GENERAL LOWER BOUNDS FOR ARITHMETIC ASIAN OPTION PRICES*

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Abstract

In this paper we provide model-independent lower bounds for prices of arithmetic Asian options expressed through prices of European call options on the same underlying that are assumed to be observable in the market, and the corresponding subreplicating strategy is identified. The first bound relies on the no-arbitrage assumption only and turns out to perform satisfactory in various situations. It is shown how the bound can be tightened under mild additional assumptions on the underlying market model. This considerably generalizes lower bounds in the literature which are only available in the Black-Scholes world. Furthermore, it is illustrated how to adapt the procedure to the case where only a finite number of strikes is available in the market. As a by-product, we rederive the finite strike solution for the upper bound on the Asian call price of Hobson et al. [15], who considered basket options. Numerical illustrations of the bounds are given together with comparisons to bounds resulting from model specifications.

1 Introduction

Let S_t denote an underlying asset price at time t and $r > 0$ the riskless interest rate, which will be assumed to be constant. The payoff of an arithmetic Asian option with strike K is then given by

$$\left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^{+},$$

where $t_1, \ldots, t_n$ are the discrete monitoring times and T is the maturity of the option (w.l.o.g. we will assume $t_n = T$; for simplicity of notation S_i will sometimes be written instead of S_{t_i}). The pricing of such arithmetic Asian options is in general a challenging task, since their payoff involves a sum of dependent random variables whose distribution is not available even in the Black-Scholes market model. During the last years a lot of research activity was devoted to both pricing and hedging of these financial products. In terms of pricing, numerical methods based on partial (integro-)differential equations, moment matching, Monte Carlo and quasi Monte Carlo

*This is a preprint of an article whose final and definitive form has been published in Applied Mathematical Finance 2007 ©Taylor & Francis; Applied Mathematical Finance is online at: http://journalsonline.tandf.co.uk/

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as well as fast Fourier transform have been developed (for a recent overview see for instance Ju [18], Klassen [19], Lord [20], Večeř [28] and Večeř & Xu [29]). However, most of the obtained results rely on rather restrictive model assumptions.
For risk management purposes, it would be preferable to have (at least) bounds for derivative prices that are solely implied by information available in the market, trying to avoid model assumptions (which also bypasses the problem of identifying the appropriate martingale measure to price the option in an incomplete market). Such model-independent bounds are sometimes implied by static (or semi-static) super- and subreplicating strategies on the product (see e.g. Hobson [14] for lookback options, Brown et al. [5] for barrier options and Davis et al. [10] for installment options).
For Asian call options (AC), upper price bounds in terms of European call prices on the underlying have been developed by Simon et al. [25] for arbitrary models (see also Nielsen & Sandmann [22] for the Black-Scholes model). The interpretation of these bounds as a superreplicating portfolio of the Asian call in terms of European call options can be found in Albrecher et al. [1]. In Albrecher & Schoutens [2], the implementation of such a static hedge portfolio was further investigated and extended to a model-independent framework, assuming only that market prices for European call options on the underlying are available for any strike and maturity.
For lower bounds on the AC price, Curran [9] and Rogers & Shi [23] pioneered a quite accurate method to determine lower price bounds in the Black-Scholes model based on the following idea: Jensen's inequality gives

$$\sum_{i=1}^{n} S_i \geq_{cx} \sum_{i=1}^{n} \mathbb{E}\left[S_i|Z\right], \tag{1}$$

where Z is an arbitrary random variable and $\geq_{cx}$ is the convex ordering relation, i.e:

$$X \geq_{cx} Y \Leftrightarrow \mathbb{E}\left[g(X)\right] \geq \mathbb{E}\left[g(Y)\right]$$

for every convex function g. Since the function $g(x) = (x-K)^+$ is convex, (1) leads to a lower price bound for the Asian call (in contrast to upper price bounds based on the concept of comonotonicity, this bound in general does not imply a subreplication strategy).
Since for the Black-Scholes model the distribution of the geometric mean $(\prod_{i=1}^{n} S_i)^{\frac{1}{n}}$ is explicitly available, the choice of Z as the geometric mean (or a function of it) is very popular and leads to tight lower price bounds, since the arithmetic and geometric average are strongly correlated (see e.g. Curran [9], Rogers & Shi [23], Nielsen & Sandmann [22], Thompson [26] and Vanmaele et al. [27]).
On the other hand, it is widely agreed that the Black-Scholes model is not an appropriate model for real markets (see e.g. Schoutens [24]), and it is natural to look for extensions of the above bounds to more realistic models. This, however, turns out to be a delicate issue in general and the available lower bounds for AC prices almost exclusively rely on the assumption of a Black-Scholes framework.

In this paper, we aim at finding arbitrage-free model-independent lower bounds for AC prices, i.e. bounds which hold uniformly, without specifying any model (hence being robust against model misspecification). The only information used are the prices of traded European call prices, which can be observed in the market, reflecting information on marginal distributions of the underlying asset. In Section 2, one such price bound based on conditioning will be developed and it will

also be shown that there is a feasible subreplicating strategy associated to it.
The resulting bound significantly improves upon the trivial bound based on the Asian call-put parity. If the option is far out of the money, however, the performance of the bound deteriorates and we investigate improvements under some mild additional assumptions on the market.

In contrast to bounds derived by conditioning, Hobson et al. [16] found model-free lower bounds for basket options (whose payoff depends on a weighted sum of different underlyings) written on exactly two different underlyings, showing that perfect negative dependence between the two assets (the lower Frechet bound) provides the minimal value (so that the lower bound is in fact attained and thus optimal). This result clearly implies a lower price bound for Asian options with two maturity days, but since there is no meaningful model with perfect negative dependence of prices of the same asset over some time period, this lower bound will not be attained in the Asian case (see Section 2 for details).
An extension of this model-free lower bound for basket (and also Asian) options with more than two assets (averaging days, respectively) seems however out of reach, since the lower Frechet bound is not a copula for more than two random variables. The lower bounds derived in this paper can be seen as a generalization of [16] that is applicable for arbitrarily many assets (averaging days, respectively).

The bounds discussed in Section 2 are derived under the assumption that European call prices with arbitrary strikes and maturities are available in the market. This assumption will be relaxed in Section 3 and it is shown how to obtain price bounds if only a finite number of strikes is available in the market. For model-independent upper bounds on the AC price that are also based on the entire option price surface, Hobson et al. [15] derived the optimal relaxation when in fact only finitely many strikes are available and this weaker bound is identified with a particular discretisation of the asset price process (in [15] the result was formulated for basket options). For an elegant re-derivation of this result based on comonotonicity, see Chen et al. [8]. As a by-product of our approach, in Section 3 we will state a simple direct algorithm for obtaining this upper bound in the spirit of [15].
Note in passing that corresponding bounds for Asian put options can be derived through the Asian call-put parity (3). Moreover, the case of discrete marginal distributions of the underlying asset S_t can also be handled (which is in fact merely a matter of appropriately defining the inverse distribution function).
Section 4 gives some numerical illustrations for the bounds obtained in the paper. Since some of the bounds hold uniformly for every arbitrage-free model, they will certainly be weaker than those obtained by specifying a particular model. We nevertheless compare them to the available model-specific lower bounds for the Black-Scholes, the Variance-Gamma Lévy model and the Heston model, and the robust bounds turn out to be not far off the latter for options in the money (recall that by specifying the model, one implicitly assumes that the full option surface is available).
Finally, we give a numerical illustration for a situation, where only a finite number of quoted European call options is available. Using data from the S&P 500 from a specified trading day, model-independent lower bounds for some AC options are determined and compared with Monte Carlo estimates for several corresponding model prices that result from calibrating each of the models to the same set of data.

2 Model-independent lower bounds for Asian options

Lower bounds by sub-replication

Denote by $\mathrm{AC}(K,n)$ the price of an Asian call option at time 0

$$\mathrm{AC}(K,n) = \mathbb{E}\left[e^{-rT}\left(\frac{1}{n}\sum_{i=1}^{n} S_i - K\right)^+\right]$$

with strike K and n averaging days, where the expectation is taken with respect to an appropriate risk-neutral pricing measure. If the conditioning variable Z is assumed to be independent of the asset price process, we obtain from Jensen's inequality (1)

$$\begin{aligned}
\mathrm{AC}(K,n) &\geq \mathbb{E}\left[e^{-rT}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[S_i|Z\right] - K\right)^+\right] \\
&= \mathbb{E}\left[e^{-rT}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}\left[S_i\right] - K\right)^+\right] \\
&= \left(\frac{1}{n}\sum_{i=1}^{n} \exp\Big(-r(T-t_i)\Big)S_0 - K\exp(-rT)\right)^+, \qquad (2)
\end{aligned}$$

which is exactly the (very rough) lower bound for $\mathrm{AC}(K,n)$ implied by the call-put parity for Asian options

$$\mathrm{AC}(K,n) + K\exp(-r(T-t)) = \mathrm{AP}(K,n) + \frac{1}{n}\sum_{i=1}^{n}\exp\Big(-r(T-\max(t,t_i))\Big)S_{t\wedge t_i}, \qquad (3)$$

(when all information up to time t is available) and setting $t=0$ and $\mathrm{AP}(K,n)=0$.
This bound may sometimes be acceptable for options deep in the money, but deteriorates as the moneyness decreases (cf. Section 4). It can be improved considerably by choosing another conditioning random variable Z that still leads to a tractable expression in a model-independent setup. In what follows the concept of comonotonicity of a random vector will be used, which, roughly speaking, means that the components of the vector are perfectly positively dependent, which is equivalent to the fact that each component is a non-decreasing function of a single random variable (for a detailed introduction to this concept we refer to Dhaene et al. [12]). The particular advantage of this approach is the fact that the stop-loss transform of a comonotonic sum of random variables can be expressed as the sum of appropriate stop-loss transforms of the individual summands.
Let us consider the choice $Z = S_1$, then

$$\sum_{i=1}^{n}\mathbb{E}\left[S_i|S_1\right] = \sum_{i=1}^{n} S_1 e^{r(t_i-t_1)} := S^l.$$

The random vector $\big(S_1, e^{r(t_2-t_1)}S_1, \ldots, e^{r(t_n-t_1)}S_1\big)$ is comonotone, because $e^{r(t_i-t_1)}S_1$ is a non-decreasing function of S_1 for every i, and hence standard comonotonicity theory (see [12]) implies:

$$\mathbb{E}\left[\left(\frac{1}{n}S^l - K\right)^+\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(e^{r(t_i-t_1)}S_1 - F^{-1}_{e^{r(t_i-t_1)}S_1}\left(F_{S^l}(nK)\right)\right)^+\right].$$

Altogether, we obtain

$$\begin{aligned}
\mathrm{AC}(K,n) &\geq \mathbb{E}\left[e^{-rT}\frac{1}{n}\left(S^l - nK\right)^+\right] \\
&= e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(e^{r(t_i-t_1)}S_1 - \mathrm{F}^{-1}_{e^{r(t_i-t_1)}S_1}\left(\mathrm{F}_{S^l}(nK)\right)\right)^+\right] \\
&= e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(e^{r(t_i-t_1)}S_1 - \mathrm{F}^{-1}_{e^{r(t_i-t_1)}S_1}\left(\mathrm{F}_{S_1}\left(\frac{nK}{\sum_{j=1}^{n}e^{r(t_j-t_1)}}\right)\right)\right)^+\right] \\
&= e^{-rT}\frac{1}{n}\sum_{i=1}^{n}\mathbb{E}\left[\left(e^{r(t_i-t_1)}S_1 - e^{r(t_i-t_1)}\frac{nK}{\sum_{j=1}^{n}e^{r(t_j-t_1)}}\right)^+\right] \\
&= \frac{1}{n}\mathrm{C}\left(\frac{nK}{\sum_{j=1}^{n}e^{r(t_j-t_1)}}, t_1\right)\sum_{i=1}^{n}e^{-r(T-t_i)} =: \mathrm{LB}_1, \qquad (4)
\end{aligned}$$

where $\mathrm{C}(K,t_1)$ denotes the price of a European call (at time 0) with strike K, maturity t_1, current asset price S_0 and is defined by $\mathrm{C}(K,t_1) = e^{-rt_1}\mathbb{E}\left[(S_1-K)^+\right]$.
LB_1 provides a lower price bound for the AC in terms of a call with maturity t_1 and strike at $n\,K/\sum_{j=1}^{n}e^{r(t_j-t_1)}$. One should keep in mind that (in contrast to other available lower price bounds) this bound holds for any arbitrage-free market model and significantly improves upon the trivial bound (2), in particular if the Asian option is in or at the money.
If the call option in the above lower bound is traded in the market, this also implies a simple sub-replicating trading strategy:
At time t_1 the value V_1 of the portfolio specified in equation (4) is given by:

$$\begin{aligned}
V_1 &= \frac{1}{n}\sum_{i=1}^{n}e^{-r(T-t_i)}\left(S_1 - \frac{nK}{\sum_{j=1}^{n}e^{r(t_j-t_1)}}\right)^+ \\
&= \left(\frac{S_1}{n}\sum_{i=1}^{n}e^{-r(T-t_i)} - Ke^{-r(T-t_1)}\right)^+,
\end{aligned}$$

which is exactly the bound obtained by the Asian call-put parity (3) evaluated at time $t = t_1$.
The subreplicating strategy is then to do nothing, when $S_1 \leq n\,K/\sum_{i=1}^{n}e^{r(t_i-t_1)}$ or to buy $\frac{1}{n}\sum_{i=1}^{n}e^{-r(T-t_i)}$ assets in the case that $S_1 > n\,K/\sum_{i=1}^{n}e^{r(t_i-t_1)}$. The cost for this trade is exactly the payoff of the options in the portfolio plus $Ke^{-r(T-t_1)}$, which one should borrow. At each monitoring time t_i sell $e^{-r(T-t_i)}/n$ assets and invest the gain in the riskless bank account. Then at maturity T of the Asian call the payoff of the trading strategy will be:

$$\left(\frac{1}{n}\sum_{i=1}^{n}S_i - K\right)1_{\left\{\sum_{i=1}^{n}e^{r(t_i-t_1)}S_1 > nK\right\}},$$

where 1_A stands for the indicator function of event A.
Clearly the payoff of the Asian call dominates the payoff of this trading strategy and hence this is indeed a sub-replicating strategy.

Tightening the lower bounds

The following inequality obviously holds for every random variable Y:

$$\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} S_i - K\right)^{+}\right] \geq \mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} S_i - K\right) 1_{\{Y\geq c\}}\right]. \tag{5}$$

For $Y = \prod_{i=1}^{n} S_i$, the evaluation of the above bound boils down to pricing the payoff $S_i 1_{\{\prod_{i=1}^{n} S_i > c\}}$. In affine Lévy market models this can be done by formulas developed in Duffie et al. in [13]. Since in this paper we are interested in a robust bound, we will investigate the bound (5) without assuming a specific model for the underlying market. To that end, set $Y = S_t$ in inequality (5) for t to be chosen later on. As in the sequel we will use calls struck at c, we assume $c \geq 0$. This leads to

$$\begin{aligned}
\mathrm{AC}(K,n) &\geq e^{-rT}\mathbb{E}\left[\left(\frac{1}{n}\sum_{i=1}^{n} S_i - K\right) 1_{\{S_t\geq c\}}\right] \\
&= e^{-rT}\frac{1}{n}\left(\sum_{i=1}^{j(t)-1}\mathbb{E}\left[S_i 1_{\{S_t\geq c\}}\right] + \sum_{i=j(t)}^{n}\mathbb{E}\left[1_{\{S_t\geq c\}}\mathbb{E}\left[S_i \middle| \mathcal{F}_t\right]\right] - \mathbb{E}\left[nK 1_{\{S_t\geq c\}}\right]\right) \\
&= e^{-rT}\frac{1}{n}\left(\sum_{i=1}^{j(t)-1}\mathbb{E}\left[S_i 1_{\{S_t\geq c\}}\right] + \sum_{i=j(t)}^{n}\mathbb{E}\left[1_{\{S_t\geq c\}} e^{r(t_i-t)} S_t\right] - \mathbb{P}\left[S_t\geq c\right] nK\right) \qquad (6) \\
&= e^{-rT}\frac{1}{n}\left(\sum_{i=1}^{j(t)-1}\mathbb{E}\left[S_i 1_{\{S_t\geq c\}}\right] + \sum_{i=j(t)}^{n} e^{rt_i}\,\mathrm{C}(c,t) - \mathbb{P}\left[S_t\geq c\right]\left(nK - c\sum_{i=j(t)}^{n} e^{r(t_i-t)}\right)\right),
\end{aligned}$$

where $j(t) = \min\{i : t_i \geq t\}$.
If we assume that

$$S_{t_i} \text{ and } 1_{\{S_t\geq c\}} \text{ are non-negatively correlated for } t > t_i \text{ and for all } c \geq 0, \tag{7}$$

we can bound the first term in the last equality from below by

$$\mathbb{E}\left[S_i\, 1_{\{S_t\geq c\}}\right] \geq S_0 \exp(rt_i)\,\mathbb{P}\left[S_t \geq c\right]$$

and subsequently

$$\mathrm{AC}(K,n) \geq e^{-rT}\frac{1}{n}\left(\sum_{i=j(t)}^{n} e^{rt_i}\,\mathrm{C}(c,t) - \mathbb{P}\left[S_t\geq c\right]\left(nK - \sum_{i=1}^{j(t)-1} e^{rt_i} S_0 - c\sum_{i=j(t)}^{n} e^{r(t_i-t)}\right)\right). \tag{8}$$

Since the tail of the asset price distribution is given by $\mathbb{P}\left[S_t \geq c\right] = -e^{rt}\frac{\partial C(K,t)}{\partial K}\Big|_{K=c} := -e^{rt}\,\mathrm{C}_K(c,t)$, the right hand side of (8) can be rewritten as

$$\mathrm{AC}(K,n) \geq e^{-rT}\frac{1}{n}\sum_{i=j(t)}^{n} e^{rt_i}\left(\mathrm{C}(c,t) + \mathrm{C}_K(c,t)\left(\frac{nK - \sum_{i=1}^{j(t)-1} e^{rt_i} S_0}{\sum_{i=j(t)}^{n} e^{r(t_i-t)}} - c\right)\right).$$

Recalling that the no-arbitrage condition implies convexity of the call price function with respect to the strike, it becomes clear that the optimal choice for c (i.e. the value for which the right-hand side becomes maximal) is given by

$$\tilde{c}_t^{(1)} = \frac{nK - \sum_{i=1}^{j(t)-1} e^{rt_i} S_0}{\sum_{i=j(t)}^{n} e^{r(t_i - t)}}. \tag{9}$$

Hence the best lower bound with the form of inequality (8) is given by:

$$\mathrm{AC}(K,n) \geq \frac{e^{-rT}}{n} \max_{0\leq t\leq T} \mathrm{C}(\tilde{c}_t^{(1)}, t) \sum_{i=j(t)}^{n} e^{rt_i} =: \mathrm{LB}_t^{(1)}, \tag{10}$$

and this bound holds whenever (7) is justified in an arbitrage-free market (if the partial derivative $\frac{\partial C(K,t)}{\partial K}$ does not exist, which for instance is the case for discrete asset price distributions, one can simply replace the partial derivative by the left-hand derivative).
The optimization problem above can further be simplified by the observation

$$\tilde{c}_t^{(1)} = \frac{\tilde{c}_{t_j}^{(1)}}{\exp\left(r(t_j - t)\right)}$$

and consequently

$$\begin{aligned} \mathrm{C}(\tilde{c}_{t_j}^{(1)}, t_j) &= e^{-rt_j}\mathbb{E}\left[\left(S_{t_j} - \tilde{c}_{t_j}^{(1)}\right)^+\right] \\ &\geq e^{-rt_j}\mathbb{E}\left[\left(\mathbb{E}\left[S_{t_j}|S_t\right] - \tilde{c}_{t_j}^{(1)}\right)^+\right] \\ &= e^{-rt}\mathbb{E}\left[\left(S_t - \frac{\tilde{c}_{t_j}^{(1)}}{\exp\left(r(t_j - t)\right)}\right)^+\right] = \mathrm{C}(\tilde{c}_t^{(1)}, t), \end{aligned}$$

where we wrote j instead of $j(t)$.
Hence the maximum in (10) must be attained at a monitoring time, which makes the evaluation of the bound computationally simple.

The performance of $\mathrm{LB}_t^{(1)}$ is of course at least as good as LB_1 for all strikes, since $\mathrm{LB}_1^{(1)} = \mathrm{LB}_1$, and in many cases superior, in particular for options at and out of the money (cf. Section 4). This stems from the fact that especially in this strike region the slope of the call option price with respect to the maturity is large implying that it can be worth more to own a smaller amount of calls with a longer maturity than more calls with shorter maturity. This is also the reason why $\max_i LB_i^{(1)}$ is LB_1 for small strikes.

Let us now consider another assumption on the asset price process:

$$\mathbb{E}\left[S_i 1_{\{S_t \geq c\}}\right] = \mathbb{E}\left[1_{\{S_t \geq c\}}\mathbb{E}\left[S_i|S_t\right]\right] \geq \mathbb{E}\left[S_0^{1-t_i/t} S_t^{t_i/t} 1_{\{S_t\geq c\}}\right] \quad \text{for } 0 \leq t_i \leq t,\ c \geq 0 \tag{11}$$

Intuitively this assumption means that the expectation of S_i, given S_0 and S_t, should be bounded by some sort of weighted geometric average of S_0 and S_t (where the weights depend on the distance of t_i to 0 and t, respectively).
Let us now give a sufficient (but not necessary) criterion for the validity of assumption (11).

Proposition 2.1 *The assumption (11) holds for arbitrary exponential Lévy models with asset price process $S_t = S_0 \exp(X_t)$, where $(X_t)_{t\geq 0}$, $X_0 = 0$ is a Lévy process.*

Proof: Due to Jensen's inequality we find for any $t \geq t_i$:

$$\begin{aligned}
\mathbb{E}\left[S_{t_i} 1_{\{S_t \geq c\}}\right] &= \mathbb{E}\left[\mathbb{E}\left[S_0 \exp(X_{t_i}) | X_t = \ln(S_t/S_0)\right] 1_{\{S_t \geq c\}}\right] \\
&\geq S_0 \mathbb{E}\left[\exp\left(\mathbb{E}\left[X_{t_i} | X_{t_i} + (X_t - X_{t_i}) = \ln(S_t/S_0)\right]\right) 1_{\{S_t \geq c\}}\right] \\
&= \mathbb{E}\left[S_0 \exp\left(\frac{t_i}{t} \ln(S_t/S_0)\right) 1_{\{S_t \geq c\}}\right] = \mathbb{E}\left[S_0 \left(\frac{S_t}{S_0}\right)^{t_i/t} 1_{\{S_t \geq c\}}\right],
\end{aligned}$$

where the first equality in the last line above is due to the stationarity and independence of increments and the right-continuity of the paths of a Lévy process (see for instance Jacod & Protter [17, pp. 624] for a detailed discussion). □

Note that

$$\mathbb{E}\left[S_0 \left(\frac{S_t}{S_0}\right)^{t_i/t} 1_{\{S_t \geq c\}}\right] = \mathbb{E}\left[S_0 \left(\left(\frac{S_t}{S_0}\right)^{t_i/t} - \left(\frac{c}{S_0}\right)^{t_i/t}\right)^+\right] + S_0 \mathbb{P}\left[S_t \geq c\right] \left(\frac{c}{S_0}\right)^{t_i/t}.$$

Hence with (11) we find for (6):

$$\mathrm{AC}(K,n) \geq e^{-rT} \frac{1}{n} \left(f(c) - \mathbb{P}\left[S_t \geq c\right] \left(nK - \sum_{i=1}^{j(t)-1} \left(\frac{c}{S_0}\right)^{t_i/t} S_0 - c \sum_{i=j(t)}^{n} e^{r(t_i - t)}\right)\right), \quad (12)$$

where

$$f(c) = \sum_{i=1}^{j(t)-1} \mathbb{E}\left[S_0 \left(\left(\frac{S_t}{S_0}\right)^{t_i/t} - \left(\frac{c}{S_0}\right)^{t_i/t}\right)^+\right] + \sum_{i=j(t)}^{n} e^{rt_i} \, \mathrm{C}(c,t).$$

As for any t we want to find the largest possible bound, we have to maximize over c.
Consider $\tilde{c}_t^{(2)}$, which solves:

$$nK - \sum_{i=1}^{j(t)-1} S_0 \left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t} - \tilde{c}_t^{(2)} \sum_{i=j(t)}^{n} e^{r(t_i - t)} = 0. \quad (13)$$

Then inequality (12) can be rewritten as:

$$\begin{aligned}
&n\, e^{rT} \, \mathrm{AC}(K,n) \\
&\geq f(c) - \mathbb{P}\left[S_t \geq c\right] \left(\sum_{i=1}^{j(t)-1} S_0 \left(\left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t} - \left(\frac{c}{S_0}\right)^{t_i/t}\right) + (\tilde{c}_t^{(2)} - c) \sum_{i=j(t)}^{n} e^{r(t_i - t)}\right) \\
&= \underbrace{\sum_{i=1}^{j(t)-1} \mathbb{E}\left[S_0 \left(\left(\frac{S_t}{S_0}\right)^{t_i/t} - \left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}\right) 1_{\{S_t \geq c\}}\right] + \sum_{i=j(t)}^{n} e^{rt_i} \left(\mathrm{C}(c,t) + \mathrm{C}_K(c,t)(\tilde{c}_t^{(2)} - c)\right)}_{=: g(c)}
\end{aligned}$$

But since for arbitrary $c > 0$

$$g(c) \leq \sum_{i=1}^{j(t)-1} \mathbb{E}\left[S_0\left(\left(\frac{S_t}{S_0}\right)^{t_i/t} - \left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}\right)^+\right] + \sum_{i=j(t)}^{n} e^{rt_i}\,\mathrm{C}(\tilde{c}_t^{(2)}, t) = g(\tilde{c}_t^{(2)}) = f(\tilde{c}_t^{(2)}),$$

the lower bound (12), for each t, is maximized by the choice $c = \tilde{c}_t^{(2)}$ (note that the left hand side of (13) is strictly decreasing in $\tilde{c}_t^{(2)}$ and hence the numerical computation of $\tilde{c}_t^{(2)}$ is trivial). Altogether we arrive at

$$\mathrm{AC}(K, n) \geq \frac{e^{-rT}}{n} f(\tilde{c}_t^{(2)}) =: \mathrm{LB}_t^{(2)}\,. \tag{14}$$

Since $\mathrm{LB}_t^{(2)}$ is a lower bound for all t the optimal lower bound along this approach is given by:

$$\begin{aligned} \mathrm{AC}(K, n) &\geq \max_{0 \leq t \leq T} \mathrm{LB}_t^{(2)} & (15) \\ &= \frac{e^{-rT}}{n} \max_{0 \leq t \leq T} \left(\sum_{i=1}^{j(t)-1} \mathbb{E}\left[S_0\left(\left(\frac{S_t}{S_0}\right)^{t_i/t} - \left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}\right)^+\right] + \sum_{i=j(t)}^{n} e^{rt_i}\,\mathrm{C}(\tilde{c}_t^{(2)}, t) \right), \end{aligned}$$

where $\tilde{c}_t^{(2)}$ solves (13). In the examples considered in Section 4, $\mathrm{LB}_t^{(2)}$ turns out to attain its maximum always at a monitoring time $t = t_i$.

Whereas the second summand of (15) is expressed in terms of European call prices which are assumed to be available, for the first summand we need to evaluate prices of contingent claims with payoff $\left(S_t^{t_i/t} - M\right)^+$ for some constant M, which can be done using the Carr-Madan formula [7], whenever the characteristic function of the asset price at time t is available (alternatively, one can use the approach outlined in Carr & Chou [6] to price contingent claims from the European option surface). Hence these power options can be priced using only the information of the call prices and $\mathrm{LB}_t^{(2)}$ provides a computationally tractable lower price bound for the Asian call under assumption (11).

Remark 2.1: Neither of the assumptions (7) and (11) is implied by the other (for instance one can construct simple discrete-time models in which only one of them is fulfilled). Nevertheless, $\mathrm{LB}_t^{(2)}$ will have a better performance than $\mathrm{LB}_t^{(1)}$ for not too small strikes K. To see this, observe that for any fixed t, $\tilde{c}_t^{(1)} \geq \tilde{c}_t^{(2)}$, if $nK \geq \sum_{i=1}^n e^{rt_i} S_0$, in which case $\tilde{c}_t^{(1)} \geq \tilde{c}_t^{(2)} \geq S_0 e^{rt}$ and

$$\begin{aligned} &\mathbb{E}\left[S_0\left(\frac{S_t}{S_0}\right)^{t_i/t} 1_{\{S_t \geq \tilde{c}_t^{(2)}\}}\right] - S_0 \exp(rt_i)\,\mathbb{P}\left[S_t \geq \tilde{c}_t^{(1)}\right] \geq \\ &\geq \mathbb{E}\left[\left(S_0\left(\frac{S_t}{S_0}\right)^{t_i/t} - S_0\left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t}\right)^+\right] + \mathbb{P}\left[S_t \geq \tilde{c}_t^{(2)}\right]\left(S_0\left(\frac{\tilde{c}_t^{(2)}}{S_0}\right)^{t_i/t} - e^{rt_i} S_0\right) > 0. \end{aligned}$$

So for $nK \geq \sum_{i=1}^n e^{rt_i} S_0$ the required properties for the second lower bound are stronger than those for the first one, which immediately implies an ordering of the obtained bounds.

In the examples in Section 4 it can be seen, that $nK \geq \sum_{i=1}^n e^{rt_i} S_0$ is not at all a necessary condition for a better performance of $\mathrm{LB}_t^{(2)}$.

On the other hand, for small K it can happen that $\mathrm{LB}_t^{(1)} < \mathrm{LB}_t^{(2)}$ for all $t > t_1$ (for example use Jensen's inequality for $K = 0$).

Remark 2.2: A close look into the above derivations shows that the assumptions (7) and (11) to hold are not necessary for $\mathrm{LB}^{(1)}$ and $\mathrm{LB}^{(2)}$, respectively. The following weaker conditions are sufficient (although perhaps less intuitive):

$$\frac{1}{n}\sum_{i=1}^{n} S_i \geq_{sl} \frac{1}{n}\left(\sum_{i=1}^{j(t)-1} S_0 e^{rt_i} + \sum_{i=j(t)}^{n} e^{r(t_i-t)} S_t\right) =: S^{l_1} \qquad \text{and} \tag{16}$$

$$\frac{1}{n}\sum_{i=1}^{n} S_i \geq_{sl} \frac{1}{n}\left(\sum_{i=1}^{j(t)-1} S_0^{(1-t_i/t)} S_t^{t_i/t} + \sum_{i=j(t)}^{n} e^{r(t_i-t)} S_t\right) =: S^{l_2}, \qquad \text{respectively,} \tag{17}$$

where for any $0 \leq t \leq T$, again $j(t) = \min\{i : t_i \geq t\}$ and $\geq_{sl}$ denotes the stop-loss ordering of two random variables $X, Y \geq 0$ defined by:

$$X \geq_{sl} Y \quad \Leftrightarrow \quad \mathbb{E}[(X-d)^+] \geq \mathbb{E}[(Y-d)^+] \qquad \forall d \geq 0.$$

Remark 2.3: Hobson et al. [15] illustrate that their model-independent upper bound for the basket option is in fact attained for comonotonic asset prices. Similarly, the lower bound LB_1 for the Asian option would be attained for comonotonicity among the prices of the underlying asset over time. This dependence structure can however not be conciliated with the martingale property of the discounted asset price (which is implied by the no-arbitrage assumption). To see this, assume the S_i to be comonotone and to fit the observable option price surface. Then the following holds:

$$E\left[S_i | S_k = s\right] = F_{S_i}^{-1}\left(F_{S_k}(s)\right) \quad i > k.$$

On the other hand, the martingale property implies

$$E\left[S_i | S_k = s\right] = s e^{r(t_i - t_k)} \quad i > k.$$

Hence,

$$F_{S_i}^{-1}\left(F_{S_k}(s)\right) = s e^{r(t_i - t_k)} \quad i > k$$

has to hold, which is equivalent to

$$\mathrm{C}_K(s, t_k) = e^{r(t_i - t_k)}\, \mathrm{C}_K(e^{r(t_i - t_k)} s, t_i), \tag{18}$$

putting a restriction on the option price surface, but the latter is actually an input parameter for the bound LB_1. Hence LB_1 is in fact not attained by some market model.
Note that the last line also shows that in the case of comonotonicity the following holds:

$$\mathrm{C}(K, t_k) = \mathrm{C}\left(\frac{K}{e^{r(t_k - t_1)}}, t_1\right),$$

which also explains, why in this case the lower bound LB_1 would be equal to the upper comonotonicity bound.

Remark 2.4: In general it is not possible to check whether conditions (7) or (11), respectively, are fulfilled by the market, if only European call prices are available, since the latter only give information on the marginal distributions and not on path properties of the underlying asset price process. However, there might be situations in which, in addition to European option prices, some prices of derivatives that depend on both, S_{t_i} and S_t, are available (for instance correlation derivatives), which could then give some confidence into the validity of (7) or (11). Moreover and more importantly, market analysts often have an intuition on properties of the market and $\text{LB}^{(1)}$ and $\text{LB}^{(2)}$ provide improvements of pure no-arbitrage bounds in case one has good reasons to believe that the correlation properties (7) or (11), respectively, are appropriate. In particular, as shown above, exponential Lévy models provide a large set of models, for which both assumptions, (7) and (11), hold.

3 The finite strike case

Up to now we have assumed that European call option prices are available for all strikes at a specified maturity. In the following we will relax this assumption to the case of having only a finite number of observable option prices available in the market. Obviously the lack of complete knowledge of the option surface (hence less information about the price process) will weaken the developed bounds, as one has to look for the next-best alternative.

For completeness, we first give a relaxation of the comonotone upper bound developed by Simon et al. [25] to the case of finitely many strikes. Note that this problem was solved by Hobson et al. [15] (formulated for basket options); here we state a simple and direct algorithm to represent the solution.

Upper bounds

Recall from [25] that an upper bound in terms of a portfolio of European options is given by

$$\text{AC}(K,n) \leq \frac{1}{n}\sum_{i=1}^{n} \exp(-r(T-t_i))\,\text{C}\left(\kappa_i, i\right), \tag{19}$$

where

$$\kappa_i = F_{X_i}^{-1}(F_{S^c}(nK)) \tag{20}$$

and $S^c = X_1^c + X_2^c + \cdots + X_n^c$ denotes the comonotone sum of the random variables $X_1, \ldots, X_n$. In particular, $\sum_{i=1}^{n} \kappa_i = nK$.

In practice the optimal choice (20) of strikes will not necessarily be available in the market. Moreover for the determination of the κ_i one needs a specification of the underlying model or at least the knowledge of the complete option price surface (cf. [2]). Given a set of traded European options, it is natural to try to find the best choice of strikes such that (19) is minimized without further specification of the model (which will usually lead to a weaker bound than in the situation of a specified model). In Chen et al. [8] it is shown how this can be done using comonotonicity arguments. Alternatively, Hobson et al. [15] derived the optimal value of the κ_i in this situation using Lagrangian optimisation. The following algorithm provides a simple representation of that latter approach.

Assume w.l.o.g. $t_n = T$ and that for maturity t_i there are $m(i)$ European call prices available in the market. The strikes of these calls may be ordered by size and denoted by $K_{i,j}$, $j = 0, \dots m(i)$, where $K_{i,0} = 0$ (due to the martingale property in an arbitrage-free market the price of an option with strike 0 is the current value S_0 of the asset price) and $K_{i,m(i)} = \max_j(K_{i,j})$.

Step 1: Since option prices are convex and non-increasing functions of the strike price, an upper bound for the price of a European option with maturity t_i and strike K in terms of the available prices $\mathrm{C}(K_{i,j}, t_i) = c_{i,j}$ is certainly given by

$$f_i(K) = \max \{g(K) | g(K) \text{ convex, non-increasing and } g(K_{i,j}) = c_{i,j} \; \forall j \in \{0, \dots, m(i)\}\}. \quad (21)$$

The maximum in (21) is obtained by connecting the given points $(K_{i,j}, c_{i,j})$, $j = 0, \dots, m(i)$ linearly and setting $f_i(K) = c_{i,m(i)}$ for $K > K_{i,m(i)}$.
Note that this bound is not sharp, i.e. there is no model consistent with the option price functions $f_i(K)$, since for any maturity t_i, $\lim_{K\to\infty} \mathrm{C}(K, t_i) = 0 \neq c_{i,m(i)}$. Nevertheless, $f_i(K)$ is the best (smallest) upper no-arbitrage bound for $\mathrm{C}(K, t_i)$ knowing only $c_{i,j}$. Hence in order to get a bound for the Asian call, which is based on the no-arbitrage principle only, we have to use $\mathrm{C}(K, t_i) = f_i(K)$ for all $K \geq 0$ in (19). The best bound in the sense of (19) is then the solution to the following problem:

$$\min_{\kappa_i} \sum_{i=1}^{n} \exp(-r(T - t_i)) f_i(\kappa_i), \quad \text{such that } \sum_{i=1}^{n} \kappa_i \leq nK. \quad (22)$$

Step 2: The slope Δf of the call price bound $f_i(K)$ with maturity t_i at each point $K \neq K_{i,j}, j = 0, \dots, m(i)$ and $0 < K < K_{i,m(i)}$ is given by:

$$\Delta f_i(K) = \frac{c_{i,j} - c_{i,j-1}}{K_{i,j} - K_{i,j-1}}, \quad K_{i,j-1} < K < K_{i,j};$$

whereas at traded strike values $K_{i,j}$, the left- and the right-hand slope are in general different: For $1 \leq j \leq m(i) - 1$

$$\Delta f_i^-(K_{i,j}) := \frac{c_{i,j} - c_{i,j-1}}{K_{i,j} - K_{i,j-1}} \neq \frac{c_{i,j+1} - c_{i,j}}{K_{i,j+1} - K_{i,j}} := \Delta f_i^+(K_{i,j}).$$

Furthermore we have $\Delta f_i^+(K_{i,m(i)}) = 0$ and $\Delta f_i(K) = 0$ for $K > K_{i,m(i)}$.
The solution of (22) can now be found by the following straight-forward greedy algorithm:

Alg-1 If $\sum_{i=1}^n K_{i,m(i)} \leq nK$, then the solution is given by $\kappa_i = K_{i,m(i)}$, else set $\kappa_i = K$.

Alg-2 Determine $\Delta f_i^+(\kappa_i)$ and $\Delta f_i^-(\kappa_i)$ for all $i = 1, \dots n$.

Alg-3 Set

$$\begin{aligned} I &= \operatorname{argmin}_i e^{-r(T-t_i)} \Delta f_i^+(\kappa_i), & \underline{\Delta} &= e^{-r(T-t_I)} \Delta f_I^+(\kappa_I) \quad \text{and} \\ J &= \operatorname{argmax}_i e^{-r(T-t_i)} \Delta f_i^-(\kappa_i), & \overline{\Delta} &= e^{-r(T-t_J)} \Delta f_J^-(\kappa_J). \end{aligned}$$

Note that for all i both $\Delta f_i^+(\kappa_i)$ and $\Delta f_i^-(\kappa_i)$ are negative, unless $\kappa_i > K_{i,m(i)}$ (in which case they are equal to zero).
If $\underline{\Delta} < \overline{\Delta}$ then set

$$\begin{aligned} u_I = K_{I,j+1} - \kappa_I, \quad & \text{where} \quad K_{I,j} \leq \kappa_I < K_{I,j+1} \\ \text{and} \quad u_J = \kappa_J - K_{J,\tilde{j}}, \quad & \text{where} \quad K_{J,\tilde{j}} < \kappa_J \leq K_{J,\tilde{j}+1} \end{aligned}$$

and

$$\kappa_I = \kappa_I + \min\{u_I, u_J\}, \quad \kappa_J = \kappa_J - \min\{u_I, u_J\}.$$

If $\underline{\Delta} \geq \overline{\Delta}$, go to Alg-4.
Else update $\Delta f_i^+(\kappa_i)$ and $\Delta f_i^-(\kappa_i)$ for $i = I, J$ and return to the start of Alg-3.

Alg-4 Set $M = \{i \mid \kappa_i \neq K_{i,j} \; \forall j = 1, \ldots, m(i)\}$. If $|M| = 1$ Stop.
Else choose any pair $\{m_1, m_2\} \in M \times M$, $m_1 \neq m_2$ and calculate

$$\begin{aligned} u_1 = K_{m_1,j+1} - \kappa_{m_1}, \quad & \text{where} \quad K_{m_1,j} < \kappa_{m_1} < K_{m_1,j+1}, \\ u_2 = \kappa_{m_2} - K_{m_2,\tilde{j}}, \quad & \text{where} \quad K_{m_2,\tilde{j}} < \kappa_{m_2} < K_{m_2,\tilde{j}+1}, \\ \kappa_{m_1} = \kappa_{m_1} + \min\{u_1, u_2\} \quad & \text{and} \quad \kappa_{m_2} = \kappa_{m_2} - \min\{u_1, u_2\}. \end{aligned}$$

Then return to the start of Alg-4.

After the step Alg-4 has terminated, there will be at most one index $i \in \{1, 2, \ldots, n\}$, for which κ_i does not equal a traded strike $K_{i,j}$ for some $j \in \{1, \ldots, m(i)\}$. For this strike, $f_i(\kappa_i)$ can be expressed as follows:

$$f_i(\kappa_i) = \frac{K_{i,j+1} - \kappa_i}{K_{i,j+1} - K_{i,j}} c_{i,j} + \frac{\kappa_i - K_{i,j}}{K_{i,j+1} - K_{i,j}} c_{i,j+1}, \quad \text{for } K_{i,j} \leq \kappa_i < K_{i,j+1}.$$

Note that the algorithm without Alg-4 would also lead to a solution of (22), but possibly with more than one non-liquid strike κ_i.

The idea of the algorithm is to augment the current strike at the maturity time t_I of the fastest price decrease and - in order to keep $\sum_{i=1}^n \kappa_i = nK$ unchanged - to reduce the strike at the maturity time t_J of the slowest price increase. This is done as long as the decrease is not smaller than the increase and hence in every step the price for the portfolio does not increase and the upper bound is improved.

Proposition 3.1 *The algorithm presented above converges and solves* (22).

Proof: Due to the convexity of the European option price function, $\Delta f_i^+(K)$ and $\Delta f_i^-(K)$ are non-decreasing in K. Hence in Alg-3 $\overline{\Delta} = \max_i e^{-r(T-t_i)} \Delta f_i^-(K)$ is lowered or/and $\underline{\Delta} = \min_i e^{-r(T-t_i)} \Delta f_i^+(K)$ is raised. Furthermore either κ_I or κ_J is set to a new traded strike and the algorithm has to converge, since there are only finitely many strikes in the market.
To show the optimality, assume that $\tilde{\kappa}_i, i = 1, \ldots, n$ is another combination fulfilling $\sum_{i=1}^n \tilde{\kappa}_i =$

nK. Then due to the convexity of the call price:

$$\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)} f_i(\tilde{\kappa}_i) - \frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)} f_i(\kappa_i) \\
\geq\ & \frac{1}{n}\sum_{i=1}^{n} e^{-r(T-t_i)} \left(f_i(\kappa_i) - \Delta f_i^-(\kappa_i)(\kappa_i - \tilde{\kappa}_i)^+ + \Delta f_i^+(\kappa_i)(\tilde{\kappa}_i - \kappa_i)^+ - f_i(\kappa_i)\right) \\
\geq\ & \frac{1}{n}\left(\min_i \left(e^{-r(T-t_i)} \Delta f_i^+(\kappa_i)\right) \sum_{i=1}^{n} (\tilde{\kappa}_i - \kappa_i)^+ - \max_i \left(e^{-r(T-t_i)} \Delta f_i^-(\kappa_i)\right) \sum_{i=1}^{n} (\kappa_i - \tilde{\kappa}_i)^+ \right) \\
\geq\ & \frac{1}{n} \min_i \left(e^{-r(T-t_i)} \Delta f_i^+(\kappa_i)\right) \sum_{i=1}^{n} (\tilde{\kappa}_i - \kappa_i) = 0,
\end{aligned}$$

where the last inequality holds due to the terminating condition of the algorithm ($\underline{\Delta} \geq \overline{\Delta}$). Hence $\sum_{i=1}^{n} f_i(\kappa_i)$ is indeed a solution to (22). □

Observe that if $\sum_{i=1}^{n} K_{i,m(i)} > nK$, the optimal super replicating portfolio calculated by the algorithm does not involve calls struck at $\kappa_i > K_{i,m(i)}$ or $\kappa_i < 0$. This holds because: If there were some $\kappa_i > K_{i,m(i)}$, then $\max_i \Delta f_i^-(\kappa_i) = 0$. On the other hand, as $\sum_{i=1}^{n} K_{i,m(i)} > nK$, there is a $\tilde{\kappa}_j < K_{j,m(j)}$ and due to the convexity of the call prices $\Delta f_i^+(\tilde{\kappa}_j) < 0$ and hence $\min_i \Delta f_i^+(\tilde{\kappa}_i) < 0 = \max_i \Delta f_i^-(\kappa_i)$. Thus the stopping condition of the algorithm is not fulfilled and it would not have terminated.

Lower bounds

For both lower bounds LB_1 and $\mathrm{LB}_t^{(1)}$ only one European call is needed, namely the call struck at $\frac{nK}{\sum_{i=1}^{n} e^{-r(t_i - t_1)}}$ and $\frac{nK - \sum_{i=1}^{j(t)-1} e^{rt_i} S_0}{\sum_{i=j(t)}^{n} e^{r(t_i - t)}}$ respectively. In practice, this particular strike might not be traded and we need to bound its price from below with traded strikes. To cover also this case we use a convex optimization result of Bertsimas & Popescu [3]:

Proposition 3.2 *The best model-independent lower bound for the price of a European call struck at $\tilde{K}_i \leq K_{i,m(i)}$, $K_{i,j^*-1} \leq \tilde{K}_i \leq K_{i,j^*}$ and maturing at t_i is given by*

$$\mathrm{C}(\tilde{K}_i, t_i) \geq \max\left\{ c_{i,j^*-1} + \Delta f_i^-(K_{i,j^*-1})(\tilde{K}_i - K_{i,j^*-1}), c_{i,j^*} - \Delta f_i^+(K_{i,j^*})(K_{i,j^*} - \tilde{K}_i)\right\}, \tag{23}$$

with $\Delta f_i^-(0) = -e^{-rt_i}$, and by

$$\mathrm{C}(\tilde{K}_i, t_i) \geq \max\left\{ c_{i,m(i)} + \Delta f_i^-(K_{i,m(i)})(\tilde{K}_i - K_{i,m(i)}), 0\right\} \tag{24}$$

for $\tilde{K}_i > K_{i,m(i)}$.

Here we will give an alternative proof of inequalities (23) and (24), which simplifies the one given in [3]:

Proof: Inequalities (23) and (24) follow in fact directly from the convexity of $\mathrm{C}(\tilde{K}_i, t_i)$. So it just remains to show that there exist arbitrage-free asset price models consistent with the observed call prices for which inequalities (23) and (24) become equalities. Note that in the case

$\tilde{K}_i = K_{i,j}$ for some j, the inequalities are obviously equalities and hence we only have to consider $\tilde{K}_i \neq K_{i,j}\ \forall j$. We distinguish three cases:

<u>$(i)\ K_{i,j^*-1} < \tilde{K}_i < K_{i,j^*} \leq K_{i,m(i)-1}$:</u> In this case one possible choice is the arbitrage-free discrete asset price model described by:

$$\mathbb{P}\left[S_i = s\right] = \begin{cases} e^{rt_i}\left(\Delta f_i^+(K_{i,j^*}) - \Delta f_i^-(K_{i,j^*-1})\right) & \text{for } s = \bar{K}_i \\ -e^{rt_i}\Delta f_i^-(K_{i,m(i)}) & \text{for } s = M(i) \\ e^{rt_i}\left(\Delta f_i^+(K_{i,j}) - \Delta f_i^-(K_{i,j})\right) & \text{for } s = K_{i,j},\ j \notin \{j^*-1, j^*, m(i)\} \\ 0 & \text{else,} \end{cases}$$

where $\bar{K}_i$ is such that

$$c_{i,j^*-1} + \Delta f_i^-(K_{i,j^*-1})(\bar{K}_i - K_{i,j^*-1}) = c_{i,j^*} - \Delta f_i^+(K_{i,j^*})(K_{i,j^*} - \bar{K}_i)$$

and $M(i)$ solves

$$c_{i,m(i)} + \Delta f_i^-(K_{i,m(i)})(M(i) - K_{i,m(i)}) = 0.$$

<u>$(ii)\ \tilde{K}_i > K_{i,m(i)}$:</u> Here a possible arbitrage-free discrete asset price model is

$$\mathbb{P}\left[S_i = s\right] = \begin{cases} -e^{rt_i}\Delta f_i^-(K_{i,m(i)}) & \text{for } s = M(i) \\ e^{rt_i}\left(\Delta f_i^+(K_{i,j}) - \Delta f_i^-(K_{i,j})\right) & \text{for } s = K_{i,j},\ j \neq m(i) \\ 0 & \text{else.} \end{cases}$$

<u>$(iii)\ K_{i,m(i)-1} < \tilde{K}_i < K_{i,m(i)}$:</u> In this case we define for any $n \in \mathbb{N}$

$$\mathbb{P}^{(n)}\left[S_i = s\right] = \begin{cases} e^{rt_i}\left(\frac{\Delta f_i^-(K_{i,m(i)})}{n} - \Delta f_i^-(K_{i,m(i)-1})\right) & \text{for } s = \bar{K}_i^{(n)} \\ -\frac{e^{rt_i}}{n}\Delta f_i^-(K_{i,m(i)}) & \text{for } s = K_{i,m(i)} + n(M(i) - K_{i,m(i)}) \\ e^{rt_i}\left(\Delta f_i^+(K_{i,j}) - \Delta f_i^-(K_{i,j})\right) & \text{for } s = K_{i,j},\ j \notin \{m(i), m(i)-1\} \\ 0 & \text{else,} \end{cases}$$

where $\bar{K}_i^{(n)}$ solves

$$c_{i,m(i)-1} + \Delta f_i^-(K_{i,m(i)-1})(\bar{K}_i^{(n)} - K_{i,m(i)-1}) = c_{i,m(i)} - \frac{\Delta f_i^-(K_{i,m(i)})}{n}(K_{i,m(i)} - \bar{K}_i^{(n)}).$$

Then the price $\mathrm{C}^{(n)}$ of a European call option struck at $\tilde{K}_i$ in the model described by $\mathbb{P}^{(n)}$ is given by

$$\mathrm{C}^{(n)}(\tilde{K}_i, t_i) = \begin{cases} c_{i,m(i)-1} + \Delta f_i^-(K_{i,m(i)-1})(\tilde{K}_i - K_{i,j^*-1}) & \text{for } \tilde{K}_i \leq \bar{K}_i^{(n)} \\ c_{i,m(i)} - \frac{e^{rt_i}}{n}\Delta f_i^-(K_{i,m(i)})(K_{i,m(i)} - \tilde{K}_i) & \text{for } K_i > \bar{K}_i^{(n)}, \end{cases}$$

which is equivalent to:

$$\begin{aligned} \mathrm{C}^{(n)}(\tilde{K}_i, t_i) \quad &= \quad \max\Big\{c_{i,m(i)-1} + \Delta f_i^-(K_{i,m(i)-1})(\tilde{K}_i - K_{i,m(i)-1}), \\ &\qquad c_{i,m(i)} - \frac{e^{rt_i}}{n}\Delta f_i^-(K_{i,m(i)})(K_{i,m(i)} - \tilde{K}_i)\Big\} \end{aligned}$$

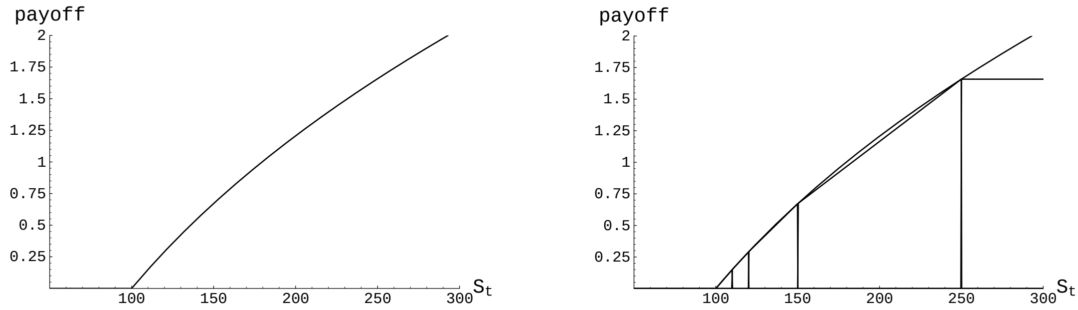

Figure 1: Payoff of the subreplicating portfolio and the contingent claim $(S_t^{1/3} - 100^{1/3})^+$ respectively.

Now letting n grow arbitrarily large, one sees that any lower bound on the price of a European call struck at $\tilde{K}_i$ maturing at t_i is bounded above by

$$\max\left\{c_{i,m(i)-1} + \Delta f_i^-(K_{i,m(i)-1})(\tilde{K}_i - K_{i,m(i)-1}), c_{i,m(i)}\right\},$$

which is the bound claimed in (23) (recall that $\Delta f_i^+(K_{i,m(i)}) = 0$). □

Note that the bound in Proposition 3.2 can also be derived using the no-arbitrage principle directly, i.e. the right hand side of inequalities (23) and (24), respectively, represent static subreplicating portfolios.

In order to adapt $\mathrm{LB}_t^{(2)}$ to the finite strike case we have to find a bound for a contingent claim of the form $(S_t^x - \tilde{K})^+$ for $0 < x < 1$. The payoff function is clearly concave, when the power option finishes in the money (see left part of Figure 1).
Hence a simple lower bound on the payoff of this contingent claim is given by the following: Assume the strikes $K_1 < K_2 < \cdots < K_n$ to be liquid for the European calls with maturity t (where $K_1^x \geq \tilde{K}$), then the payoff $(S_t^x - \tilde{K})^+$ clearly dominates the payoff

$$\frac{K_2^x - \tilde{K}}{K_2 - K_1}(S_t - K_1)^+ 1_{\{S_t \leq K_2\}} + \sum_{i=2}^{n}\left((K_i^x - \tilde{K}) + \frac{K_{i+1}^x - K_i^x}{K_{i+1} - K_i}(S_t - K_i)^+\right) 1_{\{K_i < S_t \leq K_{i+1}\}}, \quad (25)$$

where K_{n+1} is set to the maximum of possible outcomes of S_t. If S_t is unbounded, then simply the last term in the sum becomes $(K_n^x - \tilde{K})$ (see right part in Figure 1 for an example with no finite maximum and traded strikes 100,110,120,150,250). The payoff in equation (25) can be obtained by trading European type calls only, namely by the portfolio given in Table 1.

4 Numerical illustration

For a given option surface, the bounds developed in this paper provide a model-independent lower bound for the Asian option price and, for some mild additional model assumptions, improvements of this bound. Clearly, these bounds cannot be compared to the actual (model) price of the option, since the latter can only be determined by fully specifying an underlying model

strike	number
K_1	$\frac{K_2^x - \tilde{K}}{K_2 - K_1}$
K_2	$\frac{K_3^x - K_2^x}{K_3 - K_2} - \frac{K_2^x - \tilde{K}}{K_2 - K_1}$
...	...
K_i	$\frac{K_{i+1}^x - K_i^x}{K_{i+1} - K_i} - \frac{K_i^x - K_{i-1}^x}{K_i - K_{i-1}}$
...	...
K_n	$-\frac{K_n^x - K_{n-1}^x}{K_n - K_{n-1}}$

Table 1: Strikes and number of calls with maturity t needed for payoff (25) in the unbounded case

(and for incomplete models even then the price is not unique). Yet, in order to get some illustration of the numerical performance of the bound, we will first look at a Black-Scholes model with specified parameters and compare the lower bounds available for that situation with the model-independent ones of this paper (where the necessary call prices are then also calculated as Black-Scholes prices). One should however keep in mind that the strength of the bounds in this paper is its wider applicability beyond model specifications (and, clearly, for a fully specified model one can improve the bounds considerably).

In Figures 2 and 4 an AC with $S_0 = 100$, riskless interest rate $r = 0.05$, volatility $\sigma = 0.1812$ and expiry 3 years is considered, while in figures 3 and 5 the expiry is changed to 10 years (with yearly averaging in each case). The plots depict the difference between the Black Scholes price and the developed lower bounds $\text{LB}_t^{(1)}$ and $\text{LB}_t^{(2)}$, i.e.

$$\text{AC}(K,T) - \text{LB}_t^{(i)}, \quad (i = 1, 2)$$

where all necessary contingent claims for the bounds are priced in the Black-Scholes model. Note that for each averaging time t, $\text{LB}_t^{(i)}$ provides a lower bound and the best among all the t then determines the actual lower bound for the Asian option price.

In Table 2, such a comparison is given for a Black-Scholes model and an Asian option with maturity $T = 120$ days and averaging at times $t_1 = 91, t_2 = 92 \ldots, t_{30} = 120$ (daily compounded interest rate $r = \ln(1+0.09/365)$). The other lower bounds in the table are taken from Vanmaele et al. [27]. It can be seen that the optimal t for $\text{LB}_t^{(1)}$ is $t = 91$ for all considered strikes and hence there is no improvement of the bound by assuming that S_i and $1_{\{S_j \geq c\}}$ are non-negatively correlated. Surprisingly the performance of the model-independent LB_1 is nearly as good as $\text{LB}B_T$ (which is the bound of [27] obtained by conditioning on the value at the maturity date of the Brownian motion governing the asset price process) and $\text{LB}_t^{(2)}$ is even tighter than $\text{LB}B_T$. MC denotes the Monte Carlo approximation of the actual model price and is almost identical to $\text{LB}G_A$, which is the best bound given in [27] and obtained by conditioning on the standardized logarithm of the geometric average. One can see that although $\text{LB}_t^{(2)}$ is not designed to lead to tight bounds in this case, it still yields quite satisfying results.
Tables 3 & 4 compare the lower bounds for a Black-Scholes model and Asian option with monthly averaging and maturity $T = 3$ and $T = 10$ years, respectively with those of other approaches

(taken from Nielsen & Sandmann [22] and called Ni & Sa) and the parameters are given in the figures. GA refers to the price of a geometric Asian option with the same strike (which is always a lower bound and can be computed easily in the Black-Scholes model, see Vorst [30]). Moreover, MC is the Monte Carlo price of the AC and UCB refers to the comonotonic upper bound which gives a model-independent upper bound. Note that, again, the bound Ni & Sa crucially relies on the structure of the Black-Scholes model. The performance of $\mathrm{LB}_t^{(2)}$ seems satisfying again (and the difference to the actual price is less than for the model-independent upper bound UCB).
Tables 5 and 6 compare the bounds of this paper for various maturities for the Heston model

$$\begin{aligned} dS/S &= r\ dt + \sqrt{v}\ dW_1 \\ dv &= \kappa(\theta - v(t))\ dt + \sigma_v \sqrt{v}\ dW_2, \end{aligned}$$

with $v_0 = 0.0175$, $\kappa = 1.5768$, $\theta = 0.0398$, $\sigma_v = 0.5751$ and for the correlation between the Brownian motions W_1 and W_2 we choose $\rho = -0.5711$. Although it is a priori not clear whether assumptions (7) and (11), respectively, are fulfilled in the Heston model, numerical experiments give strong indications that both of them are met, at least for this parameter set.
Tables 7 and 8 compare the different bounds for a Variance Gamma model, where the characteristic function of $\ln(S_T)$ is given by

$$\phi_T(u) = \exp\left(\ln(S_0) + (r+\omega)T\right)\left(1 - i\theta\nu u + \frac{1}{2}\sigma^2 \nu u\right)^{-T/\nu},$$

with $\omega = \frac{1}{\nu}\ln\left(1 - \theta\nu - \frac{1}{2}\sigma^2\nu\right)^{-T/\nu}$ (see Madan et al. [21]). The parameters used are $\sigma = 0.2684$, $\nu = 1.1737$, $\theta = -0.1280$.
The values show that the relative errors of the bounds are comparable to the ones of the Black-Scholes model. The prices for the plain vanillas and needed contingent claims were in this case obtained by the Fast Fourier Transform approach of Carr & Madan [7].
Note that the lower bounds are sharper if the averaging starts at a later point during the lifetime of the option (forward-start options). At the same time, the performance of the bounds gets weaker if the number of averaging days increases.

Finally, we consider an illustration of the bound of Section 3, where only a finite number of option prices is available in the market (which is a main field of application of the bounds developed in this paper). Table 9 shows the bounds for an Asian option on the S&P 500 index with monthly averaging, maturity $T = 1$ year and $S_0 = 1124.47$. The bounds are based on the 77 European call prices available on April 18, 2002 (see Schoutens [24]). LB_1 provides the lower bound solely based on no-arbitrage. If there is evidence to believe that assumptions (7), (11) (or assumptions (16), (17)) hold, then the bounds $\mathrm{LB}^{(1)}$, $\mathrm{LB}^{(2)}$ apply respectively. The columns in the table give $\mathrm{LB}_t^{(1)}$ for all averaging days t. As can be seen from the table, for $\mathrm{LB}^{(1)}$ the best lower bound (bold-faced in the table) is achieved for $t = 5$, while for $\mathrm{LB}^{(2)}$ it is achieved for either $t = 5$ or $t = 8$. Finally, the three columns on the right give Monte Carlo estimates for the actual AC price based on a model specification (Black Scholes, Variance Gamma and Heston model) and calibration of each model to the available call prices using a least-square approach (see [24]). Especially when the option is not far out of the money, the lower bounds seem to be quite satisfying, keeping in mind that these bounds are model-independent and purely based on quoted European call prices.

Acknowledgement. The authors would like to thank two anonymous referees for many helpful remarks to improve the presentation of the paper.

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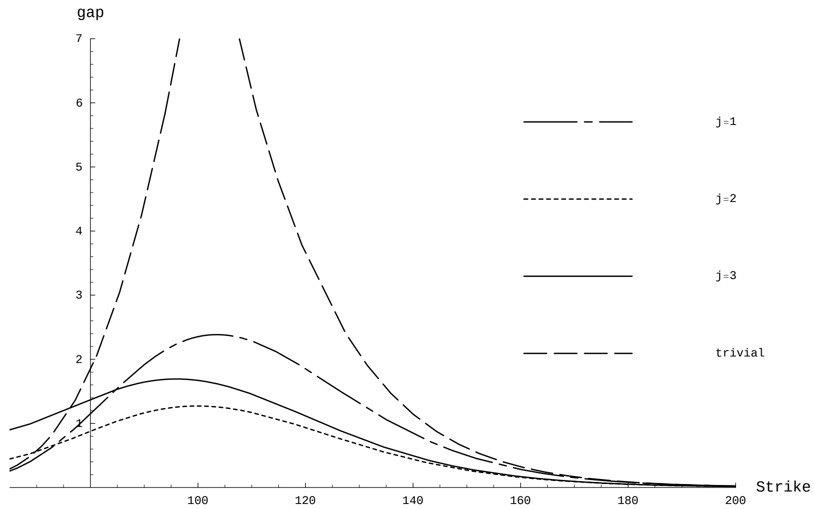

Figure 2: Comparison of some different $\mathrm{LB}_j^{(2)}$, where $j = t_j$ is a monitoring time

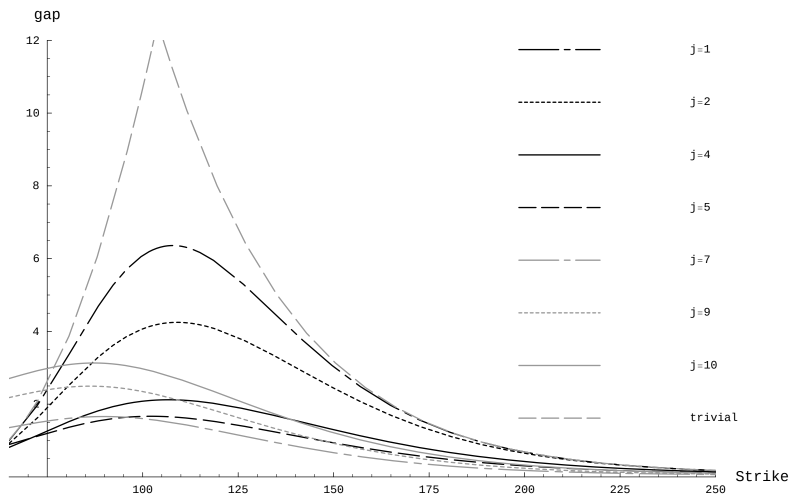

Figure 3: Comparison of some $\mathrm{LB}_j^{(2)}$, where $j = t_j$ is a monitoring time

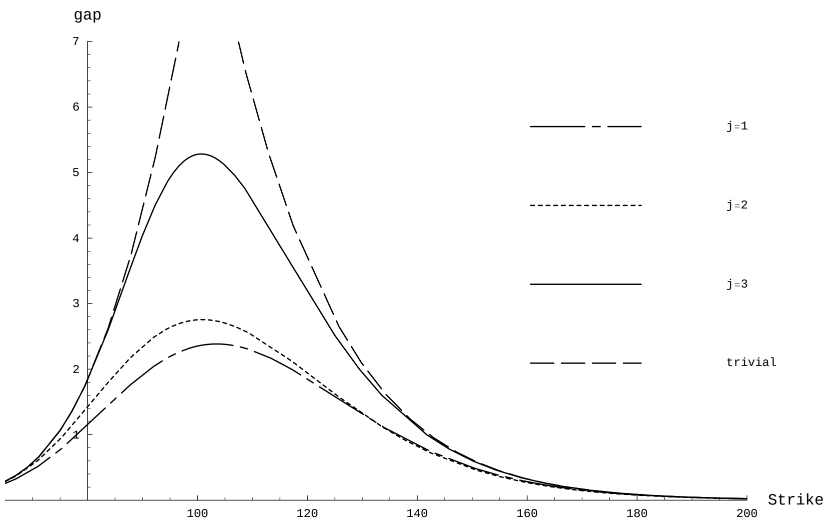

Figure 4: Comparison of different $\mathrm{LB}_j^{(1)}$, where $j = t_j$ is a monitoring time

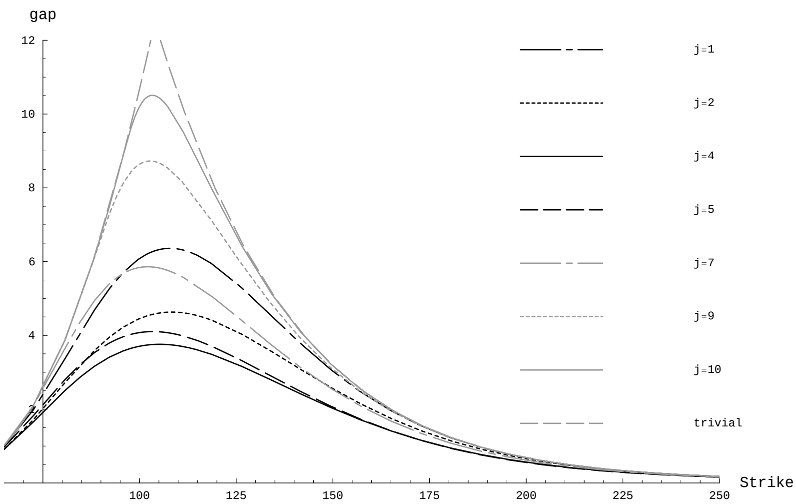

Figure 5: Comparison of different $\mathrm{LB}_j^{(1)}$, where $j = t_j$ is a monitoring time

time to maturity 120 days										
σ	strike	trivial	LB_1	$\mathrm{LB}_t^{(1)}$	t	$\mathrm{LB}B_T$	$\mathrm{LB}_t^{(2)}$	t	$\mathrm{LB}GA$	MC
0.2	80	21.9755	**21.9934**	21.9934	1	21.9948	**21.9947**	4	22.0026	22.00271
	90	12.2671	**12.6771**	12.6771	1	12.6918	**12.725**	13	12.7601	12.76012
	100	2.5586	**5.3285**	5.3285	1	5.3650	**5.4633**	15	5.5217	5.521652
	110	0	**1.4860**	1.4860	1	1.5183	**1.6069**	16	1.6528	1.652697
0.3	80	21.9755	**22.2384**	22.2384	1	22.2502	**22.2667**	9	22.3097	22.30976
	90	12.2671	**13.7284**	13.7284	1	13.7636	**13.8464**	13	13.9246	13.92461
	100	2.5586	**7.2401**	7.2401	1	7.2957	**7.4390**	14	7.5347	7.534506
	110	0	**3.2339**	3.2339	1	3.289	**3.4338**	15	3.5175	3.517352
0.4	80	21.9755	**22.866**	22.866	1	22.8945	**22.9452**	11	23.0348	23.03488
	90	12.2671	**15.1176**	15.1176	1	15.1727	**15.2998**	13	15.4238	15.42367
	100	2.5586	**9.1699**	9.1699	1	9.2444	**9.4277**	14	9.5641	9.563843
	110	0	**5.1232**	5.1232	1	5.1997	**5.3924**	15	5.5176	5.517215

Table 2: Lower bounds in [27] vs. lower bounds in this paper for an Asian call with $S_0 = 100$ and averaging days $t_1 = 91, t_2 = 92 \ldots, t_{30} = 120$

time to maturity 10 years										
strike	trivial	LB_1	$\mathrm{LB}_t^{(1)}$	t	GA	$\mathrm{LB}_t^{(2)}$	t	Ni & Sa	MC	UCB
60	42.3382	42.3382	**42.3585**	30	38.4767	**42.3382**	1	42.8949	42.9428	43.4824
70	35.635	35.635	**35.7897**	33	32.7909	**35.9635**	43	37.0393	37.0906	37.9911
80	28.9318	28.9318	**29.5498**	35	27.6674	**30.3629**	53	31.7404	31.7916	33.0538
100	15.5254	15.5285	**18.992**	39	19.2427	**21.3014**	65	22.9595	23.0116	24.8433
110	8.8222	8.9639	**14.914**	41	15.9132	**17.7675**	68	19.4444	19.4942	21.5084
120	2.1190	3.5560	**11.618**	43	13.1114	**14.8071**	71	16.4497	16.4999	18.6246
130	0	0.8042	**9.0092**	44	10.7756	**12.3422**	73	13.9148	13.9705	16.1398
140	0	0.0963	**6.9745**	46	8.8413	**10.2976**	75	11.7776	11.8417	14.0029
150	0	0.0063	**5.4005**	47	7.2469	**8.6047**	77	9.9796	10.0514	12.1666
160	0	0.0002	**4.1890**	48	5.9369	**7.2044**	78	8.468	8.5427	10.5993
180	0	$\sim 10^{-7}$	**2.5437**	50	3.9834	**5.0852**	80	6.1275	6.2062	8.0615
200	0	$\sim 10^{-11}$	**1.5698**	52	2.676	**3.6264**	82	4.4662	4.5457	6.182

Table 3: Lower bounds of Vorst and Nielsen & Sandmann vs. lower bound in this paper for an Asian call for volatility $\sigma = 0.25$, $r = 0.04$, $S_0 = 100$ and monthly averaging

time to maturity 3 years										
strike	trivial	LB_1	$\mathrm{LB}_t^{(1)}$	t	GA	$\mathrm{LB}_t^{(2)}$	t	Ni & Sa	MC	UCB
60	41.1749	41.1749	**41.1751**	9	39.7469	**41.1749**	1	41.2278	41.2315	41.3456
70	32.3057	32.3057	**32.3225**	10	31.2566	**32.3057**	1	32.6569	32.6621	33.0345
80	23.4365	23.4365	**23.6676**	11	23.4654	**24.0987**	17	24.7471	24.754	25.5085
100	5.69807	6.4074	**9.603**	12	11.4543	**11.6059**	22	12.4743	12.4799	13.854
110	0	1.4616	**5.3378**	13	7.4996	**7.5681**	23	8.383	8.3887	9.8328
120	0	0.1444	**2.7642**	13	4.7399	**4.782**	24	5.4825	5.4897	6.8529
130	0	0.0063	**1.3601**	14	2.9064	**2.952**	24	3.5097	3.5187	4.709
140	0	0.0001	**0.6454**	14	1.7355	**1.7893**	24	2.2082	2.2153	3.1997
150	0	$\sim 10^{-6}$	**0.2992**	15	1.0131	**1.0726**	25	1.3703	1.3787	2.1548
160	0	$\sim 10^{-8}$	**0.1372**	15	0.5808	**0.6375**	25	0.8417	0.8507	1.4413
180	0	$\sim 10^{-13}$	**0.0285**	16	0.1843	**0.2226**	26	0.3125	0.3196	0.6373
200	0	0	**0.006**	16	0.0573	**0.0778**	26	0.1159	0.1214	0.2811

Table 4: Lower bounds of Vorst and Nielsen & Sandmann vs. lower bound in this paper for an Asian call for volatility $\sigma = 0.25$, interest rate $r = 0.04$, $S_0 = 100$ and monthly averaging

time to maturity 120 days							
strike	trivial	LB_1	$LB_t^{(1)}$	t	$LB_t^{(2)}$	t	MC
80	21.9755	**22.0319**	22.0319	1	**22.0432**	8	22.1182
90	12.2671	**12.6074**	12.6074	1	**12.7385**	17	12.7385
100	2.5586	**4.2677**	4.2677	1	**4.3735**	18	4.4503
110	0	**0.3219**	0.3219	1	**0.3723**	17	0.3964

Table 5: Lower bounds in this paper for an Asian call in the Heston model with $r = \ln(1 + 0.09/365)$, $S_0 = 100$ and averaging days $t_1 = 91, t_2 = 92 \ldots, t_{30} = 120$. MC simulation: 10.000 iterations

time to maturity 10 years							
strike	trivial	LB_1	$LB_t^{(1)}$	t	$LB_t^{(2)}$	t	MC
60	42.0529	42.0529	**42.1551**	27	**42.2450**	34	42.5627
70	34.6447	34.6447	**34.9642**	32	**35.4483**	47	35.7649
80	27.2365	27.2365	**27.9825**	35	**28.8769**	56	29.3738
90	19.8283	19.8285	**21.3838**	39	**22.9370**	65	23.5116
100	12.4201	12.4272	**15.3951**	42	**17.5832**	72	18.3161
110	5.01196	5.2108	**10.2893**	45	**13.0323**	77	13.8686
120	0	0.3465	**6.3135**	48	**9.3466**	82	10.1925
130	0	0..0009	**3.5602**	50	**6.5047**	85	7.2785
140	0	$\sim 10^{-6}$	**1.8819**	52	**4.44041**	87	5.08126
150	0	$\sim 10^{-9}$	**0.9637**	53	**2.9128**	89	3.4924

Table 6: Lower bounds in this paper for an Asian call under the Heston model with $r = 0.03$, $S_0 = 100$ and monthly averaging. MC simulation: 10.000 iterations

time to maturity 120 days							
strike	trivial	LB_1	$LB_t^{(1)}$	t	$LB_t^{(2)}$	t	MC
80	21.9755	**22.7409**	22.7409	1	**22.7788**	10	22.9693
90	12.2671	**13.8060**	13.8060	1	**13.9086**	14	14.0586
100	2.5586	**5.6593**	5.6593	1	**5.9080**	20	6.0311
110	0	**1.4063**	1.4063	1	**1.5562**	19	1.6146

Table 7: Lower bounds in this paper for an Asian call in the Variance Gamma model with interest rate $r = \ln(1 + 0.09/365)$, $S_0 = 100$ and averaging days $t_1 = 91, t_2 = 92 \ldots, t_{30} = 120$. MC simulation: 10.000 iterations

time to maturity 10 years							
strike	trivial	LB_1	$LB_t^{(1)}$	t	$LB_t^{(2)}$	t	MC
60	42.0529	42.0737	**42.6735**	67	**42.6644**	34	44.5242
70	34.6447	34.6935	**35.5534**	29	**36.3172**	46	38.4817
80	27.2365	27.3393	**29.1556**	33	**30.6444**	54	33.0227
90	19.8283	20.0333	**23.3909**	37	**25.6772**	61	28.1534
100	12.4201	12.8175	**18.3860**	40	**21.4012**	66	23.8859
110	5.01196	5.7925	**14.1998**	43	**17.7698**	70	20.1845
120	0	0.6096	**10.8219**	45	**14.7195**	73	17.0249
130	0	0.2557	**8.1788**	47	**12.1783**	76	14.3239
140	0	0.1315	**6.1618**	49	**10.0757**	78	12.0429
150	0	0.0743	**4.6493**	50	**8.3415**	80	10.1395

Table 8: Lower bounds in this paper for an Asian call under the Variance Gamma model with $r = 0.03$, $S_0 = 100$ and monthly averaging. MC simulation: 10.000 iterations

time to maturity 1 year													
strike	trivial	LB_1	$LB_2^{(1)}$	$LB_5^{(1)}$	$LB_8^{(1)}$	$LB_{11}^{(1)}$	$LB_2^{(2)}$	$LB_5^{(2)}$	$LB_8^{(2)}$	$LB_{11}^{(2)}$	BS	VG	Hest
1000	127.846	127.846	127.846	**129.85**	127.846	127.846	124.443	**130.31**	124.191	79.7406	134.433	140.1680	138.9464
1050	78.1947	78.1947	84.5217	**88.021**	83.6685	78.1947	83.9806	87.1558	**87.4464**	66.3315	95.1107	100.0934	98.8784
1100	28.5435	37.7177	44.5938	**50.1769**	44.4893	31.7518	44.7912	52.6707	**52.7307**	47.364	64.9239	64.8947	63.8541
1150	0	10.3623	17.8382	**22.206**	16.6089	4.46463	18.3689	25.6062	**26.6335**	24.1314	39.7619	36.5324	36.0124
1200	0	0	3.75691	**7.17759**	3.32482	0	4.4824	**10.1533**	9.9608	7.00065	23.0376	17.4360	16.7947
1250	0	0	0	0	0	0	0	0.476165	**3.13161**	1.55609	12.6378	8.1695	6.2069

Table 9: Lower bounds for Asian calls solely based on traded calls (maturity $T = 1$ year, monthly averaging) on the S&P