exists some cone $V_{\boldsymbol{y}} \subset \boldsymbol{S}$ with vertex $\boldsymbol{y}$ such that $\mathbb{P}\{\boldsymbol{Z}(t) \in V_{\boldsymbol{y}} - \boldsymbol{y}\} \geq \varepsilon_{\boldsymbol{S}}(t)$. Hence, $uV_{\boldsymbol{y}} \subset u\boldsymbol{S}$. Note that using the properties of cone

$$uV_{\boldsymbol{y}} = u(\boldsymbol{y} + (V_{\boldsymbol{y}} - \boldsymbol{y})) = \boldsymbol{x} + u(V_{\boldsymbol{y}} - \boldsymbol{y}) \supset \boldsymbol{x} + (V_{\boldsymbol{y}} - \boldsymbol{y}).$$

Hence, $\boldsymbol{x} + (V_{\boldsymbol{y}} - \boldsymbol{y}) \subset u\boldsymbol{S}$ is some cone with vertex $\boldsymbol{x}$, and

$$\mathbb{P}\{\boldsymbol{Z}(t) \in uV_{\boldsymbol{y}} - \boldsymbol{x}\} \geq \mathbb{P}\{\boldsymbol{Z}(t) \in V_{\boldsymbol{y}} - \boldsymbol{y}\} \geq \varepsilon_{\boldsymbol{S}}(t).$$

□

Proof of Theorem 1.3: Consider the first inequality. Define the following stopping moment

$$\tau = \inf\{t \in L_T : \boldsymbol{Z}(t) - \boldsymbol{c}(t) \in u\boldsymbol{S}\}.$$

According to the strong Markov property

$$\begin{aligned}
&\mathbb{P}\{\boldsymbol{Z}(T) - \boldsymbol{c}(T) \in u\boldsymbol{S}\} \\
&\quad = \int_{L_T} \mathbb{P}\{\tau \in \mathrm{d}t\} \int_{u\boldsymbol{S}} \mathbb{P}\{\boldsymbol{Z}(t) - \boldsymbol{c}(t) \in \mathrm{d}\boldsymbol{x} | \tau = t\} \\
&\qquad \times \mathbb{P}\{\boldsymbol{Z}(T) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(t) - \boldsymbol{c}(t) = \boldsymbol{x}\}.
\end{aligned}$$

Using Lemma 4.1, $\boldsymbol{u}S$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$. Hence for all $\boldsymbol{x} \in u\boldsymbol{S}$, $t \in L_T$

$$\begin{aligned}
&\mathbb{P}\{\boldsymbol{Z}(T) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(t) - \boldsymbol{c}(t) = \boldsymbol{x}\} \\
&\quad \geq \mathbb{P}\{\boldsymbol{Z}(T) - \boldsymbol{c}(T) \in \boldsymbol{V}_{\boldsymbol{x}} | \boldsymbol{Z}(t) - \boldsymbol{c}(t) = \boldsymbol{x}\} \\
&\quad = \mathbb{P}\{\boldsymbol{Z}(T - t) - (\boldsymbol{c}(T) - \boldsymbol{c}(t)) \in \boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x}\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{Z}(1) - (\boldsymbol{c}(T) - \boldsymbol{c}(t))/\sqrt{T - t} \in (\boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x})/\sqrt{T - t}\right\} \\
&\quad \geq \mathbb{P}\left\{\sqrt{T}\boldsymbol{Z}(1) \in \boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x} + \sqrt{T}(\boldsymbol{c}(T) - \boldsymbol{c}(t))/\sqrt{T - t}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}\left\{\boldsymbol{Z}(T) \in \boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x} + \sqrt{T}(\boldsymbol{c}(T) - \boldsymbol{c}(t))/\sqrt{T-t}\right\} \\
&= \int_{\boldsymbol{V}_{\boldsymbol{x}}-\boldsymbol{x}} \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\tilde{\boldsymbol{x}}+\sqrt{T}\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})^\top \Sigma^{-1}(\tilde{\boldsymbol{x}}+\sqrt{T}\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})} \mathrm{d}\tilde{\boldsymbol{x}} \\
&\geq \int_{\boldsymbol{V}_{\boldsymbol{x}}-\boldsymbol{x}} \frac{1}{(2\pi)^{\frac{d}{2}}\sqrt{|\Sigma|}} e^{-T(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})^\top \Sigma^{-1}(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})} e^{-\frac{1}{2}(\sqrt{2}\tilde{\boldsymbol{x}})^\top \Sigma^{-1}(\sqrt{2}\tilde{\boldsymbol{x}})} \mathrm{d}\tilde{\boldsymbol{x}} \\
&= e^{-T(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})^\top \Sigma^{-1}(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})} \frac{\mathbb{P}\left\{\boldsymbol{Z}(T) \in \sqrt{2}(\boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x})\right\}}{2^{d/2}} \\
&\geq \frac{1}{2^{d/2}} e^{-T(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})^\top \Sigma^{-1}(\frac{\boldsymbol{c}(T)-\boldsymbol{c}(t)}{\sqrt{T-t}})} \mathbb{P}\left\{\boldsymbol{Z}(T) \in \boldsymbol{V}_{\boldsymbol{x}} - \boldsymbol{x}\right\} \\
&\geq \frac{\mathfrak{C}\varepsilon_{u\boldsymbol{S}}}{2^{d/2}} \geq \frac{\mathfrak{C}\varepsilon_{\boldsymbol{S}}}{2^{d/2}},
\end{aligned}
$$

where $\boldsymbol{V}_{\boldsymbol{x}}$ is the cone from Definition 1.1. As the right part does not depend on $\boldsymbol{x}$ and t, we can write

$$
\begin{aligned}
\mathbb{P}\left\{\boldsymbol{Z}(T) - \boldsymbol{c}(T) \in u\boldsymbol{S}\right\} \quad &\geq \quad \frac{\mathfrak{C}\varepsilon_{\boldsymbol{S}}}{2^{d/2}} \int_{L_T} \mathbb{P}\left\{\tau \in \mathrm{d}t\right\} \\
&\qquad \times \int_{u\boldsymbol{S}} \mathbb{P}\left\{\boldsymbol{Z}(t) - \boldsymbol{c}(t) \in \mathrm{d}\boldsymbol{x} | \tau = t\right\} \\
&= \quad \frac{\mathfrak{C}\varepsilon_{\boldsymbol{S}}}{2^{d/2}} \mathbb{P}\left\{\exists_{t \in L_T} \boldsymbol{Z}(t) - \boldsymbol{c}(t) \in u\boldsymbol{S}\right\},
\end{aligned}
$$

hence, the first inequality holds. Consider the second one. Define a set

$$
\boldsymbol{a}^+ = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \geq \boldsymbol{a}\}
$$

and

$$
\boldsymbol{S}_u = \frac{1}{u}\boldsymbol{F}^{-1}(u\boldsymbol{a}^+).
$$

Set $\boldsymbol{S}_u$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$ for $V_{\boldsymbol{x}} = \boldsymbol{x}^+$, as for any $\boldsymbol{y} \geq \boldsymbol{x} \in \boldsymbol{S}_u$

$$
\boldsymbol{F}(u\boldsymbol{y}) \geq \boldsymbol{F}(u\boldsymbol{x}) \geq u\boldsymbol{a}^+,
$$

hence $\boldsymbol{y} \in \boldsymbol{S}_u$ and

$$
\varepsilon_{\boldsymbol{S}_u}(t) = \mathbb{P}\left\{\boldsymbol{X}(t) \in \boldsymbol{x}^+ - \boldsymbol{x}\right\} = \mathbb{P}\left\{\boldsymbol{X}(t) \in [0, +\infty)^d\right\}
$$

does not depend on u. Applying the result above for the set S_u we obtain

$$
\mathbb{P}\left\{\exists_{t \in L_T} : \boldsymbol{X}(t) \in u\boldsymbol{S}_u\right\} \leq \frac{2^{d/2}\mathbb{P}\left\{\boldsymbol{X}(T) \in u\boldsymbol{S}\right\}}{\mathfrak{C}(L_T)\varepsilon_{\boldsymbol{S}_u}(T)} = \frac{2^{d/2}\mathbb{P}\left\{\boldsymbol{X}(T) \in u\boldsymbol{S}_u\right\}}{\mathfrak{C}(L_T)\mathbb{P}\left\{\boldsymbol{X}(T) \in [0, +\infty)^d\right\}}.
$$

As the event $\{\boldsymbol{X}(t) \in u\boldsymbol{S}_u\}$ is equal to the event $\{\boldsymbol{F}(\boldsymbol{X}(t) - \boldsymbol{c}(t)) > u\boldsymbol{a}\}$, this completes the proof. $\square$

Proof of Theorem 1.4: Define

$$\varphi_{\boldsymbol{c}_1,\dots,\boldsymbol{c}_n}(\boldsymbol{S}, \mathbb{T}) = \mathbb{P}\left\{\exists_{\boldsymbol{t}\in\mathbb{T}} : \sum_{i=1}^{n}(\boldsymbol{Z}_i(t_i) - \boldsymbol{c}_i(t_i)) \in \boldsymbol{S}\right\}$$

and

$$\psi_k(\boldsymbol{S}) := \mathbb{P}\left\{\exists_{\boldsymbol{t}\in\mathbb{T}_k} : \sum_{i=1}^{k}(\boldsymbol{Z}_i(t_i) - \boldsymbol{c}_i(t_i)) + \sum_{i=k+1}^{n}(\boldsymbol{Z}_i(T_i) - \boldsymbol{c}_i(T_i)) \in \boldsymbol{S}\right\},$$

where $\mathbb{T}_k = [0, T_1] \times \ldots \times [0, T_k]$. As in the previous section we are going to prove that the inequality

$$\psi_k(u\boldsymbol{S}) \leq \frac{2^{d/2}\psi_{k-1}(u\boldsymbol{S})}{\varepsilon_{\boldsymbol{S}}(T_k)\mathfrak{C}_k}$$

takes place for any $k \in \{1, \dots, n\}$. We can fix the trajectories of processes $\boldsymbol{Z}_i(t)$ called $\boldsymbol{x}_i(t)$, fix random vectors $\boldsymbol{Z}_i(T_i)$ called $\boldsymbol{x}_i$, and define the process

$$\boldsymbol{Z}^{*k}(t, \boldsymbol{t}^k) = \boldsymbol{Z}_k(t) - \boldsymbol{c}_k(t) + \sum_{i=1}^{k-1}(\boldsymbol{x}_i(t_i) - \boldsymbol{c}_i(t_i)) + \sum_{i=k+1}^{n}(\boldsymbol{x}_i - \boldsymbol{c}_i(T_i)),$$

where $\boldsymbol{t}^k = (t_1, \dots, t_{k-1}) \in \mathbb{T}_{k-1}$.

As $\boldsymbol{Z}_i$ are independent processes, it is enough to show that for every set of trajectories $\boldsymbol{x}_i(t)$ and points $\boldsymbol{x}_j$, the inequality

$$\psi^*(u\boldsymbol{S}) \leq \frac{2^{d/2}\nu(u\boldsymbol{S})}{\varepsilon_{\boldsymbol{S}}(T_k)\mathfrak{C}_k}$$

takes place, where

$$\begin{aligned}
\psi^*(\boldsymbol{S}) &= \mathbb{P}\left\{\exists_{t\in[0,T_k]} : \boldsymbol{Z}^{*k}(t, \boldsymbol{t}^k) \in \boldsymbol{S} \ for\ some\ \boldsymbol{t}^k \in \mathbb{T}_{k-1}\right\},\\
\nu(\boldsymbol{S}) &= \mathbb{P}\left\{\boldsymbol{Z}^{*k}(T_k, \boldsymbol{t}^k) \in \boldsymbol{S} \ for\ some\ \boldsymbol{t}^k \in \mathbb{T}_{k-1}\right\}.
\end{aligned}$$

Define the following stopping moment:

$$\tau_k = \inf\left\{t : \boldsymbol{Z}^{*k}(t, \boldsymbol{t}^k) \in u\boldsymbol{S} \ for\ some\ \boldsymbol{t}^k \in \mathbb{T}^k\right\},$$

in case such set is not empty. Otherwise we put $\tau = \infty$. Define further a random vector

$$\tilde{\boldsymbol{x}}_k = \begin{cases} \boldsymbol{x}^*, & \tau_k \le T_k, \\ \boldsymbol{0}, & otherwise, \end{cases}$$

where $\boldsymbol{x}^*$ is any point from the following set:

$$\bigcup_{\boldsymbol{t}^k \in \mathbb{T}^k} \left\{ \boldsymbol{Z}^{*k}(\tau_k, \boldsymbol{t}^k) \right\} \bigcap u\boldsymbol{S}.$$

Using the total probability formula we obtain

$$\begin{aligned} \nu(u\boldsymbol{S}) &= \int_0^{T_k} \mathbb{P}\{\tau_k \in \mathrm{d}t\} \\ &\qquad \times \mathbb{P}\left\{ \boldsymbol{Z}^{*k}(T_k, \boldsymbol{t}^k) \in u\boldsymbol{S} \textit{ for some } \boldsymbol{t}^k \in \mathbb{T}^k \,|\tau_k = t \right\} \\ &= \int_0^{T_k} \mathbb{P}\{\tau_k \in \mathrm{d}t\} \int_{\boldsymbol{S}} \mathbb{P}\{\tilde{\boldsymbol{x}}_k \in \mathrm{d}\boldsymbol{x}_0 \,|\tau_k = t\} \\ &\qquad \times \mathbb{P}\left\{ \boldsymbol{Z}^{*k}(T_k, \boldsymbol{t}^k) \in u\boldsymbol{S} \textit{ for some } \boldsymbol{t}^k \in \mathbb{T}_{k-1} \,|\tau_k = t, \tilde{\boldsymbol{x}}_k = \boldsymbol{x}_0 \right\}. \end{aligned}$$

For any $\boldsymbol{t}^k \in \mathbb{T}_{k-1}$ we have

$$\boldsymbol{Z}^{*k}(T_k, \boldsymbol{t}^k) - \boldsymbol{Z}^{*k}(t, \boldsymbol{t}^k) = \boldsymbol{Z}_k(T_k) - \boldsymbol{Z}_k(t) - (\boldsymbol{c}_k(T_k) - \boldsymbol{c}_k(t)).$$

Thus, using the same chain of inequalities as in Theorem 1.3 we obtain

$$\begin{aligned} &\mathbb{P}\left\{ \boldsymbol{Z}^{*k}(T_k, \boldsymbol{t}^k) \in u\boldsymbol{S} \textit{ for some } \boldsymbol{t}^k \in \mathbb{T}^k \,|\tau_k = t, \tilde{\boldsymbol{x}}_k = \boldsymbol{x}_0 \right\} \\ &\quad \ge \mathbb{P}\{\boldsymbol{Z}_k(T_k) - \boldsymbol{Z}_k(t) - (\boldsymbol{c}_k(T_k) - \boldsymbol{c}_k(t)) \in u\boldsymbol{S} - \boldsymbol{x}_0\} \\ &\quad \ge \frac{\mathfrak{C}_k \varepsilon_{\boldsymbol{S}}(T_k)}{2^{d/2}}, \end{aligned}$$

which completes the proof. □

Proof of Theorem 3.1: Define a stopping moment

$$\tau = \inf\{t \in L_T : \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) \in u\boldsymbol{S}\},$$

in case such set is not empty. Otherwise we put $\tau = \infty$. According to the strong Markov property

$$\begin{aligned} &\mathbb{P}\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S}\} \\ &\quad = \int_{L_T} \mathbb{P}\{\tau \in \mathrm{d}t\} \mathbb{P}\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \tau = t\} \end{aligned}$$

$$
\begin{aligned}
&= \int_{L_T} \mathbb{P}\left\{\tau \in \mathrm{d}t\right\} \int_{u\boldsymbol{S}} \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) \in \mathrm{d}\boldsymbol{x} | \tau = t\right\} \\
&\qquad \times \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) = \boldsymbol{x}, \tau = t\right\}.
\end{aligned}
$$

According to Lemma 4.1, $u\boldsymbol{S}$ satisfies the cone condition with respect to the process $\boldsymbol{Z}(t)$. Then

$$
\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) = \boldsymbol{x}, \tau = t\right\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) = \boldsymbol{x}\right\} \\
&\quad \geq \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in V_{\boldsymbol{x}} | \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) = \boldsymbol{x}\right\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(T) + \boldsymbol{c}(t) \in V_{\boldsymbol{x}} - \boldsymbol{x}\right\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T) - \boldsymbol{f}(t)) - (\boldsymbol{c}(T) - \boldsymbol{c}(t)) \in V_{\boldsymbol{x}} - \boldsymbol{x}\right\} \\
&\quad = \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{\delta}(t)) - \frac{\boldsymbol{c}(T) - \boldsymbol{c}(t)}{\sqrt{f_1(T) - f_1(t)}} \in \frac{V_{\boldsymbol{x}} - \boldsymbol{x}}{\sqrt{f_1(T) - f_1(t)}}\right\} \\
&\quad \geq \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{\delta}(t)) - \frac{\boldsymbol{c}(T) - \boldsymbol{c}(t)}{\sqrt{f_1(T) - f_1(t)}} \in \frac{V_{\boldsymbol{x}} - \boldsymbol{x}}{\sqrt{f_1(T)}}\right\} \\
&\quad = \int_{\boldsymbol{y} \in \frac{V_{\boldsymbol{x}} - \boldsymbol{x}}{\sqrt{f_1(T)}}} \varphi_{\boldsymbol{\delta}(t)}\left(\boldsymbol{y} + \frac{\boldsymbol{c}(T) - \boldsymbol{c}(t)}{\sqrt{f_1(T) - f_1(t)}}\right) \mathrm{d}\boldsymbol{y} \\
&\quad \geq \int_{\boldsymbol{y} \in \frac{V_{\boldsymbol{x}} - \boldsymbol{x}}{\sqrt{f_1(T)}}} \mathfrak{C}(L_T) \varphi_{\boldsymbol{\delta}(t)}(\sqrt{2}\boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\
&\quad \geq \frac{\mathfrak{C}(L_T)}{2^{d/2}} \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{\delta}(t)) \in \frac{V_{\boldsymbol{x}} - \boldsymbol{x}}{\sqrt{f_1(T)}}\right\} \\
&\quad \geq \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{\delta}(t)) \in V_{\boldsymbol{x}} - \boldsymbol{x}\right\} \\
&\quad = \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \mathbb{P}\left\{\boldsymbol{B}(\boldsymbol{\delta}(t)) \in A^{-1}(V_{\boldsymbol{x}} - \boldsymbol{x})\right\} \\
&\quad = \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \frac{1}{\sqrt{2\pi \prod_{i=1}^{d} \delta_i(t)}} \int_{\boldsymbol{y} \in A^{-1}(V_{\boldsymbol{x}} - \boldsymbol{x})} e^{-\frac{1}{2}\sum\limits_{i=1}^{d} \frac{y_i^2}{\delta_i(t)}} \mathrm{d}\boldsymbol{y},
\end{aligned}
$$

where $\varphi_{\boldsymbol{\delta}(t)}$ is the pdf of $\boldsymbol{Z}(\boldsymbol{\delta}(t))$. Using that all the functions $\delta_i(t)$ are bounded and separated from zero for $t \in L_T$, there exists some constants $\delta, \Delta > 0$, such that for all $i \in \{1, \dots, d\}$ and all $t \in L_T$

$$
\delta \leq \delta_i(t) \leq \Delta.
$$

Hence we obtain

$$
\begin{aligned}
\frac{1}{\sqrt{2\pi\prod_{i=1}^{d}\delta_i(t)}} &\geq \frac{1}{\sqrt{2\pi\prod_{i=1}^{d}\Delta}},\\
e^{-\frac{1}{2}\sum\limits_{i=1}^{d}\frac{x_i^2}{\delta_i(t)}} &\geq e^{-\frac{1}{2}\sum\limits_{i=1}^{d}\frac{x_i^2}{\delta}},
\end{aligned}
$$

and finally

$$
\begin{aligned}
&\mathbb{P}\left\{\boldsymbol{Z}(\boldsymbol{f}(T)) - \boldsymbol{c}(T) \in u\boldsymbol{S} | \boldsymbol{Z}(\boldsymbol{f}(t)) - \boldsymbol{c}(t) = x, \tau = t\right\}\\
&\qquad \geq \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \frac{1}{\sqrt{2\pi\prod_{i=1}^{d}\Delta}} \int_{\boldsymbol{y}\in A^{-1}(V_{\boldsymbol{x}}-\boldsymbol{x})} e^{-\frac{1}{2}\sum\limits_{i=1}^{d}\frac{y_i^2}{\delta}} \mathrm{d}\boldsymbol{y}\\
&\qquad = \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \frac{\sqrt{\prod_{i=1}^{d}\delta}}{\sqrt{\prod_{i=1}^{d}\Delta}} \mathbb{P}\left\{\boldsymbol{B}(\delta) \in A^{-1}(V_{\boldsymbol{x}} - \boldsymbol{x})\right\}\\
&\qquad \geq \frac{\mathfrak{C}(L_T)}{(2f_1(T))^{d/2}} \frac{\sqrt{\prod_{i=1}^{d}\delta}}{\sqrt{\prod_{i=1}^{d}\Delta}} \varepsilon_{\boldsymbol{S}}(\delta).
\end{aligned}
$$

Hence the claim follows. ∎

Bibliography

[1] Robert J. Adler. *An introduction to continuity, extrema, and related topics for general Gaussian processes*, volume 12 of *Institute of Mathematical Statistics Lecture Notes—Monograph Series*. Institute of Mathematical Statistics, Hayward, CA, 1990.

[2] Florin Avram, Zbigniew Palmowski, and Martijn Pistorius. A two-dimensional ruin problem on the positive quadrant. *Insurance Math. Econom.*, 42(1):227–234, 2008.

[3] Florin Avram, Zbigniew Palmowski, and Martijn R. Pistorius. Exit problem of a two-dimensional risk process from the quadrant: exact and asymptotic results. *Ann. Appl. Probab.*, 18(6):2421–2449, 2008.

[4] Florin Avram, Zbigniew Palmowski, and Martijn R Pistorius. Exit problem of a two-dimensional risk process from the quadrant: exact and asymptotic results. *The Annals of Applied Probability*, 18(6):2421–2449, 2008.

[5] Long Bai, Krzysztof Dębicki, and Peng Liu. Extremes of vector-valued Gaussian processes with trend. *J. Math. Anal. Appl.*, 465(1):47–74, 2018.

[6] Simeon M. Berman. An asymptotic formula for the distribution of the maximum of a Gaussian process with stationary increments. *J. Appl. Probab.*, 22(2):454–460, 1985.

[7] S.M. Berman. *Sojourns and Extremes of Stochastic Processes*. The Wadsworth & Brooks/Cole Statistics/Probability Series. Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992.

[8] Krzysztof Bisewski, Krzysztof Dębicki, and Nikolai Kriukov. *Simultaneous ruin probability for multivariate Gaussian risk model. Submitted.*

[9] Konstantin Borovkov and Zbigniew Palmowski. The exact asymptotics for hitting probability of a remote orthant by a multivariate Lévy process: the

Cramér case. In *2017 MATRIX annals*, volume 2 of *MATRIX Book Ser.*, pages 303–309. Springer, Cham, 2019.

[10] K. Dębicki, E. Hashorva, L. Ji, and T. Rolski. Extremal behavior of hitting a cone by correlated Brownian motion with drift. *Stoch. Proc. Appl.*, 128(12):4171–4206, 2018.

[11] K. Dębicki, E. Hashorva, and N. Kriukov. Pandemic-type failures in multivariate Brownian risk models. *Arxiv preprint arXiv:2008.07480*, 2021.

[12] K. Dębicki, E. Hashorva, and P. Liu. Extremes of γ-reflected gaussian processes with stationary increments. *ESAIM: Probability and Statistics*, 21:495–535, 2017.

[13] Krzysztof Dębicki, Enkelejd Hashorva, and Lanpeng Ji. *Parisian ruin of self-similar Gaussian risk processes.* J. Appl. Probab. 52 (2015), pp. 688–702.

[14] Krzysztof Dębicki, Enkelejd Hashorva, and LanPeng Ji. *Parisian ruin over a finite-time horizon.* Sci. China Math. 59 (2016), pp. 557–572.

[15] Krzysztof Dębicki, Enkelejd Hashorva, and Nikolai Kriukov. Pandemic-type failures in multivariate brownian risk models. *Extremes*, Accepted.

[16] Krzysztof Dębicki, Enkelejd Hashorva, and Konrad Krystecki. Finite-time ruin probability for correlated brownian motions. *Scandinavian Actuarial Journal*, pages 1–26, 2021.

[17] Krzysztof Dębicki, Enkelejd Hashorva, and Z. Michna. On continuity of Pickands constants. *arXiv:2105.10435, J. Appl. Probab, in press*, 2021.

[18] Krzysztof Dębicki, Enkelejd Hashorva, and Zbigniew Michna. *Simultaneous ruin probability for two-dimensional Brownian risk model.* J. Appl. Probab. 57 (2020), pp. 597–612.

[19] Krzysztof Dębicki, Enkelejd Hashorva, and Zbigniew Michna. Simultaneous ruin probability for two-dimensional Brownian risk model. *J. Appl. Probab.*, 57(2):597–612, 2020.

[20] Krzysztof Dębicki, Enkelejd Hashorva, and Zbigniew Michna. Simultaneous ruin probability for two-dimensional brownian risk model. *Journal of Applied Probability*, 57(2):597–612, 2020.

[21] Krzysztof Dębicki, Peng Liu, and Zbigniew Michna. *Sojourn times of Gaussian processes with trend.* J. Theoret. Probab. 33 (2020), pp. 2119–2166.

[22] Krzysztof Dębicki. Ruin probability for Gaussian integrated processes. *Stochastic Processes and their Applications*, 98(1):151–174, 2002.

[23] Krzysztof Dębicki, Enkelejd Hashorva, Lanpeng Ji, and Kamil Tabiś. Extremes of vector-valued Gaussian processes: Exact asymptotics. *Stochastic Processes and their Applications*, 125(11):4039–4065, 2015.

[24] Krzysztof Dębicki, Enkelejd Hashorva, and Longmin Wang. Extremes of vector-valued Gaussian processes. *Stochastic Processes and their Applications*, 130(9):5802–5837, 2020.

[25] Krzysztof Dębicki, Enkelejd Hashorva, and Longmin Wang. Extremes of vector-valued Gaussian processes. *Stochastic Processes and their Applications*, 130(9):5802–5837, 2020.

[26] Krzysztof Dębicki and Tomasz Rolski. A note on transient Gaussian fluid models. *Queueing Systems*, 41(4):321–342, 2002.

[27] G A Delsing, M R H Mandjes, P J C Spreij, and E M M Winands. *Asymptotics and approximations of ruin probabilities for multivariate risk processes in a Markovian environment.* preprint (2018). Available at arXiv:1812.09069.

[28] Clement Dombry and Landy Rabehasaina. High order expansions for renewal functions and applications to ruin theory. *Ann. Appl. Probab.*, 27(4):2342–2382, 08 2017.

[29] Sergey Foss, Dmitry Korshunov, Zbigniew Palmowski, and Tomasz Rolski. Two-dimensional ruin probability for subexponential claim size. *Probability and Mathematical Statistics*, 37(2):319–335, 2017.

[30] Serguei Foss, Dmitry Korshunov, Zbigniew Palmowski, and Tomasz Rolski. Two-dimensional ruin probability for subexponential claim size. *Probab. Math. Statist.*, 37(2):319–335, 2017.

[31] Yehoram Gordon. Some inequalities for Gaussian processes and applications. *Israel J. Math.*, 50(4):265–289, 1985.

[32] William W. Hager. Lipschitz continuity for constrained processes. *SIAM J. Control Optim.*, 17(3):321–338, 1979.

[33] Enkelejd Hashorva. Asymptotics and bounds for multivariate Gaussian tails. *Journal of theoretical probability*, 18(1):79–97, 2005.

[34] Enkelejd Hashorva. Approximation of some multivariate risk measures for Gaussian risks. *J. Multivariate Anal.*, 169:330–340, 2019.

[35] Enkelejd Hashorva. Approximation of some multivariate risk measures for Gaussian risks. *Journal of Multivariate Analysis*, 169:330–340, 2019.

[36] Zechun Hu and Bin Jiang. On joint ruin probabilities of a two-dimensional risk model with constant interest rate. *J. Appl. Probab.*, 50(2):309–322, 2013.

[37] Zechun Hu and Bin Jiang. On joint ruin probabilities of a two-dimensional risk model with constant interest rate. *Journal of Applied Probability*, 50(2):309–322, 2013.

[38] J Hüsler and V Piterbarg. On the ruin probability for physical fractional Brownian motion. *Stochastic Processes and their Applications*, 113(2):315–332, 2004.

[39] Jürg Hüsler and V Piterbarg. Extremes of a certain class of Gaussian processes. *Stochastic Processes and their Applications*, 83(2):257–271, 1999.

[40] Lanpeng Ji. On the cumulative Parisian ruin of multi-dimensional Brownian motion risk models. *Scand. Actuar. J.*, (9):819–842, 2020.

[41] Lanpeng Ji and Stephan Robert. Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stoch. Models*, 34(1):73–97, 2018.

[42] Lanpeng Ji and Stephan Robert. Ruin problem of a two-dimensional fractional Brownian motion risk process. *Stochastic Models*, 34(1):73–97, 2018.

[43] D. A. Korshunov, V. I. Piterbarg, and E. Hashorva. On the asymptotic Laplace method and its application to random chaos. *Mat. Zametki*, 97(6):868–883, 2015.

[44] Dmitry Korshunov and Longmin Wang. Tail asymptotics for Shepp-statistics of Brownian motion in $\mathbb{R}^d$. *Extremes*, 23(1):35–54, 2020.

[45] Dmitry Korshunov and Longmin Wang. Tail asymptotics for Shepp-statistics of Brownian motion in R^d. *Extremes*, 23(1):35–54, 2020.

[46] Nikolai Kriukov. *Uniform bounds for ruin probability in Multidimensional Risk Model. Submitted.*

[47] Nikolai Kriukov. Parisian & cumulative parisian ruin probability for two-dimensional brownian risk model. *Stochastics*, pages 1–17, 2021.

[48] P. Lieshout and M. Mandjes. Tandem Brownian queues. *Math. Methods Oper. Res.*, 66(2):275–298, 2007.

[49] R. Loeffen, I. Czarna, and Z. Palmowski. *Parisian ruin probability for spectrally negative Lévy processes.* Bernoulli. (2013).

[50] Zbigniew Michna. Self-similar processes in collective risk theory. *Journal of Applied Mathematics and Stochastic Analysis*, 11(4):429–448, 1998.

[51] Zbigniew Michna. Ruin probabilities for two collaborating insurance companies. *Probability and Mathematical Statistics*, 40(2):369–386, 2020.

[52] Yuqing Pan and Konstantin A. Borovkov. The exact asymptotics of the large deviation probabilities in the multivariate boundary crossing problem. *Adv. in Appl. Probab.*, 51(3):835–864, 2019.

[53] V. I. Piterbarg. *Asymptotic Methods in the Theory of Gaussian Processes and Fields*, volume 148 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1996. Translated from the Russian by V.V. Piterbarg, revised by the author.

[54] Gennady Samorodnitsky and Julian Sun. Multivariate subexponential distributions and their applications. *Extremes*, 19(2):171–196, 2016.